Field theory from a bundle point of view

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Contents

Chapter 1

Differential geometry

1.1 Differentiable manifolds

More precisions in [1], chapter II (§1 and 2) and III. (§1, 2, 3 and 7); [2, 3] can also be useful. An other source for this chapter is [4]. A systematic exposition of manifolds and such can be found in [5].

1.1.1 Definition and examples

A n-dimensional **differentiable manifold** is a set M and a system of charts $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ where each set $\mathcal{U}_\alpha$ is open in $\mathbb{R}^n$ and the maps $\varphi_\alpha\colon \mathcal{U}_\alpha \to M$ are injective and satisfy the three following conditions :

- every $x \in M$ is contained in at least one set $\varphi_\alpha(\mathcal{U}_\alpha)$,
- for any two charts $\varphi_\alpha\colon \mathcal{U}_\alpha \to M$ and $\varphi_\beta\colon \mathcal{U}_\beta \to M$, the set
$$\varphi_\alpha^{-1}(\varphi_\alpha(\mathcal{U}_\alpha) \cap \varphi_\beta(\mathcal{U}_\beta))$$
is an open subset of $\mathcal{U}_\alpha$,
- the map
$$(\varphi_\beta^{-1} \circ \varphi_\alpha)\colon \varphi_\alpha^{-1}(\varphi_\alpha(\mathcal{U}_\alpha) \cap \varphi_\beta(\mathcal{U}_\beta)) \to \mathcal{U}_\beta$$
is differentiable[1] as map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Each time we say "manifold", we mean "differentiable manifold". We will only consider manifolds with Hausdorff topology (see later for the definition of a topology on a manifold). Any open set of $\mathbb{R}^n$ is a differentiable manifold if we choose the identity map as chart system. Most of surfaces $z = f(x, y)$ in $\mathbb{R}^3$ are manifolds, depending on certain regularity conditions on f.

If M_1 and M_2 are two differentiable manifolds, a map $f\colon M_1 \to M_2$ is **differentiable** if f is continuous and for each two coordinate systems $\varphi_1\colon \mathcal{U}_1 \to M_1$ and $\varphi_2\colon \mathcal{U}_2 \to M_2$, the map $\varphi_2^{-1} \circ f \circ \varphi_1$ is differentiable on its domain. One can show that if $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$ are differentiable, then $g \circ f\colon M_1 \to M_3$ is differentiable.

[1] In the sequel, by "differentiable" we always mean smooth. If this map is differentiable, C^k, analytic,... then the manifold is said to be differentiable, C^k, analytic,...

Example: the sphere

The sphere S^n is the set

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \text{ st } \|x\| = 1\}$$

for which we consider the following open set in $\mathbb{R}^n$:

$$\mathcal{U} = \{(u_1, \ldots, u_n) \in \mathbb{R}^n \text{ st } \|u\| < 1\}$$

and the charts $\varphi_i : \mathcal{U} \to S$, and $\tilde{\varphi}_i : \mathcal{U} \to S$

$$\varphi_i(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, \sqrt{1 - \|u\|^2}, u_i, \ldots, u_n) \tag{1.1a}$$
$$\tilde{\varphi}_i(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, -\sqrt{1 - \|u\|^2}, u_i, \ldots, u_n). \tag{1.1b}$$

These map are clearly injective. To see that $\varphi(\mathcal{U}) \cup \tilde{\varphi}(\mathcal{U}) = S$, consider $(x_1, \ldots, x_{n+1}) \in S$. Then at least one of the x_i is non zero. Let us suppose $x_1 \neq 0$, thus $x_1^2 = 1 - (x_2^2 + \ldots + x_{n+1}^2)$ and

$$x_1 = \pm\sqrt{1 - (\ldots)}. \tag{1.2}$$

If we put $u_i = x_{i+1}$, we have $x = \varphi(u)$ or $x = \tilde{\varphi}(u)$ following the sign in relation (1.2)-10. The fact that $\varphi^{-1} \circ \tilde{\varphi}$ and $\tilde{\varphi}^{-1} \circ \varphi$ are differentiable is a "first year in analysis exercise".

Example: projective space

On $\mathbb{R}^{n+1} \backslash \{o\}$, we consider the equivalence relation $v \sim \lambda w$ for all non zero $\lambda \in \mathbb{R}$, and we put

$$\mathbb{R}P^n = \left(\mathbb{R}^{n+1} \backslash \{o\}\right) / \sim .$$

This is the set of all the one dimensional subspaces of $\mathbb{R}^{n+1}$. This is the real **projective space** of dimension n. We set $\mathcal{U} = \mathbb{R}^n$ and

$$\varphi_i(u_1, \ldots, u_n) = \operatorname{Span}\{(u_1, \ldots, u_{i-1}, 1, u_i, \ldots, u_n)\}.$$

One can see that this gives a manifold structure to $\mathbb{R}P^n$. Moreover, the map

$$\begin{aligned} A \,:\, S^n &\to \mathbb{R}P^n \\ v &\mapsto \operatorname{Span} v \end{aligned} \tag{1.3}$$

is differentiable.

Let us show how to identify $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R}P^1$, the set of directions in the plane $\mathbb{R}^2$. Indeed consider any vertical line l (which does contain the origin). A non vertical vector subspace of $\mathbb{R}^2$ intersects l in one and only one point, while the vertical vector subspace is associated with the infinite point.

1.1.2 Topology on manifold and submanifold

A subset $V \subset M$ is **open** if for every chart $\varphi : \mathcal{U} \to M$, the set $\varphi^{-1}(V \cap \varphi(\mathcal{U}))$ is open in $\mathcal{U}$.

Theorem 1.1.
This definition gives a topology on M which has the following properties :

(i) the charts maps are continuous,

(ii) the sets $\varphi_\alpha(\mathcal{U}_\alpha)$ are open.

Proof. First we prove that the open system defines a topology. For this, remark that φ_α^{-1} is injective (if not, there should be some multivalued points). Then $\varphi^{-1}(A\cap B) = \varphi^{-1}(A)\cap\varphi^{-1}(B)$. If V_1 and V_2 are open in M, then

$$\varphi^{-1}(V_1 \cap V_2 \cap \varphi(\mathcal{U})) = \varphi^{-1}(V_1 \cap \varphi(\mathcal{U})) \cap \varphi^{-1}(V_2 \cap \varphi(\mathcal{U}))$$

which is open in $\mathbb{R}^n$. The same property works for the unions.

Now we turn our attention to the continuity of $\varphi\colon \mathcal{U} \to M$; for an open set V in M, we have to show that $\varphi^{-1}(V)$ is open in $\mathcal{U} \subset \mathbb{R}^n$. But the definition of the topology on M, is precisely the fact that $\varphi^{-1}(V \cap \varphi(\mathcal{U}))$ is open. □

If M is a differentiable manifold and N, a subset of M, we say that N is a **submanifold** of dimension k if $\forall\, p \in N$, there exists a chart $\varphi\colon \mathcal{U} \to M$ around p such that

$$\varphi^{-1}(\varphi(\mathcal{U}) \cap N) = \mathbb{R}^k \cap \mathcal{U} := \{(x_1, \ldots, x_k, 0 \ldots, 0) \in \mathcal{U}\}.$$

In this case, N is itself a manifold of dimension k for which one can choose the φ of the definition as charts.

Let us consider M and N, two differentiable manifolds, $f\colon M \to N$ a C^∞ map and $x \in M$. We say that f is an **immersion** at x if $df_x\colon T_xM \to T_{f(x)}N$ is injective and that f is a **submersion** if df_x is surjective.

If M and M are two analytic manifolds, a map $\phi\colon M \to N$ is **regular** at $p \in M$ if it is analytic at p and $d\phi_p\colon T_pM \to T_{\phi(p)}N$ is injective.

Proposition 1.2.
Let M be a submanifold of the manifold N. If $p \in M$, then there exists a coordinate system $\{x_1, \ldots, x_n\}$ on a neighbourhood of p in N such that $x_1(p) = \ldots = x_n(p) = 0$ and such that the set

$$U = \{q \in V \text{ st } x_j(q) = 0\, \forall\, m + 1 \leqslant j \leqslant n\}$$

gives a local chart of M containing p.

Proof. No proof. □

The sense of this proposition is that one can put p at the center of a coordinate system on N such that M is just a submanifold of N parametrised by the fact that its last $m - n$ components are zero.

Now we can give a characterization for a submanifold: N is a submanifold of M when $N \subset M$ (as set) and the identity $\iota\colon N \to M$ is regular.

Proposition 1.3.
The own topology of a submanifold is finer than the induced one from the manifold.

Proof. Let M be a manifold of dimension n and N a submanifold[2] of dimension $k < n$. We consider V, an open subset of N for the induced topology, so $V = N \cap \mathcal{O}$ for a certain open subset $\mathcal{O}$ of M. The aim is to show that V is an open subset in the topology of N.

Let us define $\mathcal{P} = \varphi^{-1}(\mathcal{O})$. The charts of N are the projection to $\mathbb{R}^k$ of the ones of M. We have to consider $W = \varphi^{-1}(V)$, since N is a submanifold, $\varphi^{-1}(\mathcal{O} \cap N) = \mathbb{R}^k \cap \mathcal{P}$. It is clear that

[2] In the whole proof, we should say "there exists a sub-neighbourhood such that…"

$W = \mathbb{R}^k \cap \mathcal{P}$ is an open subset of $\mathbb{R}^k$ because it is the projection on the k first coordinates of an open subset of $\mathbb{R}^n$.

The subset V of N will be open in the sense of the own topology of N if $\varphi'^{-1}(V \cap \varphi'(\mathcal{U}'))$ is open in $\mathbb{R}^k$ where φ' is the restriction of φ to his k first coordinates: $\varphi'(a) = \varphi(a, 0)$ and $\mathcal{U}'$ is the projection of $\mathcal{U}$. □

Lemma 1.4.
Let V, M be two manifolds and $\varphi\colon V \to M$, a differentiable map. We suppose that $\varphi(V)$ is contained in a submanifold S of M. If $\varphi\colon V \to S$ is continuous, then it is differentiable.

Remark 1.5. The map φ is certainly continuous as map from V to M (this is in the assumptions). But this don't implies that it is continuous for the topology on S (which is the induced one from M). So the continuity of $\varphi\colon V \to S$ is a true assumption.

Proof. Let $p \in V$. By proposition 1.2, we have a coordinate system $\{x_1, \ldots, x_m\}$ valid on a neighbourhood N of $\varphi(p)$ in M such that the set

$$\{r \in N \text{ st } x_j(r) = 0 \,\forall s < j \leqslant m\}$$

with the restriction of $(x_1, \ldots x_s) \in N_S$ form a local chart which contains $\varphi(p)$. From the continuity of φ, there exists a chart (W, ψ) around p such that $\varphi(W) \subset N_S$. The coordinates $x_j(\varphi(q))$ are differentiable functions of the coordinates of q in W. In particular, the coordinates $x_j(\varphi(q))$ for $1 \leqslant j \leqslant s$ are differentiable and $\varphi\colon V \to S$ is differentiable because its expression in a chart is differentiable. □

A consequence of this lemma: if V and S are submanifolds of M with $V \subset S$, and if S has the induced topology from M, then V is a submanifold of S. Indeed, we can consider the inclusion $\iota\colon V \to S$: it is differentiable from V to M and continuous from V to S then it is differentiable from V to S by the lemma. Thus $V = \iota^{-1}(S)$ is a submanifold of S (this is a classical result of differential geometry).

Proposition 1.6.
A submanifold is open if and only if it has the same dimension as the main manifold.

Proof. Necessary condition. We consider some charts $\varphi_i\colon U_i \to M$ on some open subsets U_i of $\mathbb{R}^n$. If N is open in M, then this can be written as

$$N = \bigcup_i U_i.$$

If we choose the charts on M in such a manner that $\varphi_i\colon U_i \cap \mathbb{R}^k \to N$ are charts of N, we must have $\varphi_i(U_i \cap \mathbb{R}^k) = \varphi_i(U_i)$. Then it is clear that $k = n$ is necessary.
Sufficient condition. If N has same dimension as M, the charts $\varphi_i\colon U_i \to M$ are trivially restricted to N. □

1.1.3 Tangent vector

As first attempt, we define a tangent vector of M at the point $x \in M$ as the "derivative" of a path $\gamma\colon (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = x$. It is denoted by

$$\gamma'(0) = \frac{d}{dt}\gamma(t)\Big|_{t=0}.$$

The question is to correctly define de derivative in the right hand side. Such a definition is achieved as follows. A **tangent vector** to the manifold M is a linear map $X\colon C^\infty(M)\to\mathbb{R}$ which can be written under the form

$$Xf=(f\circ X)'(0)=\frac{d}{dt}\Big[f(X(t))\Big]_{t=0} \tag{1.4}$$

for a certain path $X\colon\mathbb{R}\to M$. Notice the abuse of notation between the tangent vector and the path which defines it.

A more formal way to define a tangent vector is to say that it is an equivalent class of path in the sense that two path are equivalent if and only if they induced maps by (1.4)-13 are equals.

Using the chain rule $d(g\circ f)(a)=dg(f(a))\circ df(a)$ for the differentiation in $\mathbb{R}^n$, one sees that this equivalence notion doesn't depend on the choice of φ. In other words, if φ and $\tilde\varphi$ are two charts for a neighbourhood of x, then $(\varphi^{-1}\circ\gamma)'(0)=(\varphi^{-1}\circ\sigma)'(0)$ if and only if $(\tilde\varphi^{-1}\circ\gamma)'(0)=(\tilde\varphi^{-1}\circ\sigma)'(0)$. The space of all tangent vectors at x is denoted by T_xM. There exists a bijection $[\gamma]\leftrightarrow(\varphi^{-1}\circ\gamma)'(0)$ between T_xM and $\mathbb{R}^n$, so T_xM is endowed with a vector space structure.

If $(\mathcal{U},\varphi)$ is a chart around $X(0)$, we can express Xf using only well know objects by defining the function $\tilde f=f\circ\varphi$ and $\tilde X=\varphi^{-1}\circ X$

$$Xf=\frac{d}{dt}\Big[(\tilde f\circ\tilde X)(t)\Big]_{t=0}=\frac{\partial\tilde f}{\partial x^\alpha}\Big|_{x=\tilde X(0)}\frac{d\tilde X^\alpha}{dt}\Big|_{t=0}.$$

In this sense, we write

$$X=\frac{d\tilde X^\alpha}{dt}\frac{\partial}{\partial x^\alpha} \tag{1.5}$$

and we say that $\{\partial_1,\ldots,\partial_n\}$ is a basis of T_xM. As far as notations are concerned, from now a tangent vector is written as $X=X^\alpha\partial_\alpha$ where X^α is related to the path $X\colon\mathbb{R}\to M$ by $X^\alpha=d\tilde X^\alpha/dt$. We will no more mention the chart φ and write

$$Xf=\frac{d}{dt}\Big[f(X(t))\Big]_{t=0}.$$

Correctness of this short notation is because the equivalence relation is independent of the choice of chart. When we speak about a tangent vector to a given path $X(t)$ without specification, we think about $X'(0)$.

All this construction gives back the notion of tangent vector when $M\subset\mathbb{R}^m$. In order to see it, think to a surface in $\mathbb{R}^3$. A tangent vector is precisely given by a derivative of a path: if $c\colon\mathbb{R}\to\mathbb{R}^n$ is a path in the surface, a tangent vector to this curve is given by

$$\lim_{t\to0}\frac{c(t_0)-c(t_0+t)}{t}$$

which is a well know limit of a difference in $\mathbb{R}^n$.

Let us precise how does a tangent vector acts on maps others than $\mathbb{R}$-valued functions. If V is a vector space and $f\colon M\to V$, we define

$$Xf=(Xf^i)e_i$$

where $\{e_i\}$ is a basis of V and the functions $f^i\colon M\to\mathbb{R}$, the decomposition of f with respect to this basis. If we consider a map $\varphi\colon M\to N$ between two manifolds, the natural definition is $Xf:=dfX$. More precisely, if we consider local coordinates x^α and a function $f\colon M\to\mathbb{R}$,

$$(d\varphi X)f=\frac{d}{dt}\Big[(f\circ\varphi\circ X)(t)\Big]_{t=0}=\frac{\partial f}{\partial x^\alpha}\frac{\partial\varphi^\alpha}{\partial x^\beta}\frac{dX^\beta}{dt}. \tag{1.6}$$

Now we are in a notational trouble: when we write $X = X^\alpha \partial_\alpha$, the "$X^\alpha$" is the derivative of the "X^α" which appears in the path $X(t) = (X^1(t), \ldots, X^n(t))$ which gives X by $X = X'(0)$. So equation (1.6)-13 gives

$$X(\varphi) := d\varphi X = X^\beta(\partial_\beta \varphi^\alpha)\partial_\alpha. \tag{1.7}$$

1.1.4 Differential of a map

Let $f\colon M_1 \to M_2$ be a differentiable map, $x \in M_1$ and $X \in T_xM_1$, i.e. $X\colon \mathbb{R} \to M_1$ with $X(0) = x$ and $X'(0) = X$. We can consider the path $Y = f \circ X$ in M_2. The tangent vector to this path is written $df_x X$.

Proposition 1.7.
If $f\colon M_1 \to M_2$ is a differentiable map between two differentiable manifolds, the map

$$\begin{aligned} df_x \,:\, T_xM_1 &\to T_{f(x)}M_2 \\ X'(0) &\mapsto (f \circ X)'(0) \end{aligned} \tag{1.8}$$

is linear.

Proof. We consider local coordinates $x\colon \mathbb{R}^n \to M_1$ and $y\colon \mathbb{R}^m \to M_2$. The maps $f\colon M_1 \to M_2$ and $y^{-1} \circ f \circ x\colon \mathbb{R}^n \to \mathbb{R}^m$ will sometimes be denoted by the same symbol f. We have $(x^{-1} \circ X)(t) = (x_1(t), \ldots, x_n(t))$ and $(y^{-1} \circ Y)(t) = \big(y_1(x_1(t), \ldots, x_n(t), \ldots, y_m(x_1(t), \ldots, x_n(t)\big)$, so that

$$Y'(0) = \left(\sum_{i=1}^n \frac{\partial y_1}{\partial x_i} x_i'(0), \ldots, \sum_{i=1}^n \frac{\partial y_m}{\partial x_i} x_i'(0) \right) \in \mathbb{R}^m$$

which can be written in a more matricial way under the form

$$Y'(0) = \left(\frac{\partial y_i}{\partial x_j} x_j'(0) \right).$$

So in the parametrisations x and y, the map df_x is given by the matrix $\partial y^i / \partial x_j$ which is well defined from the only given of f. □

Let $x\colon \mathcal{U} \to M$ and $y\colon \mathcal{V} \to M$ be two charts systems around $p \in M$. Consider the path $c(t) = x(0, \ldots, t, \ldots 0)$ where the t is at the position k. Then, with respect to these coordinates,

$$c'(0)f = \frac{d}{dt}\Big[f(c(t)) \Big]_{t=0} = \frac{\partial f}{\partial x^i} \frac{dc^i}{dt} = \frac{\partial f}{\partial x^k},$$

so $c'(0) = \partial/\partial x^k$. Here, implicitly, we wrote $c^i = (x^i)^{-1} \circ c$ where $(x^i)^{-1}$ is the ith component of x^{-1} seen as element of $\mathbb{R}^n$. We can make the same computation with the system y. With these abuse of notation,

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \tag{1.9}$$

as it can be seen by applying it on any function $f\colon M \to \mathbb{R}$. More precisely if $x\colon \mathcal{U} \to M$ and $y\colon \mathcal{U} \to M$ are two charts (let $\mathcal{U}$ be the intersection of the domains of x and y), let $f\colon M \to \mathbb{R}$ and $\overline{f} = f \circ x$, $\tilde{f} = f \circ y$. The action of the vector ∂_{x^i} of the function f is given by

$$\partial_{x^i} f = \frac{\partial \overline{f}}{\partial x^i}$$

where the right hand side is a real number that can be computed with usual analysis on $\mathbb{R}^n$. This real *defines* the left hand side. Now, $\overline{f} = \tilde{f} \circ y^{-1} \circ x$, so that

$$\frac{\partial \overline{f}}{\partial x^i} = \frac{\partial(\tilde{f} \circ y^{-1} \circ x)}{\partial x^i} = \frac{\partial \tilde{f}}{\partial y^j} \frac{\partial y^j}{\partial x^i}$$

where $\frac{\partial \tilde{f}}{\partial y^j}$ is precisely what we write now by $\partial_{y^j} f$ and $\frac{\partial y^j}{\partial x^i}$ must be understood as the derivative with respect to x^i of the function $(y^{-1} \circ x)\colon \mathbb{R}^n \to \mathbb{R}^n$.

Let $f\colon M \to N$ and $g\colon N \to \mathbb{R}$; the definitions gives

$$(df_x X)g = \frac{d}{dt}\Big[(g \circ f)(X(t))\Big]_{t=0} = \frac{\partial g}{\partial y^i} \frac{\partial f^i}{\partial x^\alpha} \frac{dX^\alpha}{dt}.$$

This shows that $\frac{\partial f^i}{\partial x^\alpha} \frac{dX^\alpha}{dt}$ is $(df_x X)^i$. But dX^α/dt is what we should call X^α in the decompositon $X = X^\alpha \partial_\alpha$ then the matrix of df is given by $\frac{\partial f^i}{\partial x^\alpha}$. So we find back the old notion of differential.

Remark 1.8. If $X \in T_x M$ and f is a *vector valued* function on M, then one can define Xf by exactly the same expression. In this case,

$$df X = \frac{d}{dt}\Big[f(v(t))\Big]_{t=0} = Xf.$$

A map $f\colon M_1 \to M_2$ is an **immersion** at $p \in M_1$ if $df_p\colon T_p M_1 \to T_{f(p)} M_2$ is injective. It is a **submersion** if df_p is surjective.

1.1.5 Tangent and cotangent bundle

Tangent bundle

If M is a n dimensional manifold, as set the tangent bundle is the *disjoint* union of tangent spaces

$$TM = \bigcup_{x \in M} T_x M.$$

Theorem 1.9.
The tangent bundle admits a $2n$ dimensional manifold structure for which the projection

$$\begin{aligned} \pi : TM &\to M \\ T_p M &\mapsto p \end{aligned} \tag{1.10}$$

is a submersion.

The structure is easy to guess. If $\varphi_\alpha\colon \mathcal{U}_\alpha \to M$ is a coordinate system on M (with $\mathcal{U}_\alpha \subset \mathbb{R}^n$), we define $\psi_\alpha\colon \mathcal{U}_\alpha \times \mathbb{R}^n \to TM$ by

$$\psi(\underbrace{x_1, \ldots x_n}_{\in \mathcal{U}_\alpha}, \underbrace{a_1, \ldots a_n}_{\in \mathbb{R}^n}) = \sum_i a_i \left.\frac{\partial}{\partial x_i}\right|_{\varphi(x_1, \ldots, x_n)}.$$

The map $\psi_\beta^{-1} \circ \psi_\beta$ is differentiable because

$$(\psi_\beta^{-1} \circ \psi_\beta)(x, a) = (y(x), \sum_i a_i \left.\frac{\partial y_j}{\partial x_i}\right|_{y(x)})$$

which is a composition of differentiable maps. The set TM endowed with this structure is called the **tangent bundle**.

Commutator of vector fields

If X, $Y \in \mathfrak{X}(M)$, one can define the **commutator** $[X,Y]$ in the following way. First remark that, if $f\colon M \to \mathbb{R}$, the object $X(f)$ is also a function from M to $\mathbb{R}$ by $X(f)(x) = X_x(f)$, so we can apply Y on $X(f)$. The definition of $[X,Y]_x$ is

$$[X,Y]_x f = X_x(Yf) - Y_x(Xf). \tag{1.11}$$

If $X = X^i\partial_i$ and $Y = Y^j\partial_j$, then $XY(f) = X^i\partial_i(Y^j\partial_j f) = X^i\partial_i Y^j\partial_j f + X^iY^j\partial^2_{ij}f$. From symmetry $\partial^2_{ij}f = \partial^2_{ji}f$, the difference $XYf - YXf$ is only $X^i\partial_i Y^j - Y^i\partial_i X^j$, so that

$$[X,Y]^i = XY^i - YX^i \tag{1.12}$$

where X^i and Y^i are seen as functions from M to $\mathbb{R}$.

Some Leibnitz formulas

See [1], chapter I, proposition 1.4.

Lemma 1.10.
If M and N are two manifolds, we have a canonical isomorphism

$$T_{(p,q)}(M\times N) \simeq T_pM + T_qN.$$

Proof. A $Z \in T_{(p,q)}(M\times N)$ is the tangent vector to a curve $(x(t), y(y))$ in $M\times N$. We can consider $X \in T_pM$ given by $X = x'(0)$ and $Y \in T_qN$ given by $Y = y'(0)$. The isomorphism is the identification $(X,Y) \simeq Z$. Indeed, let us define $\overline{X} \in T_{(p,q)}(M\times N)$, the tangent vector to the curve $(x(t), q)$, and $\overline{Y} \in T_{(p,q)}(M\times N)$, the tangent vector to the curve $(p, y(t))$. Then $Z = \overline{X} + \overline{Y}$ because for any $f\colon M\times N \to \mathbb{R}$,

$$Zf = \frac{d}{dt}f(x(t),y(t))\Big|_{t=0} = \frac{d}{dt}f(x(t),y(0))\Big|_{t=0} + \frac{d}{dt}f(x(0),y(t))\Big|_{t=0} = \overline{X}f + \overline{Y}f. \tag{1.13}$$

□

Proposition 1.11 (Leibnitz formula)**.**
Let us consider M, N, V, three manifold; a map $\varphi\colon M\times N \to V$ and a vector $Z \in T_{(p,q)}(M\times N)$ which corresponds (lemma 1.10) to $(X,Y) \in T_pM + T_qN$.

*If we define $\varphi_1\colon M \to V$ and $\varphi_2\colon N \to V$ by $\varphi_1(p') = \varphi(p',q)$ and $\varphi_2(q') = \varphi(p,q')$, we have the **Leibnitz formula** :*

$$d\varphi(Z) = d\varphi_1(X) + d\varphi_2(Y). \tag{1.14}$$

Proof. Since $Z = \overline{X} + \overline{Y}$, we just have to remark that

$$d\varphi(\overline{X}) = \frac{d}{dt}\varphi(x(t),q)\Big|_{t=0} = d\varphi_1(X),$$

so $d\varphi(Z) = d\varphi(\overline{X} + \overline{Y}) = d\varphi_1(X) + d\varphi_2(Y)$. □

One of the most important application of the Leibnitz rule is the corollary 1.32 on principal bundles.

Cotangent bundle

A form on a vector space V is a linear map $\alpha\colon V \to \mathbb{R}$. The set of all forms on V is denoted by V^* and is called the **dual space** of V. On each point of a manifold, one can consider the tangent bundle which is a vector space. Then one can consider, for each $x \in M$ the dual space $T_x^*M := (T_xM)^*$ which is called the **cotangent bundle**. A 1-**differential form** on M is a smooth map $\omega\colon M \to T^*M$ such that $\omega_x := \omega(x) \in T_x^*M$. So, for each $x \in M$, we have a 1-form $\omega_x\colon T_xM \to \mathbb{R}$.

Here, the smoothness is the fact that for any smooth vector field $X \in \mathfrak{X}(M)$, the map $x \to \omega_x(X_x)$ is smooth as function on M. One often consider vector-valued forms. This is exactly the same, but $\omega_x X_x$ belongs to a certain vector space instead of $\mathbb{R}$. The set of V-valued 1-forms on M is denoted by $\Omega(M, V)$ and simply $\Omega(M)$ if $V = \mathbb{R}$ The cotangent space T_p^*M of M at p is the dual space of T_pM, i.e. the vector space of all the (real valued) linear[3] 1-forms on T_pM. In the coordinate system $x\colon \mathcal{U} \to M$, we naturally use, on T_p^*M, the dual basis of the basis $\{\partial/\partial_{x^i}, \ldots \partial/\partial_{x^i}\}$ of T_pM. This dual basis is denoted by $\{dx_1, \ldots, dx_n\}$, the definition being as usual :

$$dx_i(\partial^j) = \delta_i^j. \tag{1.15}$$

The notation comes from the fact that equation (1.15)-17 describes the action of the differential of the projection $x_i\colon \mathcal{U} \to \mathbb{R}$ on the vector ∂^j.

If $(\mathcal{U}_\alpha, \varphi_\alpha)$ is a chart of M, then the maps

$$\begin{aligned} \phi_\alpha \ &: \mathcal{U}_\alpha \times \mathbb{R}^n \to T^*M \\ &(x, a) \mapsto a^i dx_i|_x \end{aligned} \tag{1.16}$$

give to T^*M a $2n$ dimensional manifold structure such that the canonical projection $\pi\colon T^*M \to M$ is an immersion.

When V is a finite-dimensional vector space, we denote by V^* its dual[4] and we often use the identifications $V \simeq V^* \simeq T_vV \simeq T_wV \simeq T_v^*V$ where v and w are any elements of V. Note however that there are no *canonical* isomorphism between these spaces, unless we consider some basis.

Exterior algebra

Here are some recall without proof about forms on vector space. If V is a vector space, we denote by $\Lambda^k V^*$ the space of all the k-form on V. We define $\wedge\colon \Lambda^k V^* \times \Lambda^l V^* \to \Lambda^{k+l} V^*$ by

$$(\omega^k \wedge \eta^l)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\eta(v_{\sigma(k+1)}, v_{\sigma(k+1)}) \tag{1.17}$$

If $\{e_1, \ldots, e_n\}$ is a basis of V, the dual basis $\{\sigma^1, \ldots, \sigma^n\}$ of V^* is defined by $\sigma^i(e_j) = \delta_j^i$.

If $I = \{1 \leqslant i_1 \leqslant \ldots i_k \leqslant n\}$, we write $\sigma^I = \sigma^{i_1} \wedge \ldots \sigma^{i_k}$ any k-form can be decomposed as

$$\omega = \sum_I \omega_I \sigma^I.$$

The exterior algebra is provided with the **interior product** denoted by ι. It is defined by

$$\begin{aligned} \iota(v_0)\colon \Lambda^k W \to \Lambda^{k-1} W \\ (\iota(v_0)\omega)(v_1, \ldots, v_{k-1}) = \omega(v_0, v_1, \ldots, v_{k-1}). \end{aligned} \tag{1.18}$$

[3]When we say *a form*, we will always mean *a linear form*.

[4]The vector space of all the linear map $V \to \mathbb{R}$.

Pull-back and push-forward

Let $\varphi\colon M \to N$ be a smooth map, α a k-form on N, and Y a vector field on N. Consider the map $d\varphi\colon T_xM \to T_{\varphi(x)}M$. The aim is to extend it to a map from the tensor algebra of T_xM to the one of $T_{\varphi(x)}M$. See [1] for precise definition of the tensor algebra.

The **pull-back** of φ on a k-form α is the map

$$\varphi^*\colon \Omega^k(N) \to \Omega^k(M)$$

defined by

$$(\varphi^*\alpha)_m(v_1,\ldots,v_k) = \alpha_{\varphi(m)}(d\varphi_m v_1,\ldots,d\varphi_m v_k) \tag{1.19}$$

for all $m \in M$ and $v_i \in \mathfrak{X}(M)$.

Note the particular case $k = 0$. In this case, we take –instead of α– a function $f\colon N \to \mathbb{R}$ and the definition (1.19)-18 gives $\varphi^* f\colon M \to \mathbb{R}$ by

$$\varphi^* f = f \circ \varphi.$$

The **push-forward** of φ on a k-form is the map

$$\varphi_*\colon \Omega^k(M) \to \Omega^k(N)$$

defined by $\varphi_* = (\varphi^{-1})^*$. For $v \in T_nN$, we explicitly have :

$$(\varphi_*\alpha)_n(v) = \alpha_{\varphi^{-1}(n)}(d\varphi_n^{-1} v).$$

Let now $\varphi\colon M \to N$ be a diffeomorphism. The **pull-back** of φ on a vector field is the map

$$\varphi^*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$$

defined by

$$(\varphi^*Y)(m) = [(d\varphi^{-1})_m \circ Y \circ \varphi](m),$$

or

$$(\varphi^*Y)_{\varphi^{-1}(n)} = (d\varphi^{-1})_n Y_n,$$

for all $n \in N$ and $m \in M$. Notice that

$$(d\varphi^{-1})_n\colon T_nN \to T_{\varphi^{-1}(n)}M,$$

and that $\varphi^{-1}(n)$ is well defined because φ is an homeomorphism.

The **push-forward** is, as before, defined by $\varphi_* = (\varphi^{-1})^*$. In order to show how to manipulate these notations, let us prove the following equation :

$$f_{*\xi} = (df)_\xi.$$

For $\varphi\colon M \to N$ and Y in $\mathfrak{X}(N)$, we just defined $\varphi^*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$, by

$$(\varphi^*Y)_{\varphi^{-1}(n)} = (d\varphi^{-1})_n Y_n. \tag{1.20}$$

Take $f\colon M \to N$; we want to compute $f_* = (f^{-1})^*$ with $(f^{-1})^*\colon \mathfrak{X}(M) \to \mathfrak{X}(N)$. Replacing the "$-1$" on the right places, the definition (1.20)-18 gives us

$$\Big[(f^{-1})^*X\Big]_{f(m)} = (df)_m X_m,$$

if $X \in \mathfrak{X}(M)$, and $m \in M$.

We can rewrite it without any indices: the coherence of the spaces automatically impose the indices: $(f^{-1})^*X = (df)X$. It can also be rewritten as $(f^{-1})^* = df$, and thus $f_* = df$. From there to $f_{*\xi} = (df)_\xi$, it is straightforward.

Differential of k-forms

The differential of a k-form is defined by the following theorem.

Theorem 1.12.
Let M be a differentiable manifold. Then for each $k \in \mathbb{N}$, there exists an unique map

$$d\colon \Omega^k(M) \to \Omega^{k+1}(M)$$

such that

- *(i) d is linear,*
- *(ii) for $k = 0$, we find back the $d\colon C^\infty(M) \to \Omega^(M)$ previously defined,*
- *(iii) if f is a function and ω^k a k-form, then*
$$d(f\omega^k) = df \wedge \omega^k + f d\omega^k, \tag{1.21}$$
- *(iv) $d(\omega^k \wedge \eta^l) = d\omega^k \wedge \eta^l + (-1)^k \omega^k \wedge d\eta^l$,*
- *(v) $d \circ d = 0$.*

An explicit expression for $d\omega^k$ is actually given by

$$d\omega^k = \sum d\omega_I \wedge dx^I \tag{1.22}$$

if $\omega^k = \sum \omega_I dx^I$. An useful other way to write it is the following. If ω is a k-form and $X_1, \ldots, X_{p+1}$ some vector fields,

$$\begin{aligned}(k+1)d\omega(X_1, \ldots, X_{p+1}) = &\sum_{i=1}^{p+1} (-1)^{i+1} X_i \omega(X_1, \ldots \hat{X}_i, \ldots, X_{p+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \hat{X}_j, \ldots, X_{p+1}).\end{aligned} \tag{1.23}$$

Let us show it with $p = 1$. Let $\omega = \omega_i dx^i$ and compute $d\omega(X, Y) = \partial_i \omega_j (dx^i \wedge dx^j)(X, Y)$. For this, we have to keep in mind that the ∂_i acts only on ω_j while, in equation (1.23)-19, a term $X\omega(Y)$ means –pointwise– the action of X on the function $\omega(Y)\colon M \to \mathbb{R}$. So we have to use Leibnitz formula :

$$(\partial_i \omega_j) X^i Y^j = (X\omega_j) Y^j = X(\omega_j Y^j) - \omega_j X Y^j.$$

On the other hand, we know that $[X, Y]^i = XY^i - YX^i$, so

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]). \tag{1.24}$$

Hodge operator

Let us take a manifold M endowed with a metric g. We can define a map $r\colon T_x^* M \to T_x M$ by, for $\alpha \in T_x^* M$,

$$\langle r(\alpha), v \rangle = \alpha(v).$$

for all $v \in T_x M$, where $\langle \cdot, \cdot \rangle$ stands for the product given by the metric g. If we have $\alpha, \beta \in T_x^* M$, we can define

$$\langle \alpha, \beta \rangle = \langle r(\alpha), r(\beta) \rangle.$$

With this, we define an inner product on $\Lambda^p(T_x^*M)$:

$$\langle \alpha_1 \wedge \dots \alpha_p, \beta_1 \wedge \dots \beta_p \rangle = \det_{ij} \langle \alpha_i, \beta_j \rangle.$$

The **Hodge operator** is $\star\colon \Lambda^p(T_x^*M) \to \Lambda^{n-p}(T_x^*M)$ such that for any $\phi \in \Lambda^p(T_x^*M)$,

$$\phi \wedge (\star\psi) = \langle \phi, \psi \rangle \Omega = \sqrt{|\det(g)|} dx^1 \wedge \dots \wedge dx^n. \tag{1.25}$$

Volume form and orientation

Let M be a n dimensional smooth manifold. A **volume form** on M is a nowhere vanishing n-form and the manifold itself is said to be **orientable** if such a volume form exists. Two volume forms μ_1 and μ_2 are describe the same orientation if there exists a function $f > 0$ such that[5] $\mu_1 = f\mu_2$.

Proposition 1.13.
There exists only two orientations on a connected orientable manifold.

$\boxed{\textbf{Problème et notes pour moi}}$ **1.**
Vérifier l'énoncé du théorème et trouver une référence.

One says that the *ordered* basis $(v_1, \cdots, v_n)$ of T_xM is **positively oriented** with respect to the volume form μ is $\mu_x(v_1, \cdots, v_n) > 0$.

1.1.6 Musical isomorphism

In some literature, we find the symbols $v^\flat$ and $\alpha^\sharp$. What does it mean ? For $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^2(M)$, the **flat** operation $v^\flat \in \Omega^1(M)$ is simply defined by the inner product :

$$v^\flat = i(v)\omega \tag{1.26}$$

In the same way, we define the **sharp** operation by taking a 1-form α and defining $\alpha^\sharp$ by

$$i(\alpha^\sharp)\omega = \alpha. \tag{1.27}$$

An immediate property is, for all $v \in \mathfrak{X}(M)$, $v^{\flat\sharp} = v$, and for all $\alpha \in \Omega^1(M)$, $\alpha^{\sharp\flat} = \omega$.

1.1.7 Lie derivative

Consider $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^p(M)$. Let $\varphi_t\colon M \to M$ be the flow of X. The **Lie derivative** of α is

$$\mathcal{L}_X\alpha = \lim_{t\to 0} \frac{1}{t}[(\varphi_t^*\alpha) - \alpha] = \frac{d}{dt}\varphi_t^*\alpha\Big|_{t=0}. \tag{1.28}$$

More explicitly, for $x \in M$ and $v \in T_xM$,

$$(\mathcal{L}_X\alpha)_x(v) = \lim_{t\to 0} \frac{1}{t}\left[(\varphi_t^*\alpha)_x(v) - \alpha_x(v)\right]$$

In the definition of the **Lie derivative** for a vector field, we need an extra minus sign :

$$(\mathcal{L}_XY)_x = \frac{d}{dt}\varphi_{-t*}Y_{\varphi_t(x)}\Big|_{t=0}. \tag{1.29}$$

[5]Recall that the space of n-forms is one-dimensional.

Why a minus sign ? Because $Y_{\varphi_t(x)} \in T_{\varphi_t(x)}M$, but $(d\varphi_{-t})_a \colon T_aM \to T_{\varphi_{-t}(a)}M$ so that, if we want, $\varphi_{-t*}Y_{\varphi_t(x)}$ to be a vector at x, we can't use φ_{t*}.

These two definitions can be embedded in only one. Let $X \in \mathfrak{X}(M)$ and φ_t its integral curve[6]. We know that φ_{t*} is an isomorphism $\varphi_{t*} \colon T_{\varphi^{-1}(x)}M \to T_xM$. It can be extended to an isomorphism of the tensor algebras at $\varphi^{-1}(x)$ and x. We note it $\tilde{\varphi}_t$. For all tensor field K on M, we define

$$(\mathcal{L}_XK)_x = \lim_{t\to 0}[K_x - (\tilde{\varphi}_tK)_x].$$

On a Riemannian manifold (M,g), a vector field X is a **Killing vector field** if $\mathcal{L}_Xg = 0$.

Lemma 1.14.
Let $f \colon (-\epsilon,\epsilon) \times M \to \mathbb{R}$ be a differentiable map with $f(0,p) = 0$ for all $p \in U$. Then there exists $g \colon (-\epsilon,\epsilon) \times M \to \mathbb{R}$, a differentiable map such that $f(t,p) = tg(t,p)$ and

$$g(0,q) = \left.\frac{\partial f(t,q)}{\partial t}\right|_{t=0}.$$

Proof. Take

$$g(t,q) = \int_0^1 \frac{\partial f(ts,p)}{\partial (ts)} ds,$$

and use the change of variable $s \to ts$. □

Lemma 1.15.
If φ_t is the integral curve of X, for all function $f \colon M \to \mathbb{R}$, there exists a map g, $g_t(p) = g(t,p)$ such that $f \circ \varphi_t = f + tg_t$ and $g_0 = Xf$.

Proof. Consider $\overline{f}(t,p) = f(\varphi_t(p)) - f(p)$, and apply the lemma :

$$f \circ \varphi_t = tg_t(p) + f(p).$$

Thus we have

$$Xf = \lim_{t\to 0}\frac{1}{t}[f(\varphi_t(p)) - f(p)] = \lim_{t\to 0} g_t(p) = g_0(p).$$

□

One of the main properties of the Lie derivative is the following :

Theorem 1.16.
Let $X, Y \in \mathfrak{X}(M)$ and φ_t be the integral curve of X. Then

$$[X,Y]_p = \lim_{t\to 0}\frac{1}{t}[Y - d\varphi_tY](\varphi_t(p)),$$

or

$$\mathcal{L}_XY = [X,Y].$$

Proof. Take $f \colon M \to \mathbb{R}$ and the function given by the lemma: $g_t \colon M \to \mathbb{R}$ such that $f \circ \varphi_t = f + tg_t$ and $g_0 = Xf$. Then put $p(t) = \varphi_t^{-1}(p)$. The rest of the proof is a computation :

$$(\varphi_{t*}Y)_pf = Y(f \circ \varphi_t)_{p(t)} = (Yf)_{p(t)} + t(Yg_t)_{p(t)},$$

[6] *i.e.* for all $x \in M$, $\varphi_0(x) = x$ and $\left.\frac{d}{dt}\varphi_{u+t}(x)\right|_{t=0} = X_{\varphi_u(x)}$.

so

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}[Y_p-(\varphi_{t*}Y)_p]f &= \lim_{t\to 0}\frac{1}{t}[(Yf)_p-(Yf)_{p(t)}]-\lim_{t\to 0}(Yg_t)_{p(t)}\\ &= X_p(Yf)-Y_pg_0\\ &= [X,Y]_pf.\end{aligned} \tag{1.30}$$

□

A second important property is

Theorem 1.17.
For any function $f\colon M\to V$,

$$\mathcal{L}_Xf=Xf.$$

Proof. If $X(t)$ is the path which defines the vector X, it is obvious that at $t=0$, $X(t)$ is an integral curve to X, so that we can take $X(t)$ instead of φ_t in (1.28)-20. Therefore we have :

$$\mathcal{L}_Xf=\frac{d}{dt}\varphi_t^*f\bigg|_{t=0}=Xf \tag{1.31}$$

by definition of the action of a vector on a function.

□

1.2 Example: Lie groups

A **Lie group** is a manifold G endowed with a group structure such that the inversion map $i\colon G\to G$, $i(x)=x^{-1}$ and the multiplication $m\colon G\times G\to G$, $m(x,y)=xy$ are differentiable. The **Lie algebra** of the Lie group G is the tangent space of G at the identity: $\mathcal{G}=T_eG$.

It is immediate to see that $g\mapsto g^{-1}$ is a smooth homeomorphism and that, for any fixed g_0,g_1, the maps

$$\begin{aligned}&g\mapsto g_0g,\\&g\mapsto gg_0,\\&g\mapsto g_0gg_1\end{aligned}$$

are smooth homeomorphisms. When $A\subset G$, we define $A^{-1}=\{g^{-1}\text{ st } g\in G\}$.

1.2.1 Connected component of Lie groups

Proposition 1.18.
If G is a connected Lie group and $\mathcal{U}$, a neighbourhood of the identity e, then G is generated by $\mathcal{U}$ in the sense that $\forall g\in G$, there exists a finite *number of $g_i\in\mathcal{U}$ such that*

$$g=g_1\ldots g_n.$$

Notice that the number n is function of g in general.

Proof. Eventually passing to a subset, we can suppose that $\mathcal{U}$ is open. In this case, $\mathcal{U}^{-1}$ is open because it is the image of $\mathcal{U}$ under the homeomorphism $g\mapsto g^{-1}$. Now we consider $V=\mathcal{U}\cap\mathcal{U}^{-1}$. The main property of this set is that $V=V^{-1}$. Let

$$[V]=\{g_1\ldots g_n\text{ st } g_i\in V\};$$

we will prove that $[V] = G$ by proving that it is closed and open in G (the fact that G is connected the concludes).

We begin by openness of $[V]$. Let $g_0 = g_1 \cdots g_n \in [V]$. We know that $g_0 V$ is open because the multiplication by g_0 is an homeomorphism. It is clear that $g_0 V \subset [V]$ and that $g_0 = g_0 e \in g_0 V$. Hence $g_0 \in g_0 V \subseteq [V]$. It proves that $[V]$ is open because $g_0 V$ is a neighbourhood of g_0 in $[V]$.

We now turn our attention to the closeness of $[V]$. Let $h \in \overline{[V]}$. The set hV is an open set which contains h and $hV \cap [V] \neq \emptyset$ because an open which contains an element of the closure of a set intersects the set (it is almost the definition of the closure). Let $g_0 \in hV \cap [V]$. There exists a $h_1 \in V$ such that $g_0 = hh_1$. For this h_1, we have $hh_1 = g_0 = g_1 \cdots g_n$, and therefore

$$h = g_1 \cdots g_n h_1^{-1} \in [V].$$

This proves that $h \in [V]$ because $h_1^{-1} \in V$ from the fact that $V = V^{-1}$. □

Remark that this proof emphasises the topological aspect of a Lie group: the differential structure was only used to prove thinks like that A^{-1} is open when A is open.

Proposition 1.19.
Let G be a Lie group and G_0, the identity component of G. We have the following :

(i) *G_0 is an open invariant subgroup of G,*

(ii) *G_0 is a Lie group,*

(iii) *the connected components of G are lateral classes of G_0. More specifically, if x belongs to the connected component G_1, then $G_1 = xG_0 = G_0 x$.*

Proof. We know that when M_1 is open in the manifold M, one can put on M_1 a differential structure of manifold of same dimension as M with the induced topology. Since G_0 is open, it is a smooth manifold. In order for G_0 to be a Lie group, we have to prove that it is stable under the inversion and that $gh \in G_0$ whenever $g, h \in G_0$.

First, G_0^{-1} is connected because it is homeomorphic to G_0 in G. The element e belongs to the intersection of G_0 and G_0^{-1}, so $G_0 \cup G_0^{-1}$ is connected as non-disjoint union of connected sets. Hence $G_0 \cup G_0^{-1} = G_0$ and we conclude that $G_0^{-1} \subseteq G_0$. The set $G_0 G_0$ is connected because it is the image of $G_0 \times G_0$ under the multiplication map, but $e \in G_0 G_0$, so $G_0 G_0 \subseteq G_0$ and G_0 is thus closed for the multiplication. Hence G_0 is a Lie group.

For all $x \in G$, we have $e = xex^{-1} \in xG_0 x^{-1}$, but $xG_0 x^{-1}$ is connected. Hence $xG_0 x^{-1} \subseteq G_0$, which proves that G_0 is an invariant subset of G.

Lateral classes xG_0 are connected because the left multiplication is an homeomorphism. They are moreover *maximal* connected subsets because, if $xG_0 \subset H$ (proper inclusion) with a connected H, then $G_0 \subset x^{-1}H$ (still proper inclusion). But the definition of G_0 is that this proper inclusion is impossible. Therefore, the sets of the form xG_0 are maximally connected sets. It is clear that $\cup_{g \in G} gG_0 = G$.

Notice that the last point works with $G_0 x$ too. □

1.2.2 The Lie algebra of SU(2)

Let consider $G = \mathrm{SU}(2)$; the elements are complexes 2×2 matrices U such that $UU^\dagger = \mathbb{1}$ and $\det U = 1$. An element of the Lie algebra is given by a path $u \colon \mathbb{R} \to G$ in the group with

$u(0) = \mathbb{1}$. Since for all t, $u(t)u(t)^\dagger = \mathbb{1}$,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big[u(t)u(t)^\dagger\Big]_{t=0} \\ &= u(0)\frac{d}{dt}\Big[u(t)^\dagger\Big]_{t=0} + \frac{d}{dt}\Big[u(t)\Big]_{t=0} u(0)^\dagger \\ &= [d_t u(t)]^\dagger + [d_t u(t)]. \end{aligned} \tag{1.32}$$

So a general element of the Lie algebra $\mathfrak{su}(2)$ is an anti-hermitian matrix. The same trick gives the condition of vanishing trace.

1.2.3 What is $g^{-1}dg$?

The expression $g^{-1}dg$ is often written in the physical literature. In our framework, the way to gives a sense to this expression is to consider it pointwise acting on a tangent vector. More precisely, the scheme is the data of a manifold M, a Lie group G and a map $g\colon M \to G$. Pointwise, we have to apply $g(x)^{-1}dg_x$ to a tangent vector $v \in T_xM$.

Note that $dg_x\colon T_xM \to T_{g(0)}G \neq T_eG$, so $dg_x \notin \mathcal{G}$. But the product $g(x)^{-1}dg_x v$ is defined by

$$g(x)^{-1}dg_x v = \frac{d}{dt}\Big[g(x)^{-1}g(v(t))\Big]_{t=0} \in \mathcal{G}.$$

1.2.4 Exponential map

Lemma 1.20.
Let G, H be two Lie groups with algebras $\mathcal{G}$ and $\mathcal{H}$. Let $\phi\colon G \to H$ be a homomorphism differentiable at e, the unit in G. Then for all $X \in \mathcal{G}$, the following formula holds:

$$\phi(\exp X) = \exp(d\phi_e X).$$

It can be found in [6].

1.2.5 Invariant vector fields

As convention, the **left invariant** on G associated with X in the Lie algebra $\mathcal{G}$ at $g \in G$ is given by the path

$$\tilde{X}_g(t) = g e^{tX} \tag{1.33}$$

while the **right invariant** is given by

$$\underset{\sim}{X}_g(t) = e^{tX} g \tag{1.34}$$

The invariance means that $(dL_h)_g \tilde{X}_g = \tilde{X}_{hg}$ and $(dR_h)_g \underset{\sim}{X}_g = \underset{\sim}{X}_{gh}$. The invariant vector fields are important because they carry the structure of the tangent space at identity (the Lie algebra). More precisely we have the following result :

Theorem 1.21.
The map $X \to X_e$ is a bijection between the left invariant vector fields on a Lie group and its Lie algebra T_eG.

Invariant vector fields are also often used in order to transport a structure from the identity of a Lie group to the whole group by $A_g(X_g) = A_e(dL_{g^{-1}}X_g)$ where A_e is some structure and X_g, a vector at g.

1.2.6 Adjoint map

The ideas of this short note comes from [6]. A more traumatic definition of the adjoint group can be found in [3], chapter II, §5. Let G be a Lie group, and $\mathcal{G}$, its Lie algebra. We define the **adjoint map** at the point $x \in G$ by

$$\begin{aligned} \mathbf{Ad}_x \colon G &\to G \\ \mathbf{Ad}_x\, y &= xyx^{-1} \end{aligned} \tag{1.35}$$

Then we define

$$Ad_x := (d\,\mathbf{Ad}_x)_e \colon \mathcal{G} \to \mathcal{G};$$

the chain rule applied on $\mathbf{Ad}_{xy} = \mathbf{Ad}_x \circ \mathbf{Ad}_y$ leads to $Ad_{xy} = Ad_x \circ Ad_y$, and thus we can see Ad as a group homomorphism $Ad \colon G \to GL(\mathcal{G})$, $Ad(x) = Ad_x$.

Definition 1.22.
*This homomorphism is the **adjoint representation** of the group G in the vector space $\mathcal{G}$.*

Finally, we define

$$ad := d(Ad)_1 \colon \mathcal{G} \to L(\mathcal{G}, \mathcal{G})$$

where we identify $T_1 GL(\mathcal{G})$ with $L(\mathcal{G}, \mathcal{G})$.

Lemma 1.23.
If $f \colon G \to G$ is an automorphism of G (i.e. : $f(xy) = f(x)f(y)$), then df_e is an automorphism of $\mathcal{G}$: $df[X, Y] = [df X, df Y]$

Proof. First, remark that $f(\mathbf{Ad}_x\, y) = \mathbf{Ad}_{f(x)}\, f(y)$. Now, $\mathrm{Ad}_x X = (d\,\mathbf{Ad}_x)_e X$, so that one can compute :

$$\begin{aligned} df(\mathrm{Ad}_x X) &= \frac{d}{dt}\Big[f(\mathbf{Ad}_x X(t)) \Big]_{t=0} \\ &= \frac{d}{dt}\Big[\mathbf{Ad}_{f(x)}\, f(X(t)) \Big]_{t=0} \\ &= (d\,\mathbf{Ad}_{f(x)})_{f(e)} df X \\ &= \mathrm{Ad}_{f(x)}\, df X. \end{aligned} \tag{1.36}$$

On the other hand, we need to understand how does the ad work.

$$\mathrm{ad}\, XY = \frac{d}{dt}\Big[\mathrm{Ad}_{X(t)} \Big]_{t=0} Y = \frac{d}{dt}\Big[\mathrm{Ad}_{X(t)}\, Y \Big]_{t=0}$$

because $\mathrm{Ad}_{X(t)} \colon \mathcal{G} \to \mathcal{G}$ is linear, so that Y can enter the derivation (for this, we identify $\mathcal{G}$ and $T_X\mathcal{G}$). Since $\mathrm{Ad}_{X(t)}\, Y$ is a path in $\mathcal{G}$ the *true space* is

$$(\mathrm{ad}\, X)Y = \frac{d}{dt}\Big[\mathrm{Ad}_{X(t)}\, Y \Big]_{t=0} \in T_{[X,Y]}\mathcal{G} \simeq \mathcal{G}.$$

For the same reason of linearity, df can get in the derivative in the expression $df \frac{d}{dt}\Big[\mathrm{Ad}_{X(t)}\, Y \Big]_{t=0}$.

Thus

$$\begin{aligned}(\operatorname{ad} X)Y &= \frac{d}{dt}\Big[df\big(\operatorname{Ad}_{X(t)} Y\big)\Big]_{t=0}\\ &= \frac{d}{dt}\Big[\operatorname{Ad}_{f(X(t))} dfY\Big]_{t=0}\\ &= \frac{d}{dt}\Big[\operatorname{Ad}_{f(X(t))}\Big]_{t=0} dfY\\ &= \operatorname{ad}(dfX)dfY\\ &= [dfX, dfY]\end{aligned} \tag{1.37}$$

because $f(X(t))$ is a path which gives dfX.

□

One can show that $[X, Y]$ is tangent to the curve

$$c(t) = e^{-\sqrt{s}X}e^{-\sqrt{s}Y}e^{\sqrt{s}X}e^{\sqrt{s}Y}. \tag{1.38}$$

1.3 Fundamental vector field

If $\mathcal{G}$ is the Lie algebra of a Lie group G acting on a manifold M (the action of g on x being denoted by $x \cdot g$), the **fundamental vector field** associated with $A \in \mathcal{G}$ is given by

$$A^*_x = \frac{d}{dt}\Big[x \cdot e^{-tA}\Big]_{t=0}. \tag{1.39}$$

We always suppose that the action is effective. If the action of G is transitive, the fundamental vectors at point $x \in M$ form a basis of T_xM. More precisely, we have the

Lemma 1.24.
For any $v \in T_xM$, there exists a $A \in \mathcal{G}$ such that $v = A^*_x$, in other terms

$$\operatorname{Span}\{A^*_x \; st \; A \in \mathcal{G}\} = T_xM.$$

Proof. The vector v is given by a path $v(t)$ in M. Since the action is transitive, one can write $v(t) = x \cdot c(t)$ for a certain path c in G which fulfills $c(0) = e$. We have to show that v depends only on $c'(0) \in \mathcal{G}$. We consider

$$\begin{aligned}R\colon G \times M &\to M\\ R(g, x) &= x \cdot g,\end{aligned} \tag{1.40}$$

so

$$v = \frac{d}{dt}\Big[R(c(t), x)\Big]_{t=0} = dR_{(e,x)}\big[(d_tc(t), x) + (c(0), x)\big]. \tag{1.41}$$

□

Lemma 1.25.
If A, $B \in \mathcal{G}$ are such that $A^* = B^*$, then $A = B$.

Proof. We consider once again the map (1.40)-26 and we look at

$$v = \frac{d}{dt}\Big[R(c(t),x)\Big]_{t=0} = (dR)_{(e,x)}\frac{d}{dt}\Big[(c(t),x)\Big]_{t=0},$$

keeping in mind that $c(t) = e^{-tA}$. In order to treat this expression, we define

$$R_1\colon G \to M, \quad R_1(h) = R(h,x), \tag{1.42a}$$
$$R_2\colon M \to M, \quad R_2(y) = R(g,y). \tag{1.42b}$$

So

$$v = dR_1(X) + dR_2(0) = dR_1 c'(0)$$

and the assumption $A^*_x = B^*_x$ becomes $dR_1 A = dR_1 B$. This makes, for small enough t, $R_1(e^{tA}e^{-tB}) = x\cdot e^{tA}e^{-tB} = x$; if the action is effective, it imposes $A = B$. □

Lemma 1.26.
If we consider the action of a matrix group, R_g acts on the fundamental field by

$$dR_g(A^*_\xi) = \big(\operatorname{Ad}(g^{-1})A\big)^*_{\xi\cdot g}.$$

Proof. Just notice that $e^{-t\operatorname{Ad}(g^{-1})A} = \mathbf{Ad}_{g^{-1}}(e^{-tA}) = g^{-1}e^{-tA}g$, thus

$$\big(\operatorname{Ad}(g^{-1})A\big)^*_{\xi\cdot g} = \frac{d}{dt}\Big[\xi\cdot ge^{-t\operatorname{Ad}(g^{-1})A}\Big]_{t=0} = dR_g(A^*_\xi). \tag{1.43}$$

□

1.4 Vector bundle

Let M be a smooth manifold. A V**-vector bundle** of rank r on M is a smooth manifold F and a smooth projection $p\colon F \to M$ such that

- for any $x \in M$, the fiber $F_x := p^{-1}(x)$ is a vector space of dimension r on the same field that V (let's say $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).
- for any $x \in M$, there exists an open neighbourhood $\mathcal{U}$ of x and a "chart diffeomorphism" $\phi\colon p^{-1}(\mathcal{U}) \to \mathcal{U}\times V$ such that for any $l \in p^{-1}(y)$,
  - $\phi(l) = (y, \phi_y(l))$
  - $\phi_y\colon E_y \to V$ is a vector space isomorphism.

The pair $(\mathcal{U},\phi)$ is a *local trivialization*; M is the *base space*; F, the *total space*, p the *projection* and r, the *rank* of the bundle. The denominations of total and base spaces will also be used in the same way for principal bundles.

We will sometimes use charts diffeomorphism $\phi\colon \mathcal{U}\times V \to p^{-1}(\mathcal{U})$ instead of $\phi\colon p^{-1}(\mathcal{U}) \to \mathcal{U}\times V$. Since they are diffeomorphism, this difference don't affects anything.

1.4.1 Transition functions

The trivializations will be denoted by Greek indices: $\mathcal{U}_\alpha$, ϕ_α,... The symbol $\mathcal{U}_{\alpha\beta}$ naturally denotes $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. If we consider two local trivializations $(\mathcal{U}_\alpha, \phi_\alpha)$ and $(\mathcal{U}_\beta, \phi_\beta)$, we have to look at $\phi_\alpha \circ \phi_\beta^{-1} \colon \mathcal{U}_{\alpha\beta} \times \mathbb{K}^r \to \mathcal{U}_{\alpha\beta} \times \mathbb{K}^r$. We define the **transition functions** $g_{\alpha\beta} \colon \mathcal{U}_{\alpha\beta} \to GL(r, \mathbb{K})$ by

$$\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)v). \tag{1.44}$$

These functions take their values in $GL(r, \mathbb{K})$ because $\phi_y \colon E_y \to V$ is a vector space isomorphism. Since $(\phi_\alpha \circ \phi_\beta)^{-1} = \phi_\beta \circ \phi_\alpha^{-1}$, it is clear that $g_{\alpha\beta}(x) = g_{\alpha\beta}(x)^{-1}$.

If $x \in \mathcal{U}_{\alpha\beta\gamma} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$, we have $\phi_\alpha \circ \phi_\gamma^{-1}(x, v) = (x, g_{\alpha\gamma}(x)v)$, but also $\phi_\alpha \circ \phi_\gamma^{-1} = \phi_\alpha \circ \phi_\beta^{-1} \phi_\beta \circ \phi_\gamma^{-1}$, then

$$(x, g_{\alpha\gamma}(x)v) = (\phi_\alpha \circ \phi_\beta^{-1})(x, g_{\beta\gamma}(x)v) = (x, g_{\alpha\beta}(x)_{\beta\gamma}(x)v). \tag{1.45}$$

Thus $g_{\alpha\gamma}(x) = g_{\alpha\beta}(x) g_{\beta\gamma}(x)$. So, as linear maps, we have

$$g_{\alpha\beta} \circ g_{\alpha\gamma} \circ g_{\gamma\alpha} = \mathbb{1}. \tag{1.46}$$

1.4.2 Inverse construction

Let us consider a manifold M, an open covering $\{\mathcal{U}_\alpha : \alpha \in I\}$ and some functions $g_{\alpha\beta} \colon \mathcal{U}_{\alpha\beta} \to GL(r, \mathbb{K})$ which fulfill relations (1.46)-28. We will build a vector bundle $E \xrightarrow{p} M$ whose transition functions are the $g_{\alpha\beta}$'s. Let $\tilde{E}$ be the disjoint union

$$\tilde{E} = \bigsqcup_{\alpha \in I} \mathcal{U}_\alpha \times \mathbb{K}^r,$$

i.e. triples of the form $(x, v, \alpha) \in M \times \mathbb{K}^r \times I$ with the condition that $x \in \mathcal{U}_\alpha$. We define an equivalence relation on $\tilde{E}$ by $(x, v, \alpha) \sim (y, w, \beta)$ if and only if $x = y$ and $w = g_{\alpha\beta}(x)v$. Next, we define $E = \tilde{E}/\sim$ and $\omega \colon \tilde{E} \to E$, the canonical projection. The projection $p \colon E \to M$ is naturally defined by $p([x, v, \alpha]) = x$. The chart diffeomorphism is $\varphi_\alpha \colon \mathcal{U}_\alpha \times \mathbb{K}^r \to p^{-1}(\mathcal{U}_\alpha)$,

$$\varphi_\alpha(x, v) = \omega(x, v, \alpha).$$

Now we have to prove that E endowed with the φ_α's is a vector bundle.

First we prove that φ_α is surjective. For this we remark that a general element in $p^{-1}(\mathcal{U}_\alpha)$ can be written under the form $\omega(x, v, \alpha)$ with $x \in \mathcal{U}_{\alpha\beta}$. But

$$\begin{aligned} \varphi_\alpha(x, g_{\alpha\beta}(x)w) &= \omega(x, g_{\alpha\beta}(x)w, \alpha) \\ &= \omega(x, g_{\alpha\beta}(x) g_{\alpha\beta}(x)w, \beta) \\ &= \omega(x, w\beta), \end{aligned} \tag{1.47}$$

then φ_α is surjective. Now we suppose $\varphi_\alpha(x, v) = \varphi_\alpha(y, w)$. Then $\omega(x, v, \alpha) = \omega(y, w, \alpha)$ and $x = y$, $w = g_{\alpha\alpha}v$ which immediately gives $v = w$. Then φ_α is injective.

Finally, we have

$$(\varphi\alpha \circ \varphi_\beta^{-1})(\omega(x, v, \alpha)) = \varphi_\alpha(x, g_{\alpha\beta}(x)v) = \omega(x, g_{\alpha\beta}(x)v, \alpha), \tag{1.48}$$

which proves that the maps g are the transition functions of the vector bundle E.

1.4.3 Equivalence of vector bundle

Let $E \xrightarrow{p} M$ and $F \xrightarrow{p'} M$ be two vector bundles on M. They are **equivalent** if there exists a smooth diffeomorphism $f\colon E \to F$ such that

- $p' \circ f = p$,
- $f|_{E_x}\colon E_x \to F_x$ is a vector space isomorphism.

Let E and F be two equivalent vector bundles, $\{\mathcal{U}_\alpha \text{ st } \alpha \in I\}$, an open covering which trivialize E and F in the same time and ϕ^E_α, ϕ^F_α the corresponding trivializations. A map $f\colon E \to F$ reads "in the trivialization" as $\phi^F_\alpha \circ f|_{p^{-1}(\mu_\alpha)} \circ \phi^{E}{}^{-1}_\alpha\colon \mathcal{U}_\alpha \times \mathbb{K}^r \to \mathcal{U}_\alpha \times \mathbb{K}^r$ and defines a map $\lambda_\alpha\colon \mathcal{U}_\alpha \to GL(r, \mathbb{K})$ by

$$(\phi^F_\alpha \circ f|_{p^{-1}(\mu_\alpha)} \circ \phi^{E}{}^{-1}_\alpha)(x, v) = (x, \lambda_\alpha(x)v). \tag{1.49}$$

If we denote by g^E the transition functions for E (and g^F for F),

$$\phi^F_\alpha \circ \phi^{F-1}_\beta = (\phi^F_\alpha \circ f \circ \phi^{E-1}_\alpha) \circ (\phi^E_\alpha \circ \phi^{E-1}_\beta) \circ (\phi^E_\beta \circ f^{-1} \circ \phi^{E-1}_\beta),$$

so that

$$g^F_{\alpha\beta}(x) = \lambda_\alpha(x) g^E_{\alpha\beta}(x) \lambda(x)^{-1}. \tag{1.50}$$

Once again we have an inverse construction. We consider a vector bundle E on M with transition functions g^E and some maps $\lambda_\alpha\colon \mathcal{U}_\alpha \to GL(r, \mathbb{K})$; then we define $g^F_{\alpha\beta}(x)$ by equation (1.50)-29.

From subsection 1.4.2, one can construct a vector bundle F on M whose transition functions are these g^F. With the trivializations ϕ^F of F, one can define $f\colon E \to F$ by

$$(\phi^F_\alpha \circ f \circ \phi^{E-1}_\alpha)(x, v) = (x, \lambda_\alpha(x)v).$$

When a basis space B is given, we denote by $\mathrm{Vect}(B)$ the set of isomorphism classes of vector bundles over B. In the complex case, we denote it by $\mathrm{Vect}_{\mathbb{C}}(B)$.

Proposition 1.27.
Any vector bundle over $\mathbb{R}^n$ is trivial.

Proof. Let $p\colon F \to M$ be a vector bundle on $M = \mathbb{R}^n$ and $\{\mathcal{U}_\alpha\}$ be covering of $\mathbb{R}^n$ by local trivializations. Now consider a partition of unity related to the covering $\mathcal{U}_\alpha$: a set of functions $f_\alpha\colon M \to \mathbb{R}$ such that

- $f_\alpha > 0$,
- $\forall x \in M$, one can find a neighbourhood of x in which only a *finite* number of f_α is non zero,
- $\forall x \in M$, $\sum_\alpha f_\alpha(x) = 1$.
- $f_\alpha = 0$ outside of $\mathcal{U}_\alpha$.

Using that partition of unity, we build the trivialization function $f\colon F \to \mathbb{R}^n \times V$ by $f(l) = (x, \sum_\alpha f_\alpha(x)\phi_{\alpha x}(l))$. □

The following two propositions have some importance in K-theory.

Proposition 1.28.
Let $\pi\colon E\to B$ be a complex vector bundle over a basis compact, Hausdorff, connected basis B. Then there exists a vector bundle E' such that $E\oplus E'$ is trivial.

Proposition 1.29.
Let $f\colon A\to B$ be a map between the topological spaces A and B, and consider a vector bundle $\pi\colon E\to B$. Then there exists one and only one vector bundle $\pi'\colon E'\to A$ and a map $f'\colon E'\to E$ such that $f'|_{E'_x}\colon E'_x\to E_{f(x)}$ is an isomorphism. The vector bundle E' is unique up to isomorphism.

Proofs can be found in [7]. Let us denote by $f^*(E)$ the function given by proposition 1.29. It satisfies the following properties

$$\begin{aligned}(fg)^*(E)&=g^*\big(f^*(E)\big)\\ \mathrm{id}^*(E)&=E\\ f^*(E_1\oplus E_2)&=f^*(E_1)\oplus f^*(E_2)\\ f^*(E_1\otimes E_2)&=f^*(E_1)\otimes f^*(E_2).\end{aligned}\tag{1.51}$$

1.4.4 Sections of vector bundle

A **section** of the vector bundle $p\colon E\to M$ is a smooth map $s\colon M\to E$ such that $p\circ s=\mathrm{id}|_M$. The set of all the sections is denoted by $\Gamma^\infty(M)$ or simply $\Gamma(E)$.

If $(\mathcal{U}_\alpha,\phi_\alpha)$ is a local trivialization, one can describe the section s by a function $s_\alpha\colon\mathcal{U}_\alpha\to V$ defined by $\phi_\alpha(s(x))=(x,s_\alpha(x))$, or equivalently by

$$s(x)=\phi_\alpha^{-1}(x,s_\alpha(x)).$$

As usual when we define such a local quantity, we have to ask ourself how are related s_α and s_β on $\mathcal{U}_\alpha\cap\mathcal{U}_\beta$. The best is $s_\alpha=s_\beta$, but most of the time it is not. Here, we compute

$$\phi_\beta\circ\phi_\alpha^{-1}\circ\phi_\alpha(s(s))=(x,g_{\alpha\beta}(x)s_\alpha(x)),$$

which is obviously also equal to $(x,s_\beta(x))$. Then

$$s_\beta(x)=g_{\alpha\beta}(x)s_\alpha(x)\tag{1.52}$$

without summation.

1.5 Vector valued differential forms

Let E be a vector bundle over M. A **E-valued p-form** is a section

$$e\in\Gamma\big(E\otimes\bigwedge^p T^*M\big).$$

We denote by $\Omega(M,E)=\Gamma\big(E\otimes\bigwedge^p T^*M\big)$ the set of E-valued differential forms. An element of $\Omega^1(M,E)=\Gamma\big(E\otimes\bigwedge T^*M\big)$ always reads $\sum_i s_i\otimes\omega_i$ for some sections s_i and usual differential forms ω_i.

A form of $\Omega^p(M,E)$ can be seen as a fiber morphism $\underbrace{TM\otimes\cdots\otimes TM}_{p\text{ times}}\to E$ by associating

$$s\otimes\omega(X_1,\cdots,X_p)=s(x)\omega(X_1,\cdots,X_p)\in E_x$$

to the element $(s \otimes \omega) \in \Omega^p(M, E)$. There exists a wedge product between vector-valued forms. If $e \in \Omega^p(M, E_1)$ and $f \in \Omega^q(M, E_2)$, then we define $e \wedge f \in \Omega^{p+q}(M, E_1 \otimes E_2)$ by

$$(e \wedge f)(v_1, \cdots, v_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} (-1)^\pi e(v_{\pi(1)}, \cdots v_{\pi(p)}) \otimes f(v_{\pi(p+1)}, \cdots, v_{\pi(p+q)}) \in E_1 \otimes E_2. \tag{1.53}$$

where $(-1)^\pi$ stands for the sign of the permutation π. For example when e, $f \in \Omega^1(M, E)$, we have

$$(e \wedge f)(X, Y) = e(X) \otimes f(Y) - e(Y) \otimes f(X) \in E \otimes E.$$

When M is a differentiable manifold, the **fundamental 1-form** is the element $\theta \in \Omega(M, TM)$ such that

$$\iota(X)\theta = X$$

for every $X \in \Gamma(TM)$.

1.6 Lie algebra valued differential forms

An important particular case of vector valued forms is given by Lie algebra valued forms. That case appears for example in the connection theory over principal bundle[7]. If ω and η are elements of $\Omega^1(M, \mathcal{G})$ for some Lie algebra $\mathcal{G}$, we define

$$(\omega \wedge \eta)(X, Y) = \omega(X) \otimes \eta(Y) - \omega(Y) \otimes \eta(X).$$

Combining with the Lie bracket, we define

$$[\omega \wedge \eta](X, Y) := [\omega(X), \eta(Y)] - [\omega(Y), \eta(X)]. \tag{1.54}$$

Using the proposition 4.21, we often implicitly transforms the tensor product into a product (4.104b)-153 and put

$$(\omega \wedge \omega)(X, Y) = [\omega(X), \omega(Y)]. \tag{1.55}$$

Let us point out the fact that that kind of formula only holds for a "wedge square", but not for a general product $\omega \wedge \eta$. Remark that for $\omega \in \Omega^1(M, \mathcal{G})$ and $\beta \in \Omega^2(M, \mathcal{G})$, a simple computation of definition (1.53)-31 yields

$$(\omega \wedge \beta)(X, Y, Z) = \omega(X) \otimes \beta(Y, Z) - \omega(Y) \otimes \beta(X, Z) + \omega(Z) \otimes \beta(X, Y), \tag{1.56}$$

so that, using the same trick as for equation (1.55)-31, we find

$$(\omega \wedge \beta - \beta \wedge \omega)(X, Y, Z) = [\omega(X), \beta(Y, Z)] - [\omega(Y), \beta(X, Z)] + [\omega(Z), \beta(X, Y)].$$

But that expression is exactly what we find by exchanging the tensor product by Lie bracket in expression (1.56)-31. So we define

$$[\omega \wedge \beta] = \omega \wedge \beta - \beta \wedge \omega \tag{1.57}$$

when $\omega \in \Omega^1(M, \mathcal{G})$ and $\beta \in \Omega^2(M, \mathcal{G})$. The reader should remark that this is what one would expect from generalisation of definition (1.54)-31.

[7]So in Maxwell and other gauge field theories.

1.7 Principal bundle

Let M be a manifold and G, a Lie group whose unit is denoted by e. A G-**principal bundle** on M is a smooth manifold P, a smooth map $\pi\colon P \to M$ and a right action of G on P denoted by $\xi \cdot g$ with $g \in G$ and $\xi \in P$ such that

- $\pi(\xi \cdot g) = \pi(\xi)$,
- $\forall \xi \in \pi^{-1}(x)$, $\pi^{-1}(x) = \{\xi \cdot g \text{ st } g \in G\} \simeq G$,
- $\forall x \in M$, there exists a neighbourhood $\mathcal{U}_\alpha$ of x in M, a diffeomorphism $\phi_\alpha\colon \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times G$ and a diffeomorphism $\phi_{\alpha x}\colon P \to G$ such that
  - $\phi_\alpha(\xi) = (x, \phi_{\alpha x}(\xi))$,
  - $\phi_{\alpha x}(\xi \cdot g) = \phi_{\alpha x}(\xi) \cdot g$.

The group G is often called the **structure group**. We suppose that the action is effective. We will sometimes use the notation $P(G, M)$ to precise that P is a principal bundle over M with structure group G.

The whole construction is given in figure 1.1. All is not yet defined, but in the following, the notations will follow this scheme.

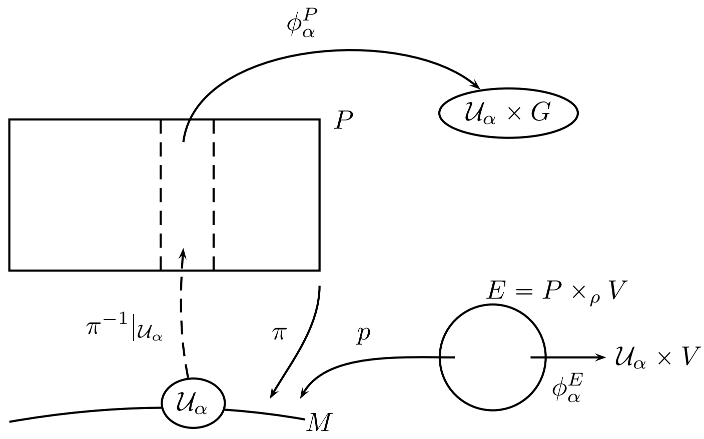


Figure 1.1: Some bundles

Lemma 1.30.
The map ϕ_α^{-1} fulfills

$$\phi\alpha^{-1}(x,h) \cdot g = \phi_\alpha^{-1}(x, hg).$$

Proof. From the definition of a principal bundle, any $\xi \in P$ can be written under the form $\xi = \phi_\alpha^{-1}(x, \phi_{\alpha x}(\xi))$ with ϕ_x satisfying $\phi_x(\xi \cdot h) = \phi_x(\xi)h$ for a certain function $\phi_x\colon P \to G$. We consider in particular $\xi = \phi_\alpha^{-1}(x,h) \cdot g$. Then $\xi \cdot g^{-1} = \phi_\alpha^{-1}(x,h)$. But $\xi \cdot g^{-1} = \phi_\alpha^{-1}(x, \phi_{\alpha x}(\xi)g^{-1})$, then $h = \phi\alpha_x(\xi)g^{-1}$ and $\phi_{\alpha x}(\xi) = hg$. So we have

$$\xi = \phi_\alpha^{-1}(x,h) \cdot g = \phi_\alpha^{-1}(x, \phi_{\alpha x}(\xi)) = \phi_\alpha^{-1}(x, hg).$$

□

Let

$$R = \{(x, y) \in P \times P \text{ st } x = y \cdot g \text{ for a certain } g \in G\}.$$

Proposition 1.31.
The function $u\colon R \to G$ defined by the condition

$$p \cdot u(p, q) = q.$$

is differentiable.

Proof. Let $\mathcal{U}$ be an open subset of M and $\sigma\colon \mathcal{U} \to P$, a section. We consider a differentiable map $\rho\colon \pi^{-1}(\mathcal{U}) \to G$ such that $\rho(\xi \cdot g) = \rho(\xi) \cdot g$ and $\rho(\sigma(x)) = e$. Such a map is given by

$$\rho(\xi) = \phi_x(\sigma(x))^{-1}\phi_x(\xi)$$

where $x = \pi(\xi)$. We naturally define $R_{\mathcal{U}} = R \cap (\pi^{-1}(\mathcal{U}) \times \pi^{-1}(\mathcal{U}))$ and we pick $(\xi, \eta) \in R_{\mathcal{U}}$. Let $s \in G$ be the one such that $\xi \cdot s = \eta$, so that $\rho(\xi) \cdot s = \rho(\eta)$. Then the restriction of u to $R_{\mathcal{U}}$ is given by $u(\xi, \eta) = \rho(\xi)^{-1}\rho(\eta)$ which makes $u|_{\mathcal{U}}$ differentiable. Since this reasoning can be made on every chart open $\mathcal{U}$, u is differentiable everywhere on P. □

The following is a corollary of Leibnitz rule.

Corollary 1.32.
If P is a G-principal bundle and v, a are curve in P and G respectively, we can consider the curve $u(t) = v(t)a(t)$. We have :

$$\frac{d}{dt}u(t)\Big|_{t=0} = \frac{d}{dt}v(t)a(0)\Big|_{t=0} + \frac{d}{dt}v(0)a(t)\Big|_{t=0}.$$

The proof is direct. This result is often written as

$$\dot{u}_t = \dot{v}_t a_t + v_t \dot{a}_t. \tag{1.58}$$

A main application is

$$\frac{d}{dt}\Big[r \cdot h(t)\Big]_{t=0} = \frac{d}{dt}\Big[r \cdot e^{th'(0)}\Big]_{t=0}. \tag{1.59}$$

1.7.1 Transition functions

Let $(\mathcal{U}_\alpha, \phi_\alpha)$ be a local trivialization of P. This induces transition functions $g_{\alpha\beta}\colon \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$ defined by

$$\begin{aligned} \phi_\alpha \circ \phi_\beta^{-1} \ \colon \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times G &\to \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times G \\ (x, a) &\mapsto (x, g_{\alpha\beta}(x)a). \end{aligned} \tag{1.60}$$

Clearly, $g_{\alpha\alpha} = e$ and $g_{\alpha\beta}g_{\alpha\beta} = e$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Then the triviality

$$\phi_\alpha \circ \phi_\beta^{-1} \circ \phi_\beta \circ \phi_\gamma^{-1} \circ \phi_\gamma \circ \phi_\alpha^{-1} = \mathrm{id}$$

implies the compatibility conditions

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = e \tag{1.61}$$

on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$.

There is an inverse construction. Let $\{\mathcal{U}_\alpha \text{ st } \alpha \in I\}$ be an open covering of M and $g_{\alpha\beta}\colon \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$ a family of functions such that $g_{\alpha\alpha} = e$, $g_{\alpha\beta}g_{\alpha\beta} = e$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = e$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$. Then the following construction gives a G-principal bundle whose transition functions are the $g_{\alpha\beta}$'s.

- $\tilde{P} = \bigsqcup_{\alpha\in I} \mathcal{U}_\alpha \times G$ (disjoint union),
- if $(x,a) \in \mathcal{U}_\alpha \times G$ and $(y,b) \in \mathcal{U}_\beta \times G$, then $(x,a) \sim (y,b)$ if and only if $x = y$ and $b = g_{\alpha\beta}(x)a$,
- $\pi\colon \tilde{P} \to M$ is defined by $\pi[(x,a)] = x$ where $[(x,a)]$ is the class of (x,a) for $\sim$,
- the action is defined by $[(x,a)] \cdot g = [(x,ag)]$.

Theorem 1.33.
Let G be a Lie group; M, a differentiable manifold; $\{\mathcal{U}_\alpha\}_{\alpha\in I}$, an open covering of M and some functions $\varphi_{\alpha\beta}\colon \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$ such that $\varphi_{\alpha\beta}(x) = \varphi_{\alpha\gamma}(x)\varphi_{\gamma\beta}(x)$. Then there exists a principal bundle P whose transition functions are the φ_α's for the covering $\{\mathcal{U}_\alpha\}_{\alpha\in I}$.

Proof. We consider the topological space

$$E = \bigcup_{\alpha\in I} (G \times \mathcal{U}_\alpha \times I) \tag{1.62}$$

where we put the discrete topology on I. Each $G\times\mathcal{U}_\alpha\times\{\alpha\}$ is a manifold. Thus E has a structure of differentiable manifold induced from the one of $G \times M$. We consider on E the equivalence relation given by the following subset of $E \times E$:

$$R = \big\{ \big((g,x,\alpha),(h,y,\beta)\big) \in E \times E \text{ st } y = x \text{ and } h = \varphi_{\alpha\beta}(x)g \big\} .$$

We will show that $P = E/R$ has a structure of principal bundle. We begin by defining an action of G on P by

$$[(g,x,\alpha)\cdot h] = [(gh,x,\alpha)].$$

In order to see that this definition is correct, let us consider $[g',x,\beta] = [g,x,\alpha]$. From the definition of the equivalence class, $g' = \varphi_{\alpha\beta}(x)g$. Then $[(g',x,\beta)] \cdot h = [(\varphi_{\alpha\beta}(g)gh,x,\beta)]$, and the form of R shows that this is well $[(gh,x,\alpha)]$. Since the map $(g,h) \to gh$ is differentiable on G, the so defined action is a differentiable action of G on P and G is a transformation group on P.

If $[(g,x,\alpha)] = [(gh,x,\alpha)]$, then $gh = \varphi_{\alpha\alpha}g = g$ and $h = e$. So the action is effective.

Now we consider the quotient P/G. A typical element is

$$\overline{(s,x,i)} = \{[s,x,i]\cdot g \text{ st } g \in G\}.$$

The projection $\pi\colon P \to M$, $[(s,x,\alpha)] \to x$ is well defined and we can consider $\varphi\colon P/G \to M$, $\varphi\overline{(s,x,\alpha)} = x$. It provides a bijection between P/G and M. So we can identify P/G and M. Now we are going to show that P endowed with the projection $\pi\colon P \to X$ is a principal bundle.

We consider the map

$$\begin{aligned} h_\alpha \,:\, G \times \mathcal{U}_\alpha &\to P \\ (g,x) &\mapsto \omega(g,x,\alpha) \end{aligned} \tag{1.63}$$

where $\omega\colon E \to P = E/R$ is the canonical projection. Since

$$(\pi\circ h_\alpha)(g,x) = (\pi\circ\omega)(g,x,\alpha) = \pi[(g,x,\alpha)] = x,$$

the map h_α actually is $h_\alpha\colon G\times\mathcal{U}_\alpha \to \pi^{-1}(\mathcal{U}_\alpha)$. In order to see that h_α is surjective on $\pi^{-1}(\mathcal{U}_\alpha)$, let us take a general element of $\pi^{-1}(\mathcal{U}_\alpha)$ under the form $\omega(g,x,\beta)$ with $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Then $(g,x,\beta) \in [(\varphi_{\alpha\beta}(x)g,x,\alpha)]$ and therefore $\omega(g,x,\beta) = h_\alpha(\varphi_{\alpha\beta}(x)g,x)$. For the injectivity, remark that $\omega(g,x,\beta) = \omega(h,y,\alpha)$ implies $x = y$ and $h = \varphi_{\beta\beta}(x)g = g$. In particular, $h_\alpha(g,x) = h_\alpha(h,y)$ implies $x = y$ and $g = h$.

Now we will prove that the inverse of h_α is continuous. For this we consider an open set $\Omega \subset G \times \mathcal{U}_\alpha$ and we have to show that $h_\alpha(\Omega)$ is open in $\pi^{-1}(\mathcal{U}_\alpha)$.

We recall the **quotient topology** : if A is a topological space with an equivalence relation $\sim$ and the canonical projection $\varphi\colon A \to A/\sim$, then $V \subset A/\sim$ is open if and only if $\varphi^{-1}(V) \subset A$ is open. So in our case, we have to check the openness of $V = \omega^{-1}(h_\alpha(\Omega))$ in E. We consider the open covering

$$\{G \times \mathcal{U}_\alpha \times \{\alpha\}\}_{\alpha\in I}$$

of E and we will show that the intersection of V with any of these open set is open. We have to show that $\omega^{-1}\big(h_\alpha(\Omega) \cap (G \times \mathcal{U}_\alpha \times \{\beta\})\big)$ is open for any $\beta \in I$. For this, we define a map $\alpha\colon G \times (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \{\beta\} \to G \times \mathcal{U}_\alpha$ by

$$\alpha_\beta(g,x,\beta) = (\varphi_{\alpha\beta}(x)g, x) \tag{1.64}$$

which is continuous. The set $(h_\alpha \circ \alpha_\beta)^{-1}(h_\alpha(\Omega)) = \alpha_\beta^{-1}(\Omega)$ is open because $h_\alpha \circ \alpha_\beta$ is the restriction of ω to $G \times (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \{\beta\}$. Then h_α is an homeomorphism from $G \times \mathcal{U}_\alpha$ tp $\pi^{-1}(\mathcal{U}_\alpha)$. Since it is build from differentiable functions, it is moreover a diffeomorphism.

So we have a chart system $\{(h_\alpha, \mathcal{U}_\alpha)\}_{\alpha\in I}$ where h_α fulfils the "good" properties with respect to π. It remains to be proved that the $\varphi_{\alpha\beta}$'s are the transition functions and that $\pi^{-1}(\pi(\xi)) = \xi \cdot G$ for every $\xi \in P$. We begin by the latter. For $\xi = [(g,x,\alpha)]$, $\pi(\xi) = x$ and we have to study the set

$$\pi^{-1}(x) = \{[(h,x,\beta)] \text{ st } h \in G, \beta \in I\}.$$

Clearly, $[(h,x,\beta)] \cdot G \subset \pi^{-1}(x)$. The fact that there is nothing else than $[(h,x,\beta)] \cdot G$ in $\pi^{-1}(x)$ is seen by

$$[h,x,\beta] = [\varphi_{\alpha\beta}(x)g, x, \alpha] \in [(h,x,\alpha)] \cdot G.$$

In order to check the change of charts, let us consider $g' = h_{\beta,x}^{-1} \circ h_{\alpha,x}(g)$ where

$$h_{\alpha,x}(g) = h_\alpha(g,x) = \omega(g,x,\alpha). \tag{1.65}$$

The fact that $h_\beta(g',x) = g_\alpha(g,x)$ concludes the proof. To see this fact, remark that $h_{\beta,x}(h_{\beta,x}^{-1} \circ h_{\alpha,x}(g)) = h_{\alpha,x}(g)$, so that $h_\alpha(g',x) = h_\alpha(g,x)$ implies $\omega(g',x,\beta) = \omega(g,x,\alpha)$ which proves that $g' = \varphi_{\alpha\beta}(g)$. □

The **trivial bundle** is simply $P = M \times G$ and $\pi(x,g) = x$ with the action $(x,a) \cdot g = (x, ag)$.

1.7.2 Morphisms and such. . .

An **homomorphism** between $P(G,M)$ and $P'(G',M')$ is a differentiable map $h\colon P \to P'$ such that $\forall \xi \in P, g \in G$,

$$h(\xi \cdot g) = h(\xi) \cdot h_G(g) \tag{1.66}$$

where $h_G\colon G \to G'$ is a Lie group homomorphism. From the definition, h maps a fiber to only one fiber, but it is not specially surjective on any fiber. So h induces a homomorphism $h_M\colon M \to M'$ such that $\pi' \circ h = h_M \circ \pi$.

An **isomorphism** is a homomorphism $g\colon P(G,M) \to P'(G',M')$ such that

- h_P is a diffeomorphism $P \to P'$,
- h_G is a Lie group homomorphism $G \to G'$, and

- h_M is a diffeomorphism $M \to M'$.

A principal bundle is **trivial** if one can find an isomorphism $h\colon G \times M \to P$ such that $\pi \circ h = \mathrm{id} \circ \mathtt{pr}_2$, i.e. the following diagram commutes :

$$
\begin{tikzcd}
G \times M \arrow[r, "h"] \arrow[d, "\mathtt{pr}_2"] & P \arrow[d, "\pi"] \\
M \arrow[r, "\mathrm{id}"'] & M
\end{tikzcd}
\tag{1.67}
$$

We say that P is **locally trivial** if for every $x \in M$, there exists an open neighbourhood $\mathcal{U}$ in M such that $\pi^{-1}(\mathcal{U})$ endowed with the induced structure of principal bundle is trivial.

1.7.3 Frame bundle: first

In the ideas, the building of a vector bundle is just to put a vector space on each point of the base manifold. A principal bundle is to put something on which a group acts on each point. If you have a vector bundle on a manifold, you can consider, on each point $x \in M$, the set of all the basis of the fiber E_x over x. The group $GL(r, \mathbb{K})$ naturally acts on this set which becomes a candidate to be a $GL(r, \mathbb{K})$-principal bundle.

More formally, we consider a vector bundle $F \xrightarrow{p} M$, and for each x, the set of the basis of the vector space $F_x = p^{-1}(x)$. We define

$$
P = \bigcup_{x \in M} (\text{basis of } F_x).
$$

We naturally consider the projection $\pi\colon P \to M$, $\pi(b_x) = x$ if b_x is a basis of F_x.

Let $\phi^F_\alpha\colon p^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{K}^r$ be a local trivialization of F, and $\{\overline{e}_1, \ldots, \overline{e}_r\}$, the canonical basis of $\mathbb{K}^r$. We naturally define

$$
\overline{S}_{\alpha i}(x) = \phi^{F\,-1}_\alpha(x, \overline{e}_i).
$$

The set $\{\overline{S}_{\alpha 1}(x), \ldots, \overline{S}_{\alpha r}(x)\}$ is a "reference" basis of F_x with respect to the trivialization ϕ_α. If we choose another basis $\{\overline{v}_1, \ldots \overline{v}_r\}$ of F_x, we can find a matrix $A \in GL(r, \mathbb{K})$ such that $\overline{v}_k = A^l_k \overline{S}_{\alpha l}(x)$. This gives a bijection

$$
\begin{aligned}
\phi^P_\alpha \ &: \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times GL(r, \mathbb{K}) \\
(\overline{v}_1, \ldots, \overline{v}_r) &\mapsto (x, A).
\end{aligned}
\tag{1.68}
$$

One can give to P a $GL(r, \mathbb{K})$-principal bundle structure such that the ϕ^P_α are diffeomorphism.

Let $(\mathcal{U}_\alpha, \phi^F_\alpha)$ be a local trivialization of F and $g^F_{\alpha\beta}\colon \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(r, \mathbb{K})$. In this case, $(\mathcal{U}_\alpha, \phi^P_\alpha)$ is a trivialization of P whose transition function is $g^P_{\alpha\beta} = g^F_{\alpha\beta}$. Indeed

$$
\phi^P_\alpha \circ \phi^{P\,-1}_\beta(x, A) = \phi^P_\alpha(\{\overline{v}_1, \ldots, \overline{v}_r\})
$$

where $\overline{v}_s = (\phi^F_\beta)^{-1}(x, A^l_s \overline{e}_l)$. In order to see it, recall that $\overline{v}_s = A^l_s \overline{S}_{\alpha l}(x)$ and that $\phi^{F\,-1}_\alpha(x, \overline{e}_s) = \overline{S}_{\alpha s}(x)$. Then

$$
\overline{v}_s = (\phi^F_\beta)^{-1}(x, A^l_s \overline{e}_l) = A^l_s \overline{S}_{\alpha s}(x).
$$

On the other hand, from the definition of ϕ^P_β, the basis $(\phi^P_\beta)^{-1}(x,A)$ is the one obtained by applying A on S. With all this,

$$\begin{aligned}\phi^P_\alpha\circ(\phi^P_\beta)^{-1}(x,A) &= \phi^P_\alpha\{(\phi^F_\beta)^{-1}(x,A^l_s\overline{e}_l)\}_{s=1,\ldots r}\\ &= \phi^P_\alpha\{(\phi^F_\alpha)^{-1}\circ(\phi^E_\alpha\circ\phi^{F\,-1}_\beta)(x,A^l_s\overline{e}_l)\}_{s=1,\ldots r}\\ &= \phi^P_\alpha\{(\phi^E_\alpha)^{-1}(x,g^F_{\alpha\beta}(x)^s_iA^l_s\overline{e}_l)\}_{i=1,\ldots r}\\ &= (x,g^F_{\alpha\beta}(x)A).\end{aligned}\tag{1.69}$$

The last product $g^F_{\alpha\beta}(x)A$ is a matricial product.

1.7.4 Frame bundle: second

Basis

If M is a m-dimensional manifold, a **frame** of T_xM is an isomorphism $b\colon\mathbb{R}^m\to T_xM$. In our purpose, we will always deal with (pseudo)Riemannian manifold. So, the tangents spaces T_xM comes with a metric, and we ask a frame to be isometric. In other words, we ask b to be an isometry from $(\mathbb{R}^m,\cdot)$ to (T_xM,g_x), where the dot denotes the (pseudo)euclidian product on $\mathbb{R}^m$. Such a frame is given by a base point x of M and a matrix S in $\mathrm{SO}(g_x)$:

$$b(v)=(Sv)^i(\partial_i)_x,\tag{1.70}$$

if the vector v is written as $v=v^i\overline{1}_i$ in the canonical orthogonal frame $\{\overline{1}_i\}$ of $\mathbb{R}^m$ and $\mathrm{SO}(g_x)$ is the set of the $m\times m$ matrix A such that $A^tg_xA=g_x$.

This frame intuitively corresponds to the basis of T_xM (see as a “true” vector space) that we would have written by $\{Se_i\}_x$ if $e_i=\frac{\partial}{\partial x^i}$. In order to follow this idea, we will effectively denote by $\{Se_i\}_x$ the map $b\colon\mathbb{R}^m\to T_xM$ given by (1.70)-37.

We will often write the frame b as $\{be_i\}_x$, making no differences in notation between the b of $\mathrm{SO}(M)$ and the b of $\mathrm{SO}(g_x)$ which implement it.

Remark 1.34. One has to distinguish a *frame* and a *basis* : a basis is only a free and generator set while a frame can be interpreted as an ordered basis.

Construction

We just saw how to build a frame bundle over a manifold. One can get another expression of the frame bundle when we express a basis of T_xM by means of an isomorphism between $\mathbb{R}^n$ and T_xM. If M is a n-dimensional manifold, a **frame** at x is an ordered basis

$$b=(\mathbf{b}_1,\ldots,\mathbf{b}_n)$$

of T_xM. It is clear that any frame defines an isomorphism (linear bijective map)

$$\begin{aligned}\tilde{b}\colon\mathbb{R}^n&\to T_xM\\ e_i&\mapsto\mathbf{e_i}\end{aligned}\tag{1.71}$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. It is also clear that any isomorphism gives rise to a frame. Then we see a frame of M at x as an isomorphism $\tilde{b}\colon\mathbb{R}^n\to T_xM$. Let $B(M)_x$ be the of all the frames of M at x; we define

$$B(M)=\bigcup_{x\in M}B(M)_x.$$

For all $b \in B(M)_x$, we define $p_B(b) = x$ and the action $B(M) \times GL(n, \mathbb{R}) \to B(M)$ by $b \cdot g = (\mathbf{b}'_1, \ldots, \mathbf{b}'_n)$ where

$$\mathbf{b}'_j = \mathbf{b}_i g_i^{\ j}. \tag{1.72}$$

It is easy to see that $\widetilde{b \cdot g} = \tilde{b} \circ g \colon \mathbb{R}^n \to T_x M$. So we can give to

$$\begin{tikzcd} GL(n, \mathbb{R}) \arrow[r, rightsquigarrow] & B(M) \arrow[d, "p_B"] \\ & M \end{tikzcd} \tag{1.73}$$

a structure of principal bundle[8]. If $(\mathcal{U}_\alpha, \varphi_\alpha)$ is a local coordinate chart on M, we define

$$\begin{aligned} \tilde{\varphi} \ : \ p_B^{-1}(\mathcal{U}_\alpha) &\to \varphi_\alpha(\mathcal{U}_\alpha) \times GL(n, \mathbb{R}) \\ b &\mapsto (\varphi_\alpha(x), A(b)) \end{aligned} \tag{1.74}$$

where $A(b) \in GL(n, \mathbb{R})$ is defined by the condition $\mathbf{b}_j = A_j^{\ i} \partial_i|_x$. The matrix $A(b)$ is the one which transforms the canonical basis (in the trivialization φ_α) into $b \in B(M)_x$. That's for the principal bundle structure.

The manifold structure of $B(M)$ is given by $\Phi_\alpha \colon p_B^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times GL(\mathbb{R})$,

$$\begin{aligned} \Phi(b) &= (\varphi_\alpha^{-1} \times \mathrm{id}\,|_{GL(n,\mathbb{R})}) \circ \tilde{\varphi}(b) \\ &= (x, A(b)) \\ &= (p_B(b), A(b)). \end{aligned} \tag{1.75}$$

It fulfils $A(b \cdot g) = A(b) \cdot g$. A section $s \colon \mathcal{U}_\alpha \to B(M)$ is sometimes called a **moving frame** over $\mathcal{U}_\alpha$.

Frame bundle over $\mathbb{R}^2$ is given as example in page 123

1.7.5 Sections of principal bundle

A **section** of a G-principal bundle is a smooth map $s \colon M \to P$ such that $s(x) \in \pi^{-1}(x)$ for any $x \in M$. A trivialization ϕ_α^P P on $\mathcal{U}_\alpha$ defines a section of P over $\mathcal{U}_\alpha$ by

$$\sigma_\alpha(x) = (\phi_\alpha^P)^{-1}(x, e)$$

where e is the neutral of the group. In the inverse sense, we have the following :

Proposition 1.35.
If $\sigma_\alpha \colon \mathcal{U}_\alpha \to P$ is local section of P over $\mathcal{U}_\alpha \subset M$, then the definition $\phi_\alpha^P(\xi) = (x, a)$ if $\xi = \sigma_\alpha(x) \cdot a$ is a local trivialization.

Proof. The function ϕ_α^P is well defined because $\xi \in \pi^{-1}(\mathcal{U}_\alpha)$ implies the existence of a $x \in \mathcal{U}_\alpha$ such that $\xi \in \pi^{-1}(x) = \{\xi \cdot g\} \simeq G$. For this x, there exists a $g \in G$ such that $\xi = \sigma_\alpha(x) \cdot g$.

Now we prove that the couple (x, a) is unique in the sense that $s_\alpha(x) \cdot a = \sigma_\alpha(y) \cdot b$ implies $(x, a) = (y, b)$. The left hand side belongs to $\pi^{-1}(x)$ while the right one belongs to $\pi^{-1}(y)$. Then $x = y$. The condition $\pi^{-1}(x) \simeq G$ imposes the unicity of the g making $\xi = \eta \cdot g$ for each couple, $\xi, \eta \in \pi^{-1}(x)$. □

[8]Much more details and proofs are given in [8].

If σ and σ' are two sections of the same principal bundle P, then there exists a differentiable map $f\colon M \to G$ such that $\sigma'(x) = \sigma(x) \cdot f(x)$. So all the sections can be deduced from only one and multiplication by such a f.

Theorem 1.36.
If $\pi\colon P(G,M) \to M$ is a principal bundle, then the four following propositions are equivalent :

(i) *P is trivial,*

(ii) *P has a global section,*

(iii) *there exists a differentiable map $\gamma\colon P \to G$ such that $\gamma(\xi \cdot g) = g^{-1}\gamma(\xi)$ for all $\xi \in P$ and $g \in G$,*

(iv) *there exists a differentiable map $\rho\colon P \to G$ such that $\rho(\xi \cdot g) = \rho(\xi)g$.*

Proof. *(i)*⇒ *(ii)*. The diagram (1.67)-36 commutes and

$$\begin{aligned} \tau : M &\to G \times M \\ x &\mapsto (e, x) \end{aligned} \tag{1.76}$$

is a local section of $G \times M$. From it we build the following global section of P:

$$\begin{aligned} \sigma : M &\to P \\ x &\mapsto h(e, x). \end{aligned} \tag{1.77}$$

This is injective because $\pi \circ h = \mathtt{pr}_2$ and differentiable because this is a composition of $x \to (e, x)$ and $(g, x) \to h(g, x)$.
(ii)⇒ *(i)*. The principal bundle P admits a global section $\sigma\colon M \to P$. From it, we can build the differentiable map

$$\begin{aligned} h : G \times M &\to P \\ (g, x) &\mapsto \sigma(x) \cdot g \end{aligned} \tag{1.78}$$

which satisfies $h(gh, x) = h(g, x) \cdot h$ and $\pi \circ (g, x) = x$. First we show that h is a fiber homomorphism and an isomorphism between P and $G \times M$ so that P is trivial. For this remark that

$$g(gh, x) = g(g, x) \cdot h = \sigma(x) \cdot gh,$$

hence equation (1.66)-35 reduces to $h((g, x) \cdot h) = h(g, x) \cdot h_G(h)$ which is true with $h_G = \mathrm{id}$. Moreover $h\colon G \times M \to P$ is bijective because $\sigma(\pi(\xi))$ belongs to the fiber of $\xi \in P$, therefore there is one and only one $\gamma(\xi) = u(\xi, \sigma(\pi(\xi)))$ such that $\xi \cdot \gamma(\xi) = (\sigma \circ \pi)\xi$. The inverse map is

$$\begin{aligned} \theta : P &\to G \times M \\ \xi &\mapsto (\gamma(\xi), \pi(\xi)) \end{aligned} \tag{1.79}$$

which is differentiable because γ and π are. So far we see that h and h^{-1} are differentiable. Then h is an isomorphism between P and $G \times M$.
(ii)⇒ *(iii)*. Let σ be the global section and define

$$\begin{aligned} \gamma : P &\to G \\ \xi &\mapsto u(\xi, (\sigma \circ \pi)\xi) \end{aligned} \tag{1.80}$$

where $u\colon R \to G$ is the map defined by the condition $\xi \cdot (\xi, \eta) = \eta$. The map γ is differentiable and we have to prove that $\gamma(\xi \cdot g) = g^{-1}\gamma(\xi)$. Since $\xi \cdot \gamma(\xi) = \sigma \circ \pi(\xi)$,

$$\gamma(\xi \cdot g) = u(\xi \cdot g, (\sigma \circ \pi)(\xi \cdot g)) = u(\xi \cdot g, (\sigma \circ \pi)(\xi)).$$

But $(\xi \cdot g)(g^{-1}\gamma(\xi)) = \xi \cdot \gamma(\xi) = x$. So $\gamma(\xi \cdot g) = u(\xi \cdot g, x)$. Thus $\gamma(\xi \cdot g) = g^{-1}\gamma(\xi)$.
(iii)⇒ *(ii)*. The given map γ fulfils $\xi \cdot g\gamma(\xi \cdot g) = \xi \cdot (\xi)$, so

$$\begin{aligned} \varphi \,:\, P &\to P \\ \xi &\mapsto \xi \cdot (\xi) \end{aligned} \tag{1.81}$$

is just function of the class of ξ, thus we have a section $\sigma'\colon P/G \to P$, but we know that P/G and M are isomorphic.
(iii)⇒ *(iv)*. Let us define $\rho\colon P \to G$ by $\rho = J \circ \gamma$ with $J(g) = g^{-1}$, thus $\rho(\xi) = \gamma(\xi)^{-1}$ and

$$\rho(\xi \cdot g) = \gamma(\xi \cdot g)^{-1} = (g^{-1}\gamma(\xi))^{-1} = \gamma(\xi)^{-1}g = \rho(\xi)g.$$

(iv)⇒ *(iii)*. The proof is just the same with $\rho = J \circ \rho$. □

Definition 1.37.
*A section $\psi \in \Gamma(P, TP)$ is G-**equivariant** when*

$$d\tau_g\psi(\xi) = \psi(\xi \cdot g).$$

Be careful: this *does not* define equivariant sections of the principal bundle.

1.7.6 Equivalence of principal bundle

Two principal bundles $\pi\colon P \to M$ and $\pi'\colon P' \to M$ are **equivalent** if there exists a diffeomorphism $\varphi\colon P \to P'$ such that

- $\pi' \circ \varphi = \pi$
- $\varphi(\xi \cdot g) = \varphi(\xi) \cdot g$.

If $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is an open covering of M on which we have trivializations ϕ_α of P and ψ_α of P', the diffeomorphism φ induces some functions $\lambda\colon \mathcal{U}_\alpha \to G$ by setting

$$(\phi_\alpha \circ \varphi^{-1} \circ \psi_\alpha^{-1})(x, a) = (x, \lambda_\alpha(x)a).$$

This definition works because from the definitions of principal bundle and equivalence, one sees that $(\phi_\alpha \circ \varphi^{-1} \circ \psi_\alpha^{-1})(x, \cdot) = (x, \cdot)$.

Transition functions

We have some transition functions for P and P' given by equations

$$\begin{aligned} (\phi_\alpha \circ \phi_\beta^{-1})(x, g) &= (x, g_{\alpha\beta}(x)g) \\ (\psi_\alpha \circ \psi_\beta^{-1})(x, g) &= (x, g'_{\alpha\beta}(x)g). \end{aligned}$$

Now, we want to know what is $g'_{\alpha\beta}$ in function of $g_{\alpha\beta}$. First remark that $(\psi_\alpha \circ \varphi \circ \phi_\alpha^{-1})(x,a) = (x, \lambda_\alpha(x)^{-1})a$, and next, compute

$$\begin{aligned}(x, g_{\alpha\beta}(x)a)a &= (\psi_\alpha \circ \varphi \circ \phi_\beta^{-1} \circ \phi_\beta \circ \varphi^{-1} \circ \psi_\beta^{-1})(x,a)\\ &= (\psi_\alpha \circ \varphi \circ \phi_\beta^{-1})(x, \lambda_\beta(x)a)\\ &= (\psi_\alpha \circ \varphi \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1})(x, \lambda_\beta(x)a)\\ &= (x, \lambda_\alpha(x)^{-1} g_{\alpha\beta}(x)\lambda_\beta(x)a).\end{aligned} \tag{1.82}$$

Then

$$g_{\alpha\beta}(x) = \lambda_\alpha(x)^{-1} g_{\alpha\beta}(x)\lambda_\beta. \tag{1.83}$$

One can show that if two principal bundle have transition functions whose fulfill this condition, they are equivalent. A G-principal bundle is **trivial** if it is equivalent to the one given by $\pi_1 \colon M \times G \to M$.

1.7.7 Reduction of the structural group

We say that a principal bundle $P(G,M)$ is **reducible** when there exists a principal bundle $P'(H,M)$ such that

- H is a subgroup of G,
- there exists an homeomorphism $h\colon P' \to P$ such that $h_G\colon H \to G$ is an injective homomorphism.

In this case we say that G is reducible to H and that P' is a reduced principal bundle.

Theorem 1.38.
If P is a principal bundle over M, the structural group G is reducible to the Lie subgroup H if and only if there exists an open covering $\{\mathcal{U}_i\}_{i\in I}$ of M and transition functions φ_{ij} taking their values in H.

Proof. No proof. □

The following comes from [9]. Let us consider the principal bundle

$$\begin{tikzcd} G \arrow[r, squiggly] & P \arrow[d, "\pi_P"] \\ & M \end{tikzcd} \tag{1.84}$$

and H, a closed subgroup of G. We denote by $j\colon H \to G$ the inclusion map. The principal bundle

$$\begin{tikzcd} H \arrow[r, squiggly] & Q \arrow[d, "\pi_Q"] \\ & M \end{tikzcd} \tag{1.85}$$

is a **reduction** of P to the group H if there exists a map $u\colon Q \to P$ such that $\pi_P \circ u = \pi_Q$ and $u(\xi \cdot h) = u(\xi) \cdot j(h)$. In this case, u is an embedding [9] of Q in P and the image is a closed submanifold of P.

[9] plongement

Let M be a n-dimensional manifold and $B(M)$ be its frame bundle. This is a $GL(n,\mathbb{R})$-principal bundle. If G is a closed subgroup[10] of $GL(n,\mathbb{R})$, a **G-structure** is a reduction of $B(M)$ to G.

1.7.8 Density

A **density** on a d-dimensional manifold M is a section of the principal bundle whose fiber P_x over $x \in M$ is the space of homogeneous non vanishing maps

$$\rho\colon \bigwedge^{d} T_xM \to \mathbb{R}^*_+ \tag{1.86}$$

such that $\rho(\lambda v) = |\lambda|\rho(v)$ for every $\lambda \in \mathbb{R}$ and $v \in \bigwedge^d T_xM$.

1.8 Associated bundle

Let $\pi\colon P \to M$ be a G-principal bundle and $\rho\colon G \to GL(V)$, a representation of G on a vector space V (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of dimension r.

The associated bundle $E = P \times_\rho V \xrightarrow{p} M$ is defined as following. On $P \times V$, we consider the equivalence relation

$$(\xi, v) \sim (\xi \cdot g, \rho(g^{-1})v)$$

for $g \in G$, $\xi \in P$ and $v \in V$. Then we define

- $E = P \times_\rho V := P \times V/\sim$,
- $p\big[(\xi, v)\big] = \pi(\xi)$

where $\big[(\xi, v)\big]$ is the class of (ξ, v) in $P \times V$.

If $\phi^P_\alpha(\xi) = (\pi(\xi), a(\xi))$ is a trivialization of P on $\mathcal{U}_\alpha$, then

$$\phi^E\big[(\xi, v)\big] = (\pi(\xi), \rho(a)v) \tag{1.87}$$

is a trivialization of E.

In order to see that it is a good definition, let us consider $(\eta, w) \sim (\xi, v)$. It immediately gives the existence of a $g \in G$ such that $\eta = \xi \cdot g$ and $w = \rho(g^{-1})v$. Then $\phi^E\big[(\xi \cdot g, \rho(g^{-1})v)\big] = (\pi(\xi \cdot g), \rho(b)\rho(g^{-1})v)$. From the definition of ϕ^E, the vector b is given by $\phi^P(\xi \cdot g) = (\pi(\xi \cdot g), b)$, and the definition of a principal bundle gives $b = \phi_{\pi(\xi)}(\xi \cdot g) = \phi_{\pi(\xi)}(\xi) \cdot g = ag$. The fact that ρ is a homomorphism makes $\rho(ag)\rho(g^{-1}) = \rho(a)v$ and ϕ^E is well defined.

Let G be a Lie group, ρ a representation of G on V and M, a manifold. We consider $P = M \times G \xrightarrow{\mathtt{pr}_1} M$, the trivial G-principal bundle on M. Then $E = P \times_\rho V \xrightarrow{p} M$ is **trivial**, i.e. we can build a $\varphi\colon P \times_\rho V \to M \times V$ such that $\mathtt{pr}_1 \circ \varphi = p$. It is rather easy: we define

$$\varphi\big[\big((x, g), v\big)\big] = (x, \rho(g)v).$$

It is easy to see that $(\mathtt{pr}_1 \circ \varphi)[(x, g), v] = x$ and $p[(x, g), v] = \mathtt{pr}_1(x, g) = x$.

[10] Typically $\mathrm{SO}(p, q)$ or $\mathrm{SO}_0(p, q)$.

1.8.1 Transition functions

Proposition 1.39.
Let $(\mathcal{U}_\alpha, \phi^P_\alpha)$ *be a trivialization of* $P \xrightarrow{\pi} M$ *whose transition functions are* $g_{\alpha\beta}\colon \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$. *Then* $(\mathcal{U}_\alpha, \phi^E_\alpha)$ *given by* (1.87)-42 *is a local trivialization of* $E \xrightarrow{p} M$ *whose transition functions* $g^E_{\alpha\beta}\colon \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(\dim V, \mathbb{K})$ *are given by*

$$g^E_{\alpha\beta}(x) = \rho(g^P_{\alpha\beta}(x)).$$

Proof. If we write $a := \phi^E_{\beta x}(\pi^{-1}(x))$, we have $\phi^P_\beta(\pi^{-1}(x)) = (x, a)$ and $\phi^E_\alpha \circ (\phi^E_\beta)^{-1}(x, v) = \phi^E_\alpha[(\pi^{-1}(x), \rho(a)^{-1}v)]$. So,

$$\begin{aligned} \phi^E_\alpha[(\pi^{-1}(x), \rho(a)^{-1}v)] &= \Big(x, \rho\big(\phi_{\alpha x}(\pi^{-1}(x))\big)\rho\big(\phi_{\beta x}(\pi^{-1}(x))\big)^{-1}v\Big) \\ &= \Big(x, \rho\big(\phi_{\alpha x}(\pi^{-1}(x))\phi_{\beta x}(\pi^{-1}(x))\big)\Big). \end{aligned} \tag{1.88}$$

Then

$$g^E_{\alpha\beta} = \rho\Big(\phi_{\alpha x}(\pi^{-1}(x))\phi_{\beta x}(\pi^{-1}(x))\Big) = \rho(g^P_{\alpha\beta}(x)). \tag{1.89}$$

□

1.8.2 Sections on associated bundle

Equivariant functions

We consider a bundle $E = P \times_\rho V \xrightarrow{p} M$ associated with the principal bundle $P \xrightarrow{\pi} M$ and a section $\psi\colon M \to E$.

$$\begin{tikzcd} P \arrow[dr, "\pi^P"'] & & E = P \times_\rho V \arrow[dl, "\pi^E"] \\ & M & \end{tikzcd}$$

A **section** of E is a map $\psi\colon M \to E$ such that $\pi^E \circ \psi = id_M$. We define the function $\hat{\psi}\colon P \to V$ by

$$\psi(\pi(\xi)) = [\xi, \hat{\psi}(\xi)]. \tag{1.90}$$

Let us see the condition under which this equation well defines $\hat{\psi}$. First, remark that a ψ defined by this equation is a section because $p[\xi, v] = \pi(\xi)$, so that $(p \circ \psi)(\pi(\xi)) = \pi(\xi)$. Now, consider a η such that $\pi(\eta) = \pi(\xi)$. Then there exists a $g \in G$ for which $\eta \cdot g = \xi$. For any g and for this one in particular,

$$\psi(\pi(\eta)) = [\eta, \hat{\psi}(\eta)] = [\eta \cdot g, \rho(g^{-1})\hat{\psi}(\eta)].$$

Then equation (1.90)-43 defines $\hat{\psi}$ from ψ if and only if

$$\hat{\psi}(\xi \cdot g) = \rho(g^{-1})\hat{\psi}(\xi). \tag{1.91}$$

This condition is called the **equivariance** of $\hat{\psi}$. Reciprocally, any equivariant function $\hat{\psi}$ defines a section of $E = P \times_\rho V$.

If $\eta = \xi \cdot g = \chi \cdot k$, one define a sum

$$[\xi, v] + [\chi, w] = [\eta, \rho(g)v + \rho(k)w]. \tag{1.92}$$

If $\psi, \eta\colon M \to E$ are two sections defined by the equivariant functions $\hat{\psi}, \hat{\eta}\colon P \to V$, then the section $\psi + \eta$ is defined by the equivariant function $\hat{\psi} + \hat{\eta}$.

For the endomorphism of sections of E

Let us now make a step backward, and take A in $\operatorname{End}\Gamma(E)$. We will now see that A defines (and is defined by) an equivariant function $\hat{A}\colon P \to \operatorname{End} V$. Let $\psi\colon M \to E$ be in $\Gamma(E)$. If $\psi(x) = [\xi, v]$, we define the new section $A\psi$ by

$$(A\psi)(x) = [\xi, \hat{A}(\xi)v] = [\xi, \hat{A}(\xi)\hat{\psi}(\xi)].$$

In order for $A\psi$ to be well defined, the function $\hat{A}$ must satisfy

$$\hat{A}(\xi \cdot g) = \rho(g^{-1})\hat{A}(\xi)\rho(g) \tag{1.93}$$

for all g in G.

Local expressions

We consider a local trivialization $\phi^P_\alpha\colon \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times G$ of P on $\mathcal{U}_\alpha$ and the corresponding section $\sigma_\alpha\colon \mathcal{U}_\alpha \to P$ given by

$$\sigma_\alpha(x) = (\phi^P_\alpha)^{-1}(x, e).$$

We saw at page 42 that a trivialization of P gives a trivialization of the associated bundle $E = P \times_\rho V$; the definition is

$$\phi^E_\alpha[(\xi, v)] = (\pi(\xi), \rho(a)v) \tag{1.94}$$

if $\phi^P_\alpha(\xi) = (\pi(\xi), a)$. With $\xi = \sigma_\alpha(x)$, we find

$$\phi^E_\alpha[(\sigma_\alpha(x), v)] = (\pi(\sigma_\alpha(x)), \rho(a)v) = (x, v). \tag{1.95}$$

The section ψ can also be seen with respect to the "reference" sections σ_α by means of the definition

$$\psi(x) = [\sigma_\alpha(x), \psi_{(\alpha)}(x)] \tag{1.96}$$

for a function $\psi_{(\alpha)}\colon M \to V$.

Lemma 1.40.
Let $\psi\colon M \to E$ be a section and $\hat{\psi}\colon P \to V$, the corresponding equivariant function. Then

$$\psi_{(\alpha)}(x) = \hat{\psi}(\sigma_\alpha(x)).$$

Proof. By definition, $\psi(x) = \psi(\pi(\xi)) = [\xi, \hat{\psi}(\xi)]$. Thus if we consider in particular $\xi = \sigma_\alpha(x)$,

$$\phi^E_\alpha(\psi(x)) = \phi^E_\alpha[\xi, \hat{\psi}(\xi)] = \phi^E_\alpha[s_\alpha(x), \hat{\psi}(\sigma_\alpha(x))] = (x, \hat{\psi}(\sigma_\alpha(x))). \tag{1.97}$$

□

Let us anticipate. A **spinor** is a section of an associated bundle $E = P \times_\rho V$ where P is a Lorentz-principal bundle, $V = \mathbb{C}^2$ and ρ is the spinor representation of Lorentz on $\mathbb{C}^2$. So a spinor $\psi\colon M \to E$ is *locally* described by a function $\psi_{(\alpha)}\colon M \to \mathbb{C}^2$. The latter is the one that we are used to handle in physics. In this picture, the transformation law of ψ under a Lorentz transformation comes naturally.

Let $\{e_i\}$ be a basis of V; we consider some "reference" sections $\gamma_{\alpha i}$ of the associated bundle $E = P \times_\rho V$ defined by

$$\gamma_{\alpha i}(x) = [\phi^{-1}_\alpha(x, e), e_i]. \tag{1.98}$$

A general section $\psi\colon M \to E$ is defined by an equivariant function $\hat{\psi}\colon P \to V$ which can be written as $\hat{\psi}(\xi) = a^i(\xi)e_i$. If $\eta = \phi_\alpha^{-1}(x, e)$ and $\xi = \eta \cdot g(\xi)$,

$$\psi(x) = [\xi, a^i e_i] = a^i[\eta, \rho(g)e_i] = a^i(\xi)\rho(g(\xi))_i{}^j[\eta, e_j] = c^j(\xi)\gamma_{\alpha j}(x). \tag{1.99}$$

Since the left hand side of this equation just depends on x, the functions c^j must actually not depend on the choice of $\xi \in \pi^{-1}(x)$. So we have $c^j : M \to \mathbb{R}$. Indeed, if we choose $\chi \in \pi^{-1}(x)$,

$$\psi(x) = c^j(\xi)\gamma_{\alpha j}(x) \stackrel{!}{=} [\xi, a^i(\chi)e_i] = \ldots = c^j(\chi)\gamma_{\alpha j}(x),$$

so that $c^j(\xi) = c^j(\chi)$. So any section $\psi\colon M \to E$ can be decomposed (over the open set $\mathcal{U}_\alpha$) as

$$\psi(x) = s_\alpha^i(x)\gamma_{\alpha i}(x). \tag{1.100}$$

1.8.3 Associated and vector bundle

General construction

We are going to see that a vector bundle is an associated bundle. For this, we consider a vector bundle $p\colon F \to M$ with a fiber $F_x = V$ of dimension m. Let $G = GL(V)$, P be the trivial principal bundle $P = M \times G$ and ρ be the definition representation of G on V. We set $E = P \times_\rho V$. Our aim is to put a vector bundle structure on E which is equivalent to the one of F. The bijection $b\colon F \to E$ will clearly be

$$b(\phi^{-1}(x, v)) = [(x, e), v]. \tag{1.101}$$

We define the projection $q\colon E \to M$ by

$$q[(x, g), w] = x$$

and we have to show that $q^{-1}(x) = \{[(x, g), w] \text{ st } g \in G \text{ and } w \in V\}$ is a vector space isomorphic to V. The following definitions define a vector space structure:

- multiplication by a scalar: $\lambda[(x, g), v] = [(x, g), \lambda v]$,
- addition: $[(x, g), v] + [(x, h), w] = [(x, e), \rho(g)v + \rho(h)w]$.

As local trivialization map, we consider

$$\begin{aligned} \chi\colon q^{-1}(\mathcal{U}) &\to \mathcal{U} \times V \\ [(x, g), v] &\mapsto (x, \rho(g)v). \end{aligned} \tag{1.102}$$

With this structure, the bijection b is an equivalence because $b|_{F_x}$ is a vector space isomorphism and $q \circ b = p$.

1.8.4 Equivariant functions for a vector field

In order to define in the same way an equivariant function for a vector field $X \in \mathfrak{X}(M)$, we need to see TM as an associated bundle.

Proposition 1.41.
If M is a n dimensional manifold, we have the following isomorphism:

$$\mathrm{SO}(M) \times_{\rho^M} \mathbb{R}^m \simeq TM$$

where $\rho^M :\ \mathrm{SO}(m) \times \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\rho^M(A)v = Av$.

Proof. Recall that an element $b \in \mathrm{SO}(M)_x$ is a map $b\colon \mathbb{R}^m \to T_xM$. The isomorphism is no difficult. It is $\psi\colon \mathrm{SO}(M) \times_{\rho^M} \mathbb{R}^m \to TM$ defined by

$$\psi[b, v] = b(v).$$

It prove no difficult to see that ψ is well defined, injective and surjective. $\square$

Now, let us consider $X \in \mathfrak{X}(M)$. We can see it as an element of $\Gamma(\mathrm{SO}(M) \times_{\rho^M} \mathbb{R}^m)$, and define an equivariant function $\hat{X}\colon \mathrm{SO}(M) \to \mathbb{R}^m$.

Let us make it more explicit. A vector field $Y \in \mathfrak{X}(M)$ is, for each x in M, the data of a tangent vector $Y_x \in T_xM$. Hence the formula $b(v) = Y_x$ defines an element $[b, v]$ in $\mathrm{SO}(M) \times_{\rho^M} \mathbb{R}^m$, and Y defines a section $\tilde{Y}(x) = [b(x), v(x)]$ of $\mathrm{SO}(M) \times_{\rho^M} \mathbb{R}^m$. The associated equivariant function is given by $\hat{Y}(b) = v$ if $b(v) = Y_x$. In other words, the equivariant function $\hat{Y}\colon \mathrm{SO}(M) \to \mathbb{R}^m$ associated with the vector field $Y \in \mathfrak{X}(M)$ is given by

$$\hat{Y}(b) = b^{-1}(Y_x), \tag{1.103}$$

where $x = \pi(b)$.

1.8.5 Gauge transformations

A **gauge transformation** of the G-principal bundle $\pi\colon P \to M$ is a diffeomorphism $\varphi\colon P \to P$ such that

- $\pi \circ \varphi = \pi$,
- $\varphi(\xi \cdot g) = \varphi(\xi) \cdot g$.

When we consider some local sections on $\sigma_\alpha\colon \mathcal{U}_\alpha \to P$, we can describe a gauge transformation with a function $\tilde{\varphi}_\alpha\colon M \to G$ by requiring

$$\varphi(\sigma_\alpha(x)) = \sigma_\alpha(x) \cdot \tilde{\varphi}_\alpha(x).$$

This formula defines φ from $\tilde{\varphi}$ as well as $\tilde{\varphi}$ from φ.

The group of gauge transformations has a natural action on the space of sections given by

$$(\varphi \cdot \psi)(x) = [\varphi(\xi), v]. \tag{1.104a}$$

if $\psi(x) = [\xi, v] = [\xi, \hat{\psi}(\xi)]$. This law can also be seen on the equivariant function $\hat{\psi}$ which defines ψ. The rule is

$$\widehat{\varphi \cdot \psi}(\xi) = \hat{\psi}(\varphi^{-1}(\xi)). \tag{1.104b}$$

Indeed, in the same way as before we find $(\varphi \cdot \psi)(x) = [\xi, \widehat{\varphi \cdot \psi}(x)] \stackrel{!}{=} [\varphi(\xi), v] = [\varphi(\xi), \hat{\psi}(\xi)]$. Taking $\xi \to \varphi^{-1}(\xi)$ as representative, $(\varphi \cdot \psi)(x) = [\xi, \hat{\psi} \circ \varphi^{-1}(\xi)]$.

1.9 Adjoint bundle

Let $\pi\colon P \to M$ be a G-principal bundle. The **adjoint bundle** is the associated bundle $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathcal{G}$. An element of that bundle is an equivalent class given by

$$[\xi, X] = [\xi \cdot g, \mathrm{Ad}(g^{-1})X]$$

for every $g \in G$. Here $\xi \in P$ and $X \in \mathcal{G}$.

1.10 Connection on vector bundle: local description

A **connection** on the vector bundle $p\colon E \to M$ is a bilinear map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(E) &\to \Gamma(E), \\ (X, s) &\mapsto \nabla_X s \end{aligned} \tag{1.105}$$

such that

- $\nabla_{fX} s = f \nabla_X s$,
- $\nabla_X (fs) = (X \cdot f)s + f\nabla_X s$

for all $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ and $s \in \Gamma(E)$. The operation ∇ is often called a **covariant derivative**.

An easy example is given on the trivial bundle $E = \mathtt{pr}_1\colon M \times \mathbb{C} \to M$. For this bundle, $\Gamma(E) = C^\infty(M, \mathbb{C})$ and the common derivation is a covariant derivation: $\nabla_X s = (ds)X$.

Proposition 1.42.
The value of $(\nabla_X s)(x)$ depends only on X_x and s on a neighbourhood of $x \in M$.

Proof. Let X, $Y \in \mathfrak{X}(M)$ such that $Y_z = f(z)X_z$ with $f(x) = 1$ and $f(z) \neq 1$ everywhere else. Then

$$(\nabla_Y s)(x) - (\nabla_X s)(x) = (f(x) - 1)(\nabla_X s)(x) = 0.$$

Since it is true for any function, the linearity makes that it cannot depend on X_z with $z \neq x$. If we consider now two sections s and s' which are equals on a neighbourhood of x, we can write $s' = fs$ for a certain function f which is 1 on the neighbourhood. Then

$$(\nabla_X s')(x) - (\nabla_X s)(x) = (f(x) - 1)(\nabla_X s)(x) + (Xf)s(x)$$

which zero because on a neighbourhood of x, f is the constant 1. □

This proposition shows that it makes sense to consider only local descriptions of connections. Let $\{e_1, \ldots, e_r\}$ be a basis of V and consider the local sections $\overline{S}_{\alpha i}\colon \mathcal{U}_\alpha \to E$,

$$\overline{S}_{\alpha i}(x) = \phi_\alpha^{-1}(x, e_i).$$

A local section $s_\alpha\colon \mathcal{U}_\alpha \to V$ can be decomposed as $s_\alpha(x) = s^i_\alpha(x)e_i$ with respect to this basis (up to an isomorphism between the different V at each point). Then on $\mathcal{U}_\alpha$,

$$s^i_\alpha \overline{S}_{\alpha i}(x) = s^i_\alpha(x)\phi_\alpha^{-1}(x, e_i) = \phi_\alpha^{-1}(x, s^i_\alpha e_i) = \phi_\alpha^{-1}(x, s_\alpha(x)) = s(x). \tag{1.106}$$

The first equality is the definition of the product $\mathbb{R} \times F \to F$.

So any $s \in \Gamma(E)$ can be (locally !) written under the form[11] $s = s^i_\alpha \overline{S}_{\alpha i}$; in particular $\nabla_X(\overline{S}_{\alpha i})$ can. We define the coefficients θ by

$$\nabla_X(\overline{S}_{\alpha i}) = (\theta_\alpha)^j_i(X)\overline{S}_{\alpha j}. \tag{1.107}$$

where, for each i and j, $(\theta_\alpha)^j_i$ is a 1-form on $\mathcal{U}_\alpha$. We can consider θ_α as a matrix-valued 1-form on $\mathcal{U}_\alpha$.

[11] be careful on the fact that the "coefficient" s^i_α depends on x : the right way to express this equation is $s(x) = s^i_\alpha(x)\overline{S}_{\alpha i}(x)$.

Proposition 1.43.
The formula
$$(\nabla_X s)_\alpha = X s_\alpha + \theta_\alpha(X) s_\alpha \tag{1.108}$$
gives a local description of the connection.

Proof. For any $s \in \Gamma(E)$, we have
$$\begin{aligned}\nabla_X s &= \nabla_X\big(\sum_j s^j_\alpha \overline{S}_{\alpha j}\big)\\ &= \sum_j \Big((X s^j_\alpha)\overline{S}_{\alpha j} + s^j_\alpha \nabla_X \overline{S}_{\alpha j}\Big)\\ &= \sum_i \Big[(X s^i_\alpha) + s^j_\alpha (\theta_\alpha)^i_j(X)\Big]\overline{S}_{\alpha i}.\end{aligned}$$
□

1.10.1 Connection and transition functions

A connection determines some local matrix-valued 1-forms θ_α on the trivialization $\mathcal{U}_\alpha$. Two natural questions raise. The first is the converse: does a matrix-valued 1-form defines a connection ? The second is to know what is θ_α in function of θ_β on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$? The answer to the latter is given by the following proposition :

Proposition 1.44.
The 1*-form* θ_α *relative to the trivialization* $(\mathcal{U}_\alpha, \phi_\alpha)$ *is related to the* 1*-form* θ_β *relative to the trivialization* $(\mathcal{U}_\beta, \phi_\beta)$ *by*
$$\theta_\beta = g^{-1}_{\alpha\beta} dg_{\alpha\beta} + g^{-1}_{\alpha\beta}\theta_\alpha g_{\alpha\beta}. \tag{1.109}$$

This equation looks like something you know ? If you think to equation (4.66)-145 or (4.74)-146 or any physical equation of gauge transformation for the bosons, then you are almost right.

Proof. We can use equation (1.52)-30 pointwise on $(\nabla_X s)_\alpha$:
$$\begin{aligned}(\nabla_X s)_\alpha &= g_{\alpha\beta}(\nabla_X s)_\beta\\ &= g_{\alpha\beta}\big(X s_\beta + \theta_\beta(X) s_\beta\big)\\ &= g_{\alpha\beta}\big(X(g_{\alpha\beta}s_\alpha) + \theta_\beta(X) g_{\alpha\beta} s_\alpha\big).\end{aligned} \tag{1.110}$$
We have to compare it with equation (1.108)-48. Note that $g_{\alpha\beta}$ and $\theta_\alpha(X)$ are matrices, then one cannot do
$$g_{\alpha\beta}\theta_\beta(X) g_{\alpha\beta} = g_{\alpha\beta} g_{\alpha\beta}\theta_\beta(X) = \theta_\beta(X)$$
by using $g_{\alpha\beta}g_{\alpha\beta} = \mathbb{1}$. Taking carefully subscripts into account, one sees that the correct form is $(g_{\alpha\beta})^i_j \theta_\beta(X)^j_k (g_{\alpha\beta})^k_l$. Applying Leibnitz formula $(X(fg) = f(Xg) + (Xf)g)$, and making the simplification $g_{\alpha\beta}g_{\alpha\beta} = \mathbb{1}$ in the first term, we find
$$\theta_\alpha(X) s_\alpha = g_{\alpha\beta}(X g_{\alpha\beta}) s_\alpha + g^{-1}_{\alpha\beta}\theta_\beta(X) g_{\alpha\beta} s_\alpha.$$
The claim follows from the fact that $X g_{\alpha\beta} = dg_{\alpha\beta}(X)$. □

Notice that formula (1.109)-48 shows in particular that θ_α takes its values in the Lie algebra $\mathfrak{gl}(V)$, see for example subsection 1.2.3.

The inverse is given in the

Proposition 1.45.
If we choose a family of $\mathfrak{gl}(V)$-valued 1-forms θ_α on $\mathcal{U}_\alpha$ satisfying (1.109)-48,then the formula

$$(\nabla_X s)_\alpha = Xs_\alpha + \theta_\alpha(X)s_\alpha$$

defines a connection on E.

Proof. Note that θ is $C^\infty(M)$-linear, thus

$$(\nabla_{fX} s)_\alpha = (fX)s_\alpha + \theta_\alpha(fX)s_\alpha = f[Xs_\alpha + \theta_\alpha(X)s_\alpha] = f(\nabla_X s)_\alpha. \tag{1.111}$$

In expressions such that $\theta_\alpha(X)(fs_\alpha)$, the product is a matrix times vector product between $\theta_\alpha(X)$ and s_α; the position of the f is not important. So we can check the second condition :

$$\begin{aligned}(\nabla_X(fs))_\alpha &= X(fs_\alpha) + \theta_\alpha(X)(fs_\alpha)\\ &= X(f)s_\alpha + f(Xs_\alpha) + f\theta_\alpha(X)s_\alpha\\ &= df(X)s_\alpha + f(\nabla_X s)_\alpha.\end{aligned} \tag{1.112}$$

This concludes the proof. □

1.10.2 Torsion and curvature

The map $T^\nabla\colon \mathfrak{X}(X) \times \mathfrak{X}(X) \to \mathfrak{X}(X)$ defined by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.113}$$

is the **torsion** of the connection ∇. When $T^\nabla(X,Y) = 0$ for every X and Y in $\mathfrak{X}(X)$, we say that ∇ is a **torsion free** connection. Let X, Y be in $\mathfrak{X}(M)$, and consider the map $R(X,Y)\colon \Gamma(E) \to \Gamma(E)$ defined by

$$\begin{aligned}R(X,Y)\,:\,\Gamma(E) &\to \Gamma(E)\\ s &\mapsto \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s.\end{aligned} \tag{1.114}$$

For each $x \in M$, R can be seen as a bilinear map $R\colon T_xM \times T_xM \to \operatorname{End}(E_x)$. It is called the **curvature** of the connection ∇. For every $f \in C^\infty(M)$, it satisfies

$$R(fX,Y)s = fR(X,Y)s = R(X,Y)fs.$$

In a trivialization $(\mathcal{U}_\alpha, \phi_\alpha)$, we have $(\nabla_X s)_\alpha = Xs_\alpha + \theta_\alpha(X)s_\alpha$. In the expression of $(R(X,Y)s)_\alpha$, the terms coming from the Xs_α part of covariant derivative make

$$XYs_\alpha - YXs_\alpha - [X,Y]s_\alpha = 0.$$

The other terms are no more than matricial product, hence the formula

$$(R(X,Y)s)_\alpha = \Omega_\alpha(X,Y)s_\alpha \tag{1.115}$$

defines a 2-form Ω_α which takes values in $GL(r,\mathbb{K})$. We can find an expression for Ω in terms of θ :

$$\Omega_\alpha(X,Y) = X\theta_\alpha(Y) - Y\theta_\alpha(X) - \theta_\alpha([X,Y]) + \theta_\alpha(X)\theta_\alpha(Y) - \theta_\alpha(Y)\theta_\alpha(X);$$

it is written as

$$\Omega_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha + \frac{1}{2}[\theta_\alpha, \theta_\alpha] \tag{1.116}$$

which is a notational shortcut for

$$\Omega_\alpha(X, Y) = d\theta_\alpha(X, Y) + [\theta_\alpha(X), \theta_\alpha(Y)]. \tag{1.117}$$

These equations are called **structure equations**. Pointwise, the second term is a matrix commutator; be careful on the fact that, when we will speak about principal bundle, the forms θ's will take their values in a Lie algebra. On $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, we have

$$\Omega_\beta(X, Y) = g_{\alpha\beta}^{-1}\Omega_\alpha(X, Y)g_{\alpha\beta}.$$

The curvature and the connection fulfill the **Bianchi identities** :

Lemma 1.46.

$$d\Omega_\alpha + [\theta_\alpha \wedge \Omega_\alpha] = 0.$$

Proof. For each matricial entry, θ_α is a 1-form on $\mathcal{U}_\alpha$, then $\theta_\alpha(X)$ is a function which to $x \in M$ assign $\theta_\alpha(x)(X_x) \in \mathbb{R}$. So we can apply d and Leibnitz on the product $\theta_\alpha(X)\theta_\alpha(Y)$.

$$d\big(\theta_\alpha(X)\theta_\alpha(Y)\big) = \theta_\alpha(X)d\theta_\alpha(Y) + d\theta_\alpha(X)\theta_\alpha(Y).$$

Differentiating equation (1.116)-49, $d\Omega_\alpha = d\theta_\alpha \wedge \theta_\alpha - \theta_\alpha \wedge d\theta_\alpha$. □

1.10.3 Divergence, gradient and Laplacian

We define the **gradient** of a function $f \in C^\infty(M)$, denoted by ∇f as the vector field such that

$$g(\nabla f, X) = X(f). \tag{1.118}$$

The **divergence** of a vector field $X \in \Gamma(TM)$, is the function $\nabla \cdot X \in C^\infty(M)$ defined by

$$(\nabla \cdot X)(x) = \mathrm{Tr}\left(v \mapsto \nabla_v X\right) \tag{1.119}$$

where the trace is the one of $v \mapsto \nabla_v X$ seen as an operator on T_xM. The **Laplacian** of the function f is the function Δf given by

$$\Delta f = \nabla \cdot (\nabla f). \tag{1.120}$$

1.11 Connexion on vector bundle: algebraic view

A **connection** on the vector bundle $\pi\colon E \to M$ is a linear map

$$\nabla\colon \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes \Omega^1(M)$$

which satisfies the Leibnitz rule

$$\nabla(\sigma f) = (\nabla\sigma)f + \sigma \otimes df \tag{1.121}$$

for any section $\sigma\colon M \to E$ and function $f\colon M \to \mathbb{C}$. If $\{\sigma_i\}$ is a local basis of E, one can write $\sigma = \sigma_i f^i$ and one defines the **Christoffel symbols** $\Gamma^j_{i\mu}$ in this basis by

$$\nabla\sigma = \nabla(\sigma_i f^i) = (\nabla\sigma_i)f^i + \sigma_i \otimes f(f^i) = f^i\Gamma^j_{i\mu}\sigma_j \otimes dx^\mu + \sigma_i \otimes d(f^i). \tag{1.122}$$

The notations $d\sigma = \sigma_i \otimes d(f^i)$ and $\Gamma\sigma = f^i\Gamma^j_{i\mu}\sigma_j \otimes dx^\mu$ lead us to the compact usual form

$$\nabla\sigma = (d + \Gamma)\sigma.$$

When $E = TM$ over a (pseudo)Riemannian manifold M, we know the Levi-Civita connection which is compatible with the metric:

$$g(\nabla X, Y) + g(X, \nabla Y) = d\big(g(X, Y)\big). \tag{1.123}$$

One can see g as acting on $\big(\mathfrak{X}(M) \otimes \Omega^1(M)\big) \times \mathfrak{X}(M)$ with

$$g\big(r^i_\nu \partial_i \otimes dx^\nu, t^j\partial_j\big) := r^i_\nu j^j g(\partial_i, \partial_j)dx^\nu,$$

which at each point is a form. From condition (1.123)-51, we see ∇ as a Levi-Civita connection on the bundle $E = T^*M$ which values in

$$\Gamma^\infty(T^*M) \otimes \Omega^1(M) \simeq \Omega^1(M) \otimes \Omega^1(M).$$

This is defined as follows. A 1-form ω can always be written under the for $\omega = X^\flat := g(X, .)$ for a certain $X \in \mathfrak{X}(M)$. Then (1.123)-51 gives

$$(\nabla X)^\flat Y + \omega(\nabla Y) = d(\omega Y),$$

and we put $\nabla\omega = (\nabla X)^\flat$, i.e

$$(\nabla\omega)Y = d(\omega Y) - \omega(\nabla Y) \tag{1.124}$$

for all $Y \in \mathfrak{X}(M)$. When $\omega = dx^i$ and $Y = \partial_j$, we find

$$(\nabla dx^i)\partial_j = d(dx^i\partial_j) - dx^i(\nabla\partial_j) = d(\delta^i_j) - \Gamma^l_{jk}\partial_l \otimes dx^k = -\Gamma^l_{jk}\delta^i_l \otimes dx^k = -\Gamma^i_{jk}\, dx^k. \tag{1.125}$$

So we get the local formula

$$\nabla dx^i = -\Gamma^i_{jk}\, dx^j \otimes dx^k. \tag{1.126}$$

If the form writes locally $\omega = dx^i f_i$,

$$\nabla\omega = \nabla(dx^i)f_i + dx^i \otimes df_i = -f_i\Gamma^i_{jk}\, dx^j \otimes dx^k + d\omega = (d - \tilde{\Gamma})\omega \tag{1.127}$$

where we taken the notations $d\omega = dx^i \otimes df_i$ and $\tilde{\Gamma}\omega = f_i\Gamma^i_{jk}dx^j \otimes dx^k$.

1.11.1 Exterior derivative

If E is a m-dimensional vector bundle over M and $s\colon M \to E$ is a section, we say that a **exterior derivative** is a map $D\colon \Gamma(E) \to \Gamma(E \otimes \Omega^1 M)$ such that for every $f \in C^\infty(M)$ we have

$$D(fs) = s \otimes df + f(Ds).$$

An exterior derivative can be extended to $D\colon \Gamma(E \otimes \Omega^k M) \to \Gamma(E \otimes \Omega^{k+1}M)$ imposing the condition

$$D(\omega \wedge \alpha) = (D\omega) \wedge \alpha + (-1)^k\omega \wedge d\alpha \tag{1.128}$$

for every $\omega \in \Gamma(E \otimes \Omega^k M)$ and $\alpha \in \Gamma(E \otimes \Omega^l M)$. The result is an element of $\Gamma(E \otimes \Omega^{k+l+1}M)$.

Coordinatewise expressions are obtained when one choose a specific section (e_i) of the frame bundle of E. In that case for each i, the derivative e_i is an element of $\Gamma(E \otimes \Omega^1 M)$ and we define $\omega^j_i \in \Omega^1(M)$ by

$$De_i = \sum_{j=1}^{k} e_j \otimes \omega^j_i. \tag{1.129}$$

For each i and j, we have an element $\omega_i^j \in \Omega^1(M)$, so that we say that $\omega \in \Omega^1(M, \mathfrak{gl}(m))$. Now a section can be expressed as $s = s^i e_i$ where s^i are functions, so we have

$$D(s) = D(s^i e_i) = e_i \otimes ds^i + s^i D(e_i) = e_i \otimes ds^i + s^i e_j \otimes \omega_i^j = e_i \otimes ds^i + e_i \otimes s^j \omega_j^i. \quad (1.130)$$

Expressed in component, we find $D(s)^i = ds^i + s^j \omega_j^i$, so that we often write

$$D = d + \omega. \quad (1.131)$$

When a section e is given, we write $s = s^i(e)e_i$, indicating the dependence of the functions s^i in the choice of the frame e :

$$D(s) = e_i \otimes ds^i(e) + e_i \otimes s^j(e)\omega(e)_j^i.$$

When we apply both sides to a vector $X \in \Gamma(TM)$, we find

$$D_X(s) = e_i \otimes \Big(X(s^i) + s^j \omega_j^i(X) \Big). \quad (1.132)$$

By convention we say that, when $f \in C^\infty(M)$, is a function, D_X reduces to the action of the vector field X:

$$D_X(f) = X(f). \quad (1.133)$$

Covariant exterior derivative

An important exterior derivative is the covariant exterior derivative. If the vector bundle E is endowed by a covariant derivative ∇, we define the corresponding **covariant exterior derivative** by the following :

1. for a section $s\colon M \to E$ (i.e. a 0-form) we define

$$(d_\nabla s)(X) = \nabla_X s, \quad (1.134)$$

2. and on the 1-form $\sum_i (s_i \otimes \omega_i) \in \Gamma(E \otimes T^*M)$,

$$d_\nabla \big(\sum_i s_i \otimes \omega_i \big) = \sum_i (d_\nabla s_i) \wedge \omega_i + \sum_i s_i \otimes d\omega_i. \quad (1.135)$$

The latter relation is the condition (1.128)-51 with $k = 0$.

Soldering form and torsion

Let us particularize to the case where E has the same dimension as the manifold. In that case, we can introduce a **soldering form**, that is an element $\theta \in \Omega^1(M, E)$ such that for every $x \in M$ the map $\theta_x\colon T_xM \to E_x$ is a vector space isomorphism. When a soldering form θ is given, the **torsion** is the exterior derivative D is

$$T = D\theta. \quad (1.136)$$

Using a local frame e, we have forms $\theta^i(e) \in \Omega^1(M)$ such that

$$\theta(X) = \theta^i(X)e_i.$$

We see θ as an element of $\Gamma(E \otimes \Omega^1(M))$ by identifying $\theta = e_i \otimes \theta^i$. Thus we have

$$D\theta = D(e_i \otimes \theta^i) = De_i \wedge \theta^i(e) + e_i \wedge d\theta^i(e) = (e_j \otimes_i^j) \wedge \theta^i(e) + e_i \wedge d\theta^i(e),$$

or in coordinates :

$$(D\theta)^i = \omega^i_j \wedge \theta^j(e) + d\theta^i(e). \tag{1.137}$$

Notice that it provides the formula

$$T = d_\omega \theta \tag{1.138}$$

for the torsion as exterior covariant derivative of the connection form.

Example : Levi-Civita

We consider the vector bundle $E = TM$ and the local basis $e_i = \partial_i$. An exterior derivative in this case is a map $D\colon \Gamma(TM) \to \Gamma\Big(TM \otimes \Omega^1 M\Big)$. In that particular case, we denote by $\nabla_X Y$ the vector field $D(Y)X$, and it is computed by first writing $D(X)_x = \sum_i Z^i_x \otimes \omega^i_x$ with $Z^i \in \Gamma(TM)$ and $\omega^i \in \Omega^1(M)$. The we have

$$D(X)_x Y_x = \omega + x^i(Y_x) Z^i_x. \tag{1.139}$$

A good choice of soldering form is $\theta_x = \mathrm{id}$ for every $x \in M$, or $\theta(X) = X$. In coordinates, that soldering form is given by $\theta^i(\partial_j) = \delta^i_j$. The **Christoffel symbols** are defined by

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k, \tag{1.140}$$

and the covariant derivative reads

$$\nabla_X Y = \nabla_{X^i \partial_i}(Y^j \partial_j) = X^i \Big((\partial_i Y^j)\partial_j + Y^j \nabla_{\partial_i} \partial_j \Big) = \Big(X(Y^k) + X^i Y^j \Gamma^k_{ij} \Big) \partial_k. \tag{1.141}$$

We can determine the Christoffel symbols in function of the connection form using the fact that on the one hand, $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$, and on the other hand,

$$\nabla_{\partial_i}\partial_j = D(\partial_j)(\partial_i) = \partial_k \otimes \omega^k_j(\omega_i),$$

so that

$$\Gamma^k_{ij} = \omega^k_j(\partial_i) \tag{1.142}$$

Now we can get the same result as equation (1.141)-53 using the exterior derivative formalism. First we have $DY = \partial_i \otimes dY^i + \partial_i \otimes X^j \omega^i_j$, so that

$$(DY)X = \partial_i \otimes dY^i(X) = \partial_i \otimes X^j \omega^i_j (X^k \partial_k),$$

in which we use the relation $\omega^i_j(X^k\partial_k) = X^k \omega^i_j(\partial_k) = X^k \Gamma^i_{jk}$ to get

$$(DY)X = \big(X(Y^i) + X^j X^k \Gamma^i_{jk} \big) \partial_i.$$

Notice that the anti-symmetric part of Γ with respect to its two lower indices does not influence the covariant derivative. Let us compute the torsion in terms of Γ. For that remark that $d\theta^i = 0$ because

$$(d\theta^i)(X,Y) = X\theta^i(Y) - Y\theta^i(X) - \theta^i\big([X,Y]\big) = X(Y^i) - Y(X^i) - [X,Y]^i = 0.$$

Thus we have

$$\begin{aligned}
(D\theta)(\partial_k \otimes \partial_l) &= \big((D\partial_i)\partial_k\big)\theta^i(\partial_l) - \big((D\partial_i)\partial_l\big)\theta^i(\partial_k) \\
&= \delta^i_l \Gamma^j_{ik}\partial_j - \delta^i_k \Gamma^j_{il}\partial_j \\
&= (\Gamma^j_{lk} - \Gamma^j_{kl})\partial_j.
\end{aligned}$$

The connection ∇ is moreover compatible with the metric because

$$\nabla_Z\big(g(X,Y)\big) = Z\big(\eta(eX,eY)\big) = \eta\big(\underbrace{D_Z(eX)}_{=e(\nabla_Z X)}, eY\big) + \eta\big(eX, D_Z(eY)\big) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

1.12 Connection on principal bundle

1.12.1 First definition: 1-form

We consider a G-principal bundle

$$\begin{array}{ccc} G & \rightsquigarrow & P \\ & & \big\downarrow \pi \\ & & M \end{array}$$

and $\mathcal{G}$, the Lie algebra of G.

Definition 1.47.
*A **connection** on P is a 1-form $\omega \in \Omega(P, \mathcal{G})$ which fulfills*

- $\omega_\xi(A^*_\xi) = A$,
- $(R^*_g \omega)_\xi(\Sigma) = \mathrm{Ad}(g^{-1})(\omega_\xi(\Sigma))$,

for all $A \in \mathcal{G}$, $g \in G$, $\xi \in P$ and $\Sigma \in T_\xi P$

Here, R_g is the right action: $R_g \xi = \xi \cdot g$ and A^* stands for the **fundamental field** associated with A for the action of G on P:

$$A^*_\xi = \frac{d}{dt}\Big[\xi \cdot e^{-tA}\Big]_{t=0}, \tag{1.143}$$

For each $\xi \in P$, we have $\omega_\xi : T_\xi P \to \mathcal{G}$. See section 1.3.

If α is a connection 1-form on P, we say that Σ is an **horizontal** vector field if $\alpha_\xi(\Sigma) = 0$ for all $\xi \in P$. If $X_x \in T_x M$ and $\xi \in \pi^{-1}(x)$, there exists an unique[12] Σ in $T_\xi P$ which is horizontal and such that $\pi_*(\Sigma) = X_x$. This Σ is called the **horizontal lift** of X_x. We can also pointwise construct the horizontal lift of a vector field. The one of X is often denoted by $\overline{X}$; it is an element of $\mathfrak{X}(P)$.

1.12.2 Second definition: horizontal space

For each $\xi \in P$, we define the **vertical space** $V_\xi P$ as the subspace of $T_\xi P$ whose vectors are tangent to the fibers: each $v \in V_\xi P$ fulfills $d\pi v = 0$. Any such vector is given by a path contained in the fiber of ξ. So, $v \in V_\xi P$ if and only if there exists a path $g(t) \in G$ such that $v = \frac{d}{dt}\Big[\xi \cdot g(t)\Big]_{t=0}$.

A **connection** Γ is a choice, for each $\xi \in P$, of an **horizontal space** $H_\xi P$ such that

- $T_\xi P = V_\xi P \oplus H_\xi P$,
- $H_{\xi \cdot g} = (dR_g)_\xi H_\xi$,
- $H_\xi P$ depends on ξ under a differentiable way.

The second condition means that the distribution $\xi \to H_\xi$ is invariant under G. Thanks to the first one, for each $X \in T_\xi P$, there exists only one choice of $Y \in H_\xi P$ and $Z \in V_\xi P$ such that $X = Y + Z$. These are denoted by vX and hX and are naturally named *horizontal* and *vertical components* of X. The third condition means that if X is a differentiable vector field on P, then

[12] See [1], chapter II, proposition 1.2.

vX and hX are also differentiable vector fields. We will often write V_ξ and H_ξ instead of $V_\xi P$ and $V_\xi P$.

The word *connection* probably comes from the fact that the horizontal space gives a way to jump from a fiber to the next one. When we consider a connection Γ, we can define a $\mathcal{G}$-valued connection 1-form by

$$\omega(X)^*_\xi = vX_\xi.$$

The existence is explained in section 1.3. It is clear that $\omega(X) = 0$ if and only if X is horizontal. The theorem which connects the two definitions is the following.

Theorem 1.48.
If Γ is a connection on a G-principal bundle, and ω is its 1-form, then

(i) for any $A \in \mathcal{G}$, we have $\omega(A^*) = A$,

(ii) $(R_g)^*\omega = \mathrm{Ad}(g^{-1})\omega$, i.e. for any $X \in T_\xi P$, $g \in G$ and $\xi \in M$,

$$\omega((dR_g)_\xi X) = \mathrm{Ad}(g^{-1})\omega_\xi(X)$$

Conversely, if one has a $\mathcal{G}$-valued 1-form on P which fulfills these two requirement, then one has one and only one connection on P whose associated 1-form is ω.

Proof. (i) The definition of ω is $\omega(X)^*_\xi = vX$. Then $\omega(A^*)^*_\xi = vA^*_\xi = A^*_\xi$ because A^* is vertical. From lemma 1.25, $\omega(A^*) = A$.

(ii) Let $X \in \mathfrak{X}(P)$. If X is horizontal, the definition of a connection makes dR_dX also horizontal, then the claim becomes $0 = 0$ which is true. If X is vertical, there exists a $A \in \mathcal{G}$ such that $X = A^*$ and a lemma shows that dR_gX is then the fundamental field of $\mathrm{Ad}(g^{-1})A$. Using the properties of a connection,

$$(R_g^*\omega)_\xi(X) = \omega_{\xi\cdot g}(dR_gX) = \mathrm{Ad}(g^{-1})A = \mathrm{Ad}(g^{-1})\omega_\xi(X). \tag{1.144}$$

Now we turn our attention to the inverse sense: we consider a 1-form which fulfills the two conditions and we define

$$H_\xi = \{X \in T_\xi P \text{ st } \omega(X) = 0\}. \tag{1.145}$$

We are going to show that this prescription is a connection. First consider a $X \in V_\xi$, then $X = A^*$ and $\omega(X) = A$. So $H_\xi \cap V_\xi = 0$. Now we consider $X \in T_\xi P$ and we decompose it as

$$X = A^* + (X - A^*)$$

where A^* is the vertical component of X. If $\omega(dR_gX) = 0$ for all $g \in G$, then $\omega(X) = 0$, then a vector $X \in H_\xi$ fulfills at most $\dim G$ independent constraints $\omega(dR_gX) = 0$ and $\dim H_\xi$ is at least $\dim P - \dim G$. On the other hand, $\dim V_\xi = \dim G$; then

$$\dim V_\xi + \dim H_\xi \geqslant \dim G + \dim P - \dim G.$$

Then the equality must holds and $V_\xi \oplus H_\xi = T_\xi P$.

We have now to prove that ω is the connection form of H_ξ, i.e. that $\omega(X)$ is the unique $A \in \mathcal{G}$ such that A^*_ξ is the vertical component of X. Indeed if $X \in T_\xi P$, it can be decomposed as into $A^* \in V_\xi$ and $Y \in H_\xi$ and

$$\omega(X) = \omega(A^* + Y) = \omega(A^*) = A.$$

It remains to be proved that the horizontal space H_ξ of any connection Γ is related to the corresponding 1-form ω by $H_\xi = \{X \in T_\xi P \text{ st } \omega_\xi(X) = 0\}$. From the connection Γ, the 1-form is defined by the requirement that $\omega(X)^*_\xi = vX_\xi$. For $X \in H_\xi$, it is clear that $vX = 0$, so that $\omega(X)^* = 0$. This implies $\omega(X) = 0$ because we suppose that the action of G is effective.

□

The projection $\pi\colon P \to M$ induces a linear map $d\pi\colon T_\xi P \to T_x M$. We will see that, when a connection is given, it is an isomorphism between H_ξ and $T_x M$ (if $x = \pi(\xi)$). The **horizontal lift** of $X \in \mathfrak{X}(M)$ is the unique horizontal vector field (i.e. it is pointwise horizontal) such that $d\pi(\overline{X}_\xi) = X_{\pi(\xi)}$. The proposition which allows this definition is the following.

Proposition 1.49.
For a given connection on the G-principal bundle P and a vector field X on M, there exists an unique horizontal lift of X. Moreover, for any $g \in G$, the horizontal lift is invariant under dR_g.

The inverse implication is also true: any horizontal field on P which is invariant under dR_g for all g is the horizontal lift of a vector field on M.

This proposition comes from [1], chapter II, proposition 1.2.

Proof. We consider the restriction $d\pi\colon H_\xi \to T_{\pi(\xi)}M$. It is injective because $d\pi(X-Y)$ vanishes only when $X - Y$ is vertical or zero. Then it is zero. It is cleat that $d\pi\colon T_\xi P \to T_{\pi(\xi)}M$ is surjective. But $d\pi X = 0$ if X is vertical, then $d\pi$ is surjective from only H_ξ.

So we have existence and unicity of an horizontal lift. Now we turn our attention to the invariance. The vector $dR_g\overline{X}_\xi$ is a vector at $\xi \cdot g$. From the definition of a connection, $dR_g H_\xi = H_{\xi\cdot g}$, then $dR_g\overline{X}_\xi$ is the unique horizontal vector at $\xi \cdot g$ which is sent to X_x by $d\pi$. Thus it is $\overline{X}_{\xi\cdot g}$.

For the inverse sense, we consider $\overline{X}$, an horizontal invariant vector field on P. If $x \in M$, we choose $\xi \in \pi^{-1}(x)$ and we define $X_x = d\pi(\overline{X}_\xi)$. This construction is independent of the choice of ξ because for $\xi' = \xi \cdot g$, we have

$$d\pi(\overline{X}_{\xi'}) = \pi(dR_g\overline{X}_\xi) = \pi(\overline{X}_\xi).$$

□

An other way to see the invariance is the following formula:

$$\overline{X}_{\xi\cdot g} = (dR_g)_\xi\overline{X}_\xi.$$

By definition, $\overline{X}_{\xi\cdot g}$ is the unique vector of $T_{\xi\cdot g}P$ which fulfils $d\pi\overline{X}_{\xi\cdot g} = X_x$ if $\xi\pi^{-1}(x)$, so the following computation proves the formula:

$$(d\pi)_{\xi\cdot g}((dR_g)_\xi\overline{X}_\xi) = d(\pi \circ R_g)_\xi\overline{X}_\xi = d\pi_\xi\overline{X}_\xi = X_x. \tag{1.146}$$

1.12.3 Curvature

The curvature of a vector or associated bundle satisfies $\Omega_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha$. So we naturally define the **curvature** of the connection ω on a principal bundle as the $\mathcal{G}$-valued 2-form

$$\Omega = d\omega + \omega \wedge \omega. \tag{1.147}$$

When we consider a local section $\sigma_\alpha\colon \mathcal{U}_\alpha \to P$ on $\mathcal{U}_\alpha \subset M$, we can express the curvature with a 2-form on M instead of P by the formula

$$F_{(\alpha)} = \sigma_\alpha^*\Omega,$$

or, more explicitly, by $F_{(\alpha)x}(X,Y) = \Omega_{\sigma_\alpha(x)}(d\sigma_\alpha X, d\sigma_\alpha Y))$. Note that if $\mathcal{G}$ is abelian, $\Omega = d\omega$ and $d\Omega = 0$.

1.13 Exterior covariant derivative and Bianchi identity

Let $\omega \in \Omega^1(P, \mathcal{G})$ be a connection 1-form on the G-principal bundle P. Using the operation $[. \wedge .]$ defined in section 1.6, we define the **exterior covariant derivative** by

$$d_\omega \alpha = d\alpha + \frac{1}{2}[\omega \wedge \alpha] \qquad \text{when } \alpha \in \Omega^1(P, \mathcal{G}), \tag{1.148}$$

$$d_\omega \beta = d\beta + [\omega \wedge \beta] \qquad \text{when } \beta \in \Omega^2(P, \mathcal{G}), \tag{1.149}$$

The **curvature** is the 2-form defined by

$$\Omega = d_\omega \omega = d\omega + \omega \wedge \omega \tag{1.150}$$

where d_ω is the exterior covariant derivative associated with the connection form ω, and the wedge has to be understood as in equation (1.55)-31.

Proposition 1.50.
The curvature form satisfies the identity

$$d_\omega \Omega = 0 \tag{1.151}$$

which is the Bianchi identity

Proof. taking the differential of $\Omega = d\omega + \omega \wedge \omega$, we find

$$d\Omega = d^2\omega + d\omega \wedge \omega - \omega \wedge d\omega$$

in which $d^2\omega = 0$ and we replace $d\omega$ by $\Omega - \omega \wedge \omega$, so that

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega,$$

which becomes the Bianchi identity using the definition of d_ω and the notation (1.57)-31. □

Remark that the Bianchi identity reads $d_\omega^2 \omega = 0$, but that in general d_ω does not square to zero.

1.14 Covariant derivative on associated bundle

Now we consider a general G-principal bundle $\pi\colon P \to M$ and an associated bundle $E = P \times_\rho V$. We define a product $\mathbb{R} \times E \to E$ by

$$\lambda[\xi, v] = [\xi, \lambda v]. \tag{1.152}$$

It is clear that the equivariant function $\widehat{\lambda\psi}$ defines the section $\lambda\psi$. A **covariant derivative** is a map

$$\begin{aligned} \nabla \,:\, \mathfrak{X}(M) \times \Gamma(M, E) &\to \Gamma(M, E) \\ (X, \psi) &\mapsto \nabla_X \psi \end{aligned} \tag{1.153}$$

such that

$$\nabla_{fX}\psi = f\nabla_X\psi, \tag{1.154a}$$

$$\nabla_X(f\psi) = (X \cdot f)\psi + f\nabla_X\psi \tag{1.154b}$$

where products have to be understood by formula (1.152)-57.

Theorem 1.51.
A connection on a principal bundle gives rise to a covariant derivative on any associated bundle by the formula

$$\widehat{\nabla^E_X\psi}(\xi) = \overline{X}_\xi(\hat{\psi}) \tag{1.155}$$

where $\hat{\psi}\colon P \to V$ *is the function associated with the section* $\psi\colon M \to E$.

We have to prove that it is a good definition: the function $\widehat{\nabla^E_X\psi}$ must define a section $\nabla^R_X\psi\colon M \to E$ and the association $\psi \to \nabla^E_X\psi$ must be a covariant derivative.

With the discussion of page 13 about the application of a tangent vector on a map between manifolds, we have $(d\varphi X)f = X(f \circ \varphi)$. By using this equality in the case of $\overline{X}$ with $\hat{\psi}$ and R_g, we find $(dR_g\overline{X})(\hat{\psi}) = \overline{X}(\hat{\psi} \circ R_g)$ and thus

$$\overline{X}_{\xi\cdot g}(\hat{\psi}) = \overline{X}_\xi(dR_g\hat{\psi}).$$

We prove the theorem step by step.

Proposition 1.52.
The function $\widehat{\nabla^E_X\psi}$ *defines a section of* P.

Proof. We have to see that $\widehat{\nabla^E_X\psi}$ is an equivariant function. The equivariance of $\hat{\psi}$ gives $\hat{\psi} \circ R_g = \rho(g^{-1})\hat{\psi}$, thus

$$\widehat{\nabla^E_X\psi}(\xi\cdot g) = \overline{X}_{\xi\cdot g}(\hat{\psi}) = \big((dR_g)_\xi\overline{X}_\xi\big)(\hat{\psi}) = \overline{X}_\xi(\hat{\psi} \circ R_g) = \overline{X}_\xi(\rho(g^{-1})\hat{\psi}) = \rho(g^{-1})\overline{X}_\xi(\hat{\psi}). \tag{1.156}$$

The last equality comes from the fact that the product $\rho(g^{-1})\hat{\psi}$ is a linear product "matrix times vector" and that $\overline{X}_\xi$ is linear. □

Theorem 1.53.
The definition

$$\widehat{\nabla^E_X\psi}(\xi) = \overline{X}_\xi(\hat{\psi})$$

defines a covariant derivative.

Proof. We have to check the two conditions given on page 47.
First condition. By definition, $\widehat{\nabla^E_{fX}\psi}(\xi) = \overline{fX}_\xi(\hat{\psi})$. Now we prove that

$$\overline{fX}_\xi(\hat{\psi}) = (f \circ \pi)(\xi)\overline{X}_\xi(\hat{\psi}). \tag{1.157}$$

This formula is coherent because $\overline{X}_\xi(\hat{\psi}) \in V$ and $(f \circ \pi)(\xi) \in \mathbb{R}$. By definition of the horizontal lift, $\overline{fX}_\xi$ is the unique vector such that

- $d\pi_\xi(\overline{fX}_\xi) = (fX)_x = f(x)d\pi\overline{X}_\xi = (f \circ \pi)(\xi)d\pi\overline{X}_\xi$,
- $\omega_\xi(\overline{fX}_\xi) = 0$.

We check that $(f \circ \pi)(\xi)\overline{X}_\xi$ also fulfills these two conditions because $d\pi$ and ω are $C^\infty(P)$-linear. Equation (1.157)-58 immediately gives

$$\widehat{\nabla^E_{fX}\psi}(\xi) = (f \circ \pi)(\xi)\widehat{\nabla^E_X\psi}(\xi). \tag{1.158}$$

Now we show that $\widehat{f\nabla^E_X\psi}$ is the same. The section $f\nabla^E_X\psi\colon M\to E$ is given by $(f\nabla^E_X\psi)(x)=f(x)(\nabla^E_X\psi)(x)$, and by definition of the associated equivariant function,

$$f(x)(\nabla^E_X\psi)(x)=[\xi,f(x)\widehat{\nabla^E_X\psi}(\xi)].$$

Then

$$\widehat{f\nabla^E_X\psi}(\xi)=f(x)\widehat{\nabla^E_X\psi}(\xi)=(f\circ\pi)(\xi)\widehat{\nabla^E_X\psi}(\xi).\tag{1.159}$$

All this shows that $\nabla^E_{fX}\psi=f\nabla^E_X\psi$.
Second condition. This is a computation using the Leibniz rule:

$$\begin{aligned}\widehat{\nabla^E_X(f\psi)}(\xi)&=\overline{X}_\xi(\widehat{f\psi})\overset{(a)}{=}\overline{X}_\xi((\pi\circ f)\hat{\psi})\\&\overset{(b)}{=}\overline{X}_\xi(\pi^*f)\hat{\psi}(\xi)+(\pi^*f)(\xi)\overline{X}_\xi\hat{\psi}=d(f\circ\pi)_\xi\overline{X}_\xi\hat{\psi}(\xi)+f\widehat{\nabla^E_X\psi}(x)\\&=df_{\pi(\xi)}d\pi_\xi\overline{X}_\xi\hat{\psi}(\xi)+f\widehat{\nabla^E_X\psi}(x)=X_x(f)\hat{\psi}(\xi)+f\widehat{\nabla^E_X\psi}(x)\\&=\widehat{(Xf)\psi}(\xi)+\widehat{f\nabla^E_X\psi}(\xi)\end{aligned}\tag{1.160}$$

where (a) is because $\widehat{f\psi}=\pi^*f\hat{\psi}$, and (b) is an application of the Leibniz rule. □

Theorem 1.54.
Using the local coordinates related to the sections $\sigma_\alpha\colon\mathcal{U}_\alpha\to P$, the covariant derivatives reads:

$$(\nabla_X\psi)_{(\alpha)}(x)=X_x\psi_{(\alpha)}-\rho_*(\sigma^*_\alpha\omega_x(X))\psi_{(\alpha)}(x)\tag{1.161}$$

where $\rho_*\colon\mathcal{G}\to\mathrm{End}(V)$ is defined by

$$\rho_*(A)=\frac{d}{dt}\Big[\rho(e^{tA})\Big]_{t=0}\tag{1.162}$$

Proof. The problem reduces to the search of $\overline{X}$ because

$$(\nabla_X\psi)_{(\alpha)}(x)=\widehat{\nabla_X\psi}(\sigma_\alpha(x))=\overline{X}_{\sigma_\alpha(x)}(\hat{\psi}).$$

We claim that $\overline{X}_{\sigma_\alpha(x)}=d\sigma_\alpha X_x-\omega(d\sigma_\alpha X_x)^*$. We have to check that $d\pi\overline{X}=X$ and $\omega(\overline{X})=0$. The latter comes easily from the fact that $\omega(A^*)=A$. For the first one, remark that s_α is a section, then $d(\pi\circ s_\alpha)=\mathrm{id}$, and $d\pi(ds_\alpha X_x)=X_x$, while

$$d\pi(A^*_\xi)=d\pi\frac{d}{dt}\Big[\xi\cdot e^{-tA}\Big]_{t=0}=\frac{d}{dt}\Big[\pi(\xi)\Big]_{t=0}=0.\tag{1.163}$$

Since the horizontal lift is unique, we deduce

$$(\nabla_X\psi)_{(\alpha)}(x)=\big(d\sigma_\alpha X_x-\omega(d\sigma_\alpha X_x)^*\big)\hat{\psi}.\tag{1.164}$$

From the definition of a fundamental vector field,

$$\begin{aligned}\omega(d\sigma_\alpha X_x)^*_{\sigma_\alpha(x)}\hat{\psi}&=\frac{d}{dt}\Big[\hat{\psi}\big(\sigma_\alpha(x)\cdot e^{-t\omega(d\sigma_\alpha X_x)}\big)\Big]_{t=0}\\&=\frac{d}{dt}\Big[\rho(e^{t\omega(d\sigma_\alpha X_x)})\hat{\psi}(\sigma_\alpha(x))\Big]_{t=0}&&\text{from (1.91)-43}\\&=(d\rho)_e(\omega\circ d\sigma_\alpha)X_x(\hat{\psi}\circ\sigma_\alpha)(x)\\&=\rho_*\big((\sigma^*_\alpha\omega)(X_x)\big)\psi_{(\alpha)}(x)&&\text{by (1.162)-59}\end{aligned}\tag{1.165}$$

□

We can express the covariant derivative by means of some maps $\theta_\alpha\colon \mathfrak{X}(M)\times M\to \operatorname{End}(V)$ given by

$$\nabla_X\gamma_{\alpha i}=\theta_\alpha(X)_i^{\ j}\gamma_{\alpha j}. \tag{1.166}$$

where the $\gamma_{\alpha i}$'s were given in equation (1.98)-44. By the definition (1.154b)-57,

$$\begin{aligned}(\nabla_X\psi)(x)&=(X\cdot s_\alpha^i)_x\gamma_{\alpha i}(x)+s_\alpha^i(x)(\nabla_X\gamma_{\alpha i})(x)\\&=(X\cdot s_\alpha^i)_x\gamma_{\alpha i}(x)+s_\alpha^i(x)\theta_\alpha(X)_i^{\ j}\gamma_{\alpha j}(x).\end{aligned}$$

On the othre hand with the notations of equation (1.96)-44, $\gamma_{\alpha j}=e_i$ and $X_x\gamma_{\alpha j}=0$. Then equation (1.161)-59 gives $\theta_\alpha(X)=\rho_*(\sigma_\alpha^*\omega_x(X))$, or

$$\theta_\alpha=\rho_*(\sigma_\alpha^*\omega_x). \tag{1.167}$$

1.14.1 Curvature on associated bundle

From the definition (1.92)-43, it makes sense to define the curvature 2-form by

$$R(X,Y)\psi=\nabla_X\nabla_Y\psi-\nabla_Y\nabla_X\psi-\nabla_{[X,Y]}\psi.$$

It is also clear that $\psi_{(\alpha)}$ defines a section of the trivial vector bundle $F=M\times V$ by $x\to(x,\psi_{(\alpha)}(x))$, so one can define $\Omega_\alpha(X,Y)\colon\Gamma(M,E)\to\Gamma(M,E)$ by

$$\big(R(X,Y)\psi\big)_{(\alpha)}=\Omega_\alpha(X,Y)\psi_{(\alpha)}$$

and take back all the work around Bianchi because of the relation (1.161)-59 which can be written as $(\nabla_X\psi)_{(\alpha)}(x)=X_x\psi_{(\alpha)}+\theta_\alpha(X)\psi_{(\alpha)}(x)$ and which is the same as in proposition 1.45.

1.14.2 Connection on frame bundle

General framework

The frame bundle was defined at page 36. Let $F\xrightarrow{p}M$ be a $\mathbb{K}$-vector bundle with some local trivialization $(\mathcal{U}_\alpha,\phi_\alpha^E)$ and the corresponding transition functions $g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\to GL(r,\mathbb{K})$. We consider $\pi\colon P\to M$, the frame bundle of F; it is a $GL(r,\mathbb{K})$-principal bundle. Let ∇ be a covariant derivative on F and θ_α, the associated matrices 1-form. The frame bundle is

$$P=\bigcup_{x\in M}(\text{frame of }F_x).$$

A connection is a $\mathcal{G}$-valued 1-form; in our case it is a map

$$\omega_\xi^\alpha\colon T_\xi\big(\pi^{-1}(\mathcal{U}_\alpha)\big)\to\mathfrak{gl}(r,\mathbb{K}).$$

We define our connection by, for $g\in GL(r,\mathbb{K})$, $x\in\mathcal{U}_\alpha$, $X_x\in T_xM$ and $A\in\mathfrak{gl}(r,\mathbb{K})$,

$$\omega^\alpha_{S_\alpha(x)\cdot g}\big(R_{g*}s_\alpha(x)_*X_x+A^*_{S_\alpha(x)\cdot g}\big):=A+\operatorname{Ad}(g^{-1})\theta_\alpha(X_x). \tag{1.168}$$

where $S_\alpha\colon\mathcal{U}_\alpha\to P$ is the section defined by the trivialization ϕ_α^P:

$$S_\alpha(x)=\{\overline{v}_\alpha={\phi_\alpha^E}^{-1}(x,e_i)\}_{i=1,\ldots,r}.$$

Since $\theta_\alpha(X_x)\in\operatorname{End}(\mathbb{K}^r)\subset\mathfrak{gl}(r,\mathbb{K})$, the second term of (1.168)-60 makes sense. This formula is a good definition of ω because of the following lemma:

Lemma 1.55.
If $\xi = S_\alpha(x)\cdot g$ and $\Sigma \in T_\xi P$, there exists a choice of $A \in \mathcal{G}$, and $X_x \in T_x M$ such that

$$\Sigma = R_{g*} s_\alpha(X)_* X_x + A^*_{S_\alpha(x)\cdot g}. \tag{1.169}$$

Proof. If $\xi \in P$ is a basis of E at y, there exists only one choice of $x \in M$ and $g \in G$ such that $\xi = S_\alpha(x)\cdot g$.

Let us consider a general path $c\colon \mathbb{R} \to P$ under the form $c(t) = s_\alpha(x(t))\cdot g(t)$ where x and g are path in M and $GL(r,\mathbb{K})$. The frame $c(t)$ is the one of $F_{x(t)}$ obtained by the transformation $g(t)$ from $s_\alpha(x(t))$. It is a set of r vectors, and each of them can be written as a combination of the vectors of $s_\alpha(x(t))$, so we write

$$c^i(t) = s^j_\alpha(x(t)) g^i_j(t) \tag{1.170}$$

where $s^j_\alpha(x(t)) \in F_{x(t)}$ and $g^i_j(t) \in \mathbb{K}$. We compute $\Sigma = c'(0)$ by using the Leibnitz rule and we denote $x'(0) = X_x$, $x(0) = x$ and $g^i_j(0) = g^i_j$ (the matrix of g):

$$\begin{aligned}\Sigma^i &= \frac{d}{dt}\Big[s^j_\alpha(x(t))\Big]_{t=0} g^i_j + s^j_\alpha(x)\frac{d}{dt}\Big[g^i_j(t)\Big]_{t=0}\\ &= (ds^j_\alpha)_x X_x g^i_j + {g^i_j}'(0) s^j_\alpha(x).\end{aligned} \tag{1.171}$$

Going to more compact matrix form, it gives

$$\Sigma = (ds_\alpha)_x X_x \cdot g + s_\alpha(x) g'(0).$$

The second term, $s^j_\alpha(x) g'^i_j(0)$, is a general vector tangent to a fiber. So it can be written as a fundamental field A^*_ξ.

□

Lemma 1.56.
On $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, the form fulfills $\omega^\alpha = \omega^\beta$.

Proof. Let $\gamma\colon \mathbb{R} \to M$ be a path whose derivative is X_x. Then

$$\begin{aligned}(R_g)_* s_\alpha(x)_* X_x &= \frac{d}{dt}\Big[s_\alpha(\gamma_t)\cdot g\Big]_{t=0} = \frac{d}{dt}\Big[s_\beta(\gamma_t) g_{\alpha\beta}(\gamma_t)\cdot g\Big]_{t=0}\\ &= \frac{d}{dt}\Big[s_\alpha(\gamma_t) g_{\alpha\beta}(x)\cdot g\Big]_{t=0} + \frac{d}{dt}\Big[s_\beta(x)\cdot g_{\alpha\beta}(\gamma_t)\cdot g\Big]_{t=0}.\end{aligned} \tag{1.172}$$

What is in the derivative of the first term is $R_{g_{\alpha\beta}(x)g}(s_\beta(\gamma_t))$. Taking the derivative, we find the expected $R_{g_{\alpha\beta}(x)g\,*} s_{\beta *} X_x$.

For the second term, we note $r := s_\beta(x)\cdot g_{\alpha\beta}(g) g$, and we have to compute the following, using equation (1.59)-33,

$$\begin{aligned}&\frac{d}{dt}\Big[r\cdot \mathrm{Ad}_{g^{-1}}(g^{-1}_{\alpha\beta}(x) g_{\alpha\beta}(\gamma_t))\Big]_{t=0}\\ &\qquad = \frac{d}{dt}\Big[r\cdot \exp t\big((d\,\mathbf{Ad}_{g^{-1}})_e(g^{-1}_{\alpha\beta}(x)(dg_{\alpha\beta})_x X_x)\big)\Big]_{t=0}\\ &\qquad = \frac{d}{dt}\Big[r\cdot \exp t\big(\mathrm{Ad}_{g^{-1}}\, g^{-1}_{\alpha\beta}(x) dg_{\alpha\beta}\Big]_{t=0}\\ &\qquad = \Big(\mathrm{Ad}_{g^{-1}}\, g^{-1}_{\alpha\beta}(x) dg_{\alpha\beta} X_x\Big)^*_r.\end{aligned} \tag{1.173}$$

Using this, we can perform the computation:

$$\begin{aligned}
\omega^{\beta}_{S_\alpha(x)\cdot g}\big(R_{g*}s_\alpha(x)_*X_x + A^*_{S_\alpha(x)\cdot g}\big) &= \omega^{\beta}_{S_\beta(x)\cdot g_{\alpha\beta}(x)g}\Big(R_{g_{\alpha\beta}}(x)g_{*}s_\beta(x)_*X_x \\
&\qquad + (\mathrm{Ad}_{g^{-1}}\, g^{-1}_{\alpha\beta}(x)dg_{\alpha\beta}X_x)^*_r + A^*\Big) \\
&= \mathrm{Ad}_{(g_{\alpha\beta}(x)g)^{-1}}\,\theta_\beta(X_x) \\
&\quad + \mathrm{Ad}_{g^{-1}}\, g^{-1}_{\alpha\beta}(x)dg_{\alpha\beta}(X_x) + A \\
&= \mathrm{Ad}_{g^{-1}}\left((g^{-1}_{\alpha\beta}\theta_\beta g_{\alpha\beta} + g^{-1}_{\alpha\beta}dg_{\alpha\beta})(X_x)\right) + A \\
&= \omega^{\alpha}_{S_\alpha(x)g}\big(R_{g*}s_\alpha(x)_*X_x + A^A_{S_\alpha(x)\cdot g}\big).
\end{aligned} \tag{1.174}$$

□

Proposition 1.57.
The ω defined by formula (1.168)-60 is a connection 1-form.

Proof. The first condition, $\omega(A^*_\xi) = A$, is immediate from the definition. The lemma 1.26 gives the second condition in the case $\Sigma = A^*_\xi$. It remains to be checked that $\omega(dR_g\Sigma) = \mathrm{Ad}(g^{-1})\omega(\Sigma)$ in the case $\Sigma = dR_h ds_\alpha X_x$. This is obtained using the fact that Ad is a homomorphism. □

Levi-Civita connection

Let (M, g) be a Riemannian manifold. We look at a connection 1-form $\alpha \in \Omega^1(\mathrm{SO}(M), so(\mathbb{R}^m))$ on $\mathrm{SO}(M)$, and we define a covariant derivative $\nabla^\alpha\colon \mathfrak{X}(M) \times T(M) \to T(M)$, where $T(M)$ is the tensor bundle on M by (cf. theorem (1.53)-58)

$$\widehat{\nabla^\alpha_X s} = \overline{X}\hat{s}, \tag{1.175}$$

for any $s \in T(M)$. Our purpose now is to prove that an automatic property of this connection is $\nabla^\alpha g = 0$. The unique such connection which is torsion-free is the **Levi-Civita** one.

The metric g is a section of the tensor bundle $T^*M \otimes T^*M$. So we have, in order to find $\hat{g}$ and to use equation (1.175)-62, to see $T^*M \otimes T^*M$ as an associated bundle. As done in 1.8.4, we see that

$$T^*M \otimes T^*M \simeq \mathrm{SO}(M) \times_\rho (V^* \otimes V^*),$$

with the following definitions:

- The isomorphism is given by $\psi[b, \alpha \otimes \beta](X \otimes Y) = \alpha(b^{-1}X)\beta(b^{-1}Y)$,
- $\rho(A)\alpha = \alpha \circ A$,
- $b \cdot A = b \circ A$.

Here, $V = \mathbb{R}^m$; $b\colon V \to T_xM$; $\alpha, \beta \in V^*$; $X, Y \in T_xM$ and $A \in \mathrm{SO}(m)$ is seen as $A\colon V \to V$.

The following shows that ψ is well defined:

$$\begin{aligned}
\psi[b \cdot A, \rho(A^{-1})\alpha \otimes \beta](X \otimes Y) &= (\alpha \circ A)(A^{-1} \circ b^{-1}X)(\beta \circ A)(A^{-1} \circ b^{-1}Y) \\
&= \psi[b, \alpha \otimes \beta](X \otimes Y)
\end{aligned} \tag{1.176}$$

Proposition 1.58.
The function $\hat{g}$ is given by

$$\hat{g}(b)(v \otimes w) = g_x(b(v) \otimes b(w)) = v \cdot w.$$

Proof. The second equality is just the fact that $b\colon (\mathbb{R}^m, \cdot) \to (T_xM, g_x)$ is isometric. On the other hand, if $\hat{g}(b) = \alpha \otimes \beta$, we have:

$$\begin{aligned} g_x(X \otimes Y) &= \psi[b, \alpha \otimes \beta](X \otimes Y) = \alpha(b^{-1}X)\beta(b^{-1}Y) \\ &= \alpha \otimes \beta(b^{-1}X \otimes b^{-1}Y) = \hat{g}(b)(b^{-1}X \otimes b^{-1}Y). \end{aligned} \tag{1.177}$$

Since b is bijective, X and Y can be written as bv and bw respectively for some $v, w \in V$, so that

$$g_x(bv \otimes bw) = \hat{g}(b)v \otimes w.$$

□

It is now easy to see that $\overline{X}\hat{g} = 0$. As $\hat{g}$ takes its values in $V^* \otimes V^*$, $\overline{X}\hat{g}$ belongs to this space and can be applied on $v \otimes w \in V \otimes V$. Let $\overline{X}(t)$ be a path in $\mathrm{SO}(M)$ which defines $\overline{X}$; if $\overline{X} \in T_b\,\mathrm{SO}(M)$, $\overline{X}(0) = b$. We have

$$\overline{X}\hat{g}(v \otimes w) = \frac{d}{dt}\hat{g}(\overline{X}(t))v \otimes w\Big|_{t=0} = \frac{d}{dt}v \cdot w\Big|_{t=0}, \tag{1.178}$$

which is obviously zero.

1.14.3 Holonomy

Let the principal bundle

$$\begin{tikzcd} G \arrow[r, squiggly] & P \arrow[d, "\pi"] \\ & M \end{tikzcd} \tag{1.179}$$

and ω a connection on G. Let $\gamma\colon [0,1] \to M$, a closed curve piecewise smooth; $\gamma(0) = \gamma(1) = x$. For each $p \in \pi^{-1}(x)$, there exists one and only one horizontal lift $\tilde{\gamma}\colon [0,1] \to P$ such that $\tilde{\gamma}(0) = p$. There exists of course an element $g \in G$ such that $\tilde{\gamma}(1) = p \cdot g$.

We define the following equivariance relation on P: we say that $p \sim q$ if and only if p and q can be joined by a piecewise smooth path. The **holonomy group** at the point p is

$$\mathrm{Hol}_p(\omega) = \{g \in G \text{ st } p \sim p \cdot g\}.$$

1.14.4 Connection and gauge transformation

Proposition 1.59.
If ω is a connection on a G-principal bundle and φ, a gauge transformation, the form $\beta = \varphi^*\omega$ is a connection 1-form too.

Proof. It is rather easy to see that $\varphi_* A^*_\xi = A^*_{\varphi(x)}$:

$$\varphi_* A^*_\xi = \frac{d}{dt}\Big[\varphi(\xi e^{-tA})\Big]_{t=0} = \frac{d}{dt}\Big[\varphi(\xi)e^{-tA}\Big]_{t=0} = A^*_{\varphi(x)}.$$

The same kind of reasoning leads to $\varphi_* R_{g*} = R_{g*}\varphi_*$. From here, it is easy to see that

$$(\varphi^*\omega)_\xi(A^*_\xi) = \omega_{\varphi(\xi)}(\varphi_* A^*_\xi) = A,$$

and

$$\big(R^*_g(\varphi^*\omega)_\xi\big)(\Sigma) = (R^*_g\omega)_{\varphi(\xi)}(\varphi_*\Omega) = \mathrm{Ad}(g^{-1})\big((\varphi^*\omega)_\xi(\Sigma)\big).$$

□

So, the "gauge transformed" of a connection is still a connection. It is hopeful in order to define gauge invariants objects (Lagrangian) from connections (electromagnetic fields).

Local description

Let $\pi\colon P \to M$ be a G-principal bundle given with some trivializations $\phi^P_\alpha\colon \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times G$ over $\mathcal{U}_\alpha \subset M$ and $s_\alpha\colon \mathcal{U}_\alpha \to \pi^{-1}(\mathcal{U}_\alpha)$, a section. In front of that, we consider an associated bundle $p\colon E = P \times_\rho V \to M$ with a trivialization $\phi^E_\alpha\colon E \to \mathcal{U}_\alpha \times V$. One can choose a section s_α compatible with the trivialization in the sense that $\phi^P_\alpha(s_\alpha(x)\cdot g) = (x, g)$; the same can be done with E by choosing $\phi^E_\alpha([s_\alpha(x), v]) = (x, v)$. All this given in figure 1.1.

A section $\psi\colon M \to E$ is described by a function $\psi_\alpha\colon \mathcal{U}_\alpha \to V$ defined by $\phi^E_\alpha(\psi(x)) = (x, \psi_\alpha(x))$. In the inverse sense, ψ is defined (on $\mathcal{U}_\alpha$) from ψ_α by $\psi(x) = [s_\alpha(x), \psi_\alpha(x)]$. In the same way, a gauge transformation $\varphi\colon P \to P$ is described by functions $\tilde\varphi_\alpha\colon \mathcal{U}_\alpha \to G$,

$$\varphi(s_\alpha(x)) = s_\alpha(x)\cdot\tilde\varphi_\alpha(x). \tag{1.180}$$

The function $\tilde\varphi_\alpha$ also fulfil

$$(\phi^P_\alpha \circ \varphi \circ {\phi^P_\alpha}^{-1})(x, g) = (x, \tilde\varphi(x)\cdot g) \tag{1.181}$$

because

$$\begin{aligned}(\phi^P_\alpha \circ \varphi \circ {\phi^P_\alpha}^{-1})(x, g) &= (\phi^P_\alpha \circ \varphi)(s_\alpha(x)\cdot g)\\ &= \phi^P_\alpha(\varphi(s_\alpha(x))\cdot g)\\ &= \phi^P_\alpha(s_\alpha(c)\cdot\tilde\varphi_\alpha(x)g)\\ &= (x, \tilde\varphi_\alpha(x)g).\end{aligned} \tag{1.182}$$

We know that a connection on P is given by its 1-form ω. Moreover we have the following:

Proposition 1.60.
A connection on P is completely determined on $\pi^{-1}(\mathcal{U}_\alpha)$ from the data of the $\mathcal{G}$-valued 1-form $\sigma^*_\alpha\omega$ on $\mathcal{U}_\alpha$.

Proof. We consider a 1-form ω which fulfils the two conditions of page 54. Our purpose is to find back $\omega_\xi(\Sigma)$, $\forall\xi \in P, \Sigma \in T_\xi P$ from the data of $\sigma^*_\alpha\omega$ alone. For any ξ, there exists a g such that $\xi = \sigma_\alpha(x)\cdot g$. We have

$$\mathrm{Ad}_{g^{-1}}(\omega_{\sigma_\alpha(x)\Sigma}) = (R^*_g\omega)_{\sigma_\alpha(x)}(\Sigma) = \omega_{\sigma_\alpha(x)\cdot g}\big((dR_g)_{\sigma_\alpha(x)}\Sigma\big). \tag{1.183}$$

If we know $s^*_\alpha\omega$, then we know $\omega\big((ds_\alpha)_x v\big)$ for any $v \in T_x M$. So

$$\omega_{\sigma_\alpha(x)\cdot g}\big((dR_g)_{\sigma_\alpha(x)}\Sigma\big)$$

is given from $\sigma^*_\alpha\omega$ for every Σ of the form $\Sigma = (d\sigma_\alpha)_x v$. From the form (1.169)-61 of a vector in $T_\xi P$, it just remains to express $\omega_{\sigma_\alpha(x)\cdot g}(A^*_{\sigma_\alpha(x)\cdot g})$ in terms of s^*_α. The definition of a connection makes that it is simply A.

∎

Covariant derivative

If we have a connection on P, we can define a covariant derivative on the associated bundle E by

$$(\nabla_X\psi)_{(\alpha)}(x) = X_x(\psi_\alpha) + \rho_*(s_\alpha^*\omega_x(X))\psi_{(\alpha)}(x),$$

the matricial 1-form being given by $\theta_\alpha = \rho_*\sigma_\alpha^*\omega$. The gauge transformation φ acts on the connection ω by defining $\omega^\varphi := \varphi^*\omega$.

Proposition 1.61.
If $\beta = \varphi^*\omega$, then

$$s_\alpha^*(\beta) = \mathrm{Ad}_{\tilde\varphi_\alpha(x)^{-1}}\, s_\alpha^*(\omega) + \tilde\varphi_\alpha(x)^{-1}d\tilde\varphi_\alpha.$$

Proof. Let $\gamma\colon \mathbb{R}\to M$ be a path such that $\gamma(0) = x$ and $\gamma'(0) = X_x$. We have to compute the following:

$$(s_\alpha^*\beta)(X_x) = (s_\alpha^*\varphi^*\omega)(X_x) = \omega_{(\varphi\circ s_\alpha)(x)}\big((\varphi\circ s_\alpha)_*X_x\big). \tag{1.184}$$

What lies in the derivative is:

$$\begin{aligned}
(\varphi\circ s_\alpha)_*(X_x) &= \frac{d}{dt}\Big[(\varphi\circ s_\alpha\circ\gamma)(t)\Big]_{t=0}\\
&= \frac{d}{dt}\Big[s_\alpha(\gamma(t))\cdot\tilde\varphi_\alpha(\gamma(t))\Big]_{t=0}\\
&= \frac{d}{dt}\Big[s_\alpha(\gamma(t))\cdot\tilde\varphi_\alpha(\gamma(0))\Big]_{t=0} + \frac{d}{dt}\Big[s_\alpha(\gamma(0))\cdot\tilde\varphi_\alpha(\gamma(t))\Big]_{t=0}\\
&= R_{\tilde\varphi_\alpha(x)*}s_{\alpha*}X_x + \frac{d}{dt}\Big[s_\alpha(x)\cdot\tilde\varphi_\alpha(x)e^{t\tilde\varphi_\alpha(x)^{-1}(d\tilde\varphi_\alpha)_x\gamma'(0)}\Big]_{t=0}.
\end{aligned} \tag{1.185}$$

A justification of the remplacement $\tilde\varphi_\alpha(\gamma(t))\to\tilde\varphi_\alpha(x)e^{t\tilde\varphi_\alpha(x)^{-1}(d\tilde\varphi_\alpha)_x\gamma'(0)}$ is given in the corresponding proof at page 145. If we put this expression into equation (1.184)-65, the first term becomes

$$\begin{aligned}
\omega_{(\varphi\circ s_\alpha)(x)}\big(R_{\tilde\varphi_\alpha(x)*}s_{\alpha*}X_x\big) &= (R^*_{\tilde\varphi_\alpha(x)}\omega)_{s_\alpha(x)}(s_{\alpha*}X_x)\\
&= \mathrm{Ad}_{\tilde\varphi_\alpha(x)^{-1}}\big(\omega_{s_\alpha(x)}(s_{\alpha*}X_x)\big)\\
&= \mathrm{Ad}_{\tilde\varphi_\alpha(x)^{-1}}(s_\alpha^*\omega)(X_x).
\end{aligned}$$

The second term is the case of a connection applied to a fundamental vector field.

□

1.15 Product of principal bundle

In this section, we build a $G_1\times G_2$-principal bundle from the data of a G_1 and a G_2-principal bundle. The physical motivation is clear: as far as electromagnetism is concerned, particles are sections of $U(1)$-principal bundle while the relativistic invariance must be expressed by means of a $\mathrm{SL}(2,\mathbb{C})$-associated bundle. So the physical fields must be sections of something as the product of the two bundles. See subsection 4.7.

1.15.1 Putting together principal bundle

Let us consider two principal bundle over the same base space

$$G_1 \rightsquigarrow P_1 \xrightarrow{p_1} M,$$

and

$$G_2 \rightsquigarrow P_2 \xrightarrow{p_2} M.$$

First we define the set

$$P_1 \circ P_2 = \{(\xi_1, \xi_2) \in P_1 \times P_2 \text{ st } p_1(\xi_1) = p_2(\xi_2)\} \tag{1.186}$$

which will be the total space of our new bundle. The projection $p\colon P_1 \circ P_2 \to M$ is naturally defined by

$$p(\xi_1, \xi_2) = p_1(\xi_1) = p_2(\xi_2),$$

while the right action of $G_1 \times G_2$ on $P_1 \circ P_2$ is given by

$$(\xi_1, \xi_2) \cdot (g_1, g_2) = (\xi_1 \cdot g_1, \xi_2 \cdot g_2)$$

With all these definitions,

$$\begin{tikzcd} G_1 \times G_2 \arrow[r, rightsquigarrow] & P_1 \circ P_2 \arrow[d, "p"] \\ & M \end{tikzcd}$$

is a $G_1 \times G_2$-principal bundle over M. We define the natural projections

$$\begin{aligned} \pi_i \;:\; P_1 \times P_2 &\to P_i \\ (\xi_1, \xi_2) &\mapsto \xi_i, \end{aligned} \tag{1.187}$$

and if e_i denotes the identity element of G_i, we can identify G_1 to $G_1 \times \{e_2\}$ and G_2 to $G_2 \times \{e_1\}$; in the same way, $\mathcal{G}_1 = \mathcal{G}_1 \times \{0\} \subset \mathcal{G}_1 \times \mathcal{G}_2$. So we get the following principal bundles :

$$G_2 \rightsquigarrow P_1 \circ P_2 \xrightarrow{\pi_1} P_1$$

$$G_1 \rightsquigarrow P_1 \circ P_2 \xrightarrow{\pi_2} P_2.$$

It is clear that the following diagram commutes :

$$\begin{tikzcd} P_1 \arrow[dr, "P_1"'] & P_1 \circ P_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "p"] & P_2 \arrow[dl, "p_2"] \\ & M & \end{tikzcd}$$

1.15.2 Connections

Let ω_i be a connection on the bundle $p_i\colon P_i \to M$. Using the identifications, $\pi_1^*\omega_1$ is a connection on $\pi_2\colon P_1 \circ P_2 \to P_2$ (the same is true for $1 \leftrightarrow 2$), and $\pi_1^*\omega_1 \oplus \pi_2^*\omega_2$ is a connection on $p\colon P_1 \circ P_2 \to M$. Let us prove the first claim.

Let $A \in \mathcal{G}_1$. We first have to prove that $\pi_1^*\omega_1(A^*) = A$. For this, remark that $A = (A, 0) \in \mathcal{G}_1 \oplus \mathcal{G}_2$ and

$$A^*_\xi = \frac{d}{dt}\Big[\xi \cdot e^{-t(A,0)}\Big]_{t=0} = \frac{d}{dt}\Big[(\xi_1, \xi_2) \cdot (e^{-tA}, e_2)\Big]_{t=0} = \frac{d}{dt}\Big[(\xi_1 \cdot e^{-tA}, \xi_2)\Big]_{t=0}, \tag{1.188}$$

so $d\pi_1 A^* = \frac{d}{dt}\Big[\pi_1(\ldots)\Big]_{t=0} = \omega_1(A^*) = A$. Let now $\Sigma \in T_{(\xi_1,\xi_2)}(P_1 \circ P_2)$ be given by the path $(\xi_1(t), \xi_2(t))$. In this case we have

$$\begin{aligned}
\big(R^*_{(g,e_2)\pi_1^*\omega_1}\big)_{(\xi_1,\xi_2)}\Sigma &= (\pi_1^*\omega_1)(dR_{(g,e_2)}\Sigma)\\
&= \omega_1(\frac{d}{dt}\Big[\xi_1(t)\cdot g\Big]_{t=0})\\
&= \omega_1(dR_g\frac{d}{dt}\Big[\xi_1(t)\Big]_{t=0})\\
&= \mathrm{Ad}(g^{-1})\pi_1^*\omega_1(\frac{d}{dt}\Big[(\xi_1(t),\xi_2(t))\Big]_{t=0})\\
&= \mathrm{Ad}(g^{-1})\pi_1^*\omega_1\Sigma.
\end{aligned} \tag{1.189}$$

1.15.3 Representations

Let V be a vector space and $\rho_i\colon G_i \to GL(V)$ be some representations such that

$$[\rho_1(g_1), \rho_2(g_2)] = 0 \tag{1.190}$$

for all $g_1 \in G_1$ and $g_2 \in G_2$ (in the sense of commutators of matrices). In this case, one can define the representation $\rho_1 \times \rho_2\colon G_1 \times G_2 \to GL(V)$ by

$$(\rho_1 \times \rho_2)(g_1, g_2) = \rho_1(g_1) \circ \rho_2(g_2) = \rho_2(g_2) \circ \rho_1(g_1). \tag{1.191}$$

The relation (1.190)-67 is needed in order for $\rho_1 \times \rho_2$ to be a representation, as one can check by writing down explicitly the requirement

$$(\rho_1 \times \rho_2)\big((g_1, g_2)(g_1', g_2')\big) = (\rho_1 \times \rho_2)(g_1g_1', g_2g_2')$$

Chapter 2

Decompositions of Lie algebras

2.1 Root spaces

References for Lie algebras and their modules are [10–16].

2.1.1 Cartan subalgebra

Since the Killing form on the Cartan subalgebra $\mathfrak{h}$ is nondegenerate, we can introduce, for each linear function $\phi\colon \mathfrak{h} \to \mathbb{R}$, an element t_ϕ of $\mathfrak{h}$ such that

$$\phi(h) = B(t_\phi, h) \tag{2.1}$$

for every $h \in \mathfrak{h}$.

Proposition 2.1.
We have

$$[x, y] = B(x, y)t_\alpha \tag{2.2}$$

whenever $x \in \mathfrak{g}_\alpha$ *and* $y \in \mathfrak{g}_{-\alpha}$.

Proof. For the proof, we show that the Killing for of $[x, y]$ and $B(x, y)t_\alpha$ with any element $h \in \mathfrak{h}$ are the same. Indeed, using the invariance of the Killing form,

$$B\big(h, [x, y]\big) = B\big([h, x], y\big) = \alpha(h)B(x, y) = B(t_\alpha, h)B(x, y) = B\big(B(x, y)t_\alpha, h\big). \tag{2.3}$$

□

Now, for each root α, we pick $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$B(e_\alpha, f_\alpha) = \frac{2}{B(t_\alpha, t_\alpha)}, \tag{2.4}$$

and then we pose

$$h_\alpha = \frac{2}{B(t_\alpha, t_\alpha)} t_\alpha. \tag{2.5}$$

In that case, for each root, the set $\{e_\alpha, f_\alpha, h_\alpha\}$ generates an algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ that is denoted by $\mathfrak{sl}(2, \mathbb{R})_\alpha$.

The space $\mathfrak{h}^*$ is endowed with an inner product defined by

$$(\alpha, \beta) = B(t_\alpha, t_\beta). \tag{2.6}$$

2.1.2 Cartan-Weyl basis

Let us study the eigenvalue equation

$$\mathrm{ad}(A)X = \rho X. \tag{2.7}$$

The number of solutions with $\rho = 0$ depends on the choice of $A \in \mathfrak{g}$.

Lemma 2.2.
If A is chosen in such a way that $\mathrm{ad}(A)X = 0$ has a maximal number of solutions, then the number of solutions is equal to the rank of $\mathfrak{g}$ and the eigenvalue $\alpha = 0$ is the only degenerated one in equation (2.7)-70.

We suppose A to be chosen in order to fulfill the lemma. Thus we have linearly independent vectors H_i $(i = 1, \ldots l)$ such that

$$[A, H_i] = 0 \tag{2.8}$$

where l is the rank of $\mathfrak{g}$. Since $[A, A] = 0$, the vector A is a combination $A = \lambda^i H_i$. Since $\mathrm{ad}(A)$ is diagonalisable, one can find vectors E_α with

$$[A, E_\alpha] = \alpha E_\alpha, \tag{2.9}$$

and such that $\{H_i, E_\alpha\}$ is a basis of $\mathfrak{g}$. Using the fact that $\mathrm{ad}(A)$ is a derivation, we find

$$[A, [H_i, E_\alpha]] = \alpha[H_i, E_\alpha], \tag{2.10}$$

The eigenvalue $\alpha = 0$ being the only one to be degenerated, one concludes that $[H_i, E_\alpha]$ is a multiple of E_α:

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{2.11}$$

Replacing $A = \lambda^i H_i$, we have

$$\alpha E_\alpha = [\lambda^i H_i, E_\alpha] = \lambda^i \alpha_i E_\alpha, \tag{2.12}$$

thus $\alpha = \lambda^i \alpha_i$ (with a summation over $i = 1, \ldots, l$).

Before to go further, notice that the space spanned by $\{H_i\}_{i=1,\ldots,l}$ is a maximal abelian subalgebra of $\mathfrak{g}$, so that it is a Cartan subalgebra that we, naturally denote by $\mathfrak{h}^*$. Thus, what we are doing here is the usual root space construction. In order to stick the notations, let us associate the form $\sigma_\alpha \in \mathfrak{h}^*$ defined by $\sigma_\alpha(H_i) = \alpha_i$. In that case,

$$\sigma_\alpha(A) = \sigma_\alpha(\lambda^i H_i) = \lambda^i \alpha_i = \alpha \tag{2.13}$$

and we have

$$[A, E_\alpha] = \sigma_\alpha(A) E_\alpha. \tag{2.14}$$

On the other hand, we have $[H_i, E_\alpha] = \alpha_i E_\alpha = \sigma_\alpha(H_i) E_\alpha$, so that the eigenvalue α is identified to the root α, and we have $E_\alpha \in \mathfrak{g}_\alpha$.

Let us now express the vectors t_α in the basis of the H_i. The definition property is $B(t_\alpha, H_i) = \alpha(H_i) = \alpha_i$. If $t_\alpha = (t_\alpha)^i H_i$, we have

$$\alpha_i = B(t_\alpha, H_i) = B_{kl}(t_\alpha)^k \underbrace{(H_i)^l}_{=\delta_i^l} = B_{ki}(t_\alpha)^k. \tag{2.15}$$

If (B^{ij}) are the matrix elements of B^{-1}, we have

$$(l_\alpha)^l = \alpha_i B^{il} = \alpha^l \tag{2.16}$$

where α^l is defined by the second equality. Using proposition 2.1, we have

$$[E_\alpha, E_{-\alpha}] = B(E_\alpha, E_{-\alpha})\alpha^l H_l. \tag{2.17}$$

Thus one can renormalise E_α in such a way to have

$$\begin{aligned} [H_i, H_j] &= 0, \\ [E_\alpha, E_{-\alpha}] &= \alpha^i H_i \\ [H_i, E_\alpha] &= \alpha_i E_\alpha = \alpha(H_i)E_\alpha \\ [E_\alpha, E_\beta] &= N_{\alpha\beta}E_{\alpha+\beta} \end{aligned} \tag{2.18}$$

where the constant $N_{\alpha\beta}$ are still undetermined. A basis $\{H_i, E_\alpha\}$ of $\mathfrak{g}$ which fulfill these requirements is a basis of **Cartan-Weyl**.

2.1.3 Cartan matrix

We follow [13]. We denote by Π the system of simple roots of $\mathfrak{g}$. All the positive roots have the form

$$\sum_{\alpha\in\Pi} k_\alpha \alpha \tag{2.19}$$

with $k_\alpha \in \mathbb{N}$.

Theorem 2.3.
Let α and β be simple roots Thus

(i) $\alpha - \beta$ is not a simple root

(ii) we have

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -p \tag{2.20}$$

where p is a strictly positive integer.

Partial proof. We are going to prove that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer. Let α and γ be non vanishing roots such that $\alpha + \gamma$ is not a root, and define

$$E'_{\gamma-j\alpha} = \operatorname{ad}(E_{-\alpha})^k E_\gamma \in \mathfrak{g}_{\gamma-k\alpha}. \tag{2.21}$$

Since there are a finite number of roots, there exists a minimal positive integer g such that $\operatorname{ad}(E_{-\alpha})^{g+1}E_\gamma = 0$. We define the constants μ_k (which depend on γ and α) by

$$[E_\alpha, E'_{\gamma-k\alpha}] = \mu_k E'_{\gamma-(k-1)\alpha}. \tag{2.22}$$

Using the definition of $E'_{\gamma-k\alpha}$ and Jacobi, one founds

$$\mu_k E'_{\gamma-(k-1)\alpha} = \big[E'_\alpha, [E_{-\alpha}, E'_{\gamma-(k-1)\alpha}]\big] = \alpha^i[H_i, E'_{\gamma-(k-1)\alpha}] + \mu_{k-1}E'_{\gamma-(k-a)\alpha}, \tag{2.23}$$

so that $\mu_k = \alpha^i\gamma_i - (k-1)\alpha^i\alpha_i + \mu_{k-1}$, and we have the induction formula

$$\mu_k = (\alpha,\gamma) - (k-1)(\alpha,\alpha) + \mu_{k-1} \tag{2.24}$$

for $k \geqslant 2$. If we define $\mu_0 = 0$, that relation is even true for $k = 1$. The sum for $k = 1$ to $k = j$ is easy to compute and we get

$$\mu_j = j(\alpha, \gamma) - \frac{j(j-1)}{2}(\alpha, \alpha). \tag{2.25}$$

Since $\mu_{g+1} = 0$, we have

$$(\alpha, \gamma) = g(\alpha, \alpha)/2, \tag{2.26}$$

and thus

$$\mu_j = \frac{j(g - j + 1)(\alpha, \alpha)}{2}. \tag{2.27}$$

Let β be any root and look at the string $\beta + j\alpha$. There exists a maximal $j \geqslant 0$ for which $\beta + j\alpha$ is a root while $\beta + (j+1)\alpha$ is not a root. Now we consider $\gamma = \beta + j\alpha$ with that maximal j. Putting $\gamma = \alpha + j\beta$ in (2.26)-72, one finds

$$(\alpha, \beta) = \frac{(g - 2j)(\alpha, \alpha)}{2}, \tag{2.28}$$

and finally,

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = g - 2j, \tag{2.29}$$

which is obviously an integer.

□

From the inner product on $\mathfrak{h}^*$, we deduce a notion of **angle**:

$$\cos(\theta_{\alpha,\beta}) = \frac{(\alpha, \beta)}{\sqrt{(\alpha, \alpha)(\beta, \beta)}}. \tag{2.30}$$

The **length** of the root α is the number $\sqrt{(\alpha, \alpha)}$.

Lemma 2.4.
If α and β are roots, then

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}, \tag{2.31}$$

and

$$\beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \tag{2.32}$$

is a root too.

If α and β are non vanishing, then the α-string which contains β contains at most 4 roots. Finally, the ratio

$$\frac{2(\alpha, \beta)}{(\alpha, \beta)} \tag{2.33}$$

takes only the values 0, ± 1, ± 2 or ± 3.

Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a system of simple roots. The **Cartan matrix** is the $l \times l$ matrix with entries

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}. \tag{2.34}$$

Notice that, in the literacy, one find also the convention $A_{ij} = 2(\alpha_i \alpha_j)/(\alpha_j, \alpha_j)$, as in [15], for example.

2.1.4 Dynkin diagram

Proposition 2.5.
If α and β are simple roots, then the angle $\theta_{\alpha,\beta}$ can only take the values $90°$, $120°$, $135°$ or $150°$.

Proof. No proof. □

In order to draw the **Dynkin diagram** of a Lie algebra, one draws a circle for each simple root, and one joins the roots with 1, 2 or 3 lines, following that the value of the angle is 120°, 135° or 150°. If the roots are orthogonal (angle 90°), they are not connected. If the length of a root is maximal, the circle is left empty. If not, it is filled.

One easily determines the number of lines between two roots by the following proposition.

Proposition 2.6.
If α and β are two simple roots with $(\alpha,\alpha) \leqslant (\beta,\beta)$, then

$$\frac{(\alpha,\alpha)}{(\beta,\beta)} = \begin{cases} 1 & \text{if } \theta_{\alpha,\beta} = 120° \\ 2 & \text{if } \theta_{\alpha,\beta} = 135° \\ 3 & \text{if } \theta_{\alpha,\beta} = 150°. \end{cases} \tag{2.35}$$

Proof. No proof. □

If M is a weight of a representation, its **Dynkin coefficients** are

$$M_i = \frac{2(M,\alpha_i)}{(\alpha_i,\alpha_i)}, \tag{2.36}$$

and we can compute the Dynkin coefficients from one weight to another by the simple formula

$$(M-\alpha_j)_i = M_i - A_{ij}. \tag{2.37}$$

A weight is **dominant** if all its Dynkin coefficients are strictly positive.

2.1.5 Chevalley basis

It $\{\alpha_i\}$ are the simple roots, we consider the following new basis for $\mathfrak{h}$:

$$H_{\alpha_i} = \frac{2\alpha_i^*}{(\alpha_i,\alpha_i)} \tag{2.38}$$

where α_i^* is the dual of α_i. This is the element of $\mathscr{H}$ defined by $\alpha_j(\alpha_i^*) = \delta_{ij}$. As usual in $\mathfrak{h}$, we have

$$[H_{\alpha_i}, H_{\alpha_j}] = 0. \tag{2.39}$$

Each root is a combination of the simple roots. If $\beta = \sum_{i=1}^{l} k_i\alpha_i$, we generalise the definition of H_{α_i} to

$$H_\beta = \frac{2\beta^*}{(\beta,\beta)} = \sum_i k_i \frac{(\alpha_i,\alpha_i)}{(\beta,\beta)} H_{\alpha_i}. \tag{2.40}$$

The element H_β is the **co-weight** associated with the weight β.

Using the inner product $(.,.)$, we have the decomposition $\beta = \sum_i (\beta,\alpha_i)\alpha_i$ of the roots. An immediate consequence is that

$$\beta(\alpha_i^*) = (\alpha_i,\beta). \tag{2.41}$$

If β is any root, we denote by β_i the result of β on H_{α_i}:

$$\beta_i = \beta(H_{\alpha_i}) = \frac{2(\alpha_i, \beta)}{(\alpha_i, \alpha_i)}.. \tag{2.42}$$

Theorem 2.7 (Chevalley basis)**.**
For each root β, one can found an eigenvector E_β of $\mathrm{ad}(H_\beta)$ such that

$$\begin{aligned}
[H_\beta, H_\gamma] &= 0 \\
[E_\beta, E_{-\beta}] &= H_\beta \\
[E_\beta, E_\gamma] &= \begin{cases} \pm(p+1)E_{\beta+\gamma} & \textit{if } \beta+\gamma \textit{ is a root} \\ 0 & \textit{otherwise} \end{cases} \\
[H_\beta, E_\gamma] &= 2\frac{(\beta,\gamma)}{(\beta,\beta)}E_\gamma
\end{aligned} \tag{2.43}$$

where p is the biggest integer j such that $\gamma + j\beta$ is a root. Moreover, if α_i and α_j are simple roots, the latter becomes

$$[H_{\alpha_i}, E_{\pm\alpha_j}] = \pm A_{ij} E_{\pm\alpha_j} \tag{2.44}$$

where A is the Cartan matrix.

An important point to notice is that, for each positive root α, the algebra generated by $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ is $\mathfrak{sl}(2)$. This is the reason why the representation theory of $\mathfrak{g}$ reduces to the representation theory of $\mathfrak{sl}(2)$.

2.2 Representations

Since $\mathfrak{h}$ is abelian, the operators H_{α_j} $(j = 1, \ldots, l)$ are simultaneously diagonalisable. In that basis of the representation space W, the basis vectors are denoted by $|u_\Lambda\rangle$ and have the property

$$H_{\alpha_i}|u_\Lambda\rangle = \Lambda(H_{\alpha_i})|u_\Lambda\rangle, \tag{2.45}$$

and, as notation, we note $\Lambda_i = \Lambda(H_{\alpha_i})$. The root Λ is a **weight** of the vector $|u_\Lambda\rangle$. The vector $E_\beta|u_\Lambda\rangle$ is of weight $\beta + \Lambda$, indeed,

$$H_{\alpha_i}E_\beta|u_\Lambda\rangle = \big([H_{\alpha_i}, E_\beta] + E_\beta H_{\alpha_i}\big)|u_\Lambda\rangle = \left(\frac{2(\alpha_i,\beta)}{(\alpha_i,\alpha_i)} + \Lambda_i\right) E_\beta|u_\Lambda\rangle. \tag{2.46}$$

Thus the eigenvalue of $E_\beta|u_\Lambda\rangle$ for H_{α_i} is, according to the relation, (2.42)-74, $\beta(H_{\alpha_i}) + \Lambda(H_{\alpha_i})$.

We suppose that the roots α_i are given in increasing order:

$$\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_l, \tag{2.47}$$

and one says that a weight is **positive** if its first non vanishing component is positive. Then one choose a basis of W

$$|u_{\Lambda^{(1)}}\rangle, \ldots, |u_{\Lambda^{(N)}}\rangle \tag{2.48}$$

of weight vectors. One say that this basis is **canonical** if

$$\Lambda^{(1)} \geqslant \ldots \geqslant \Lambda^{(N)}. \tag{2.49}$$

Theorem 2.8.
A vector if weight Λ which is a combination of vectors of weight $\Lambda^{(k)}$ all different of Λ vanishes.

Proof. No proof. □

A consequence of that theorem is that, if W is a representation of dimension N of $\mathfrak{g}$, there are at most N different weights. When several vectors have the same weight, the number of linearly independent such vectors is the **multiplicity** of the weight. A weight who has only one weight vector is **simple**.

Proposition 2.9.
The weights Λ and $\Lambda - 2\alpha(\Lambda,\alpha)/(\alpha,\alpha)$ have the same multiplicity for every root α.

Theorem 2.10.
Two representation are equivalent when they have the same highest weight.

Proposition 2.11.
For any weight M and root α,

$$\frac{2(M,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}, \tag{2.50}$$

and

$$M - \frac{2(M,\alpha)}{(\alpha,\alpha)}\alpha \tag{2.51}$$

is a weight.

Notice, in particular, that for every weight M, the root $-M$ is also a weight.

2.2.1 About group representations

Let π be a representation of a group G. The **character** of π is the function

$$\begin{aligned} \chi_\pi \colon G &\to \mathbb{C} \\ g &\mapsto \operatorname{Tr}\big(\pi(g)\big). \end{aligned} \tag{2.52}$$

From the cyclic invariance of trace, it fulfils $\chi_\pi(gxg^{-1}) = \chi_\pi(x)$, so that the character is a central function.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We denote by $Z_\pm$ the subgroup of G generated by $\mathfrak{n}^\pm$. The **Cartan subgroup** D of G is the maximal abelian subgroup of G which has $\mathfrak{h}$ as Lie algebra.

A **character** of an abelian group is a representation of dimension one.

Let T be a representation of G on a complex vector space V. One say that $\xi \in V$ is a **highest weight** if

- $T(z)\xi = \xi$ for every $z \in Z_+$,
- $T(g)\xi = \alpha(g)\xi$ for every $g \in D$.

The function $\alpha \colon D \to \mathbb{C}$ is the **highest weight** of the representation T.

Lemma 2.12.
The function α is a character of the group D.

Proof. The number $\alpha(gg')$ is defined by $T(gg')\xi = \alpha(gg')\xi$. Using the fact that T is a representation, one easily obtains $T(gg')\xi = \alpha(g)\alpha(g')\xi$. □

2.2.2 Weyl group

For each root α, we define

$$\begin{aligned} S_\alpha \colon \mathfrak{h}^* &\to \mathfrak{h}^* \\ \Lambda &\mapsto \Lambda - \frac{2\alpha(\Lambda,\alpha)}{(\alpha,\alpha)}. \end{aligned} \tag{2.53}$$

This is an affine reflexion in $\mathfrak{h}^*$ around the direction of the root α because $S_\alpha(\alpha) = -\alpha$ and $S_\alpha(\beta) = \beta$ when (α,β). The **Weyl reflexion group** is the group generated by $\{S_{\alpha_i}\}$ $(i = 1,\ldots l)$ and the identity.

Theorem 2.13.
There exists an irreducible representation of highest weight Λ if and only if

$$\Lambda_\alpha = \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{N} \tag{2.54}$$

for every simple root α. Moreover, if ξ is a highest weight vector and if α is a simple root, then

$$E^k_{-\alpha}\xi \begin{cases} \neq 0 & \text{if } k \leqslant \Lambda_\alpha \\ = 0 & \text{if } k > \Lambda_\alpha. \end{cases} \tag{2.55}$$

Proof. No proof. □

2.2.3 List of the weights of a representation

We consider a representation of highest weight Λ. For each weight M, we define

$$\delta(M) = 2\sum_{\alpha_i\in\Pi} M_{\alpha_i} \tag{2.56}$$

where, as usual, $M_\alpha = 2(M,\alpha)/(\alpha,\alpha)$. For any root α, we define

$$\gamma(\alpha) = \frac{1}{2}\big(\delta(\Lambda) - \delta(\alpha)\big). \tag{2.57}$$

Proposition 2.11 shows in particular that $\gamma(\alpha)$ is an integer.

Proposition 2.14.
When M is a weight, $\gamma(M)$ is the number of simple roots that have to be subtracted from the highest weight Λ in order to get M.

Proof. No proof. □

Let us consider the sets

$$\Delta^k_\phi = \{M \text{ st } \gamma(M) = k\}. \tag{2.58}$$

That set is the **layer** of order k. Of course, there exists a $T(\phi)$ such that

$$\Delta_\phi = \Delta^0_\phi \cup \Delta^1_\phi \cup \ldots \cup \Delta^{T(\phi)}_\phi. \tag{2.59}$$

That $T(\phi)$ is the **height** of the representation ϕ. If Λ is the highest weight and Λ' is the lowest weight, then we have $\gamma(\Lambda) = 0$ and $\gamma(\Lambda') = T(\phi)$.

A corollary of proposition 2.14 is that, if $M \in \Delta_\phi^r$ and if α is a simple root, then $M+\alpha \in \Delta_\phi^{r-1}$, and $M - \alpha \in \Delta_\phi^{r+1}$.

Let us denote by $S_k(\phi)$ the multiplicity of the layer of order k; we have

$$S_0 + S_1 + \ldots + S_T = N, \tag{2.60}$$

where N is the dimension of the representation ϕ. The number

$$III(\phi) = \max S_k(\phi) \tag{2.61}$$

is the **width** of the representation.

Lemma 2.15.
If Λ is the highest weight and Λ' is the lowest weight, then $\delta(\Lambda) + \delta(\Lambda') = 0$.

Proof. No proof. □

From that lemma and the definition of $\gamma(M)$, we deduce that $\delta(\Lambda) - \delta(\Lambda') = 2\gamma(\Lambda') = T(\phi)$, so that $\delta(\Lambda) = T(\phi)$ and

$$\delta(M) = T(\phi) - 2\gamma(M). \tag{2.62}$$

In particular, $\delta(M)$ has a fixed parity for a given representation ϕ. It is the **parity** (even or odd) of the representation.

Theorem 2.16.
If Λ is the highest weight of the irreducible representation ϕ, then

$$T(\phi) = \sum_{\alpha_i \in \Pi} r_{\alpha_i} \Lambda_\alpha \tag{2.63}$$

where the coefficients r_{α_i} only depend on the algebra, and in particular not on the representation.

Proof. No proof. □

The coefficients r_{α_i} are known for all the simple Lie algebra, see for example page 105 of [13].

Finding all the weights of a representation

The following can be found in [13, 15].

Theorem 2.17.
If Δ_ϕ is the weight system of the irreducible representation ϕ, then

$$S_k = S_{T-k} \tag{2.64}$$

and

$$S_r \geqslant S_{r-1} \geqslant \ldots \geqslant S_2 \geqslant S_1 \tag{2.65}$$

where $r = \frac{T}{2} + 1$.

The theorem says that when $T(\phi)$ is even (let us say $T(\phi) = 2r$), then $III(\phi) = S_r(\phi)$ and when $T(\phi)$ is odd (let us say $T(\phi) = 2r + 1$), then

$$III(\phi) = S_r(\phi) = S_{r+1}(\phi). \tag{2.66}$$

Let α be a root. The **α-series** trough the weight M is the sequence of weights

$$M - r\alpha, \ldots, M + q\alpha \tag{2.67}$$

such that $M - (r+1)\alpha$ and $M + (q+1)\alpha$ do not belong to Δ_ϕ.

Proposition 2.18.
Let M be a weight of the representation ϕ and α, any root of $\mathfrak{g}$. If the α-series trough M begins at $M - r\alpha$ and ends at $M + q\alpha$, then

$$\frac{2(M,\alpha)}{(\alpha,\alpha)} = r - q, \tag{2.68}$$

or, more compactly, $M_\alpha + q = r$.

Notice that, in that proposition, q and r are well defined functions of M and α.

We are now able to determine all the weights of the representation ϕ. Let us suppose that we already know all the layers $\Delta_\phi^0, \ldots, \Delta_\phi^{r-1}$. We are going to determine the weights in the layer Δ_ϕ^r.

An element of Δ_ϕ^r has the form $M - \alpha$ with $M \in \Delta_\phi^{r-1}$ and α, a root. Thus, in order to determine Δ_ϕ^r, we have to test if $M - \alpha$ is a weight for each choice of $M \in \Delta_\phi^{r-1}$ and $\alpha \in \Pi$. Using proposition 2.18, if [1]

$$M_\alpha + q \geqslant 1, \tag{2.69}$$

then $M - \alpha \in \Delta_\phi$. The number $M_\alpha - q(M,\alpha)$ is the **lucky number** of the root $M - \alpha$. The root is a weight if its lucky number is bigger or equal to 1. Notice that $q(M,\alpha)$ depends on the representation we are looking at.

Since $M + k\alpha \in \Delta_\phi^{r-k}$, the value of q is known when one knows the "lower" layers. We are thus able to determine, by induction, all the layers from Δ_ϕ^0 which only contains the highest weight. For this one, by definition, we always have $q = 0$.

The Dynkin coefficients of one weights can be more easily computed using the following formula, which is a direct consequence of definition of the Cartan matrix:

$$(M - \alpha_j)_i = M_i - A_{ji}. \tag{2.70}$$

As example, let us determine the weights of the representation $\circ\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}$ of $\mathfrak{su}(3)$. The algebra $\mathfrak{su}(3)$ has two simple roots α and β whose inner products are $(\alpha,\alpha) = (\beta,\beta) = 1$ and $(\alpha,\beta) = -1/2$. The highest weight of $\phi = \circ\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}$ is $\Lambda = (\alpha + 2\beta)/3$.

We first test if $\Lambda - \alpha$ is a weight. Easy computations show that $\Lambda_\alpha = 0$ wile $q = 0$; thus $\Lambda - \alpha$ is not a weight. The same kind of computations show that $\Lambda_\beta = 1$, so that $\Lambda_\beta = q(\Lambda,\beta) = 1$. That shows that $\Delta_\phi^1 = \{\Lambda - \alpha\}$.

Let now $M = \Lambda - \beta = (\alpha - \beta)/3$. Since $M + \alpha \notin \Delta_\phi$, we have $q(M,\alpha) = 0$. On the other hand, $M_\alpha = 1$, so that $M - \alpha \in \Delta_\phi^2$. The last one to have to be tested is $M - \beta$. Since $M + \beta = \Lambda$, we have $q(M,\beta) = 1$, but $M_\beta = -1$. Thus $M_\beta + q(M,\beta) = 0$ and $M - \beta$ is not a weight.

We can obviously continue in that way up to find $\Delta_\phi^r = 0$, but there is an escape to be more rapid. Indeed, using theorem 2.16 with coefficients r_α that can be found in tables (for example in [13]), we find

$$T(\phi) = 2\Lambda_\alpha + 3\Lambda_\beta = 2, \tag{2.71}$$

[1] At page 104 of [13], that condition is (I think) wrongly written $M_\alpha + q \geqslant 0$; that mistake is repeated in the example of page 106.

thus we immediately know that Δ^3_ϕ does not exist.

On the other hand, one knows the width $III(\phi) = \max S_k(\phi)$ because (since $T(\phi) = 2r$, with $r = 1$), we have $III(\phi) = S_1(\phi)$. Thus, once $\Delta^1(\phi)$ is determined, we know that the next ones will never have more elements.

In the example, when we know that $M - \alpha$ is a weight, we do not have to test $M - \beta$.

2.2.4 Tensor product of representations

Tensor and weight

Let ϕ and ϕ' be representations of $\mathfrak{g}$ on the vector spaces R and R' of dimensions n and m. If $A \in \mathbb{M}_n(R)$ and $B \in \mathbb{M}_m(R')$, the **tensor product**, also know as the **Kronecker product** of A and B is the matrix $A \otimes B \in \mathbb{M}_{mn}(R \otimes R')$ whose elements are given by

$$C_{ik,jl} = A_{ij} B_{kl}. \tag{2.72}$$

The principal properties of that product are

$$(A_1 A_2) \otimes (B_1 B_2) = (A_1 \otimes B_1)(A_2 \otimes B_2) \tag{2.73a}$$
$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \tag{2.73b}$$
$$\mathbb{1}_R \otimes \mathbb{1}_{R'} = \mathbb{1}_{R \otimes R'} \tag{2.73c}$$

If φ_1 and φ_2 are two representations of a group G, the **tensor product** is defined by

$$(\varphi_1 \otimes \varphi_2)(g) = \varphi_1(g) \otimes \varphi_2(g). \tag{2.74}$$

If ϕ and ϕ' are two representations of a Lie algebra $\mathfrak{g}$, the **tensor product** representation is defined by

$$(\phi \otimes \phi')(X)(v \otimes v') = \big(\phi(X)v\big) \otimes v' + v \otimes \big(\phi'(X)v'\big). \tag{2.75}$$

If $\{\phi_k\}$ are the irreducible representations, a natural question that arise is to determine the coefficients Γ which decompose $\phi \otimes \phi'$ into irreducible representations:

$$\phi \otimes \phi' = \sum_k \Gamma_k(\phi, \phi') \phi_k \tag{2.76}$$

Let W and W' be the representation spaces and consider the following decompositions in weight spaces:

$$W = \bigoplus_{\Lambda \in \Delta_1} W_\Lambda, \qquad W' = \bigoplus_{\Lambda \in \Delta_2} W'_\Lambda. \tag{2.77}$$

By definition,

$$(W \otimes W')_\alpha = \{v \otimes v' \text{ st } (\phi \otimes \phi')(h)(v \otimes v') = \alpha(h)(v \otimes v')\}. \tag{2.78}$$

If $\big(\phi(h)v\big) \otimes v' + v \otimes \big(\phi'(h)v'\big)$ is a multiple of $v \otimes v'$, one requires that

$$\phi(h)v = \alpha_1(h)v, \tag{2.79a}$$
$$\phi'(h)v = \alpha_2(h)v' \tag{2.79b}$$

for the weights α_1 and α_2 of ϕ and ϕ'. Thus we have

$$(W \otimes W')_{\alpha_1 + \alpha_2} = W_{\alpha_1} \otimes W_{\alpha_2}. \tag{2.80}$$

We have in particular that the simple root system $\Delta_{\phi\otimes\phi'}$ of the representation $\phi\otimes\phi'$ is given by

$$\Delta_{\phi\otimes\phi'} = \Delta_{\phi} + \Delta_{\phi'}. \tag{2.81}$$

What we proved is [2]

Proposition 2.19.
If ϕ is a representation of highest weight Λ and ϕ' is a representation of highest weight Λ', then $\phi\otimes\phi'$ is a representation of height weight $\Lambda+\Lambda'$.

If, moreover, ϕ and ϕ' are irreducible, then $\phi\otimes\phi'$ is irreducible.

An irreducible representation that cannot be written under the form of a tensor product of irreducible representations is a **basic representation**.

Lemma 2.20.
A representation is basic if and only if its highest weight Λ is such that the Λ_{α_i} are all zero but one which is 1.

The basic representations of $\mathfrak{so}(10)$ are given by the Dynkin diagrams of figure 2.1. All the irreducible representations are obtained by tensor products of the basic ones. An **elementary** is a basic representation which has his "1" on a terminal point of the Dynkin diagram.

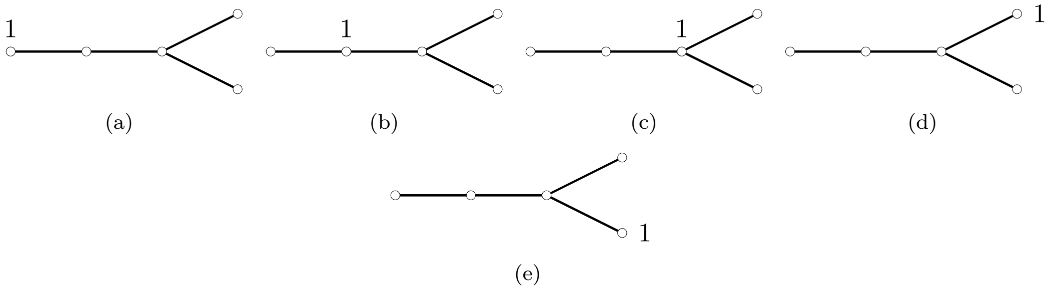
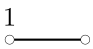
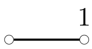


Figure 2.1: Basic representations of $\mathfrak{so}(10)$

Decomposition of tensor products of representations

Proposition 2.19 allows us to decompose a tensor product of representations into irreducible representations. Let us do it on a simple example in $\mathfrak{su}(3)$. We consider the representations $\phi =$  and $\phi' =$ . The first representation has weights

$$\Delta_{\phi} = \left\{ \frac{\alpha+2\beta}{3}, \frac{\alpha-\beta}{3}, \frac{-(2\alpha+\beta)}{3} \right\}, \tag{2.82}$$

and the second one has

$$\Delta_{\phi'} = \left\{ \frac{\alpha+2\beta}{3}, \frac{\alpha-\beta}{3}, \frac{-(2\alpha+\beta)}{3} \right\}. \tag{2.83}$$

[2] The second part is not proved.

According to equation (2.81)-80, we have 9 weights in the representation $\phi \otimes \phi'$ (all the sums of one element of Δ_ϕ with a one of $\Delta_{\phi'}$). The highest one is

$$\frac{2\alpha + 4\beta}{3},$$

which is the double of the highest weight in $\circ\!\!-\!\!\!-\!\!\overset{1}{\circ}$, so $\phi \otimes \phi'$ contains the representation $\circ\!\!-\!\!\!-\!\!\overset{2}{\circ}$. Now, we remove from the list of weights of $\phi \otimes \phi'$ the list of weight of $\circ\!\!-\!\!\!-\!\!\overset{2}{\circ}$; the result is

$$\frac{2\alpha + \beta}{3}, \frac{-(\alpha - \beta)}{3}, \frac{-(\alpha + 2\beta)}{3}, \tag{2.84}$$

which are the weights of $\overset{1}{\circ}\!\!-\!\!\!-\!\!\circ$. The conclusion is that

$$\circ\!\!-\!\!\!-\!\!\overset{1}{\circ} \otimes \circ\!\!-\!\!\!-\!\!\overset{1}{\circ} = \circ\!\!-\!\!\!-\!\!\overset{2}{\circ} \oplus \overset{1}{\circ}\!\!-\!\!\!-\!\!\circ. \tag{2.85}$$

That procedure of decomposition is quite long because it requires to compute the complete set of weights for some intermediate representations.

Symmetrization and anti symmetrization

Let ϕ be a irreducible representation. We want to compute the symmetric and antisymmetric parts of the representation $\phi^{\otimes k} = \underbrace{\phi \otimes \ldots \otimes \phi}_{k \text{ times}}$. These symmetric and antisymmetric parts are denoted by $\phi_s^{\otimes k}$ and $\phi_a^{\otimes k}$ respectively.

Proposition 2.21.
If $\{\xi_1, \ldots, \xi_N\}$ is a canonical basis of ϕ and if we denote by Λ_i the weight of the vector ξ_i, the followings hold:

(i) the weight system of $\phi_a^{\otimes k}$ is

$$\Lambda_{i_1} + \Lambda_{i_2} + \ldots + \Lambda_{i_k} \tag{2.86}$$

with $i_k > \ldots > i_2 > i_1$, and the highest weight is

$$\Lambda_1 + \ldots + \Lambda_k. \tag{2.87}$$

The dimension of the representation $\phi_a^{\otimes k}$ is

$$N(\phi_a^{\otimes k}) = \binom{n}{k}. \tag{2.88}$$

(ii) The weight system of the representation $\phi_s^{\otimes k}$ is

$$\Lambda_{i_1} + \Lambda_{i_2} + \ldots + \Lambda_{i_k} \tag{2.89}$$

with $i_k \geqslant \ldots \geqslant i_2 \geqslant i_1$, and the highest weight is

$$k\Lambda_1 \tag{2.90}$$

The dimension of the representation $\phi_s^{\otimes k}$ is

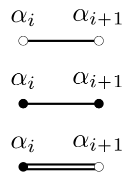
$$N(\phi_s^{\otimes k}) = \binom{n+k}{k}. \tag{2.91}$$

Proof. No proof. □

The representations $\phi_a^{\otimes k}$ and $\phi_s^{\otimes k}$ might be decomposable and we denote by $\phi_{s>}^{\otimes k}$ and $\phi_{a>}^{\otimes k}$ their highest weight parts.

Let α be a terminal point in a Dynkin diagram. The **branch** of α is the sequence of point of the Dynkin diagram $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k$ defined by the following properties.

- The point α_i is connected with (and only with) the points α_{i-1} and α_{i+1},
- the connexion between α_i and α_{i+1} is of one of the following forms



(2.92)

- the sequence $\alpha_1, \ldots, \alpha_k$ is maximal in the sense that no α_{k+1} can be added without violating one of the two first rules.

Proposition 2.22.
Let α be a terminal point in a Dynkin diagram and $\alpha_1, \ldots, \alpha_k$ be the corresponding branch. Then we have

$$\phi_{\alpha_r} \simeq \phi_{\alpha\, a>}^{\otimes r} \tag{2.93}$$

for every $r = 1, 2, \ldots, k$.

2.3 Verma module

Let us give the definition of [17]. When $\mathfrak{g}$ is a semisimple Lie algebra, we have the usual decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \tag{2.94}$$

where each of the three components are Lie algebras. In particular, the universal enveloping algebra $\mathcal{U}(\mathfrak{n}^-)$ makes sense. Let $\mu \in \mathfrak{h}^*$. We build a representation π_μ of $\mathfrak{g}$ on $V_\mu = \mathcal{U}(\mathfrak{n}^-)$ in the following way

- If $Y_\alpha \in \mathfrak{n}^-$, we define

$$\pi_\mu(Y_\alpha)1 = Y_\alpha \tag{2.95a}$$
$$\pi_\mu(Y_{\alpha_1} \ldots Y_{\alpha_n}) = Y_\alpha Y_{\alpha_1} \ldots Y_{\alpha_n}, \tag{2.95b}$$

- if $H \in \mathfrak{h}$, we define

$$\pi_\mu(H)1 = \mu(H) \tag{2.96a}$$
$$\pi_\mu(Y_{\alpha_1} \ldots Y_{\alpha_k}) = \big(\mu(H) - \sum_{j=1}^{k} \alpha_j(H)\big) Y_{\alpha_1} \ldots Y_{\alpha_k}, \tag{2.96b}$$

- and if $X_\alpha \in \mathfrak{n}^+$, we define

$$\pi_\mu(X_\alpha)1 = 0 \tag{2.97a}$$

$$\pi_\mu(X_\alpha)Y_{\alpha_1}\ldots Y_{\alpha_k} = Y_{\alpha_1}\big(\pi_\mu(X_\alpha)Y_{\alpha_2}\ldots Y_{\alpha_k}\big) \tag{2.97b}$$

$$- \delta_{\alpha,\alpha_1}\sum_{j=1}^{k}\alpha_j(H_\alpha)Y_{\alpha_1}\ldots Y_{\alpha_k}. \tag{2.97c}$$

In the last one, we do an inductive definition.

Lemma 2.23.
The couple (π_μ, V_μ) is a representation of $\mathfrak{g}$ on V_μ.

Proof. No proof. □

That representation is one **Verma module** for $\mathfrak{g}$. If the algebra $\mathfrak{g}$ is an algebra over the field $\mathbb{K}$, the field $\mathbb{K}$ itself is part of $\mathcal{U}(\mathfrak{n})^-$, so that the scalars are vectors of the representation. In that context, the multiplicative unit $1 \in \mathbb{K}$ is denoted by v_0.

Theorem 2.24.
The representation (π_μ, V_μ) of the semisimple Lie algebra $\mathfrak{g}$ is a cyclic module of highest weight, with highest weight μ and where v_0 is a vector of weight μ.

Proof. No proof. □

The Verma module is, *a priori*, infinite dimensional and non irreducible, thus one has to perform quotients of the Verma module in order to build finite dimensional irreducible representations.

2.4 The group SO(3) and its Lie algebra

We follow [18] in which more proofs can be found.

Proposition 2.25.
An element of SO(3) has exactly one eigenvector with eigenvalue 1. That vector is the ***rotation axis****.*

The generator of rotation around the axis n (unit vector) is given by the matrix

$$\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}. \tag{2.98}$$

That form results form the requirement that $Nr = n \times r$. If we denote by $R(n,\theta)$ the operator of rotation in $\mathbb{R}^3$ by an angle θ around the axis n, one shows that

$$R(b,\theta) = \mathbb{1} + \sin(\theta)N + \big(1-\cos(\theta)\big)N^2. \tag{2.99}$$

2.4.1 Rotations of functions

Consider any function $f\colon \mathbb{R}^3 \to \mathbb{C}$; we define the **rotation operator** $U(n,\theta)$ by

$$\big(U(n\theta)f\big)(r) = f\big(R(n,\theta)^{-1}r\big). \tag{2.100}$$

These operators form a group, and we have in particular that

$$U(n,\theta_1)U(n,\theta_2) = U(n,\theta_1+\theta_2).$$

We are interested in *infinitesimal* rotations, that is rotations of angle $d\theta$ for which $(d\theta)^2 \ll d\theta$, or in other words, we are interested in a development of equation (2.100)-84 restricted to linear terms in θ. What one obtains is

$$\big(U(n,d\theta)f\big)(r) = \big((1 - id\theta\, n\cdot l)f\big)(r) \tag{2.101}$$

where the operator l is defined by

$$l = -ir \times \nabla. \tag{2.102}$$

Its components $l_i = -i\epsilon_{ijk}r_j\partial_k$ satisfy commutation relations

$$[l_i, l_j] = i\epsilon_{ijk}l_k. \tag{2.103}$$

The operator $n\cdot l$ is refereed as the **generator of infinitesimal rotations**. One can derive an expression of $U(n,\theta)$ in terms of $n\cdot l$ by the following:

$$U(n,\theta+d\theta)f = U(n,\theta)U(n,d\theta)f = U(n,\theta)(1-id\theta\, n\cdot l)f,$$

so that we have the differential equation

$$\frac{dU}{d\theta}(n,\theta) = -iU(n,\theta)n\cdot l \tag{2.104}$$

with the initial condition $U(n,0)=1$. The solution is

$$U(n,\theta) = e^{-i\theta\, n\cdot l}. \tag{2.105}$$

2.4.2 Representations of SO(3)

The group SO(3) is strongly linked with $SU(2)$ by the following property :

$$\mathrm{SO}(3) = \frac{SU(2)}{\mathbb{Z}_2}. \tag{2.106}$$

Lemma 2.26.
A representation ρ_j of $SU(2)$ is a representation of SO(3) *if and only if $\rho_j(X) = \mathrm{id}$ for any X in the kernel of the homomorphism $SU(2) \to \mathrm{SO}(3)$, namely: $\rho_j(\pm\mathbb{1}) = \mathrm{id}$.*

Proof. We consider $\rho_j\colon SU(2) \to \operatorname{End} V_j$ and $\psi\colon SU(2) \to \mathrm{SO}(3)$. The latter fulfils $\psi(\mathbb{1}) = \psi(-\mathbb{1}) = \mathbb{1}$, which is an important equation because it ensures us that the rest of the expressions are well defined with respect to the class representative.

If $\rho_j(-\mathbb{1}) = \mathbb{1}$, we define $d_j\colon\ \mathrm{SO}(3) \to \operatorname{End} V$ by $d_j([x]) = \rho_j(x)$ (check that this is well defined). With this,

$$d_j([x])d_j([y]) = \rho_j(x)\rho_j(y) = \rho_j(xy) = d_j([xy]).$$

Now let us suppose that $d_j([x]) = \rho_j(x)$ is a representation. Thus

$$\rho_j(x) = d_j([x]) = d_j([-x]) = \rho_j(-x) = \rho_j(-\mathbb{1})\rho_j(x),$$

so $\rho_j(-\mathbb{1}) = \mathrm{id}_{V_j}$.

□

Moreover, any representation of $\mathrm{SO}(3)$ comes from a representation $\tilde{\rho}$ of $SU(2)$ by setting $\tilde{\rho}(-\mathbb{1}) = \mathrm{id}$ and $\tilde{\rho}(x) = \rho([x])$.

Now, we research the representations of $SU(2)$ for which the matrix $-\mathbb{1}$ is represented by the identity operator. These will be representations of $\mathrm{SO}(3)$. The spin j representations of $SU(2)$ is given by

$$\rho_j(X)\phi_{pq}(\xi) = \phi_{pq}(X^{-1}\xi).$$

With $X = -\mathbb{1}$, this gives: $\phi_{pq}(-\xi) = (-1)^{p+q}\phi_{pq}(\xi)$. If we want it to be equal to $\phi_{pq}(\xi)$, we need $p + q = 2j$ even. This is true if and only if $j \in \mathbb{N}$.

The conclusion is that the irreducible representations of SO(3) are the integer spin irreducible representations of SU(2). Note that the non relativistic mechanics has SO(3) as group of space symmetry. Thus there are no hope to find any half integer spin in a non relativistic theory.

2.4.3 Representations of the algebra $\mathfrak{su}(2) = \mathfrak{so}(3)$

Determination of the representations

In the case of $\mathfrak{so}(3)$, the Cartan subalgebra is one dimensional, and one has only one simple root: $\alpha = J_{12}^*$. If $\Lambda = aJ_{12}^*$, one has $(\Lambda, \alpha) = a$, and theorem 2.13 says that Λ is highest weight of an irreducible representation if and only if $a \in \mathbb{N}/2$.

Ladder operators

We are now going to determine the irreducible representations in a more explicit way. From the relation (2.106)-84, we know that the study of $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are the same. The algebra $\mathfrak{su}(2)$ is the real algebra generated by the matrices of the form $\begin{pmatrix} \alpha & \beta \\ -\beta^* & -\alpha \end{pmatrix}$ with α, $\beta \in \mathbb{C}$. A convenient basis is given by

$$u_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad u_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.107}$$

That algebra satisfies the commutation relations

$$[u_i, u_j] = \epsilon_{ijk}u_k. \tag{2.108}$$

The trick to build finite dimensional representations of that algebra is common (see [19] for example). The first step is to perform a change of basis $J_k = iu_k$ that brings the algebra under the form (see section 2.4 to understand why)

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{2.109}$$

We are going to construct all the finite dimensional irreducible representations of the algebra (2.109)-85. The key point of that new basis is that one can define the **ladder operators**

$$J_\pm = J_1 \pm iJ_2 \tag{2.110}$$

that have the property that

$$[J_3, J_\pm] = \pm J_\pm. \tag{2.111}$$

Notice that for every i, we have $(J_i)^* = J_i$, so that $(L^\pm)^* = L^\mp$. An other important property is that, defining $J^2 = J_1^2 + J_2^2 + J_3^2$, we have

$$[J_i, J^2] = 0, \tag{2.112}$$

which show that J^2 is a Casimir operator, and is thus by Schur's lemma a multiple of identity. Notice that we are using an abuse of notation between J_i as element of $\mathfrak{su}(2)$ and J_i as the operator that represent J_i. In the first case, products like J_iJ_j make no sense[3], but it makes sense as operator composition.

The subalgebra $\{J^2, J_3\}$ being abelian, we can diagonalise J^2 and J_3 in the same time. Let $|m,\sigma\rangle$ be an orthonormal basis of the eigenspace of J_3 associated with the eigenvalue m. The index σ is for a possible degenerateness to be studied later. We have

$$J_3|m,\sigma\rangle = m|m,\sigma\rangle.$$

Using the commutation relations between J_3 and the ladder operators, we have

$$J_3J_\pm|m,\sigma\rangle = \big(\pm J_\pm + J_\pm J_3\big)|m,\sigma\rangle = (m \pm 1)J_\pm|m,\sigma\rangle. \tag{2.113}$$

Thus $J_\pm|m,\sigma\rangle$ is an eigenvector of J_3 with the eigenvalue $m \pm 1$, which means that $J_\pm|m,\sigma\rangle$ is a linear combination of the vectors $|m \pm 1, \sigma\rangle$ with different values of σ. This is the reason of the name of the *ladder* operators: they raise and lower the eigenvalue of J_3.

We can now prove that one has to drop the index σ because eigenvalues of J_3 cannot be degenerated. For, compute

$$J_+J_- = (J_1 + iJ_2)(J_1 - iJ_2) = J^2 - J_3^2 + i[J_2, J_1] = J^2 - J_3^2 + J_3, \tag{2.114}$$

so that

$$J_+J_-|m,\sigma\rangle = (\alpha - m^2 + m)|m,\sigma\rangle$$

where α is defined by $J^2 = \alpha\mathbb{1}$. That proves that the space generated by $|m,\sigma\rangle$ and the action of J_3, J_+ and J_- is invariant under the representation, while one cannot obtain $|m,\sigma'\rangle$ by action of $J_\pm$ on $|m,\sigma\rangle$. Since we are looking for *irreducible* representations, that space must actually be all the representation space. That rules out the possibility to have two different vectors $|m,\sigma_1\rangle$ and $|m,\sigma_2\rangle$.

The explicit matrix form of $J_\pm$ are:

$$J_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.115}$$

Since we are searching for finite dimensional representations, there exists a maximal eigenvalue of J_3. Let us denote by j that maximal eigenvalue and by $|j\rangle$ the corresponding eigenvector. The

[3] In fact, one has to understand these products as elements of the universal enveloping algebra. What we are building is a reprensentation of that algebra, which, obviously, restricts to a representation of the algebra. When we use the Schur's lemma, in fact we invoke it in $\mathcal{U}(\mathfrak{so}(3))$

relation (2.113)-86 shows that if $J_+|j\rangle \neq 0$, then $J_+|j\rangle$ is an eigenvector for J_3 with eigenvalue $j+1$, which contradicts maximality. Then we have $J_+|j\rangle = 0$.

Since we know the action of J_3 and J_+ on $|j\rangle$, it is convenient to write J^2 in terms of these two operators. This is done in the same way as probing equation (2.114)-86:

$$J^2 = J_3^2 + J_3 + J_-J_+, \tag{2.116}$$

so that

$$J^2|j\rangle = j(j+1)|j\rangle. \tag{2.117}$$

We know that $J^2 = \alpha\mathbb{1}$ and that α is a characteristic of the representation. What equation (2.117)-87 tells us is that the maximal eigenvalue of J_3 is related to α by $j(j+1) = \alpha$.

We are now able to determine the proportionality constant of relation $J_\pm|m\rangle \propto |m\pm 1\rangle$. Since $(J_-)^* = J_+$, we have

$$\|J_-|m\rangle\|^2 = \langle m|J_+J_-|m\rangle = j(j+1) - m^2 + m. \tag{2.118}$$

Then one has

$$J_-|m\rangle = \sqrt{j(j+1) - m(m-1)}|m-1\rangle, \tag{2.119a}$$

$$J_+|m\rangle = \sqrt{j(j+1) - m(m+1)}|m+1\rangle. \tag{2.119b}$$

As expected, $J_-|-j\rangle = 0$ and $J_+|j\rangle = 0$. Notice that we avoid the possibility $J_-|m\rangle = -\sqrt{\cdots}|m-1\rangle$ by a simple redefinition $|m-1\rangle \to -|m-1\rangle$.

Equation (2.118)-87 shows that the norm of $|m\rangle$ becomes negative for $m < -j$ and $m > j+1$. We conclude that the minimal eigenvalue of J_3 is $-j$. Since $|j\rangle$ has to be reached from $|-j\rangle$ by action of J_+, the difference $j - (-j)$ must be an integer. Thus $j \in \mathbb{N}/2$. The number j is the **spin** of the representation.

Let us give the explicit example with spin one half. When $j = \frac{1}{2}$, the vector space is generated by the vectors $|1/2\rangle$ and $|-1/2\rangle$, and the operators are given by

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.120}$$

from which we deduce

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Notice that we have $J_i = \frac{1}{2}\sigma_i$ with the **Pauli matrices**,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.121}$$

These matrices fulfil the relation

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k. \tag{2.122}$$

Weight vectors

The algebra $\mathfrak{so}(3)$ does not contain abelian subalgebra of dimension bigger than one, so a Cartan subalgebra is generated by J_3. The unique (up to dilatation) element of $\mathscr{H}^*$ is thus given by $\alpha(J_3) = 1$. The relation $[J_z, J_\pm]$ provides the root spaces:

$$\begin{aligned} \mathfrak{so}(3)_1 &= \{J_+\} \\ \mathfrak{so}(3)_{-1} &= \{J_-\}, \end{aligned} \tag{2.123}$$

thus $\mathfrak{n}^\pm$ is generated by $J_\pm$.

2.5 Thinks about $\mathfrak{su}(3)$

Using the Cartan matrix of $\mathfrak{su}(3)$ and formula (2.70)-78, we will determine the Dynkin coefficients of the representation $\overset{1}{\circ}\!\!-\!\!-\!\!-\!\!\circ$ without even explicitly compute the weights. For that, we follow the construction of [15]. The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.124}$$

The Dynkin coefficients of the highest weight is given by

$$\Lambda_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.125}$$

Since Λ is highest weight, we have $q(\Lambda, \alpha_i) = 1$, so that $\Lambda_1 + q(\Lambda, \alpha_1) = 1$ and $\Lambda_2 + q(\Lambda, \alpha_2) = 0$. Thus the only weight of the first layer is $M = \Lambda - \alpha_1$. Using formula (2.70)-78, we find

$$(\Lambda - \alpha_1)_i = \Lambda_i - A_{1i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{2.126}$$

We also have, by construction, $p(M, \alpha_1) = 1$ and $p(M, \alpha_2) = 0$, so that $M_1 + p(M, \alpha_1) = -1 + 1 = 0$ and $M_2 + p(M, \alpha_2) = 1$. We conclude that $M - \alpha_2$ is a weight, and its Dynkin coefficients are given by

$$(M - \alpha_2)_i = \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{2.127}$$

Chapter 3

From Clifford algebra to spin manifold

Bibliography for Clifford algebras, spin group and related topics are [20–24]. More agebraic point of view can be found in [25, 26]. More details about "square rooting" second order differential operators are in [27]. For physical concerns, the reader should refer to [28–30].

3.1 Invitation : Clifford algebra in quantum field theory

3.1.1 Schrödinger, Klein-Gordon and Dirac

The origin of the Klein-Gordon equation is almost the same as the one of the Schrödinger: one replace physical functions by operators. For a free particle, the correspondence are

$$\begin{aligned} &\text{energy} && E \to i\hbar\frac{\partial}{\partial t}, \\ &\text{momentum} && \mathbf{p} \to -i\hbar\nabla. \end{aligned}$$

The Schrödinger equation (which is the non relativistic quantum wave equation) comes from replacement in the non non relativistic expression of the Hamiltonian

$$E = \frac{\mathbf{p}^2}{2m} \longrightarrow \left(\partial_t - \frac{i\hbar}{2m}\nabla^2\right)\psi = 0,$$

while the Klein-Gordon one (which is the relativistic quantum wave equation) comes from the relativistic corresponding equation:

$$E^2 = \mathbf{p}^2c^2 + m^2c^4 \longrightarrow \left(\partial^\mu\partial_\mu + (\frac{mc}{\hbar})^2\right)\psi = 0.$$

This is a second order differential equation; there are however no "law of nature" which forbid a first order equation. We try

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{\hbar c}{i}\alpha^k\partial_k + \beta mc^2\right)\psi \equiv \hat{H}\psi.$$

There are some physical constraints on the coefficients α^k and β. We will study one of them: we want the components of ψ to satisfy the Klein-Gordon equation, so that the plane waves fulfill the fundamental relation $E^2 = p^2c^2 + m^2c^4$.

In order to see the implications of this constraint on the coefficients, we apply two times the operator $\hat{H}$, and we compare the result with the Klein-Gordon equation. We find:

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}\mathbb{1}, \tag{3.1a}$$

$$\alpha^i\beta + \beta\alpha^i = 0, \tag{3.1b}$$

$$(\alpha^i)^2 = \beta^2 = \mathbb{1}. \tag{3.1c}$$

If we define $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$, we find that the matrices γ^μ have to give a representation of the Clifford algebra[1]:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}. \tag{3.2}$$

The Dirac equation reads

$$\left(-i\gamma^\mu\partial_\mu + \frac{mc}{\hbar}\right)\psi = 0.$$

If we want to perform some computation with the quantum field theory, we need an explicit form for the γ's; that's the reason why we study representations of the Clifford algebra. The **Dirac operator** $\mathcal{D}$ is the operator which lies in the Dirac equation:

$$\mathcal{D} = \sum_{\mu=0}^{3}\gamma^\mu\frac{\partial}{\partial x^\mu}. \tag{3.3}$$

3.1.2 Lorentz algebra

There is an other physical reason (which is in fact the same, but differently presented) justifying the study of the Clifford algebra. The quantum field theory need representation of the Lorentz algebra[2]

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}).$$

A proof of these relations is given in lemma 3.1. Dirac had a trick to find such J matrices from a representation of the Clifford algebra. If we have $n\times n$ matrices γ_μ such that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_{n\times n},$$

a n-dimensional representation of the Lorenz algebra is obtained by

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right].$$

Lemma 3.1.
The matrices of $\mathfrak{so}(p,q)$ satisfy the definition relation

$$M^t\eta + \eta M = 0, \tag{3.4}$$

and if M^{ab} is the "rotation" in the place of directions a and b (i.e. a trigonometric or an hyperbolic rotation following that a and b are of the same type or not), then the action on $\mathbb{R}^{(p,q)}$ is given by $(x')^\mu = (M^{ab})^\mu_\nu x^\nu$ with

$$(M^{ab})^\mu_\nu = \eta^{a\mu}\delta^b_\nu - \eta^{b\mu}\delta^a_\nu. \tag{3.5}$$

[1] Don't be afraid with the extra minus sign: the quantum field theory is most written with the metric $(+,-,-,-)$ instead of $(-,+,+,+)$.

[2] When one think to real infinitesimal rotation matrices, the presence of i seems not natural, but one redefines $J \to iJ$ for formalism reasons.

The commutation relations are given by

$$[M^{ab}, M^{cd}] = -\eta^{ac}M^{bd} + \eta^{ad}M^{bc} + \eta^{bc}M^{ad} - \eta^{bd}M^{ac}. \tag{3.6}$$

Notice that $M^{ab} = -M^{ba}$.

See section 12.5 of [31]. By a simple redefinition $J = iM$, one obtains

$$[J, J] = i\eta J \tag{3.7}$$

instead of $[M, M] = \eta M$, and the matrices J are Hermitian. Here η is the matrix $\eta = diag(\underbrace{+,\ldots,+}_{p \text{ times}}, \underbrace{-,\ldots,-}_{q \text{ times}})$. As convention, we say that a direction corresponding to a *positive* entry in the metric is a *time* direction, while the spatial directions are negative.

3.2 Clifford algebra

3.2.1 Definition and universal problem

Definition 3.2.
*Let V be a (finite dimensional) vector space and q, a bilinear quadratic form over V. The **Clifford algebra** $\mathrm{Cl}(V, q)$ is the unital associative algebra generated by V subject to the relation*

$$v \cdot v = q(v) \tag{3.8}$$

for all v in $\mathrm{Cl}(V, q)$. Here the dot denotes the algebra product and $q(v)$ means $q(v, v)$.

Theorem 3.4 proves unicity of such an algebra, so that it makes sense.

Remark 3.3. The relation (3.8)-91 is no more a restriction for the elements in $\mathrm{Cl}(V, q)$ than a restriction on the choice of the algebra product.

Theorem 3.4.
Let E be an unital associative algebra and $j : V \to E$ a linear map such that

$$j(v) \cdot j(v) = q(v)1. \tag{3.9}$$

Then we have an unique extension of j to a homomorphism $\tilde{j} \colon \mathrm{Cl}(V, q) \to E$. Moreover, $\mathrm{Cl}(V, q)$ is the unique associative algebra which have this property for all such E.

$$\begin{tikzcd}
\mathrm{Cl}(V,q) \arrow[dr, "\tilde{j}"] & \\
V \arrow[u, hook', "i"'] \arrow[r, "j"'] & D
\end{tikzcd}$$

This theorem can be seen as a definition of $\mathrm{Cl}(V, q)$.

Proof. The proof shall belongs two parts: the first one will show how to extend j and why it is unique, and the second one will prove the unicity of $\mathrm{Cl}(V, q)$.

We begin by define the extension of j. First note that any linear map $f : V \to E$ can be extended to an algebra homomorphism $\overline{f} \colon T(V) \to E$ in only one way. Indeed, the homomorphism condition require that $\overline{f}(v\otimes w) = f(v)\cdot f(w)$. The whole map $\overline{f}$ is then well defined by the data of f alone.

As far as the map j is concerned, we have the relation (3.9)-91 which says that $\overline{j}(\mathcal{I}) = 0$. Indeed,

$$\overline{j}(v\otimes v - q(v)\cdot(1)) = \overline{j}(v)\cdot\overline{j}(v) - q(v)\overline{j}(1) = j(v)\cdot j(v) - q(v)1 = 0. \tag{3.10}$$

Thus $\overline{j}\colon T(V)\to E$ is a class map for $\mathcal{I}$, and we can descent $\overline{j}$ from $T(V)$ to $\mathrm{Cl}(V,q)$, We define $\tilde{j}\colon\ \mathrm{Cl}(V,q)\to E$ by

$$\tilde{j}[x] = \overline{j}(x) \tag{3.11}$$

where $[x]$ is the class of x. That's for the existence part.

The unicity is clear: $f_1 = f_2$ on V implies that $\overline{f_1} = \overline{f_2}$ on $T(V)$. Thus $\tilde{f}_1 = \tilde{f}_2$ on $\mathrm{Cl}(V,q)$.

We turn now our attention to the unicity of $\mathrm{Cl}(C,q)$. Let D be an unital associative algebra such that

(i) $V\subset D$,

(ii) For any unital associative algebra E and for any $f\colon D\to E$ such that $f(v)\cdot f(v) = -q(v)1$, there exists only one homomorphic map $\tilde{f}\colon D\to E$ which extend f.

We should find a homomorphic map $\tilde{k}\colon D\to\mathrm{Cl}(V,q)$. Let i be the canonical injection $i\colon V\to D$. Clearly, we have a homomorphism $V\to i(V)$. Now, as a space E, we can take $\mathrm{Cl}(V,q)$; i can be seen as a linear map $i\colon V\to\mathrm{Cl}(V,q)$ such that $i(v)\cdot i(v) = q(v)1$. The assumptions say that i can be extended (in only one way) to a homomorphic map $\tilde{i}\colon D\to\mathrm{Cl}(V,q)$.

The Clifford algebra is thus unique up to a homomorphism.

□

What we proved is the following: if for any E and for any $j\colon V\to E$ such that $j(v)\cdot j(v) = q(v)1$, there exist an unique $\tilde{j}\colon D\to E$ which extend j, then $D = \mathrm{Cl}(V,q)$ up to a homomorphism. One ays that $\mathrm{Cl}(V,q)$ solve an **universal problem**.

An explicit construction of $\mathrm{Cl}(V,q)$ can be achieved in the following way. We consider the tensor algebra $T(V) = \bigoplus_{n\geqslant 0}(\otimes^n V) = \mathbb{C}\oplus V\oplus(V\otimes V)\oplus\ldots$ over V the two-sided ideal $\mathcal{I}$ generated by elements of the form $v\otimes v - q(v)1$. The **Clifford algebra** for (V,q) is given by

$$\mathrm{Cl}(p,q) := T(V)/\mathcal{I} \tag{3.12}$$

in which product of $\mathrm{Cl}(V,q)$ is naturally defined by $[a]\otimes[b] = [a\otimes b]$ if $[a]$ is the class of $a\in T(V)$.

Let us now fix some notations more adapted to what we want to do. Let $V = \mathbb{R}^{p,q}$ the vector space $\mathbb{R}^{p+q}$ endowed with a diagonal metric which contains p plus sign and q minus signs. For $v,\ w\in V$, the inner product with respect to the metric η of v by w will be denoted by $\eta(v,w)$. The norm on V will be defined by $\|v\|^2 = -\eta(v,v)$. It is neither positive defined, nor negative defined. The explanation of the minus sign will come soon. The Clifford algebra is the quotient $\mathrm{Cl}(p,q) := T(V)/\mathcal{I}$ of the tensor algebra by the two-sided ideal $\mathcal{I}$ generated by elements of the form

$$(v\otimes w)\oplus(w\otimes v)\oplus 2\eta(v,w)1$$

for v,w in V. Depending on the context, we will often use the notations $\mathrm{Cl}(\eta)$ or $\mathrm{Cl}(V)$ or $\mathrm{Cl}(p,q)$. The algebra product is $[x]\cdot[y] = [x\otimes y]$, $x,\ y\in T(V)$. As long as $z\in V\subset\mathrm{Cl}(p,q)$, the expression $\eta(z,z)$ is meaningful. The definition of Cl is such that $z\cdot z = -\eta(z,z)$. This leads to the somewhat surprising formula $z^2 = \|z\|^2 = -\eta(z,z)$.

3.2.2 First representation

Let (V, g) be a metric vector space and $\mathrm{Cl}(V, g)$ its Clifford algebra. For each $v \in V$, we define the two following elements of $\mathrm{End}_{\mathbb{R}}(\bigwedge V)$:

$$\epsilon(v)\big(u_1 \wedge \cdots \wedge u_k\big) = v \wedge u_1 \wedge \cdots \wedge u_k \tag{3.13a}$$

$$\iota(v)\big(u_1 \wedge \cdots \wedge u_k\big) = \sum_{j=1}^{k} (-1)^{j-1} g(u, u_j) u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_j. \tag{3.13b}$$

One has $\epsilon(v)^2 = 0$ and $\iota(v)^2 = 0$ because $v \wedge v = 0$. In order to understand the latter, we wonder what are the terms with $g(v, u_i)g(v, u_j)$ are in

$$\iota(v)^2\big(u_1 \wedge \cdots \wedge u_k\big) = \sum_{l=1}^{k} (-1)^{j-1} g(v, u_j) \sum_{l=1}^{k-1} (-1)^{l-1} g(v, u_l) u_1 \wedge \hat{u}_l \wedge \hat{u}_j \wedge \cdots \wedge u_k.$$

Let's suppose $i < j$. The first term comes when the first $\iota(v)$ acts on u_j, its sign is given by $(-1)^{j-1}(-1)^{i-1}$. The second term has the same $(-1)^{i-1}$, but in this term, u_j is on the position $j - 1$ because u_i has disappeared.

Now we use $c(v) = \epsilon(v) + \iota(v)$ which fulfils for all u, $v \in V$:

$$\begin{aligned} c(v)^2 &= g(v, v,)1 \\ c(u)v(v) + c(v)c(u) &= 2g(u, v)1. \end{aligned}$$

Therefore c can be extended to a representation $c\colon \mathrm{Cl}(V, g) \to \mathrm{End}(\bigwedge V)$. If $\{e_0, \cdots e_n\}$ is an orthonormal basis of V (i.e. $g(e_i, e_j) = \eta_{ij}$); in this case the $c(e_j)$ are anticommuting and a basis of $\mathrm{Cl}(V, g)$ is given by

$$\{c(e_{k_1}) \cdots c(e_{k_r}) \text{ st } 1 \leqslant k_1 < \cdots < k_r \leqslant n\}. \tag{3.14}$$

3.2.3 Some consequences of the universal property

The map $-\operatorname{id}|_V$ extends to $\alpha \in \mathrm{Aut}\big(\mathrm{Cl}(V)\big)$,

$$\alpha(v_1 \cdots v_r) = (-1)^r v_1 \cdots v_r$$

$(v_i \in V)$ and provides a graduation

$$\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V).$$

The map $\tau\colon \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ extends $\operatorname{id}|_V$ to an anti-homomorphism:

$$\tau(v_1 \cdots v_r) = v_r \cdots v_1. \tag{3.15}$$

The **complexification** of $\mathrm{Cl}(V, g)$ is

$$\mathrm{Cl}^{\mathbb{C}}(V, g) := \mathrm{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Cl}(V^{\mathbb{C}}, g^{\mathbb{C}}),$$

the isomorphism being a $\mathbb{C}$-algebra isomorphism. The $\mathbb{R}$-linear operator $v \mapsto \overline{v}$ in $V^{\mathbb{C}}$ of complex conjugation extends to a $\mathbb{R}$-linear automorphism $a \mapsto \overline{a}$. We define the **adjoint** by

$$a^* = \tau(\overline{a}) \tag{3.16}$$

3.2.4 Trace

Theorem 3.5.
There exists one an only one trace $\mathrm{Tr}\colon \mathrm{Cl}^{\mathbb{C}}(V) \to \mathbb{C}$ *such that*

(i) $\mathrm{Tr}(1) = 1$,

(ii) $\mathrm{Tr}(a) = 0$ *when* a *is odd.*

Proof. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of (V, g) and $a \in \mathrm{Cl}^{\mathbb{C}}(V)$. When one decomposes a into the basis of e_i, one finds a lot of terms of each order. Since Tr is a trace, when the k_i are all different,

$$\mathrm{Tr}(e_{k_1} \cdots e_{k_{2r}}) = \mathrm{Tr}(-e_{k_2} \cdots e_{k_{2r}} e_{k_1} = \mathrm{Tr}(-e_{k_1} \cdots e_{k_{2r}})$$

So the trace of any even element is zero. We decompose a into

$$a = \sum_K a_K \prod_{i \in K} e_i$$

where the sum is taken on the subsets of $\{1, \ldots, n\}$. A trace which fulfils the conditions must vanishes on even (but non zero) elements as well as on odd elements, so the only possible form is

$$\mathrm{Tr}\, a = a_{\varnothing}.$$

Notice that in order to get this precise form, we used $\mathrm{Tr}(1) = 1$ and linearity. This proves unicity and existence. Now we have to prove that this is a good definition in the sense that an other choice of basis gives the same result. So we take a new orthonormal basis

$$e'_j = \sum_{k=1}^{n} H_{jk} e_k$$

with $H^t H = \mathbb{1}_{n \times n}$. Now we have

$$a = \sum_K a_K \prod_{i \in K} e_i = \sum_K a'_K \prod_{i \in K} e'_i,$$

and we will prove that $a_{\varnothing} = a'_{\varnothing}$. Let's compute a lot:

$$\begin{aligned} e'_i e'_j &= \sum_k \sum_l H_{ik} H_{jl} e_k e_l \\ &= \sum_{k=l} H_{ik} H_{jl} e_k e_l + \sum_{k \neq l} H_{ik} H_{jl} e_k e_l \\ &= \sum_k H_{ik} H_{jk} 1 + \sum_{k \neq l} H_{ik} H_{jl} e_k e_l \\ &= (HH^t)_{ij} 1 + \sum_{k \neq l} H_{ik} H_{jl} e_k e_l. \end{aligned}$$

The sense of this formula is that when $i \neq j$, the product $e'_i e'_j$ has no term of order zero. In other terms, as long as we only have terms of order zero, one and two, a change $e \to e'$ does not change the term of order zero. We are now going to an induction proof: we want to prove that $e'_{j_1}\dots e'_{j_{2r}}e'_l e'_k$ has no scalar term assuming that no even combination has scalar terms up to $2(r-1)$. It reads

$$\sum_{K \text{ even}} a_K \prod_{i\in K} e_i e'_l e'_k,$$

therefore we just have to look at terms of the form

$$e_{j_1}\dots e_{j_{2r}}\Big((HH)^t_{kl}1 - \sum_{i\neq j} C^{ij}_{kl} e_i e_j\Big)$$

where the e_{j_l} are all different. The first term cannot produce a scalar term. In order to find a scalar term in $e'_{j_1}\dots e'_{j_{2r}}e_k e_l$, we begin to look at terms whose decomposition of $e'_{j_1}\dots e'_{j_{2r}}$ ends by $e_l e_k$, i.e.

$$H_{j_{2r-2}l}H_{j_{2r-1}k}e'_{j_1}\dots e'_{2r-3}e_l e_k e_k e_l.$$

The induction assumption says that there are no scalar term in $e'_{2r-3}e_l e_k e_k e_l$.

□

One can prove that $\mathrm{Cl}^{\mathbb{C}}(C)$ is a Hilbert space with the scalar product

$$\langle a|b\rangle = \mathrm{Tr}(a^*b). \tag{3.17}$$

Let $v\in V$ with $g(v,v)=1$ (thus in $\mathrm{Cl}(V)$, we have $v^2=1$); since $v=\overline{v}$, we have

$$a^*v = vv^* = v^2 = 1.$$

Lemma 3.6.
The maps $a\mapsto ua$ and $a\mapsto au$ are unitary if and only if $uu^* = u^*u = 1$.

Proof. We pick $\lambda\in U(1)$ and $w=\lambda v\in V^{\mathbb{C}}$ which fulfils $w^*w=1$. This is the most general element such that $ww^*=w^*w=1$. Now for an arbitrary a, $b\in \mathrm{Cl}^{\mathbb{C}}(V)$, we compute the two followings:

$$\langle wa|wb\rangle = \mathrm{Tr}\left((wa)^*wb\right) = \mathrm{Tr}\left(a^*w^*wb\right) = \mathrm{Tr}(a^*b) = \langle a|b\rangle,$$

and

$$\langle aw|bw\rangle = \mathrm{Tr}\left(w^*a^*bw\right) = \mathrm{Tr}(ww^*a^*b) = \mathrm{Tr}(a^*b) = \langle a|b\rangle.$$

This proves that $a\mapsto wa$ and $a\mapsto aw$ are two unitary operators on the Hilbert space $\mathrm{Cl}^{\mathbb{C}}(V)$.

For the converse, we impose for all a, $b\in\mathrm{Cl}^{\mathbb{C}}(V)$:

$$\langle ua|ub\rangle = \mathrm{Tr}(ba^*u^*u) \stackrel{!}{=} \mathrm{Tr}(ba^*).$$

In particular with $a^*b=1$, $\mathrm{Tr}(u^*u)=\mathrm{Tr}(1)=1$, thus the scalar part of u^*u is 1. So we write $u^*u = 1+f$ where f is non scalar, and for any $x\in\mathrm{Cl}^{\mathbb{C}}(V)$, we have

$$\mathrm{Tr}(x) = \mathrm{Tr}(xu^*u) = \mathrm{Tr}(x) + \mathrm{Tr}(xf).$$

We conclude that $\mathrm{Tr}(xf)=0$, and therefore that $f=0$. □

3.3 Spinor representation

For the spinor representation, we restrict ourself to the even case $p + q = 2n$.

The aim of this subsection is to find some faithful representations of the complex Clifford algebra $\mathrm{Cl}^{\mathbb{C}}(p, q)$. In order to achieve this, we first consider $V^{\mathbb{C}}$, the complex vector space of V with an orthonormal basis $\{e_1, \cdots, e_{p-1}, e_p, \cdots, e_q\}$. The metric is $\eta(e_k, e_k) = 1$ and $\eta(e_{p+k}, e_{p+k}) = -1$ for $k = 0, \cdots, p-1$. We use the following basis:

$$f_k = \frac{1}{2}(e_k + e_{p+k}), \qquad g_k = \frac{1}{2}(e_k - e_{p+k}), \tag{3.18}$$

$$f_{p+s} = \frac{1}{2}(e_{2p+2s} + ie_{2p+2s+1}), \qquad g_{p+s} = \frac{1}{2}(e_{2p+2s} - ie_{2p+2s}) \tag{3.19}$$

for $k = 0, \cdots, p-1$. We note that $\{f_0, g_0\}$ spans a $\mathbb{C}^2$-space which is η-orthogonal to the one which is spanned by $\{f_1, g_1\}$. The following two spaces will prove to be useful:

$$W = \mathrm{Span}_{\mathbb{C}}\{f_0, f_1\} \simeq \mathbb{C}^2, \tag{3.20a}$$

$$\underline{W} = \mathrm{Span}_{\mathbb{C}}\{g_0, g_1\} \simeq \mathbb{C}^2. \tag{3.20b}$$

It is easy to compute the various products; among others we find

$$\eta(f_0, f_0) = 0, \quad \eta(f_1, f_0) = 0, \quad \eta(f_1, f_1) = 0; \tag{3.21}$$

so that for any $w \in W$, we have $\langle w, w \rangle = 0$; for this reason, we say that W is a **completely isotropic** subspace of $(V^{\mathbb{C}}, \eta^{\mathbb{C}})$. The space $\underline{W}$ has the same property.

Proposition 3.7.
We have

$$\underline{W} \simeq W^*, \tag{3.22}$$

where W^* is the dual space of W. By $\simeq$ we mean that there exists a linear bijective map $\psi \colon \underline{W} \to W^*$.

Proof. For each $\underline{w} \in \underline{W}$, we define $\psi(\underline{w}) \colon W \to \mathbb{C}$ by

$$\psi(\underline{w})(w) = \eta(w, \underline{w}).$$

We first show that the map ψ is injective. Let $\underline{w} \in \underline{W}$ be so that $\psi(\underline{w}) = 0$. Thus for all $v \in W$, we have

$$\psi(\underline{w})v = \eta(\underline{w}, v) = 0. \tag{3.23}$$

By decomposing $\underline{w} = ag_0 + bg_1$ and taking successively $v = f_0$ and $v = f_1$, we see that $a = b = 0$.

The next step is to see that the map ψ is surjective. We know that $dim_{\mathbb{C}}\underline{W} = dim_{\mathbb{C}}W^* = 2$ and that $\psi(g_0) \neq 0$. Let's prove that $\{\psi(g_0), \psi(g_1)\}$ is a basis of W^*. It is clear by linearity that $\{\psi(ag_0) : a \in \mathbb{C}\} = \mathrm{Span}\{\psi(g_0)\}$. The fact that ψ is injective imposes that $\psi(g_1)$ doesn't belong to $\mathrm{Span}\{\psi(g_0)\}$. So $\{\psi(g_0), \psi(g_1)\}$ is a two-dimensional free subset of W^*, and therefore is a basis of W^*. □

We turn our attention to the exterior algebra $\Lambda W = \mathbb{C} \oplus W \oplus (W \wedge W) \oplus \cdots \oplus \wedge^{p+q} W$ of W.

Definition 3.8.
We define the homomorphism $\tilde{\rho}\colon V^{\mathbb{C}} \to \mathrm{End}(\Lambda W)$ *by*

$$\begin{aligned}\tilde{\rho}(f_i)\alpha &= f_i \wedge \alpha,\\ \tilde{\rho}(g_i)\alpha &= -\iota(g_i)\alpha\end{aligned} \tag{3.24}$$

($v \in V^{\mathbb{C}}$, $\alpha \in \Lambda W$) where ι denotes the interior product defined in page 17.

More explicitly, for all $z \in \mathbb{C}$ and for all $w, w' \in W$, we have

$$\tilde{\rho}(f_i)z = zf_i, \qquad \tilde{\rho}(g_i)z = 0, \tag{3.25a}$$
$$\tilde{\rho}(f_i)w = f_i \wedge w, \qquad \tilde{\rho}(g_i)w = -\eta(g_i, w)1, \tag{3.25b}$$
$$\tilde{\rho}(f_i)(w \wedge w') = 0, \qquad \tilde{\rho}(g_i)(w \wedge w') = -\eta(g_i, w)w' + \eta(g_i, w')w. \tag{3.25c}$$

We will see that, *via* some manipulations, $\tilde{\rho}$ provides a faithful representation of the Clifford algebra, the **spinor representation**.

Remark 3.9. By "endomorphism of ΛW", we mean an endomorphism for the *linear* structure of ΛW. We obviously not have $\tilde{\rho}(x)(\alpha \wedge \beta) = \tilde{\rho}(x)\alpha \wedge \tilde{\rho}(x)\beta$.

Proposition 3.10.
The map $\tilde{\rho}$ is injective.

Proof. We have to show that $\tilde{\rho}(v) = 0$ (v in $V^{\mathbb{C}}$) implies $v = 0$. Any $v \in V^{\mathbb{C}}$ can be written as $v = a^i f_i + b^i g_i$ with a sum over i. We first have that

$$\tilde{\rho}(a^i f_i + b^i g_i)z = za^i f_i = 0,$$

but the f_i are independents and then $a^i = 0$. We can also write

$$\tilde{\rho}(b^0 g_0 + b^1 g_1) f_1 = -b^0 \eta(g_0, f_1) - b^1 \eta(g_1, f_1) = -\frac{b^1}{2} = 0,$$

then $b^1 = 0$. The same with f_0 proves that $b^0 = 0$. □

The homomorphism $\tilde{\rho}$ extends to the whole the tensor algebra of $V^{\mathbb{C}}$ by the following definitions:

$$\tilde{\rho}(1) = \mathrm{id}_{\Lambda W}, \tag{3.26a}$$
$$\tilde{\rho}(e_k) = \tilde{\rho}(e_k), \tag{3.26b}$$
$$\tilde{\rho}(e_{k_1} \otimes \ldots \otimes e_{k_r}) = \tilde{\rho}(e_{k_1}) \circ \ldots \circ \tilde{\rho}(e_{k_r}). \tag{3.26c}$$

So we get $\tilde{\rho}\colon T(V^{\mathbb{C}}) \to \mathrm{End}(\Lambda W)$. The following proposition will allow us to descent $\tilde{\rho}$ to a representation of the Clifford algebra.

Proposition 3.11.
The homomorphism $\tilde{\rho}$ maps $\mathcal{I}$ to 0: $\tilde{\rho}(\mathcal{I}) = 0$.

Problème et notes pour moi 2.
Il faut vérifier les signes dans cette démonstration. En effet, regarde la première lignes, et remarque que le signe n'est pas celui utilisé pour définir l'algèbre de Clifford.

This proposition is wrong: there is a double covering.

Proof. We have to check the following:

$$\tilde{\rho}(v \otimes w \oplus w \otimes v - 2\eta(v,w)1) = 0$$

for any choice of v, w in $\{e_0, e_1, e_2, e_3\}$. Here we will just check it explicitly for $v = e_0$ and $w = e_1$. The computation uses the definition (3.26c)-97:

$$\begin{aligned} \tilde{\rho}(e_0 \otimes e_1 \oplus e_1 \otimes e_0 - 2\eta(e_0, e_1) &= \tilde{\rho}(e_0) \circ \tilde{\rho}(e_1) + \tilde{\rho}(e_1) \circ \tilde{\rho}(e_0) \\ &= 2\left[\tilde{\rho}(f_0)^2 - \tilde{\rho}(g_0)^2\right]. \end{aligned} \tag{3.27}$$

It is easy to see that $\tilde{\rho}(f_0)^2 = 0$:

$$\tilde{\rho}(f_0)^2\left[z \oplus w \oplus w_1 \wedge w_2\right] = \tilde{\rho}(f_0)[zf_0 \oplus f_0 \wedge w] = zf_0 \wedge f_0, = 0. \tag{3.28}$$

The proof that $\tilde{\rho}(g_0)^2 = 0$ is almost the same:

$$\tilde{\rho}(g_0)^2\left[z \oplus w \oplus w_1 \wedge w_2\right] = \tilde{\rho}(g_0)[-\eta(g_0, w)1 \oplus -\eta(g_0, w_1)w_2 \oplus \eta(g_0, w_2)w_1].$$

□

We can now see $\tilde{\rho}$ as a map $\tilde{\rho}\colon \mathrm{Cl}^{\mathbb{C}}(p,q) \to \mathrm{End}(\Lambda W)$. By construction, it is a homomorphism and, thus, is a representation of $\mathrm{Cl}^{\mathbb{C}}(p,q)$ on ΛW. For compactness, we use the notation

$$\gamma_a := \sqrt{2}\tilde{\rho}(e_a). \tag{3.29}$$

Lemma 3.12.
The γ's operators satisfy the following relation:

$$\gamma_a \gamma_b + \gamma_b \gamma_a = -2\eta_{ab}\mathbb{1}. \tag{3.30}$$

Proof. We have to check this equality on any element of ΛW. If we choose $w_1 = af_0 + bf_1$ and $w_2 = a'f_0 + b'f_1$, we find $w_1 \wedge w_2 = (ab' - ba')f_0 \wedge f_1$.

For example, we will explicitly check (3.30)-98 with $a = b = 0$, i.e. $\tilde{\rho}(e_0) \circ \tilde{\rho}(e_0) = \frac{1}{2}\,\mathrm{id}$, which proves that $\gamma_0 \circ \gamma_0 = \mathrm{id}$.

$$\begin{aligned} \tilde{\rho}(e_0)^2[z \oplus w \oplus (ab' - ba')f_0 \wedge f_1] &= \tilde{\rho}(f_0 + g_0)^2[z \oplus w \oplus (ab' - ba')f_0 \wedge f_1] \\ &= \tilde{\rho}(f_0 + g_0)\Big[zf_0 \oplus f_0 \wedge w \oplus -\eta(g_0, w)1 \\ &\qquad - (ab' - ba')\eta(g_0, f_0)f_1 \\ &\qquad + (ab' - ba')\eta(g_0, f_1)f_0\Big] \\ &= \frac{1}{2}(z \oplus w \oplus (ab' - ba')f_0 \wedge f_1). \end{aligned} \tag{3.31}$$

□

Lemma 3.13.
For any sequence $i_0, \ldots i_3$ of 0 and 1 (with at least one of them equals to 1), we have

$$\mathrm{Tr}(\gamma_0^{i_0} \cdots \gamma_{2n-1}^{i_{2n-1}}) = 0. \tag{3.32}$$

We take the convention that $\gamma_a^0 = \mathbb{1}$.

Proof. If the number of nonzero i_k is even (say $2m$), we have:

$$\mathrm{Tr}(\gamma_{a_1}\ldots\gamma_{a_{2m}}) = \mathrm{Tr}(\gamma_{a_{2n}}\gamma_{a_1}\ldots\gamma_{a_{2m-1}})$$

because the trace is invariant under cyclic permutations. But we can also permute $\gamma_{a_{2m}}$ with the $2m-1$ other γ's. $\mathrm{Tr}(\gamma_{a_1}\ldots\gamma_{a_{2m}}) = (-1)^{2n-1}\,\mathrm{Tr}(\gamma_{a_{2m}}\gamma_{a_1}\ldots\gamma_{a_{2m-1}})$ because each permutation gives an extra minus sign (lemma 3.12). Then the trace is zero.

If the number of nonzero i_k is odd (say $2m-1$). Let $i_a = 0$ (we restrict ourself to the even dimensional case). We have $\mathrm{Tr}(A) = -\eta_{aa}\,\mathrm{Tr}(A\gamma_a\gamma_a)$. Using once again the cyclic invariance of the trace, $\mathrm{Tr}(\gamma_{a_1}\ldots\gamma_{a_{2m-1}}\gamma_a\gamma_a) = \mathrm{Tr}(\gamma_a\gamma_{a_1}\ldots\gamma_{a_{2m-1}}\gamma_a)$. But, if we permute the *first* γ_a with the $2m-1$ first γ's, we find $\mathrm{Tr}(\gamma_{a_1}\ldots\gamma_{a_{2m-1}}\gamma_a\gamma_a) = -\,\mathrm{Tr}(\gamma_a\gamma_{a_1}\ldots\gamma_{a_{2m-1}}\gamma_a)$, and the trace is zero again. □

Proposition 3.14.
The subset

$$\{\mathbb{1}, \gamma_a\gamma_b\,(a<b), \gamma_a\gamma_b\gamma_c\,(a<b<c), \cdots, \gamma_0\cdots\gamma_{2n}\}$$

is free in $\mathrm{End}(\Lambda W)$.

Proof. We consider a general linear combination of these operators:

$$E = \lambda\mathbb{1} + \sum_a \lambda_a\gamma_a + \sum_{a<b}\lambda_{ab}\gamma_a\gamma_b + \ldots + \sum_{a<b<c<d}\lambda_{abcd}\gamma_a\gamma_b\gamma_c\gamma_d.$$

The claim is that if $E=0$, then all the coefficients $\lambda_{(\ldots)}$ must be zero. First note that $Tr(E) = 0 = \lambda$ by lemma 3.13. It is also clear that $Tr(\gamma_i E) = 0 = \lambda_i$. In order to see that $\lambda_{ij}=0$, we compute $Tr(\gamma_j\gamma_i E) = 0 = \lambda_{ij}$. And so on. □

How many operators does we have in this free system ? Any operators in this system can be written as $\gamma_0^{i_0},\cdots\gamma_{2n-1}^{i_{2n-1}}$ with i_k equal to zero or one. Thus we have 2^{2n} operators. On the other hand, we know that $dim_{\mathbb{C}}\Lambda W = 2p+2$, and then that $dim_{\mathbb{C}}\,\mathrm{End}(\Lambda W) = 4^2 = 16$. The result is that $\{\gamma_0^{i_0},\cdots\gamma_{2n-1}^{i_{2n-1}}$ st $i_k = 0\,or\,1\}$ is a basis of $\mathrm{End}(\Lambda W)$. In other words (if we suppose a suitable ordering), the image by $\tilde\rho$ of

$$B = \{1, e_a, e_a\otimes e_b, e_a\otimes e_b\otimes e_c, e_a\otimes e_b\otimes e_c\otimes e_d\}$$

is a basis of $\mathrm{End}(\Lambda W)$.

If B is a basis of $C^{\mathbb{C}}_{(p,q)}$, then $\tilde\rho$ is bijective and thus isomorphic. Therefore, we expect $\tilde\rho\colon C^{\mathbb{C}}_{(p,q)} \to \mathrm{End}(\Lambda W)$ to be a faithful representation. It is not difficult to see that B is indeed a basis thanks to the equivalence relation.

3.3.1 Explicit representation

First, we choose a basis for ΛW:

$$1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad f_0 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad f_1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad f_0\wedge f_1 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{3.33}$$

Here is the explicit computation for the matrix γ_0 in this basis. First remark that $\tilde{\rho}(e_0)1 = f_0$, $\tilde{\rho}(e_0)f_0 = \frac{1}{2}$, $\tilde{\rho}(e_0)f_1 = f_0 \wedge f_1$, $\tilde{\rho}(e_0)(f_0 \wedge f_1) = \frac{1}{2}f_1$. Then

$$
\begin{aligned}
\gamma_0 \begin{pmatrix}1\\0\\0\\0\end{pmatrix} &= \sqrt{2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \gamma_0 \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \sqrt{2}\begin{pmatrix}\frac{1}{2}\\0\\0\\0\end{pmatrix},\\
\gamma_0 \begin{pmatrix}0\\0\\1\\0\end{pmatrix} &= \sqrt{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \gamma_0 \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \sqrt{2}\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}.
\end{aligned}
\tag{3.34}
$$

This allows us to write down γ_0; the same computation gives the other matrices.

$$
\begin{aligned}
\gamma_0 &= \sqrt{2}\begin{pmatrix}0 & \frac{1}{2} & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 1 & 0\end{pmatrix}, \qquad \gamma_1 = \sqrt{2}\begin{pmatrix}0 & -\frac{1}{2} & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2}\\ 0 & 0 & 1 & 0\end{pmatrix},\\
\gamma_2 &= \sqrt{2}\begin{pmatrix}0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 0 & \frac{1}{2}\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}, \qquad \gamma_3 = \sqrt{2}\begin{pmatrix}0 & 0 & -\frac{i}{2} & 0\\ 0 & 0 & 0 & \frac{i}{2}\\ -i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{pmatrix}.
\end{aligned}
\tag{3.35}
$$

It is easy to check that these matrices satisfies (3.30)-98.

Notice that, up to a suitable change of basis in ΛW, these are the usual Dirac matrices. Indeed we actually solved the physical problem to find a representation of the algebra (3.2)-90. We understand by the way why do physicists work with 4-components spinors: the γ's are operators on the four-dimensional space ΛW; hence the Dirac operator will naturally acts on four-components objects.

The main result of this section is an explicit faithful representation of $\mathrm{Cl}^{\mathbb{C}}(p, q)$. This allows us to write a **Dirac operator** which solve (see the invitation 3.1 and [27]) the problem to find a "square root" of the d'Alembert operator: the differential operator $\mathcal{D} = \gamma^\mu \partial_\mu$ satisfies $\mathcal{D}^2 = \Box$.

3.3.2 A remark

Let us compare the two faithful representations

$$
\begin{aligned}
c\colon\ \mathrm{Cl}(V) &\to \mathrm{End}_{\mathbb{R}}(\wedge V)\\
\tilde{\rho}\colon\ \mathrm{Cl}^{\mathbb{C}} &\to \mathrm{End}_{\mathbb{R}}(\wedge W).
\end{aligned}
$$

They obviously comes from the same ideas. One common point is that

$$
c(e_1)(e_1 \wedge e_2) = 2\tilde{\rho}(e_1)(e_1 \wedge e_2) = e_2,
$$

but they are different:

$$
\begin{aligned}
&\tilde{\rho}(e_3)(e_0 \wedge e_2) = 0\\
&c(e_3)(e_0 \wedge e_2) = e_3 \wedge e_0 \wedge e_1.
\end{aligned}
$$

3.3.3 General two dimensional Clifford algebra

The Clifford algebra for the metric
$$g = \begin{pmatrix} \alpha & \delta \\ \delta & \beta \end{pmatrix}$$
is realised by matrices
$$\gamma_1 = \epsilon \begin{pmatrix} \sqrt{\alpha} & \\ & -\sqrt{\alpha} \end{pmatrix}, \quad \gamma_2 = \epsilon \begin{pmatrix} \delta/\sqrt{\alpha} & \beta - \delta^2/|\alpha| \\ 1 & -\delta/\sqrt{\alpha} \end{pmatrix}$$
where $\epsilon = \pm 1$ is chosen in such a way that $\epsilon|\alpha| = \alpha$.

3.4 Spin group

We will not immediately go on with Dirac operators on Riemannian manifolds because we still have to build some theory about the Clifford algebra itself. In particular, we have to define the spin group which will play a central role in the definition of the Dirac operator. Almost all –and (too ?) much more– the concepts we will introduce in this section can be found in [26]; a more physical oriented but useful approach can be found in [32].

Let define the map $\chi\colon \Gamma(p,q) \to GL(\mathbb{R}^{1,3})$ by
$$\chi(x)y = \alpha(x) \cdot y \cdot x^{-1}. \tag{3.36}$$
Let
$$\Gamma(p,q) = \{x \in \mathrm{Cl}(p,q) \text{ st } x \text{ is invertible and } \chi(x)y \in V \text{ for all } y \in V\}.$$
It should be remarked that this definition comes back to the real Clifford algebra. The Clifford algebra product gives this subset a group structure which is called the **Clifford group**. Any $x \in V$ is invertible since $x \cdot x = -\eta(x,x)1$, the inverse of x is given by $x^{-1} = x/\|x\|^2$.

The subset $\mathrm{Cl}(p,q)^+$ (resp. $\mathrm{Cl}(p,q)^-$) of $\mathrm{Cl}(p,q)$ is the image of even (resp. odd) tensors of $T(V)$ by the canonical projection $T(V) \to \mathrm{Cl}(p,q)$. With these definitions, we have a natural grading of Cl:
$$\mathrm{Cl}(p,q) = \mathrm{Cl}(p,q)^+ \oplus \mathrm{Cl}(p,q)^-, \tag{3.37}$$
and the subgroups
$$\Gamma(p,q)^+ = \Gamma(p,q) \cap \mathrm{Cl}(p,q)^+, \qquad \Gamma(p,q)^- = \Gamma(p,q) \cap \mathrm{Cl}(p,q)^-. \tag{3.38}$$

For $x_1, \ldots, x_n \in V$, we have $\tau(x_1 \cdots x_n) = x_n \cdots x_1$. The **spin group** is
$$\mathrm{Spin}(p,q) = \{x \in \Gamma(p,q)^+ | \tau(x) = x^{-1}\} \tag{3.39}$$
while the **spin norm** is the map $N\colon \Gamma(p,q) \to \Gamma(p,q)$ defined by
$$N(x) = x\tau(\alpha(x)).$$
We will see in proposition 3.25 that N actually takes its values in $\mathbb{R}$ and is therefore a homomorphism $N\colon \Gamma(p,q) \to \mathbb{R}$

Remark 3.15. The elements of $\mathrm{Spin}(p,q)$ are spin-normed at 1. Indeed, take a s in $\mathrm{Spin}(p,q)$. We have $N(s) = s \cdot \tau(s) = 1$ because $\alpha(s) = s$ and $\tau(s) = s^{-1}$. In particular $\mathrm{Spin}(p,q) \cap \mathbb{R} = \mathbb{Z}_2$.

3.4.1 Studying the group structure

Proposition 3.16.
The set $\Gamma(p,q)$ admits a Lie group structure.

Proof. During this proof, μ denotes the Clifford multiplication: $\mu(x,y) = x \cdot y$. We know that $\mathrm{Cl}^{\mathbb{C}}(p,q)$ is isomorphic to $\mathrm{End}(\Lambda W)$ in which the multiplication is a continuous map. Thus μ is continuous on $\mathrm{Cl}^{\mathbb{C}}(p,q)$. But $\mathrm{Cl}(p,q)$ is a closed subset of $\mathrm{Cl}^{\mathbb{C}}(p,q)$, so μ is a continuous map in $\mathrm{Cl}(p,q)$. This proves that χ seen as a map from $\Gamma(p,q) \times V$ to V is a continuous map.

The space V is closed in $\mathrm{Cl}(p,q)$, thus $\sigma^{-1}(V)$ is also closed. But $\sigma^{-1}(V) = \Gamma(p,q) \times \mathrm{Cl}(p,q)$. So $\Gamma(p,q)$ is closed in $\mathrm{Cl}(p,q)$.

Now the result is just a consequence of theorems .26 and .27. Indeed, let us study the subset $\mathcal{I}$ which appears in the definitions of the Clifford algebra. It makes no difficult to convince ourself that it is a closed subgroup of $T(V)$. The theorem .27 thus makes $\mathrm{Cl}(p,q) = T(V)/\mathcal{I}$ a Lie group. But we just say that $\Gamma(p,q)$ is closed in $\mathrm{Cl}(p,q)$, and the fact that $\Gamma(p,q)$ is a subgroup of $\mathrm{Cl}(p,q)$ is clear. By theorem .26 we conclude that there exists a Lie group structure on $\Gamma(p,q)$. □

Lemma 3.17.
The map χ is a homomorphism, in other words χ is a representation of $\Gamma(p,q)$.

Proof. The following computation uses the fact that α is a homomorphism:

$$\begin{aligned}\chi(a \cdot b)y = \alpha(a \cdot b) \cdot y \cdot (a \cdot b)^{-1} &= \alpha(a) \cdot \alpha(b) y \cdot b^{-1} \cdot a^{-1} \\ &= \alpha(a) \cdot \chi(b) y \cdot a^{-1} = \chi(a)\chi(b)y.\end{aligned}$$

□

Let $y \in \Gamma(p,q)^-$ and $v \in V$. Where is $y \cdot v$? First note that $(y \cdot v)^{-1} = v^{-1} \cdot y^{-1}$, so that

$$\begin{aligned}\alpha(y \cdot v) \cdot w \cdot (y \cdot v)^{-1} &= -\alpha(y) \cdot v \cdot w \cdot v^{-1} \cdot y^{-1} \\ &= -\alpha(y)\big(2\eta(v,w) - w \cdot v\big) \cdot v^{-1} \cdot y^{-1} \\ &= -2\eta(v,w)\alpha(y) \cdot v^{-1} \cdot y + \alpha(y) \cdot w \cdot y^{-1}\end{aligned} \tag{3.40}$$

which belongs to V because $y \in \Gamma(p,q)$. This reasoning shows that (apart for 0), $y \cdot v \in \Gamma(p,q)^+$ if and only if $y \in \Gamma(p,q)^-$.

Lemma 3.18.
If $x \in V$ is non-isotropic (i.e. $\eta(x,x) \neq 0$), the automorphism $\chi(x)$ is the orthogonal symmetry with respect to $x^\perp$.

We recall that

$$x^\perp = \{y \in V \text{ st } \eta(x,y) = 0\}.$$

We will denote by σ^x the orthogonal symmetry with respect to $x^\perp$.

Proof. When the operator σ^x acts on y, it just change the sign of the "x-part" of y. So we can write $\sigma^x y = y - 2\eta(x,y)1_x$, where $1_x := x/\|x\|$. It should be checked if $\chi(x)y = \alpha(x) \cdot y \cdot x^{-1}$ is equal to $y - 2\eta(x,y)1_x$ or not. We know that $x \cdot x = \eta(x,x)1 = -\|x\|$. It follows that

$$x \cdot y + y \cdot x = 2\eta(x,y)\frac{x \cdot x}{\|x\|}.$$

If we multiply this at right by x^{-1}, using the fact that $\alpha(x) = -x$, we find

$$-\alpha(x) \cdot y \cdot x^{-1} = -y + 2\eta(x, y)1_x,$$

which is precisely the identity we wanted to check. □

The following result will help us to identify subgroups of Clifford group as isometry groups.

Theorem 3.19 (Cartan-Dieudonné theorem)**.**
Each σ in $O(1,3)$ can be written as $\sigma = \tau_1 \circ \ldots \circ \tau_m$, where the τ's are orthogonal symmetries with respect to hyperplanes which are orthogonal to non-isotropic vectors.

Proposition 3.20.

$$\chi(\Gamma(p, q)) = O(p, q).$$

Proof. In order to show that $\chi(\Gamma(p, q)) \subset O(p, q)$ take $z \in V$ and $x \in \Gamma(p, q)$. Since $\alpha(x) \cdot z \cdot x^{-1}$ lies in V, we can write:

$$\alpha(x) \cdot z \cdot x^{-1} = -\alpha\left(\alpha(x) \cdot z \cdot x^{-1}\right) = -x \cdot \alpha(z) \cdot \alpha(x^{-1}) = x \cdot z \cdot \alpha(x^{-1}).$$

In order to see that $\chi(x) \in O(p, q)$, we have to prove that $\|\chi(x)y\|^2_{(p,q)} = \|y\|^2_{(p,q)}$. This is achieved by the following computation:

$$\begin{aligned}\|\chi(x)y\|^2_{(p,q)} &= -\left(\alpha(x) \cdot y \cdot x^{-1}\right)^2 = \left(\alpha(x) \cdot y \cdot x^{-1}\right)\left(x \cdot y \cdot \alpha(x^{-1})\right) \\ &= -\alpha(x) \cdot y^2 \cdot \alpha(x^{-1}) = \|y\|^2_{(p,q)}.\end{aligned} \tag{3.41}$$

The last step is simply the fact that $y^2 \in \mathbb{R}$ and therefore commutes with anything. We now know that $\chi(x) \in O(p, q)$ for all $x \in \Gamma(p, q)$. Thus $\chi(\Gamma(p, q)) \subset O(p, q)$.

For the second part, let σ be in $O(p, q)$. The Cartan-Dieudonné theorem(theorem 3.19) says that $\sigma = \sigma^{x_1} \circ \ldots \circ \sigma^{x_r}$ for some $x_1, \ldots, x_r$ in V. Thus $\sigma = \chi(x_1 \cdots x_r)$, and $O(p, q) \subset \chi(\Gamma(p, q))$. □

Proposition 3.21.

$$\ker \chi = \mathbb{R}^\times \tag{3.42}$$

where the right hand side is the set of invertible elements of $\mathbb{R}$.

Proof. Before beginning the proof, we want to insist on the fact that $x \in \ker \chi$ does not mean that $\chi(x)y = 0$ for all y in V. The "zero" of an algebra is the element e which satisfies $e \cdot y = y \cdot e = y$ for all y in the algebra. In other words, x is in the kernel of χ if and only if $\chi(x) = \mathrm{id}$.

First we show that $\mathbb{R}_0 \subset \ker \chi$. If $x \in \mathbb{R}$, then $\alpha(x) = x$. Therefore, when $x \neq 0$,

$$\chi(x)y = \alpha(x) \cdot y \cdot x^{-1} = y,$$

because the algebra product $\cdot$ between an element of $\mathrm{Cl}(p, q)$ and a real is commutative. Note that this does not work with $x = 0$.

We are now going to show that $\ker \chi \subset \mathbb{R}$. Let $z \in \ker \chi$. We decompose (definitions (3.38)-101) it into his odd and even part: $z = z^+ + z^-$, with $z^\pm \in \Gamma(p, q)^\pm$. These two can be written as $z^+ = e_{j_1} \cdots e_{j_{2r}}$ and $z^- = e_{i_1} \cdots e_{i_{2r-1}}$ with no two i_k or j_k equals. This is almost the general form of elements in even and odd part of $\Gamma(p,q)$: the only other possibility is z in $\mathbb{R}$. Obviously $\alpha(z^\pm) = \pm z^\pm$. Multiplying the condition $\chi(z)y = y$ at right by $(z^+ + z^-)$, we find

$$(z^+ - z^-)y = y(z^+ + z^-).$$

Thanks to equation (3.37)-101, we can split this condition into even and odd parts:

$$z^+ y = yz^+, \qquad z^- y = -yz^-. \tag{3.43}$$

The first equation with $y = e_{j_1}$ gives $e_{j_1} \cdots e_{j_{2r}} \cdot e_{j_1} = e_{j_1} e_{j_1} \cdots e_{j_{2r}}$. In the left hand side, permute the last e_{j_1} from last to second position. So we find the right hand side, with an extra minus sign. This means that $z^+ = 0$. In the same way, the second equation gives $z^- = 0$. We are left with the last possibility: $z \in \mathbb{R}$. □

Corollary 3.22.
For any $s \in \Gamma(p,q)$, there exists some non-isotropic vectors $x_1, \ldots, x_r$, and $c \in \mathbb{R}$ such that $s = cx_1 \cdots x_r$.

Proof. Let us take a $s \in \Gamma(p,q)$; we just saw (theorem 3.20) that $\chi(s)$ is an element of $O(p,q)$. It can be written $\chi(s) = \sigma_1 \circ \ldots \circ \sigma_m$. But we had shown that $\sigma_i = \chi(x_i)$ for any x_i normal to the hyperplane defining σ_i. We thus have

$$\chi(s) = \chi(x_1 \cdots x_m),$$

where s belongs to $\Gamma(p,q)$ and is therefore invertible. This leads us to write $\mathrm{id} = \chi(s^{-1} \cdot x_1 \cdots x_m)$. But the kernel of χ is $\mathbb{R}$ (proposition 3.21); so one can find a $r \in \mathbb{R}$ such that $s^{-1} \cdot x_1 \cdots x_m = r$. The claim follows. □

Lemma 3.23.
If $v \in V$,

$$\det \chi(v) = -1. \tag{3.44}$$

Proof. We already know that $det\chi(v) = \pm 1$. To check that the right sign is plus, take the following basis of V: $\{v, v_i^\perp\}$ where $\{v_i^\perp\}$ is a basis of $v^\perp$. Calculating the action of $\chi(v)$ on this basis, we find:

$$\begin{aligned} \chi(v)v &= -v \cdot v \cdot v^{-1} = -v, \\ \chi(v)v_i^\perp &= -v \cdot v_i^\perp \cdot v^{-1} = v_i^\perp \cdot v \cdot v^{-1} = v_i^\perp. \end{aligned} \tag{3.45}$$

In this computation, we used the relation $v \cdot w = -w \cdot v - 2\langle v, w \rangle$ which is true for all v, w in V. The action of $\chi(v)$ on this basis is thus to let unchanged three vectors and to change the sign of the fourth. This proves the claim. □

Theorem 3.24.

$$\chi(\Gamma(p,q)^+) = \mathrm{SO}(p,q). \tag{3.46}$$

Proof. From corollary 3.22, and definition 3.38, an element $s \in \Gamma(p,q)^+$ reads $s = cv_1 \cdots v_{2r}$. Thus

$$\det \chi(s) = \det \chi(v_1 \cdots v_{2r}) = \det \left[\chi(v_1) \ldots \chi(v_{2r})\right]. \tag{3.47}$$

But we know that, for all v_i in V, $det\chi(v_i) = -1$. So $\det \chi(s) = 1$ and $\chi(\Gamma(p,q)^+) \subseteq \mathrm{SO}(p,q)$. As set,

$$\Gamma(p,q) = \Gamma(p,q)^+ \cup \Gamma(p,q)^-,$$

but the lemma shows that $\det \chi(\Gamma(p,q)^-) = -1$ so, from theorem 3.20, $\chi(\Gamma(p,q)^+)$ must be the whole $\mathrm{SO}(p,q)$. □

Proposition 3.25.
The map N takes values in $\mathbb{R}$ and the formula

$$N(x \cdot y) = N(x)N(y), \tag{3.48}$$

holds for all x, $y \in \Gamma(p,q)$.

Proof. We write as usual $x \in \Gamma(p,q)$ as $x = cv_1 \cdots v_r$. So,

$$N(x) = cv_1 \cdots v_r \tau(\alpha(cv_1 \cdots v_r)) = (-1)^r c^2 v_1 \cdots v_r \cdot v_r \cdots v_1. \tag{3.49}$$

The first equality comes from the fact that $\alpha(cv_1 \cdots v_r) = (-1)^r cv_1 \cdots v_r$. Now $N(x) \in \mathbb{R}$ because $v_i \cdot v_i = -\langle v_i, v_i \rangle \in \mathbb{R}$ for all i. Hence the following hold:

$$\begin{aligned} N(x \cdot y) &= v \cdot y \cdot \tau(\alpha(v \cdot y)) \\ &= v \cdot y \cdot \tau(\alpha(y)) \cdot \tau(\alpha(v)) \\ &= v \cdot N(y)\tau(\alpha(v)) \\ &= N(y)N(x). \end{aligned} \tag{3.50}$$

This is the claim. □

Theorem 3.26.
We have the following isomorphism of groups

$$\mathrm{Spin}(p,q) = \mathrm{SO}_0(p,q).$$

provided by the map χ.

Problème et notes pour moi **3.**
That result (and the proof) is wrong : there is a double covering. The next result is correct, and I should merge the two proofs.

Proof. Let $\{e_1, \cdots, e_p, f_1, \cdots, f_p\}$ be a basis of $\mathbb{R}^{p+q}$ where the e_i's are time-like and the f_j's are space-like. We have

$$\mathrm{SO}(p,q) = \mathrm{SO}_0(p,q) \cup \xi\, \mathrm{SO}_0(p,q)$$

where ξ is defined as follows: $\xi e_1 = -e_1$, $\xi f_1 = -f_1$ and $\xi e_k = e_k$, $\xi f_k = f_k$ for $k \neq 1$. This element can be implemented as $\xi = \chi(g)$ for $g = e_1 f_1$. It is easy to see that $g^{-1} = -f_1 e_1$ and that $\tau(g) = f_1 e_1$, so that $g \notin \mathrm{Spin}(p,q)$.

Is it possible to find another $h \in \Gamma(p,q)$ such that $\chi(h) = \xi$? If $\chi(a) = \chi(b)$ for $a, b \in \Gamma(p,q)$, then $a = rb$ for a certain $r \in \mathbb{R}$. So we find that $h = g^{-1}/r$ is the general form of an element in $\Gamma(p,q)$ such that $\chi(h) = \xi$. This is an element of $\mathrm{Spin}(p,q)$ if and only if $\tau(h) = h^{-1}$, or $-e_1 f_1/r = re_1 f_1$ which has no solutions. We conclude that no element of $\mathrm{Spin}(p,q)$ is send on ξ by χ. So

$$\chi\big(\mathrm{Spin}(p,q)\big) \subset SO_0(p,q).$$

Problème et notes pour moi **4.**
Surjectivity of χ *from* $\mathrm{Spin}(p,q)$ *to* $\mathrm{SO}(p,q)$ *is still to be proved.*

□

Theorem 3.27.

$$\chi(\mathrm{Spin}(p,q)) = \mathrm{SO}_0(p,q) \tag{3.51}$$

where the index 0 *means the identity component.*

Proof. Proposition 3.21, theorem 3.24 and remark 3.15 show that the map $\chi\colon \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$ is a homomorphism with $\mathbb{Z}_2$ as kernel. We begin to prove that $\chi\colon \mathrm{Spin}(p,q) \to \mathrm{SO}_0(p,q)$ is surjective. On the one hand, elements of $\mathrm{Spin}(p,q)$ satisfy one more condition than the ones of $\Gamma(p,q)^+$. Thus the algebra $\mathrm{Spin}(p,q)$ has codimension one in $\Gamma(p,q)^+$.

On the other hand, we know that $\mathrm{SO}(p,q) = \mathrm{SO}_0(p,q) \cup h\,\mathrm{SO}_0(p,q)$ where h is the matrix such that $he_i = -e_i$ for $i = 0,\ldots,3$. Since $\mathrm{Spin}(p,q)$ has codimension one in $\Gamma(p,q)^+$, there is at most one more generator in $\chi(\Gamma(p,q)^+)$ than in $\chi(\mathrm{Spin}(p,q))$ (because χ is a homomorphism). In order to prove this theorem, we just need to show that elements of $\chi(\Gamma(p,q)^+)$ which do not belong to $\chi(\mathrm{Spin}(p,q))$ is h.

Is is no difficult to see that $\chi(e_0\cdot e_1\cdot e_2\cdot e_3)e_i = -e_i$ for $i = 0\ldots 3$: just write $\chi(e_0\cdot e_1\cdot e_2\cdot e_3)e_i = e_0\cdot e_1\cdot e_2\cdot e_3\cdot e_i\cdot e_3^{-1}\cdot e_2^{-1}\cdot e_1^{-1}\cdot e_0^{-1}$ and use the commutation relations. An easy computation gives $N(e_0\cdot e_1\cdot e_2\cdot e_3) = -1$; then this is not in $\mathrm{Spin}(p,q)$ by remark 3.15. □

We write it by the exact sequence

$$\mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(p,q) \xrightarrow{\chi} \mathrm{SO}_0(p,q) \tag{3.52}$$

we say that the group $\mathrm{Spin}(p,q)$ is a **double covering** of $\mathrm{SO}_0(p,q)$.

Lemma 3.28.
If $\pi\colon \tilde{X} \to X$ *is a covering which satisfies*

(i) X *is path connected,*

(ii) $\forall x \in X$, $\tilde{X}_x := \pi^{-1}(x)$ *is path connected in* $\tilde{X}$ i.e. *for all* $a, b \in \tilde{X}$, *there exist a path in* $\tilde{X}$ *which joins* a *and* b,

then $\tilde{X}$ *is path connected.*

Proof. If $\tilde{x}$ and $\tilde{y}$ are in $\tilde{X}$, we can suppose that $\pi(\tilde{x}) \neq \pi(\tilde{y})$ (because if $\pi(\tilde{x}) = \pi(\tilde{y})$, the second assumption gives the thesis). We define x and y as their projections: $x = \pi(\tilde{x})$ and $y = \pi(\tilde{y})$. Let γ be a path such that $\gamma(0) = x$ and $\gamma(1) = y$, and $\tilde{\gamma}$ be the lift of γ in $\tilde{X}$ which contains $\tilde{x}$: $\tilde{\gamma}(0) = \tilde{x}$ and $\pi(\tilde{\gamma}(1)) = \gamma(1) = y$. Then $\tilde{\gamma}(1)$ lies in $\tilde{X}_y$. Therefore, we can consider γ' which joins $\tilde{\gamma}(1)$ and $\tilde{y}$.

So, $\gamma' \circ \tilde{\gamma}$ is a path which contains $\tilde{x}$ and $\tilde{y}$. □

Proposition 3.29.
The group $\mathrm{Spin}(p,q)$ *is connected.*

Proof. We will prove that the covering $\chi\colon \mathrm{Spin}(p,q) \to \mathrm{SO}_0(p,q)$ fulfils lemma 3.28. We just have to show that $\mathrm{Spin}(p,q)$ fulfills the second assumption of the lemma. First note that $\chi(\tilde{x}) = \chi(\tilde{y})$

implies $\chi(\tilde{x}\tilde{y}^{-1}) = e$, and then $\tilde{x} = \pm\tilde{y}$ because of proposition 3.21. Since the other case is trivial, we can suppose $\tilde{x} = -\tilde{y}$.

It remains to prove that for every $g \in \mathrm{Spin}(p,q)$, there is a path in $\mathrm{Spin}(p,q)$ which joins g and $-g$. The answer is given by the path $t \mapsto \gamma(t)g$ where

$$\gamma(t) = \exp(te_1 \cdot e_2) = \cos(t)(-1) + \sin(t)e_1 \cdot e_2$$

which satisfies $\gamma(0) = 1$ and $\gamma(\pi) = -1$. □

Proposition 3.30.
The homomorphism $\tilde{\rho}$ restricts to a homomorphism $\tilde{\rho}\colon \mathrm{Spin}(p,q) \to GL(\Lambda^+ W)$.

Proof. An element in $\mathrm{Spin}(p,q)$ reads $s = cv_1 \cdots v_{2r}$ and its image by $\tilde{\rho}$ is

$$\tilde{\rho}(s) = c\tilde{\rho}(v_1) \circ \cdots \circ \tilde{\rho}(v_{2r}).$$

When one applies $\tilde{\rho}(v_1)$ to an element $\alpha \in \Lambda^k W$, one obtains a linear combination of an element of $\Lambda^{k-1}W$ and one of $\Lambda^{k+1}W$. The element $\tilde{\rho}(s)$ being an even composition of such maps, its transforms an element of $\Lambda^+ W$ into an element of $\Lambda^+ W$. □

Notice that an element of V —no $V^{\mathbb{C}}$— is represented on $\Lambda^+ W$ by complex matrices. This is not a problem. In the case of $\mathbb{R}^{1,3}$, we have $\dim \Lambda^+ W = 2$ and thus

$$\tilde{\rho}\big(\mathrm{Spin}(1,3)\big) \subset GL(2,\mathbb{C}).$$

The following is the lemma 8.5 (page 57) of [21].

Lemma 3.31.
Let $\rho\colon \mathrm{Cl}(p,q) \to \mathrm{Hom}_{\mathbb{C}}(E,E)$ be a representation of the Clifford algebra on a vector space E. If $p+q \geqslant 2$, then for all $s \in \mathrm{Spin}(p-1,q) \subset \mathrm{Cl}(p,q)$,

$$\det{}_{\mathbb{C}}\big(\rho(s)\big) = \pm 1.$$

Proof. No proof. □

Theorem 3.32.
The representation $\tilde{\rho}$ provides a group isomorphism

$$\mathrm{Spin}(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$$

Proof. In the case $p = 2$, $q = 3$, the lemma assures us that for each s in the spin group, $\det \tilde{\rho}(s) = 1$. Since $\mathrm{Spin}(1,3)$ is connected and the determinant function is continuous, we deduce that $\det \tilde{\rho}(s) \equiv 1$. This proves that $\tilde{\rho}\big(\mathrm{Spin}(1,3)\big) \subset \mathrm{SL}(2,\mathbb{C})$. The proposition 1.18 thus implies that

$$\tilde{\rho}\big(\mathrm{Spin}(1,3)\big) = \mathrm{SL}(2,\mathbb{C}),$$

but from $\mathrm{Cl}(1,3)$, the representation $\tilde{\rho}$ is yet injective. *A forciori*, the representation $\tilde{\rho}$ is injective from $\mathrm{Spin}(1,3)$. This finishes the proof. □

3.4.2 Redefinition of $\mathrm{Spin}(V)$

As it, this new definition only holds when g is positive defined [3] . Let us take v, $x \in V$ with $g(v,v) = 1$. We have

$$-vxv^{-1} = -vxv = -2g(x,v)v + xv^2 = x - 2g(x,v)v \in V.$$

The effect was to reverse the v component of x; the map $x \mapsto -vxv^{-1}$ is σ^v. Now, when $\lambda \in U(1)$ and $w = \lambda v$, we also have that $x \mapsto -wxw^{-1}$ is σ^v. Now we look at $\chi(a)\colon x \mapsto \alpha(a)xa^{-1}$ with $a = w_1 \ldots w_r$, a product of unitary vectors in $V^{\mathbb{C}}$. Explicitly,

$$\chi(a)x = (-1)^r w_1 \ldots w_r x w_r^{-1} \ldots w_1^{-1},$$

a composition of reflexions in V. When r is even, it is a rotation. We conclude that when a is an even product of unitary vectors in $V^{\mathbb{C}}$, then $\chi(a) \in \mathrm{SO}(V)$. Theorem 3.19 states that any rotation of V is a composition of reflexions. So we define

$$\mathrm{Spin}^c(V) = \{w_1 \ldots w_{2k} \text{ st } w_j \in V^{\mathbb{C}},\ w_j^* w_j = 1\} \subset {\mathrm{Cl}^{\mathbb{C}}}^0(V), \tag{3.53}$$

and $\chi\colon \mathrm{Spin}^c(V) \to \mathrm{SO}(V)$ is a surjective group homomorphism. The inverse in $\mathrm{Spin}^c(V)$ is given by

$$(w_1 \ldots w_{2k})^{-1} = w_{2k}^* \ldots w_1^* = \overline{w_{2k}} \ldots \overline{w_1}.$$

In the real case, proposition 3.21 says that $\ker\chi = \mathbb{R}^\times$. In the complex case we get $\ker\chi = \mathbb{C}^\times$ and, when we look at $\ker\chi|_{\mathrm{Spin}^c(V)}$, we find

$$\ker\chi = U(1). \tag{3.54}$$

Then we find the short exact sequence

$$1 \xrightarrow{\mathrm{id}} U(1) \xrightarrow{\mathrm{id}} \mathrm{Spin}^c(V) \xrightarrow{\chi} \mathrm{SO}(V) \xrightarrow{\mathrm{id}} 1. \tag{3.55}$$

Let $u = w_1 \ldots w_{2k} \in \mathrm{Spin}^c(V)$ with $w_j = \lambda_j v_j$ and $\lambda_j \in V$, so $\tau(u) = w_{2k} \ldots w_1$ and

$$\tau(u)u = w_{2k} \ldots w_1 w_1 \ldots w_{2k} = \lambda_1^2 \ldots \lambda_{2k}^2 \in U(1).$$

This proves that $\tau(u)u$ is central in $\mathrm{Spin}^c(V)$. We define the homomorphism

$$\begin{aligned} \nu\colon\ \mathrm{Spin}^c(V) &\to U(1) \\ u &\mapsto \tau(u)u. \end{aligned} \tag{3.56}$$

This is a homomorphism because

$$\begin{aligned} \nu(u_1u_2) = \tau(u_1u_2)u_1u_2 &= \tau(u_2)\underbrace{\tau(u_1)u_1}_{\text{central}}u_2 = \tau(u_2)u_2\tau(u_1)u_1 \\ &= \nu(u_2)\nu(u_1) = \nu(u_1)\nu(u_2). \end{aligned}$$

The map ν naturally restricts to $U(1)$ as

$$\nu(\lambda) = \lambda^2.$$

The combined map $(\chi,\nu)\colon \mathrm{Spin}^c(V) \to \mathrm{SO}(V) \times U(1)$ has kernel $\{\pm 1\}$. We define

$$\mathrm{Spin}(V) = \ker\nu|_{\mathrm{Spin}^c(V)}. \tag{3.57}$$

[3] I think that only the identity component of $\mathrm{SO}(p,q)$ is obtained when one works with a signature (p,q).

Lemma 3.33.
This group is the same as the one defined in equation (3.39)-101.

Proof. Let $u \in \mathrm{Spin}(V)$ (in the sense of equation (3.57)-108). The fact for u to belongs to $\mathrm{Spin}(V)$ implies the two following:

(i) $u \in \mathrm{Spin}^c(V) \Rightarrow u^*u = 1$,

(ii) $u \in \ker \nu \Rightarrow \tau(u)u = 1$.

The second point says that $u^{-1} = \tau(u)$, which is a first good point to fit the first definition of $\mathrm{Spin}(V)$. Now we have to prove that $u \in \Gamma^+(V)$: u must be invertible and $\chi(u)x$ must belongs to V for all $x \in V$. These two points are contained in the definition of $\mathrm{Spin}^c(V)$. □

Let us see in the new definition how is $\chi\colon \mathrm{Spin}(V) \to \mathrm{SO}(V)$. On $\mathrm{Spin}^c(V)$, we have $\ker\chi = U(1)$, but on $\mathrm{Spin}(V)$ we require moreover $\tau(u)u = 1$, thus an element of $\ker\chi$ in $\mathrm{Spin}(V)$ fulfils $\tau(\lambda)\lambda = 1$, so that $\lambda = \{\pm 1\}$. We conclude that $\ker\chi|_{\mathrm{Spin}(V)} = \{\pm 1\}$, and then that $\mathrm{Spin}(V)$ is a double covering of $\mathrm{SO}(V)$.

3.4.3 A few about Lie algebra

Proposition 3.34.
We have an isomorphism

$$\mathfrak{spin}(p,q) \simeq \mathfrak{so}(p,q)$$

between the Lie algebras of $\mathrm{Spin}(p,q)$ *and* $\mathrm{SO}(p,q)$.

Proof. Using the second part of lemma .28, with the map $\chi\colon \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$, we find that $d\chi_e(\mathfrak{spin}(p,q)) = \mathfrak{so}(p,q)$. Then we know (lemma .29) that

$$\mathfrak{so}(p,q) = \mathfrak{spin}(p,q)/\ker\, d\chi_e.$$

On the other hand, the first part of the same lemma gives us that $\chi^{-1}(e)$ is a Lie subgroup of $\mathrm{Spin}(p,q)$ whose Lie algebra is $\ker\, d\chi_e$. But $\chi^{-1}(e) = \mathbb{Z}_2$, so $\ker\, d\chi_e = \{0\}$. □

Let us now shortly speak about the Lie algebra of $\Gamma(p,q)^+$. A basis of $\mathrm{Cl}(p,q)^+$ is

$$\{1, \gamma_0\cdot\gamma_1, \gamma_0\cdot\gamma_1, \gamma_0\cdot\gamma_3, \gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3\}.$$

Thanks to the anticommutation relations, we don't need $\gamma_1\cdot\gamma_2$ in the basis.

Remember that $\Gamma(p,q)^+$ is the set of the $x \in \mathrm{Cl}^+(p,q)$ such that $x\cdot v\cdot\alpha(x^{-1})$ lies in V for all $v \in V$. Let $x(t)$ be a path in $\Gamma(p,q)^+$ such that $x(0) = e$ and $\dot{x}(0) = X$. Differentiating the definition relation, we find

$$\dot{x}\cdot v\cdot\alpha(x^{-1})|_0 + x\cdot v\cdot(-)\alpha(\dot{x})|_0 = X\cdot v - v\cdot X,$$

therefore

$$\mathfrak{Lie}(\Gamma(p,q)^+) = \left\{X \in \mathrm{Cl}^+(p,q) \text{ such that } X\cdot v - v\cdot X \in V,\ \forall v \in V\right\}.$$

It is clear that $\mathbb{C}$ is a subset of $\mathfrak{Lie}(\Gamma(p,q)^+)$, and that V is not. The following computation shows that $V\cdot V$ is a subset $\mathfrak{Lie}(\Gamma(p,q)^+)$:

$$a\cdot b\cdot v - v\cdot a\cdot b = 2\eta(v,a)b - 2\eta(v,b)a.$$

We can also check that $V\cdot V\cdot V\cdot V\cap\mathfrak{Lie}(\Gamma(p,q)^+)=\varnothing$. A basis of $\mathfrak{Lie}(\Gamma(p,q)^+)$ is

$$\{1, e_\alpha\cdot e_\beta \text{ st } \alpha<\beta\}$$

We know that $\ker[\chi\colon\Gamma(p,q)^+\to\mathrm{SO}(p,q)]=\mathbb{R}_0$. So the kernel of the restriction of $d\chi_e$ to $\mathfrak{Lie}(\Gamma(p,q)^+)$ is the Lie algebra of $\mathbb{R}_0$ (see lemma .28), which is $\mathbb{R}$. Therefore, a basis of $\mathfrak{spin}(p,q)$ is

$$\{e_\alpha\cdot e_\beta \text{ st } \alpha<\beta\}.$$

3.4.4 Grading ΛW

We already know that $\Lambda W=\mathbb{C}\oplus W\oplus\Lambda^2 W$. This space can be written as

$$\Lambda W=\Lambda W^+\oplus\Lambda W^-,$$

with $\Lambda W^+=W$ and $\Lambda W^-=\mathbb{C}\oplus\Lambda^2 W$. The interest of such a decomposition lies in the definition of an action of $\mathrm{Cl}^+(p,q)$ on ΛW. This action will be defined by $\bullet\colon\ \mathrm{Cl}^+(p,q)\times\Lambda W\to\Lambda W$,

$$x\bullet\alpha=\tilde\rho(x)\alpha$$

for any x in $\mathrm{Cl}^+(p,q)$ and any α in ΛW (see definition 3.8).

Proposition 3.35.
This action preserves the grading of ΛW:

$$\begin{aligned}\mathrm{Cl}^+(p,q)\bullet\Lambda W^+&=\Lambda W^+\\ \mathrm{Cl}^+(p,q)\bullet\Lambda W^-&=\Lambda W^-.\end{aligned}\tag{3.58}$$

Proof. For $x\in\mathbb{C}$, theses equalities are obvious. We have to check it for $x=e_i\cdot e_j$. Here, we will just check that $(e_1\cdot e_0)\bullet(v\wedge w)\in\Lambda W^+$. This follows from a simple computation:

$$\begin{aligned}\tilde\rho(e_1)\tilde\rho(f_0+g_0)(v\wedge w)&=\tilde\rho(f_1+g_1)\left[-\eta(g_0,v)w+\eta(g_0,w)v\right]\\ &=-\eta(g_0,v)f_1\wedge w+\eta(g_0,w)f_1\wedge v\\ &\quad+\eta(g_0,v)\eta(g_1,w)-\eta(g_0,w)\eta(g_1,v).\end{aligned}\tag{3.59}$$

□

Since $\mathrm{Spin}(p,q)$ is a subgroup of $\mathrm{Cl}^+(p,q)$, we can construct two new representation of $\mathrm{Spin}(p,q)$. These are $\rho^\pm\colon\ \mathrm{Spin}(p,q)\times\Lambda W^\pm\to\Lambda W^\pm$,

$$\begin{aligned}\rho^-(s)w^-&=\tilde\rho(s)w^-,\\ \rho^+(s)w^+&=\tilde\rho(s)w^+,\end{aligned}\tag{3.60}$$

for $w^\pm$ in $\Lambda W^\pm$. This is no more than the fact that $\tilde\rho$ is reducible and that two invariant subspaces are ΛW^+ and ΛW^-.

3.4.5 Clifford algebra for $V=\mathbb{R}^2$

General definitions

The whole construction can also be applied to $V=\mathbb{R}^2$ with the Euclidean metric. This is our business now. We take the complex vector space $V^{\mathbb{C}}$ and an orthonormal basis $\{e_1,e_2\}$. As before, we define

$$f_1=\frac{1}{2}(e_1+ie_2),\qquad g_1=\frac{1}{2}(e_1-ie_2).$$

There are no difficulties to see that $Span(f_1)$ is a completely isotropic subspace of $V^{\mathbb{C}}$. Thus we define $W = \mathbb{C}f_1$, $\Lambda W = \mathbb{C} \oplus W$, $\Lambda W^+ = \mathbb{C}$, and $\Lambda W^- = W$. The homomorphism $\tilde{\rho}\colon V^{\mathbb{C}} \to \mathrm{End}(\Lambda W)$ in ΛW is defined by

$$\begin{aligned}\tilde{\rho}(f_1)\alpha &= f_1 \wedge \alpha,\\ \tilde{\rho}(g_1)\alpha &= -i(g_1)\alpha,\end{aligned} \tag{3.61}$$

where α is any element of ΛW. In the basis $1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $f_1 = \begin{pmatrix}0\\1\end{pmatrix}$, we easily find that

$$\tilde{\rho}(e_1) = \begin{pmatrix}0 & -\frac{1}{2}\\ 1 & 0\end{pmatrix}, \quad \tilde{\rho}(e_2) = \begin{pmatrix}0 & -\frac{i}{2}\\ -i & 0\end{pmatrix}.$$

For $c \in \mathbb{R}$ we also have $\tilde{\rho}(c)f_1 = cf_1$ and $\tilde{\rho}(c)1 = c$, thus we assign the matrix $\begin{pmatrix}c & 0\\ 0 & c\end{pmatrix}$ to the number c.

As before, we define $\gamma_i = \sqrt{2}\tilde{\rho}(e_i)$. We immediately have $\gamma_1\gamma_2 + \gamma_2\gamma_1 = 0$ and $\gamma_i\gamma_i = -2\mathbb{1}$, so that the γ's satisfy the Clifford algebra for the euclidian metric.

For notational conveniences, it proves useful to make a change of basis so that we get

$$\gamma_1 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \gamma_2 = -\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \tag{3.62}$$

The algebra Cl(2) is isomorphic to the algebra which is generated by direct sum $\mathrm{Cl}(2) \simeq \mathbb{R} \oplus \gamma_1 \oplus \gamma_2 \oplus \mathbb{R}\gamma_1\gamma_2$. A general element of Cl(2) can be written as $x\gamma_1 + y\gamma_2 + x'\mathbb{R} + y'\gamma_1\gamma_2$. In the representation of $\tilde{\rho}$, a general element of Cl(2) is therefore

$$\begin{pmatrix}x' + iy' & x + iy\\ -x + iy & x' - iy'\end{pmatrix},$$

so that we can write the Clifford algebra of $\mathbb{R}^2$ as

$$\mathrm{Cl}(2) = \left\{ \begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} \,:\, \alpha, \beta \in \mathbb{C} \right\}.$$

The following four matrices provide a basis:

$$1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad i = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}, \qquad j = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \qquad k = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{3.63}$$

We can check that these matrices satisfies the quaternionic algebra :

$$\begin{aligned}i^2 = j^2 = k^2 &= -1\\ ij = -ji &= k,\\ jk = -kj &= i,\\ ki = -ik &= j.\end{aligned} \tag{3.64}$$

The algebra $\mathrm{Cl}(2) = \mathbb{H}$ is represented by $\tilde{\rho}$ on $\mathbb{C}^2$ by the **Pauli matrices** $1, i, j, k$ which are given by (3.63)-111.

The maps α and τ

What are the matrices which represent V ? These are $\tilde{\rho}(e_1)$ and $\tilde{\rho}(e_2)$. Thus we can write $V = \mathrm{Span}_{\mathbb{R}}\{\gamma_1, \gamma_2\} = \mathrm{Span}_{\mathbb{R}}\{j, k\}$, or

$$V = \left\{ \begin{pmatrix} 0 & \xi \\ -\overline{\xi} & 0 \end{pmatrix} : \xi \in \mathbb{C} \right\}.$$

As before, α is the unique homomorphic extension to $\mathrm{Cl}(2)$ of $-\,\mathrm{id}$ on V. From the definitions, we get $\alpha(j) = -j$, $\alpha(k) = -k$. The extension present no difficult. For example: $\alpha(i) = \alpha(jk) = \alpha(j)\alpha(k) = jk = i$, but $\alpha(jk) = \alpha(i)$; then $\alpha(i) = i$. The same gives $\alpha(1) = 1$.

The case of τ is treated in similar way. We find: $\tau(j) = j$, $\tau(k) = k$, $\tau(i) = -i$, $\tau(1) = 1$.

Now, we can find the group $\Gamma_{(2)}$. The condition for $x \in \mathrm{Cl}(2)$ to be in $\Gamma_{(2)}$ is $\alpha(x)yx^{-1}$ to belongs to V for all $y \in V$. We put

$$x = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \qquad \alpha(x) = \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

A typical y in V is

$$y = \begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix}.$$

A few computation gives:

$$\alpha(x)yx^{-1} = \frac{1}{|\alpha|^2 + |\beta|^2} \begin{pmatrix} \alpha\eta\overline{\beta} + \beta\overline{\eta}\overline{\alpha} & \alpha\alpha\eta - \beta\beta\overline{\eta} \\ \overline{\beta}\overline{\beta}\eta - \overline{\alpha}\overline{\alpha}\eta & \eta\alpha\overline{\beta} + \overline{\alpha}\overline{\eta}\beta \end{pmatrix}.$$

If we impose it to be of the form $\begin{pmatrix} 0 & \xi \\ -\overline{\xi} & 0 \end{pmatrix}$ for all $\eta \in \mathbb{C}$, we get, for all $\eta \in \mathbb{C}$, $\mathrm{Re}(\overline{\alpha}\beta\overline{\eta}) = 0$, which implies $\overline{\alpha}\beta = 0$. So we conclude:

$$\Gamma_{(2)} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ -\overline{\beta} & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \text{ not both equals zero} \right\}.$$

Be careful on a point: $\Gamma_{(2)}$ is the *multiplicative* group generated by these two matrices, not the additive one.

The spin group

It present no difficult to find that

$$\Gamma^+_{(2)} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} : \alpha \neq 0 \right\}. \tag{3.65}$$

The **spin group** is made of elements of $\Gamma^+_{(2)}$ which satisfy $\tau(x) = x^{-1}$. We know that $\tau \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}$ and that $\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}^{-1} = \frac{1}{\alpha\overline{\alpha}} \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}$. Thus the condition $\tau(x) = x^{-1}$ becomes $|\alpha|^2 = 1$. The first conclusion is that

$$\mathrm{Spin}(2) = U(1). \tag{3.66}$$

A typical s in $\mathrm{Spin}(2)$ is

$$s = e^{i\theta} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

The next point is to see the action of Spin(2) on V. The action of $s \in \mathrm{Spin}(2)$ on a vector $v \in V$ is still defined by $s \bullet v = \chi(s)v = \alpha(s) \cdot v \cdot s^{-1}$. More explicitly:

$$\chi(s)v = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} 0 & e^{2i\theta} z \\ -e^{-2i\theta}\overline{z} & 0 \end{pmatrix}, \tag{3.67}$$

where the matrix $\begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix}$ denotes the representation of the vector v of V. This equality can be written $e^{i\theta} \cdot v = e^{2i\theta} v$. If we note $v = v_1 + iv_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, we get

$$e^{2i\theta} \bullet v = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Therefore, we can write

$$\chi(e^{i\theta}) = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

So χ projects $U(1)$ into SO(2) with a kernel $\mathbb{Z}_2$, for this reason, we say that $U(1)$ is a **double covering** of SO(2). We note it

$$\mathbb{Z}_2 \to U(1) \xrightarrow{\chi} \mathrm{SO}(2). \tag{3.68}$$

3.5 Clifford modules

References: [23, 24].

Let M be a manifold. We denote by $\mathrm{Cl}^{\mathbb{C}}(M)$ the bundle whose fibre at $x \in M$ is the complex Clifford algebra of the metric g_x : $\mathrm{Cl}^{\mathbb{C}}(M)_x = \mathrm{Cl}^{\mathbb{C}}(g_x)$. We define the important map

$$\begin{aligned} \gamma\colon \Gamma(M, \mathrm{Cl}^{\mathbb{C}}(M)) &\to \mathfrak{B}(\mathscr{H}) \\ \gamma(dx^\mu) &\mapsto \gamma^\mu(x) \end{aligned} \tag{3.69}$$

which can be extended to the whole Clifford algebra.

Let V be a vector space endowed with a bilinear symmetric form. We consider $\mathrm{Cl}(V)$, the corresponding Clifford algebra. A **Clifford module** is a real vector space E with a $\mathbb{Z}_2$-graduation and a morphism

$$\rho_E\colon\ \mathrm{Cl}(V) \to \mathrm{End}(E)$$

of $\mathbb{Z}_2$-graded vector spaces. It is defined by a linear map $\rho_E\colon V \to \mathrm{End}(V)$ such that

$$\rho_E(v)\rho_E(w) + \rho_E(w)\rho_E(v) = B(v,w)\,\mathrm{id} \tag{3.70}$$

for every v, $w \in E$. The element $\rho_E(x)v$ will often be denoted by $x \cdot v$ and the operation ρ_E is the **Clifford multiplication**. The **dual module** E^* is defined by $\rho_{E^*}(x) = \rho_E(x^t)^*$, i.e.

$$\langle \rho_{E^*}(x)\psi, v\rangle = (-1)^{|\psi||x|}\langle \psi, \rho_E\big(\tau(x)\big)v\rangle \tag{3.71}$$

for every $\psi \in E^*$ and $v \in E$. Here

Let $\mathfrak{A}$ be a $\mathbb{Z}_2$-graded subalgebra of $\mathrm{Cl}(V)$ and E_1, a $\mathfrak{A}$-module. Then the space

$$E = \mathrm{Ind}_{\mathfrak{A}}^{\mathrm{Cl}(V)}(E_1) = \mathrm{Cl}(V) \otimes_{\mathfrak{A}} E_1$$

has a structure of Clifford module, the **induced module**. The tensor product $\otimes_{\mathfrak{A}}$ is the usual one modulo the subspace spanned by elements of the form

$$x \otimes a \cdot y - xa \otimes y$$

for every x, $a \in \mathrm{Cl}(V)$ and $y \in E_1$. In a similar way, if E is a complex vector space we have a notion of $\mathrm{Cl}^{\mathbb{C}}(V)$-module.

Let $x \in \mathrm{Cl}(V)$ be such that $x^2 = 1$. In that case the Clifford multiplication $\rho_E(x)$ decomposes E in eigenspaces

$$E^{\pm} = \frac{1}{2}\big(1 \pm \rho_E(x)\big)E.$$

If V is a n-dimensional vector space with an oriented orthonormal basis $\{e_1, \ldots, e_n\}$, the algebra $\mathrm{Cl}(V)$ has a **volume element** $\omega = e_1 e_2 \ldots e_n$ which does not depend on the choice of the basis. The volume element squares to

$$\omega^2 = (-1)^{n(n+1)/2}. \tag{3.72}$$

In the complex case, we consider the complex vector space $V^{\mathbb{C}}$ and the complex Clifford algebra $\mathrm{Cl}^{\mathbb{C}}(V) = \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$, and the volume element is defined as

$$\omega_{\mathbb{C}} = i^{[(n+1)/2]}\omega. \tag{3.73}$$

where $[x]$ is denotes the integer part of x. Performing a separate computation for n even or odd, it is easy to see that in both case,

$$\omega_{\mathbb{C}}^2 = 1. \tag{3.74}$$

So in the complex case we always have an element in $\mathrm{Cl}(V)$ which squares to 1, and a $\mathrm{Cl}^{\mathbb{C}}(V)$-module W always accepts a decomposition as $W^{\pm} = \frac{1}{2}(1 + \omega_{\mathbb{C}})W$.

One says that a representation ρ of $\mathrm{Cl}(V)$ on W is **reducible** if there exists a splitting $W = W_1 \oplus W_2$ such that $\rho(\mathrm{Cl}(V))W_i \subset W_i$. If the representation is not reducible, it is said to be irreducible. Two representations $\rho_j \colon \mathrm{Cl}(V) \to \mathrm{End}(W_j)$ are **equivalent** if there exists a linear isomorphism $F \colon W_1 \to W_2$ such that $F \circ \rho_1(x) \circ F^{-1} = \rho_2(x)$ for every $x \in \mathrm{Cl}(V)$.

Proposition 3.36.
The real Clifford algebra has

$$\begin{cases} 2 & \text{if } n+1 = 0 \mod 4 \\ 1 & \text{otherwise} \end{cases}$$

inequivalent irreducible representations. The complex Clifford algebra $\mathrm{Cl}^{\mathbb{C}}(V)$ *has*

$$\begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

inequivalent irreducible representations.

Proof. No proof. □

If M is a manifold, we denote by $\mathrm{Cl}(M) = \mathrm{Cl}(TM)$ the bundle whose fiber at x is the Clifford algebras of T_xM. We consider an orthonormal basis $\{e_i\}$ and if Σ is a multi-index $\{1 \leqslant \sigma_1, \ldots, \leqslant \sigma_t \leqslant m\}$, we pose $e_\Sigma = e_{\sigma_1} \ldots e_{\sigma_t} \in \mathrm{Cl}(M)$. By convention, $e_\varnothing = 1$. Since the elements e_i are ordered, they provide an orientation:

$$d\,\mathrm{Vol} = e_1 \wedge \ldots \wedge e_m \in \bigwedge^m(M). \tag{3.75}$$

Since the map $e_{\sigma_1} \wedge \ldots \wedge e_{\sigma_t} \mapsto e_{\sigma_1 \ldots e_{\sigma_t}}$ is an isomorphism between $\mathrm{Cl}(M)$ and $\bigwedge(M)$, we say that $d\,\mathrm{Vol} \in \mathrm{Cl}(M)$. Now we define

$$\kappa = i^{-[(m+1)/2]} d\,\mathrm{Vol},$$

which is nothing else that the volume form normalised in such a way that $\kappa^2 = 1$. If m is even, it anti-commutes with TM, and if m is odd, it commutes with TM.

Let V be a m-dimensional real vector space, and $\mathrm{Cl}^{\mathbb{C}}(V)$, the corresponding complex Clifford algebra.

Lemma 3.37.
Every $\mathrm{Cl}^{\mathbb{C}}(V)$-module accepts an unique decomposition as sum of irreducible representations as follows

(i) if $m = 2n$, there exists one and only one irreducible $\mathrm{Cl}^{\mathbb{C}}(V)$-module Δ and $\dim(\Delta) = 2n$,

(ii) if $m = 2n + 1$, we have two inequivalent irreducible modules $\Delta_\pm$ with $\gamma(\kappa) = \pm 1$ on $\Delta_\pm$ and $\dim(\Delta_\pm) = 2^n$.

Proof. No proof. □

Let V be a vector bundle over M. A structure of $\mathrm{Cl}(M)$-module over V is a morphism of unital algebra $\gamma\colon \mathrm{Cl}(M) \to \mathrm{End}(V)$. When one has a basis $\{e_i\}$ of V, we pose $\gamma_i = \gamma(e_i)$. The following lemma is the lemma 1.2 of [24].

Lemma 3.38.
Let V be a $\mathrm{Cl}(V)$-module and $\{e_i\}$, an orthonormal basis for TM on a contractible open set V. Then there exists a local frame for V such that the matrices $\gamma(e_i)$ are constant.

We also define $\gamma^i = \gamma(dx^i) = g^{ij}\gamma_j$. One easily proves that

$$\gamma^i\gamma^j + \gamma^j\gamma^i = -2g^{ij} \tag{3.76}$$

where (g^{ij}) is the inverse matrix of (g_{ij}). If the endomorphisms γ_i are constant in the basis $\{e_i\}$, then the endomorphisms γ^i are constant in the basis $\{f_i = g_{ki}e_k\}$.

3.6 Spin structure

We consider a (pseudo-)Riemannian manifold (M, g) with metric signature (p, q), and $\mathrm{SO}(M)$, its frame bundle; it admits a $\mathrm{SO}(p, q)$-principal fibre bundle structure which is well defined by the metric g (see 1.7.4).

Definition 3.39.
*We say that (M, g) is a **spin manifold** if there exists a $\mathrm{Spin}(p, q)$-principal bundle P over M and a principal bundle homomorphism $\varphi\colon P \to \mathrm{SO}(M)$ which induced covering for the structure groups is χ, i.e. $\varphi(\xi \cdot s) = \varphi(\xi) \cdot \chi(s)$. A choice of P and φ is a **spin structure** on M.*

$$\begin{tikzcd}
\mathrm{Spin}(p,q) \arrow[r, squiggly] & P \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathrm{SO}(M) \arrow[dl, "p"] & \mathrm{SO}(p,q) \arrow[l, squiggly] \\
& & M & &
\end{tikzcd}$$

The wavy arrows mean “structural group of”.

Remark 3.40. When we will use the concept of spin structure in the physical oriented chapters, we will naturally use $\mathrm{SL}(2,\mathbb{C})$ as group instead of $\mathrm{Spin}(p,q)$. The isomorphism $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Spin}(1,3)$ is proved in [21]. A physical motivation of such a structure is given at page 160.

3.6.1 Example: spin structure on the sphere S^2

It is no difficult to see that $\mathrm{SO}(S^2) \simeq \mathrm{SO}(3)$. Indeed, each element of $\mathrm{SO}(S^2)$ is described by three orthonormal vectors: one which point to an element x of S^2 and two which gives a basis of T_xS^2. The action $\mathrm{SO}(3) \times S^2 \to S^2$ is transitive, and the stabilizer of any element is $\mathrm{SO}(2)$.

We define $\alpha\colon \mathrm{SO}(3)/\mathrm{SO}(2) \to S^2$ by $\alpha(g\,\mathrm{SO}(2)) = g$. One can show, using proposition 4.3 in [3] that α is a diffeomorphism. Then

$$S^2 = \frac{\mathrm{SO}(3)}{\mathrm{SO}(2)}.$$

On the other hand, we know that

$$T_eSU(2) = su(2) = \left\{ \begin{pmatrix} ix & \xi \\ -\overline{\xi} & -ix \end{pmatrix} : \xi \in \mathbb{C}, x \in \mathbb{R} \right\}. \tag{3.77}$$

It is a classical result that $\mathfrak{su}(2) \simeq \mathbb{R}^3$ not only as set but also as metric space with the identification

$$\langle X, Y \rangle = -\frac{1}{2}\,\mathrm{Tr}(XY),$$

for all X, $Y \in su(2)$. As we are in matrix groups, we know (see [6] to get more details) that $Ad_xY = xYx^{-1}$. In our case, this gives the formula

$$\langle Ad(g)X, Ad(g)Y \rangle = \langle X, Y \rangle.$$

We can now state the result for S^2.

Proposition 3.41.
The manifold S^2 with the usual metric induced from $\mathbb{R}^3$ admits the following spin structure:

$$\begin{tikzcd} \mathrm{Spin}(2) \arrow[r, squiggly] & SU(2) \arrow[rr, "\varphi = Ad"] \arrow[dr, "\pi", "U(1)"'] & & \mathrm{SO}(3)\ , \arrow[dl, "p"', "\mathrm{SO}(2)"] \\ & & S^2 & \end{tikzcd} \tag{3.78}$$

where the arrow $X \xrightarrow[G]{f} Y$ *means that G is the kernel of the map $f\colon X \to Y$.*

Proof. First, let us precise the concept of frame bundle for S^2, and how it is well described by $\mathrm{SO}(3)$. Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$. To $A \in \mathrm{SO}(3)$, we make correspond the basis $\{Ae_2, Ae_3\}$ at the point Ae_1 of S^2. The projection $p\colon \mathrm{SO}(3) \to S^2$ is then defined by $p(A) = Ae_1$. It is clear that we will define the map $\pi\colon SU(2) \to S^2$ in the same way: $\pi(U) = p(Ad(U))$.

For the rest of the demonstration, we will use the "$su(2)$ description" of $\mathbb{R}^3$ given by (3.77)-116 with $\xi = y + iz$.

Now, let us show that $\pi\colon SU(2) \to S^2$ is a $\mathrm{Spin}(2)$-principal bundle. Since we had already shown that $\mathrm{Spin}(2) \simeq U(1)$, we define the right action of $\mathrm{Spin}(2)$ on $SU(2)$ by right multiplication:

$U \cdot s = Us$ with $s = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. It is clear that $\pi(Us) = \pi(U)$:

$$Ad(Us)e_1 = (Us)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} s^{-1}U^{-1} = Us\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} s^{-1}U^{-1}, \tag{3.79}$$

because $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ is the vector e_1 in the "$su(2)$ description" of $\mathbb{R}^3$.

In order for $\pi\colon SU(2) \to S^2$ to be a Spin(2)-principal bundle, we still need to show that for all $x \in S^2$,

$$\pi^{-1}(x) = \left\{ \xi \cdot g \text{ st } g \in \mathrm{Spin}(2)\, \forall \xi \in \pi^{-1}(x) \right\}.$$

Take A, $B \in \pi^{-1}(x)$, i.e. $Ae_1 = Be_1 = x$. We need to find a $s \in \mathrm{Spin}(2)$ such that

$$A = B \cdot s. \tag{3.80}$$

The matrices A and B are such that

$$B^{-1}A\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} A^{-1}B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.81}$$

This implies that $B^{-1}A \in \mathrm{Spin}(2)$. As Ad is surjective from $SU(2)$ into SO(3), a general C in SO(3) which acts on e_1 can be written Ue_1U^{-1} for $U \in SU(2)$ such that $Ad(U) = C$. The condition (3.81)-117 becomes

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

which implies $\alpha = e^{i\theta}$, $\beta = 0$. Then $B^{-1}A$ belongs to Spin(2), and $s = B^{-1}A$ fulfills the condition (3.80)-117.

What about the induced covering for the structural groups ? The structural group of $\pi\colon SU(2) \to S^2$ is Spin(2), while the one of $p\colon \mathrm{SO}(3) \to S^2$ is SO(2). Indeed, for each $x \in S^2$, SO(2) acts on T_xS^2, leaving x unchanged. We have the following associations:

$$U \in SU(2) \xrightarrow{\varphi} A \in \mathrm{SO}(3),$$

the matrix A being defined by $A \cdot X = UXU^{-1}$. For $s \in \mathrm{Spin}(2)$ we of course also have

$$Us \in SU(2) \xrightarrow{\varphi} As \in \mathrm{SO}(3),$$

with $As \cdot X = UsXs^{-1}U^{-1}$. As we act by Spin(2) on $SU(2)$, in the fibres of SO(3), the action of Spin(2) is –via φ– the composition with $X \to sXs^{-1}$. But this is exactly $\chi(s)X$ because $\alpha(s) = s$, since $s \in \mathrm{Spin}(2)$. $\square$

3.6.2 Spinor bundle

Let us take once again the spin structure on the (pseudo-)Riemannian manifold (M, g):

$$\begin{tikzcd} \mathrm{Spin}(p,q) \arrow[r, rightsquigarrow] & P \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathrm{SO}(M) \arrow[dl, "p"] & \mathrm{SO}(p,q) \arrow[l, leftsquigarrow] \\ & & M & & \end{tikzcd}$$

with $\varphi(\xi \cdot g) = \varphi(\xi) \cdot \chi(g)$.

Let us define $S = \Lambda W$, and $\mathcal{S} = P \times_\rho S$. Take $\rho\colon \mathrm{Spin}(p,q) \times \mathcal{S} \to \mathcal{S}$, $\rho(g,s) = \tilde{\rho}(g)s$, where $\tilde{\rho}$ is the spinor representation of $\mathrm{Spin}(p,q)$ on S. We also have $\chi\colon \mathrm{Spin}(p,q) \to \mathrm{SO}_0(p,q)$, $\chi(g)v = \alpha(g) \cdot v \cdot g^{-1}$, with $\alpha(g) = g$ for $g \in \mathrm{Spin}(p,q)$.

The **spinor bundle** is the associated bundle

$$\mathcal{S} = P \times_\rho S \to M \tag{3.82}$$

A **spinor field** is an element of $\Gamma(\mathcal{S})$, the space of section of the spinor bundle.

On $\mathrm{SO}(M)$, we look at a connection 1-form $\alpha \in \Omega^1(\mathrm{SO}(M), so(\mathbb{R}^m))$, and, if $T(M)$ is the tensor bundle over M, we define a covariant derivative $\nabla^\alpha\colon \mathfrak{X}(M) \times T(M) \to T(M)$ by

$$\widehat{\nabla^\alpha_X s} = \overline{X}\hat{s},$$

for any $s \in T(M)$. See theorem 1.53, and the fact that $T(M)$ can be see as an associated bundle; it is explicitly done for $\mathfrak{X}(M)$ at page 45.

As seen in point 1.14.2, an automatic property of this connection is $\nabla^\alpha g = 0$ if g is the metric of M. The **Levi-Civita connection** is the unique[4] such connection which is torsion-free: $T^{\nabla^\alpha} = 0$.

Proposition 3.42.
The 1-form $\tilde{\alpha} = \varphi^*\alpha \in \Omega^1(P, so(\mathbb{R}^m))$ defines a connection on P. See definition 1.47 and theorem 1.53.

Proof. Let us denote by R_g the right action of $g \in \mathrm{Spin}(p,q)$ on P (*id est* $R_g\xi = \xi \cdot g$), and by $R_u^{\mathrm{SO}(M)}$ the right action of $u \in \mathrm{SO}(p,q)$ on $\mathrm{SO}(M)$. We have to check the usual two conditions of a connection.
First condition. The first one is:

$$(R_g^*\tilde{\alpha})_\xi(\Sigma) = Ad(g^{-1})(\tilde{\alpha}_\xi(\Sigma)),$$

for all $\xi \in P$, and $\Sigma \in T_\xi P$. In order to check this, we first remark that $\varphi \circ R_g = R^{\mathrm{SO}(M)}_{\chi(g)} \circ \varphi$. Indeed, for all $\xi \in P$, definition 3.39 gives us $\varphi(R_g\xi) = \varphi(\xi \cdot g) = \varphi(\xi) \cdot \chi(g)$. With this, we can make the following computation:

$$\begin{aligned} R_g^*\tilde{\alpha} &= R_g^*\varphi^*\alpha = (\varphi \circ R_g)^*\alpha = (R^{\mathrm{SO}(M)}_{\chi(g)} \circ \varphi)^*\alpha \\ &= \varphi^* R^{\mathrm{SO}(M)*}_{\chi(g)}\alpha = \varphi^*(Ad(\chi(g)^{-1}) \circ \alpha). \end{aligned} \tag{3.83}$$

The last equality comes from the fact that α is a connection 1-form. As we are in matrix groups, we have $Ad(g)x = gxg^{-1}$, so

$$[Ad(\chi(g))x]v = [\chi(g)x\chi(g)^{-1}]v = \chi(g)[xg^{-1}vg] = gxg^{-1}. \tag{3.84}$$

In the first line, the product is the usual matrix product which can be seen as operator composition.

But $(Ad(g)x)v = gxg^{-1}v$. Then $Ad(g) = Ad(\chi(g))$, if we identify $\mathfrak{spin}(p,q) \simeq \mathfrak{so}(p,q)$ by proposition 3.34. Moreover, the action of Ad is linear, so it commutes with φ^*. With these remarks, we can continue the computation (3.83)-118:

$$\varphi^*(Ad(\chi(g)^{-1}) \circ \alpha) = \varphi^*(Ad(g^{-1}) \circ \alpha) = Ad(g^{-1}) \circ \varphi^*\alpha = Ad(g^{-1}) \circ \tilde{\alpha}. \tag{3.85}$$

[4] We will not prove unicity.

This proves the first condition.
Second condition. The second one is $\tilde{\alpha}(A^*_\xi) = -A$ with the definition (1.143)-54. This is also a computation. First remark

$$\tilde{\alpha}_\xi(A^*_\xi) = (\varphi^*\alpha)_\xi(A^*_\xi) = \alpha_{\varphi(\xi)}(\varphi_{*\xi}A^*_\xi).$$

We compute $\varphi_{*\xi}A^*$ with lemma 1.20:

$$\begin{aligned}
\varphi_{*\xi}A^* &= \frac{d}{dt}\varphi(\xi\cdot\exp -tA)\Big|_{t=0} = \frac{d}{dt}(R^{\mathrm{SO}(M)}_{\chi(\exp -tA)}\circ\varphi)(\xi)\Big|_{t=0} \\
&= \frac{d}{dt}\varphi(\xi)\cdot\chi(\exp -tA)\Big|_{t=0} = \frac{d}{dt}\varphi(\xi)\cdot\exp(-td\chi_e A)\Big|_{t=0} = (d\chi_e A)^*_{\varphi(\xi)}.
\end{aligned} \tag{3.86}$$

But $d\chi_e = \mathrm{id}_{so(p,q)}$, thus $\varphi_{*\xi}A^* = A^*_{\varphi(\xi)}$. The whole makes that:

$$\tilde{\alpha}_\xi(A^*_\xi) = \alpha_{\varphi(\xi)}(\varphi_{*\xi}A^*_\xi) = \alpha_{\varphi(\xi)}(A^*_{\varphi(\xi)}) = -A.$$

This completes the proof. □

Definition 3.43.
This connection 1-form on P is called the ***spinor connection****. It gives us a covariant derivative on any associated bundle and in particular on the spinor bundle, $\tilde{\nabla}\colon \mathfrak{X}(M)\times\Gamma(\mathcal{S})\to\Gamma(\mathcal{S})$.*

Proposition 3.44.
If X, $Y \in \mathfrak{X}(M)$ are such that $X_x = Y_x$, then for all $s\in\Gamma(\mathcal{S})$,

$$(\tilde{\nabla}_X s)(x) = (\tilde{\nabla}_Y s)(x).$$

Proof. We just have to show that for all vector field Z such that $Z_x = 0$, $(\tilde{\nabla}_Z s)(x) = 0$. Such a Z can be written as $Z = fZ'$ for a function f on M which satisfies $f(x) = 0$. We have:

$$\tilde{\nabla}_Z s = \tilde{\nabla}_{fZ'}s = f\tilde{\nabla}_{Z'}s,$$

which is obviously zero at x. □

Let $x\in M$ and $\{e_{\alpha x}\}$ be an orthonormal basis of T_xM. We can extend it to $\{e_\alpha\}$, a local basis field around x such that e_α is a section of the frame bundle (in other words, we ask the extension to be smooth). The claim of proposition 3.44 is that $\tilde{\nabla}_{e_\alpha}(x)$ is an element of $\mathcal{S}_x$ which doesn't depend on the extension.

3.7 Dirac operator ☺

3.7.1 Preliminary definition

Let M be a m-dimensional (pseudo)Riemannian manifold with its spin structure

$$\begin{tikzcd}
\mathrm{Spin}(p,q) \arrow[r, rightsquigarrow] & P \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathrm{SO}(M) \arrow[dl, "p"] & \mathrm{SO}(p,q) \arrow[l, leftsquigarrow] \\
& & M & &
\end{tikzcd}$$

where φ satisfies $\varphi(\xi \cdot g) = \varphi(\xi) \cdot \chi(g)$.

Recall that for any vector space, one can see $\operatorname{End} V = V^* \otimes V$ with the definition $(v^* \otimes v)w = (v^* w)v$. This allows us to define an action of $\mathrm{Spin}(p, q)$ on $\operatorname{End} S$ by defining an action of $\mathrm{Spin}(p, q)$ on S and S^* separately. We know the action

$$\begin{aligned} \mathrm{Spin}(p, q) \times S &\to S \\ (g, v) &\mapsto \tilde{\rho}(g)v, \end{aligned} \tag{3.87}$$

and as action on S^*, we take the dual one

$$\begin{aligned} \mathrm{Spin}(p, q) \times S^* &\to S^* \\ g \cdot \alpha &= \alpha \circ \tilde{\rho}(g^{-1}) \end{aligned} \tag{3.88}$$

for all $g \in \mathrm{Spin}(p, q)$ and $\alpha \in S^*$.

Now we can make the following computation with $g \in \mathrm{Spin}(p, q)$, $\alpha \in S^*$ and $v \in S$, using the fact that $\tilde{\rho}$ is linear:

$$\begin{aligned} [g \cdot (\alpha \otimes v)]w &= [(\alpha \circ \tilde{\rho}(g^{-1}))w]\tilde{\rho}(g)v \\ &= \tilde{\rho}\left([(\alpha \circ \tilde{\rho}(g^{-1}))w]g\right) v \\ &= \big[\tilde{\rho}(g) \circ (\alpha \otimes v) \circ \tilde{\rho}(g^{-1})\big]w. \end{aligned} \tag{3.89}$$

Then we write the action of $\mathrm{Spin}(p, q)$ on $\operatorname{End} S$ by ($A \in \operatorname{End} S$)

$$g \cdot A = \tilde{\rho}(g) \circ A \circ \tilde{\rho}(g^{-1}). \tag{3.90}$$

Notice that this definition is the one required in condition (1.93)-44.

The tangent bundle $T_x M$ is given with a metric g_x. As usual, we build $S_x = \Lambda W_x$, a completely isotropic subspace of $T_x M$ with respect to the metric g_x, and a representation

$$\tilde{\rho}_x \colon T_x M \to \operatorname{End}(\Lambda W_x)$$

The first step in the definition of $\gamma(X)$ is to build $\hat{a}_X \colon P \to \operatorname{End}(\Lambda W)$ setting[5] $\hat{a}_X(p) = \tilde{\rho}(\hat{X}_{\varphi(p)})$.

Lemma 3.45.
The function $\hat{a}$ is equivariant, i.e. it satisfies

$$\hat{a}_X(p \cdot g) = g^{-1} \cdot \hat{a}_X(p) \tag{3.91}$$

for all $g \in \mathrm{Spin}(p, q)$.

Proof. It is no more than a simple computation using the equivariance of $\hat{X}$. Indeed:

$$\begin{aligned} \hat{a}_X(p \cdot g) &= \tilde{\rho}(\hat{X}_{\varphi(p \cdot g)}) = \tilde{\rho}(\hat{X}_{\varphi(p)\chi(g)}) = \tilde{\rho}(\chi(g^{-1}) \cdot \hat{X}_{\varphi(p)}) \\ &= \tilde{\rho}(g^{-1} \cdot \hat{X}_{\varphi(p)} \cdot g) = \tilde{\rho}(g^{-1}) \circ \tilde{\rho}(\hat{X}_{\varphi(p)}) \circ \tilde{\rho}(g) \\ &= g^{-1} \cdot \hat{a}_X(p). \end{aligned} \tag{3.92}$$

In the fourth line, the dots mean the Clifford product, and the last equality comes from the definition of the action (3.90)-120 of $\mathrm{Spin}(p, q)$ on $\operatorname{End} S$. □

[5]See subsection 1.8.4 for the definition of $\hat{X}$.

From the discussion of section 1.8.2, the function $\hat{a}_X : P \to \operatorname{End} S$ defines a section $a_X : M \to \operatorname{End} \mathcal{S}$. We define $\gamma \colon \mathfrak{X}(M) \to \operatorname{End} \Gamma(\mathcal{S})$ by

$$\gamma(X) = a_X. \tag{3.93}$$

We immediately have

$$\widehat{\gamma(X)}(p) = \tilde{\rho}(\hat{X}_{\varphi(p)})$$

for any $p \in P$. If we define

$$\widehat{\gamma \cdot a_X}(p) = \widehat{\gamma(X)}(p), \tag{3.94}$$

the map γ can be seen as an action on the section of $\mathcal{S}$. Indeed, $\widehat{\gamma \cdot s_X}$ is an equivariant function:

$$\begin{aligned}
\hat{\gamma}(p \cdot g)(\hat{a}_X(p \cdot g)) &= \rho(g)^{-1}\hat{\gamma}(p)\rho(g)\rho(g^{-1})\hat{a}_X(p) \\
&= \rho(g)^{-1}\hat{\gamma}(p)\hat{a}_X(p) \\
&= \rho(g^{-1})\widehat{\gamma \cdot a_X}(p),
\end{aligned} \tag{3.95}$$

so that

$$\widehat{\gamma \cdot a_X}(p) = \rho(g^{-1})\widehat{\gamma \cdot a_X}(p).$$

The map $\widehat{\gamma \cdot a_X} : P \to \operatorname{End} \Lambda W$ defined by (3.94)-121 is equivariant, and thus defines a section $\gamma \cdot a_X \in \Gamma(\mathcal{S})$, as seen in the section 1.8.2.

3.7.2 Definition of Dirac

If we consider a basis $\{e_\alpha\}$ of TM, *i.e.* m sections $e_\alpha \colon M \to TM$ such that for all x in M, the set $\{e_{\alpha x}\}$ is a basis of T_xM, we note $\gamma^\alpha := \gamma(e_\alpha) \in \operatorname{End}(\mathcal{S})$.

For any $s \in \Gamma(\mathcal{S})$, we consider the local[6] section ψ of $\mathcal{S}$ given by

$$\psi(x) = \sum_{\alpha\beta} g_x(e_\alpha, e_\beta)\gamma_x^\beta(\tilde{\nabla}_{e_\alpha} s)(x).$$

For each $x \in M$, take a A_x in[7] $\mathrm{SO}(g_x)$, and consider the new basis $e'_\alpha = A_\alpha{}^\beta e_\beta$. As A is an isometry, $g_x(e'_\alpha, e'_\beta) = g_x(e_\alpha, e_\beta)$; and since $\tilde{\rho}$ is linear, $\gamma_x'^\alpha = \tilde{\rho}_x(e'_{\alpha x}) = A_\alpha{}^\beta\tilde{\rho}(e_{\beta x}) = A_\alpha{}^\beta\gamma_x^\beta$. In the new basis, the section reads:

$$\begin{aligned}
\psi(x) &= \sum_{\alpha\beta\eta\sigma} g_x(e_\alpha, e_\beta)A_\beta{}^\sigma\gamma_x^\sigma(\tilde{\nabla}_{A_\alpha{}^\eta e_\eta} s)(x) \\
&= \sum_{\alpha\beta\eta\sigma} (A^t)^\eta{}_\alpha g_{\alpha\beta}(x)A_\beta{}^\sigma\gamma_x^\sigma(\tilde{\nabla}_{e_\eta} s)(x) \\
&= \sum_{\eta\sigma} g_x(e_\eta, e_\sigma)\gamma_x^\sigma(\tilde{\nabla}_{e_\eta} s)(x),
\end{aligned} \tag{3.96}$$

where we used the fact that $A^tgA = g$ and that all the $A_\alpha{}^\beta$ are C^∞ functions on M, so that $\tilde{\nabla}_{A_\alpha{}^\beta X} = A_\alpha{}^\beta\tilde{\nabla}_X$. This shows that $\psi(x)$ doesn't depend on the choice of the basis, so it defines a section from the data of s alone.

The **Dirac operator** $\mathcal{D} \colon \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ acting on a spinor field is defined by

$$(\mathcal{D}s)(x) = g_x(e_\alpha, e_\beta)\gamma_x^\beta(\tilde{\nabla}_{e_\alpha} s)(x). \tag{3.97}$$

[6] Extensions of e_α do not always globally exist.

[7] By $\mathrm{SO}(g_x)$, we mean the set of all the matrix A such that $A^tg_xA = g$; A_x is an isometry of (T_xM, g_x). In other words, we consider A as a section of what we could call the "isometry bundle".

Proposition 3.46.
If the field of basis $e_\alpha \in \mathfrak{X}(M)$ is everywhere an orthonormal basis, the Dirac operator reads

$$(\mathcal{D}s)(x) = g_{\alpha\beta}\gamma^\alpha(\tilde{\nabla}_{e_\beta} s)(x) \tag{3.98}$$

where γ^α is a constant numeric matrix acting on ΛW.

Proof. The building of the Dirac operator begins by considering the vector space T_xM endowed with the metric g_x; then the spinor representation $\tilde{\rho}_x\colon T_xM \to \operatorname{End}(\Lambda W_x)$ where ΛW_x is build from isotropic vectors of T_xM is defined. If the vector fields $e_\alpha \in \mathfrak{X}(M)$ are everywhere orthonormal for the metric g, then we have the matricial equality

$$\tilde{\rho}_x\big((e_\alpha)_x\big)_{ij} = \tilde{\rho}(v_\alpha)_{ij} \tag{3.99}$$

where the left hand side describe the matrix component of a linear operator acting on ΛW_x while in the right hand side we have the matrix component of a linear operator acting on ΛW and v_α is a basis on $\mathbb{R}^n$ with respect to which the metric is the same as the metric g_x in the basis $(e_\alpha)_x$. Let $\hat{\psi}\colon P \to \Lambda W$ be an equivariant function; from definition (3.93)-121 of γ we have

$$\big(\gamma(e_\alpha\hat{\psi})\big)(\xi) = (a_\alpha\hat{\psi})(\xi)$$

where $a_\alpha(\xi) = \tilde{\rho}\Big(\tilde{e}_\alpha\big(\phi(\xi)\big)\Big)$. In this expression, $\tilde{e}_\alpha$ is the equivariant function associated with the vector field $e_\alpha \in \mathfrak{X}(M)$. It is defined in subsection 1.8.4 as

$$\begin{aligned} \tilde{e}_\alpha\colon\ \mathrm{SO}(M) &\to \mathbb{R}^m \\ b &\mapsto b^{-1}\big((e_\alpha)_{\pi(b)}\big). \end{aligned} \tag{3.100}$$

So we have $\hat{a}_\alpha\colon P \to \operatorname{End}(\Lambda W)$ defined by

$$\hat{a}_\alpha(\xi) = \tilde{\rho}\big(\varphi(\xi)^{-1}e_\alpha(x)\big)$$

with $x = \pi(\xi)$. Now if ξ is any element of $\pi^{-1}(x)$, we have

$$\big(\gamma(e_\alpha)\psi\big)(x) = (a_\alpha\psi)(X) = \big[\xi, \hat{a}_\alpha(\xi)\hat{\psi}(\xi)\big] = \big[\xi, \tilde{\rho}\big(\varphi(\xi)^{-1}e_\alpha(x)\big)\hat{\psi}(\xi)\big].$$

There exists a $g \in \mathrm{Spin}(p,q)$ such that $\varphi(\xi\cdot g) = \mathbb{1}$; taking this element and using equivariance of the latter expression,

$$\big(\gamma(e_\alpha)\psi\big)(x) = \big[\xi\cdot g, \tilde{\rho}\big(e_\alpha(x)\big)\hat{\psi}(\xi\cdot g)\big] = \big[\xi\cdot g, \gamma^\alpha\hat{\psi}(\xi)\big] = [\xi, \gamma^\alpha\hat{\psi}(\xi)]. \tag{3.101}$$

What we proved is that $\big(\gamma e_\alpha\psi\big)(x) = \gamma^\alpha\psi(x)$ is the sense that

$$\widehat{\gamma(e_\alpha)\psi} = \gamma^\alpha\hat{\psi}. \tag{3.102}$$

Hence the Dirac operator reads

$$(\mathcal{D}s)(x) = g_{\alpha\beta}\gamma^\alpha\big(\tilde{\nabla}_{e_\beta}s\big)(x)$$

in the sense that

$$\widehat{\mathcal{D}s} = g_{\alpha\beta}\gamma^\alpha\,\widehat{\tilde{\nabla}_{e_\beta}s}. \tag{3.103}$$

∎

An often more convenient way to write the Dirac operator is to consider an orthonormal basis (so that the metric g and the matrices γ are constant) and to consider the equivariant functions:

$$\widehat{\mathcal{D}\psi} = g_{\alpha\beta}\gamma^{\alpha}\widehat{\nabla_{e_\alpha}\psi}.$$

This formulation is typically used when one search for Dirac operator on Lie groups. In this case, we choose left invariant vector fields generated by an orthonormal basis of the Lie algebra. The resulting field of basis is everywhere Killing-orthonormal.

Acting on a function $f\colon M \to \mathbb{R}$, it is defined by $\mathcal{D}\colon C^\infty(M) \to C^\infty(M)$,

$$(\mathcal{D}f)(x) = g_x(e_\alpha, e_\beta)\gamma_x^\beta(e_{\alpha x}\cdot f). \tag{3.104}$$

With these definitions, one has

$$(\mathcal{D}(fs))(x) = (f\mathcal{D}s)(x) + (\mathcal{D}f)(x).$$

Indeed,

$$\begin{aligned}
(\mathcal{D}(fs))(x) &= g_{\alpha\beta}\gamma_x^\beta(\tilde{\nabla}_{e_\alpha}fs)(s)\\
&= g_{\alpha\beta}\Big((e_\alpha\cdot f)s(x) + f(x)(\tilde{\nabla}_{e_\alpha}s)(x)\Big)\\
&= f(x)(\mathcal{D}s)(x) + g_{\alpha\beta}\gamma_x^\beta(e_{\alpha x}\cdot f)\\
&= (f\mathcal{D}s)(x) + (\mathcal{D}f)(x).
\end{aligned} \tag{3.105}$$

With that definition, the Dirac operator becomes a derivation of the spinor bundle.

3.8 Example: Dirac operator on $\mathbb{R}^2$ with the euclidian metric

Since the frame bundle $B(M)$ is a principal bundle (see subsection 1.7.4), one can consider some associated bundles on it. We are now going to see that the one given by the definition representation $\rho\colon GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ on $\mathbb{R}^n$ is the tangent bundle. So we study $B(M)\times_\rho \mathbb{R}^n$. By choosing a basis on each point of M, we identify each T_xM to $\mathbb{R}^n$. An element of $B(M)\times\mathbb{R}^n$ is a pair (b,v) with $b = (\mathbf{b}_1,\ldots,\mathbf{b}_n)$ and $v = (v^1,\ldots,v^n)$. We can identify v to the element of T_xM given by $v = v^i\mathbf{b}_i$.

In order to build the associated bundle, we make the identifications

$$(b,v)\cdot g \sim (b\cdot g, g^{-1}v).$$

Here, by gv we mean the vector whose components are given by $(gv)^i = v^j {g_j}^i$. The tangent vector given by $(b\cdot g, g^{-1}v)$ is $(g^{-1}v)^i(b\cdot g)_i = v^j {(g^{-1})_j}^i {g_i}^k\mathbf{b}_k = v^k\mathbf{b}_k$ So the identification map $\psi\colon B(M)\times_\rho\mathbb{R}^n \to TM$ given by

$$\psi([b,v]) = v^i\mathbf{b}_i$$

is well defined.

The following step is to consider the following spin structure:

$$\mathrm{Spin}(2) \rightsquigarrow \mathbb{R}^2\times\mathrm{SO}(2) \xrightarrow{\varphi} \mathrm{SO}(\mathbb{R}^2) \leftsquigarrow \mathrm{SO}(2)$$

Problème et notes pour moi 5.
Spin(2) *is not* $U(1)$ *because* $U(1)$ *is* SO(2) *while* Spin *covers it two times.*

We have to define the two actions and φ. One of the main result of section 3.4.5 is that $\chi\colon \mathrm{Spin}(2) = U(1) \to \mathrm{SO}(2)$ is surjective. So, we can define the action of Spin(2) on P by

$$(x, b) \cdot s = (x, \chi(s)^{-1}b).$$

On the other hand, an element A in $\mathrm{SO}(\mathbb{R}^2)$ can be written as $A = \{ae_i\}_x$ where e_i is the canonical basis of $T_x\mathbb{R}^2$, and a is a matrix of SO(2). See subsection 1.7.4. For $g \in \mathrm{SO}(2)$, we define

$$A \cdot g = \{g^{-1}ae_i\}_x. \tag{3.106}$$

and $\varphi\colon \mathbb{R}^2 \times \mathrm{SO}(2) \to \mathrm{SO}(\mathbb{R}^2)$ by

$$\varphi(x, b) = \{be_i\}_x.$$

The following shows that these definitions give a spin structure:

$$\varphi((x, b) \cdot s) = \varphi(x, \chi(s)^{-1}b) = \{\chi(s)^{-1}be_i\}_x = \{be_i\}_x \cdot \chi(s) = \varphi(x, b) \cdot \chi(s). \tag{3.107}$$

3.8.1 Connection on $\mathrm{SO}(\mathbb{R}^2)$

We are searching for a torsion-free connection on the simplest metric space: the euclidian $\mathbb{R}^2$. Thus we will try the simplest choice of horizontal space: we want an horizontal vector to be tangent to a curve of the form $X(t) = \{be_i\}_{x(t)}$. For this reason, we want to define the connection 1-form by $\omega(X) = b'(0)$. For technical reasons which will soon be apparent, we will not exactly proceed in this manner. For $X(t) = \{be_i\}_{x(t)}$, we define

$$\omega(X) = -(b(t)b(0)^{-1})'(0). \tag{3.108}$$

We of course have $\omega(X) = 0$ if and only if $b'(0) = 0$: this choice of ω follows our first idea. In order for ω to be a connection form, we have to verify the two conditions of definition 1.47.

Proposition 3.47.
The 1-form defined by

$$\omega(X) = -(b(t)b(0)^{-1})'(0)$$

for $X = \dfrac{d}{dt}\{b(t)e_i\}_{x(t)}\Big|_{t=0}$ *is a connection 1-form.*

Proof. Let $A \in \mathrm{SO}(2)$. If $u = \{be_i\}_x$, equation (3.106)-124 gives:

$$A^*_u = \frac{d}{dt}\{e^{-tA}be_i\}_x\Big|_{t=0},$$

so that $\omega(A^*_u) = -(e^{-tA}bb^{-1})'(0) = A$. This checks the first condition. For the second, one remarks that the path in $\mathrm{SO}(\mathbb{R}^2)$ which defines the vector $R_{g*}X$ is $(R_{g*}X)(t) = \{g^{-1}b(t)e_i\}_x$. It follows that

$$\begin{aligned}
\omega(R_{g*}X) &= -(g^{-1}b(t)b(0)^{-1}g)'(0)\\
&= -\left(\mathbf{Ad}_{g^{-1}}(b(t)b(0)^{-1})\right)'(0)\\
&= -Ad_{g^{-1}}(b(t)b(0)^{-1})'(0)\\
&= Ad_{g^{-1}}\omega(X).
\end{aligned} \tag{3.109}$$

□

Proposition 3.48.
The covariant derivative induced on M by this connection is

$$\nabla_X Y = X(Y). \tag{3.110}$$

Proof. In this demonstration, we will use the equivariant functions defined in 1.8.4. In order to compute $(\nabla_X Y)_x$, we have to use the definition of theorem 1.53. We first have to compute the horizontal lift of X. It is no difficult to see that $\overline{X}_{\{be_i\}_x}$ is given by the path

$$\overline{X}(t) = \{be_i\}_{X(t)}$$

if the vector field X is given by the path $X(t)$ in M. Indeed, it is trivial that $\omega(\overline{X}) = 0$, and

$$d\pi_* \overline{X} = \frac{d}{dt}\pi\{be_i\}_{X(t)}\Big|_{t=0} = \frac{d}{dt}X(t)\Big|_{t=0} = X.$$

Now, we compute $(\overline{X}\hat{s})(b)$ for $b = \{Se_i\}_x$. We begin using the basic definitions and notations:

$$(\overline{X}\hat{s})(b) = \overline{X}_b\hat{s} = \frac{d}{dt}\hat{s}(\overline{X}_b(t))\Big|_{t=0} = \frac{d}{dt}\hat{s}(\{Se_i\}_{X(t)})\Big|_{t=0}.$$

We can rewrite it with $\hat{Y}$ instead of $\hat{s}$. By construction (see (1.103)-46), if $b = \{Se_i\}_x$, $\hat{Y}(b) = S^{-1}(Y_x)$. Thus

$$(\overline{X}\hat{Y})(b) = \frac{d}{dt}S^{-1}(Y_{X(t)})\Big|_{t=0},$$

where, if $\{\overline{1}_i\}$ is a basis of $\mathbb{R}^m$, then S is

$$\begin{aligned} S\colon \mathbb{R}^m &\to T_{X(t)}M \\ v^i\overline{1}_i &\mapsto S^i_j v^j(\partial_j)_{X(t)} \end{aligned} \tag{3.111}$$

So if we write $Y_x = Y^i(x)\partial_i$, we have

$$S^{-1}(Y_{X(t)}) = (S^{-1})^i_j Y^j(X(t))\overline{1}_i$$

and

$$\frac{d}{dt}S^{-1}(Y_{X(t)})\Big|_{t=0} = (S^{-1})^i_j \frac{d}{dt}Y^j(X(t))\Big|_{t=0} \overline{1}_i = (S^{-1})^i_j X(Y^j)\overline{1}_i.$$

Since b is an isomorphism, we can apply b on both side of $\hat{X}(b) = b^{-1}(X_x)$, and take $\nabla_X Y$ instead of X:

$$(\nabla_X Y)(x) = b\big((S^{-1})^i_j X(Y^j)\overline{1}_i\big) = S^k_i(S^{-1})^i_j X(Y^j)(\partial_k)_x = X(Y^j)(\partial_j)_x = X(Y)_x. \tag{3.112}$$

□

From this and definition 1.113, we immediately conclude that our connection is torsion-free. In a certain manner, one can say that our covariant derivative is the usual one.

3.8.2 Construction of γ

Now, we construct the map γ of subsection 3.7. The first step is to define $\hat{a}_X\colon P \to \operatorname{End}(\Lambda W)$ by

$$\hat{a}_X(p) = \tilde{\rho}(\hat{X}_{\varphi(p)}).$$

Here, ΛW is the completely isotropic subspace of $(\mathbb{R}^2)^{\mathbb{C}}$ with euclidian metric; thus we can use the result of section 3.4.5. In particular, we know the representation $\tilde{\rho}$.

To see it more explicitly, we need the expression of $\hat{X}$. It is given in subsection 1.8.4: if b is the basis $\{be_i\}_x$, $\hat{Y}(b) = b^{-1}(Y_x)$. As $\varphi(b,x) = \{be_i\}_x$, we have

$$\hat{a}_X(b,x) = \tilde{\rho}(b^{-1}(X_x)).$$

The subsection 1.8.2 explains how to explicitly get $\gamma(X)$ with the definition $\gamma(X) = a_X$. If ψ is a section of $\mathcal{S}$ and $\psi(x) = [\xi, v]$, the general definition gives us $(a_X\psi)(x) = [\xi, \hat{a}_X(\xi)v]$ and in our particular case, if $\xi = (b,x)$, we get:

$$(\gamma(X)\psi)(x) = [\xi, \tilde{\rho}(b^{-1}(X_x))v]. \tag{3.113}$$

3.8.3 Covariant derivative on $\Gamma(\mathcal{S})$

Remember the spin structure of $\mathrm{SO}(\mathbb{R}^2)$: $\varphi(x,S) = \{Se_i\}_x$. We now construct the connection on $P = \mathbb{R}^2 \times \mathrm{SO}(2)$. It is defined by the 1-form $\tilde{\omega} = \varphi^*\omega$. If v is a vector of P, it is described by a path $v(t) = (x(t), b(t))$, then the path of $d\varphi(v)$ is $\{b(t)e_i\}_{x(t)}$ and $\tilde{\omega}(v) = \omega(d\varphi(v)) = -(b(t)b(0)^{-1})'(0)$.

The next step defining the Dirac operator is to find out an explicit form for the map $\tilde{\nabla}\colon \mathfrak{X}(M) \times \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$. A section $s \in \Gamma(\mathcal{S})$ is a map $s\colon M \to \mathcal{S} = (\mathbb{R}^2 \times \mathrm{SO}(2)) \times_\rho \Lambda W$; it is defined by an equivariant function $\hat{s}\colon P \to \Lambda W$. In order to find the value of $(\tilde{\nabla}_X s)(x)$ for $X \in \mathfrak{X}(M)$, we use the definition

$$\widehat{\tilde{\nabla}_X s}(\xi) = \overline{X}_\xi(\hat{s})$$

where $\overline{X}$ is the horizontal lift in the sense of $\tilde{\omega}$. For the same reason as in the proof of proposition 3.48, $\overline{X}_{(b,x)}$ is given by the path $\overline{X}(t) = (b, X(t))$ where $X(t)$ is the path which defines X. So we have

$$\widehat{\tilde{\nabla}_X s}(\xi) = \overline{X}_{(b,x)}(\hat{s}) = \frac{d}{dt}\hat{s}(b, X(t))\Big|_{t=0}.$$

Remark that ΛW is a vector space; so for every $\alpha \in \Lambda W$, the identification $T_\alpha \Lambda W = \Lambda W$ is correct.

Our first form of $\tilde{\nabla}$ is

$$(\tilde{\nabla}_X s)(x) = \Big[\xi, \frac{d}{dt}\hat{s}(b, X(t))\Big|_{t=0}\Big],$$

but we can modify this in order to get simpler expressions. Remark that we have an equivalence class, so that we can always choose the element of the class such that $\xi = (\mathbb{1}, x)$. We define $\overline{s}\colon \mathbb{R}^2 \to \Lambda W$, $\overline{s}(v) = \hat{s}(\mathbb{1}, v)$. Our second and final form for $\tilde{\nabla}$ is:

$$(\tilde{\nabla}_X s)(x) = \Big[(\mathbb{1}, x), \frac{d}{dt}\overline{s}(X(t))\Big|_{t=0}\Big] \tag{3.114a}$$

$$= [(\mathbb{1}, x), X(\overline{s})], \tag{3.114b}$$

where $X(\overline{s})$ is well defined because $\overline{s}$ is a map from $\mathbb{R}^2$ into a vector space (namely: ΛW).

3.8.4 Dirac operator on the euclidian $\mathbb{R}^2$

We continue to write explicitly the definition (3.97)-121. Putting together (3.113)-126 and (3.114b)-126, one finds

$$\gamma_x^\alpha(\tilde{\nabla}_{e_\beta} s)(x) = \gamma(e_{\alpha x})[\xi, e_\beta(\overline{s})] = [\xi, \tilde{\rho}(b^{-1}(e_{\alpha x}))e_\beta(\overline{s})]. \tag{3.115}$$

Here, $e_\beta = \partial_\beta$ and $b = \mathbb{1}$, then

$$\gamma_x^\alpha(\tilde{\nabla}_{e_\beta} s)(x) = [(\mathbb{1}, x), \tilde{\rho}(e_\alpha)\partial_\beta\overline{s}].$$

Now, the Dirac operator reads

$$(\mathcal{D}s)(x) = [(\mathbb{1}, x), \gamma^\alpha\partial_\alpha\overline{s}].$$

We can obtain a more compact expression by defining "Ys" and "As" when $s \in \Gamma(\mathcal{S})$, $Y \in \mathfrak{X}(\mathbb{R}^2)$ and $A \in \operatorname{End}\Lambda W$. The definitions are

$$\begin{aligned}(Ys)(x) &= [(\mathbb{1}, x), (Y\overline{s})(x)],\\ (As)(x) &= [(\mathbb{1}, x), A\overline{s}(x)].\end{aligned}$$

With these conventions, one writes:

$$(\mathcal{D}s)(x) = \gamma^\alpha(\partial_\alpha s)(x).$$

This justifies the expression (3.3)-90: $\mathcal{D} = \gamma^\alpha\partial_\alpha$ on flat spaces. With a good choice of basis of ΛW, the matrices γ^α are given by (3.62)-111, and

$$\gamma^\alpha\partial_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\partial_x - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\partial_y.$$

If we identify $\mathbb{R}^2$ with $\mathbb{C}$ we have the following definitions:

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

so that

$$\mathcal{D} = \begin{pmatrix} 0 & -\partial_{\overline{z}} \\ \partial_z & 0 \end{pmatrix}.$$

Chapter 4

Relativistic field theory

4.1 Mathematical framework of field theory

This is a short review; the aim is to see why the quantum theory of fields needs representations of the Poincaré group. It will be mostly physics oriented. References dealing with field theory including gauge theory and representations are [8, 18, 19, 28, 28, 32–35].

4.1.1 Axioms of the (quantum) relativistic field theory mechanics

The quantum mechanics is based on a few number of axioms:

(i) We have a Hilbert space $\mathscr{H}$. A physical state is given by a **ray** in $\mathscr{H}$, i.e. a set

$$\mathcal{R} = \{\xi\psi : |\xi| = 1\}$$

for a certain $\psi \in \mathscr{H}$ with $\langle\psi|\psi\rangle = 1$. In other words, the set of physical sates is the quotient of the set of unital vectors in $\mathscr{H}$ by the relation $\psi \sim \psi'$ if and only if $\psi = \xi\psi'$ for some unimodular complex number ξ. We denote by $\operatorname{Ray}\mathscr{H}$ the set of all rays in $\mathscr{H}$.

(ii) The observables are represented by hermitian linear operators on $\mathscr{H}$. A state $\mathcal{R}$ has value α for the observable A if $A\mathcal{R} = \alpha\mathcal{R}$, where the action of A on the ray is obvious (and well defined because A is linear).

(iii) If one has a system described by a state $\mathcal{R}$, and if one want to measure if it is in one of the state $\mathcal{R}_1, \ldots, \mathcal{R}_n$ (orthogonal rays), the answer will be $\mathcal{R}_i$ with probability

$$P(\mathcal{R} \to \mathcal{R}_i) = |\langle\mathcal{R}|\mathcal{R}_i\rangle|^2.$$

If the $\mathcal{R}_n$ form a complete system, one has a theorem which states that

$$\sum_i P(\mathcal{R} \to \mathcal{R}_i) = 1.$$

(iv) The rays of $\mathscr{H}$ furnish a representation of the (identity component of) Poincaré group.

This last point can look strange; we will see later (page 139) how it comes. It is the expression of a relativistic theory. That axiom is the reason why one make intensive use of representation theory in relativistic (quantum) field theory . . . or maybe the intensive use of representation theory is the reason of that axiom.

4.1.2 Symmetries and Wigner's theorem

Consider the following situation: someone observes a system in a state $\mathcal{R}$, and makes measures $P(\mathcal{R} \to \mathcal{R}_i)$. An other person observes the same system which is, for him, in a state $\mathcal{R}'$ and observes $P(\mathcal{R}' \to \mathcal{R}'_i)$.

If two observers are related by a transformation of the Hilbert state which induces $\mathcal{R} \to \mathcal{R}'$ and $\mathcal{R}_i \to \mathcal{R}'_i$, there are said **equivalent** if

$$P(\mathcal{R} \to \mathcal{R}_i) = P(\mathcal{R}' \to \mathcal{R}'_i). \tag{4.1}$$

Let us say it more precisely from a mathematical point of view. A **symmetry** is an invertible operator $T\colon \operatorname{Ray}\mathscr{H} \to \operatorname{Ray}\mathscr{H}$ such that for any $\phi_i \in \mathcal{R}_i$, $\phi'_i \in T\mathcal{R}_i$ and $\phi''_i \in T^{-1}\mathcal{R}_i$,

$$|\langle\phi'_1|\phi'_2\rangle|^2 = |\langle\phi_1|\phi_2\rangle|^2 = |\langle\phi''_1|\phi''_2\rangle|^2 \tag{4.2}$$

Remark 4.1. Here, neither $\mathcal{R}$ nor $\mathcal{R}'$ are measurable: the P's only are measurable.

The following can be found in [28] p.91, [33] p.354.

Theorem 4.2 (Wigner)**.**
Any symmetry T is induced by an operator U on $\mathscr{H}$ such that $\psi \in \mathcal{R}$ implies $U\psi \in \mathcal{R}'$. This operator is either unitary and linear, either anti-unitary and antilinear.

So, the symmetry operator must satisfy

$$\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle \tag{4.3a}$$
$$U(\xi\psi + \eta\psi) = \xi U\psi + \eta U\phi, \tag{4.3b}$$

or

$$\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle^* \tag{4.4a}$$
$$U(\xi\psi + \eta\psi) = \xi^* U\psi + \eta^* U\phi. \tag{4.4b}$$

In the anti-linear case operator, we do not define $U^\dagger$ by $\langle\phi|U^\dagger\psi\rangle = \langle U\phi|\psi\rangle$ because the left-hand side should be anti-linear with respect to ψ while the right-had should be linear. In place, for an antilinear operator A, we define $A^\dagger$ by

$$\langle\phi|A^\dagger\psi\rangle = \langle A\phi|\psi\rangle^* = \langle\psi|A\phi\rangle. \tag{4.5}$$

In this way, the definitions of unitary and anti-unitary in term of dagger are the same: $U^\dagger = U^{-1}$.

For any transformation $T\colon \operatorname{Ray}\mathscr{H} \to \operatorname{Ray}\mathscr{H}$, the Wigner's theorem provides an operator $U(T)\colon \mathscr{H} \to \mathscr{H}$ which induces T on Ray. If the operator T depends on a parameter θ, the operator $U(T(\theta))$ depends on θ. If T depends continuously on the parameter then the family $U(T(\theta))$ only contains unitary/linear operators or only antiunitary/antilinear operators.

In physical cases, $T(\theta)$ is mostly a Poincaré transformation: $\theta = (\Lambda, p)$. But $T(\mathbb{1}, 0)$ is the identity which is represented by $U(\mathbb{1}, 0) = \mathbb{1}$. Then all the (connected to identity) Poincaré transformations are represented by linear and unitary operators on $\mathscr{H}$.

We will follow the proof given in [28]. An other form of the proof can be found in [33]. The latter use a slightly different formalism in the axioms of the quantum mechanics; this is explained in appendix .1. It is now time to prove the theorem.

Proof of Wigner's theorem. We consider an orthonormal basis $\{\psi_k\}$ of $\mathscr{H}$ with $\psi_k \in \mathcal{R}_k$, and a choice of $\psi'_k \in T\mathcal{R}_k$. From this and the assumptions, we have

$$|\langle\psi'_k|\psi'_l\rangle|^2 = |\langle\psi_k|\psi_l\rangle|^2 = \delta_{kl}.$$

Then $\langle\psi'_k|\psi'_k\rangle = 0$ whenever $k \neq l$ and, since $\langle\psi'_k|\psi'_k\rangle$ is real and positive, $\langle\psi'_k|\psi'_k\rangle = 1$. So $\langle\psi'_k|\psi'_l\rangle = \delta_{kl}$.

The set ψ'_k is also complete in $\mathscr{H}$. Indeed suppose that we have a vector $\psi' \in \mathscr{H}$ such that $\langle\psi'|\psi'_k\rangle = 0$ for all k. If $\psi' \in \mathcal{R}$, we consider a $\psi'' \in T^{-1}\mathcal{R}$ and we have

$$|\langle\psi''|\psi_k\rangle|^2 = |\langle\psi'|\psi'_k\rangle|^2 = 0,$$

which contradicts the fact that the ψ_k's form a complete set. Now we have to fix a phase convention for the ψ_k. Since there are no canonical choice of phase, we fix with respect to an arbitrary one of the ψ_k, say ψ_1. We put

$$\gamma_k = \frac{1}{\sqrt{2}}(\psi_1 + \psi_k) \in \mathcal{C}_k \tag{4.6}$$

for $k \neq 1$. Any $\gamma'_k \in T\mathcal{C}_k$ can be written in the basis $\{\psi'_k\}$:

$$\gamma'_k = \sum_l c_{kl}\psi'_l. \tag{4.7}$$

From assumption (4.1)-130 and the fact that $|c_{kl}|^2 = |\langle\gamma'_k|\psi'_l\rangle|^2$, we find, for $k, l \neq 1$

$$|c_{kl}|^2 = \frac{1}{2}\delta_{kl}.$$

We can choose the phase of γ'_k and ψ'_k in order to get $c_{kk} = c_{k1} = 1/\sqrt{2}$. For this, we begin to fix γ'_k in such a manner to get $c_{k1} = 1/\sqrt{2}$ (from $|c_{k1}| = |\langle\gamma'_k|\psi'_1\rangle|$), and next we fix ψ'_k for the c_{kk}. From now on, the so chosen γ'_k and ψ'_k are denoted by $U\gamma_k$ and $U\psi_k$.

What we did until now is to take a basis $\{\psi_k\}$ of $\mathscr{H}$ and define $\gamma_k = 1/\sqrt{2}(\psi_1 + \psi_k)$. Next we had chosen the phases of $\psi'_k \in T\mathcal{R}_k$ and $\gamma'_k \in T\mathcal{C}_k$ in order to have

$$\begin{aligned} c_{kk} = c_{k1} &= 1/\sqrt{2} \quad && \forall k, \\ c_{kl} &= 0 && \text{if } l \neq k \text{ and } l \neq 1. \end{aligned} \tag{4.8}$$

This allows us to check a certain linearity for the operator U:

$$\begin{aligned} U\left(\frac{1}{\sqrt{2}}(\psi_k + \psi 1)\right) &= U\gamma_k \\ &= \gamma'_k \\ &= \frac{1}{\sqrt{2}}\psi'_1 + \frac{1}{\sqrt{2}}\psi'_k \qquad \text{from (4.7)-131 and (4.8)-131} \\ &= \frac{1}{\sqrt{2}}\big(U\psi_1 + U\psi_k\big). \end{aligned} \tag{4.9}$$

Now we have to build U on a general vector $\psi = \sum_k \psi_k \in \mathcal{R}$. Any vector $\psi' \in T\mathcal{R}$ can be decomposed with respect to the basis $\{\psi'_k = U\psi_k\}$:

$$\psi' = \sum_k C'_k U\psi_k. \tag{4.10}$$

From the conservation of probability $|\langle\psi_k|\psi\rangle|^2 = |\langle U\psi_k|\psi'\rangle|^2$ and $|\langle\gamma_k|\psi\rangle|^2 = |\langle U\gamma_k|\psi'\rangle|^2$, we find

$$|C_k|^2 = |C'_k|^2, \tag{4.11a}$$
$$|C_k + C_1|^2 = |C'_k + C'_1|^2. \tag{4.11b}$$

If one writes $C_k = a_k + ib_k$, one finds $\mathrm{Re}(C_k/C_1) = (a_k a_1 + b_k b_1)/|C_1|^2$. By doing the same with C'_k and using (4.11)-132,

$$\mathrm{Re}(C_k/C_1) = \mathrm{Re}(C'_k/C'_1). \tag{4.12}$$

Equation (4.11a)-132 also imposes

$$|C_k/C_1|^2 = |C'_k/C'_1|^2, \tag{4.13}$$

while compatibility between (4.13)-132 and (4.12)-132 requires

$$\mathrm{Im}(C_k/C_1) = \pm\,\mathrm{Im}(C'_k/C'_1). \tag{4.14}$$

Equations (4.12)-132 and (4.14)-132 show that C_k and C'_k must satisfy

$$C_k/C_1 = C'_k/C'_1 \tag{4.15a}$$

xor

$$C_k/C_1 = (C'_k/C'_1)^*. \tag{4.15b}$$

For a given ψ we have to show that the choice must be the same for all the C_k[1]. Let $l \neq k$ and suppose that $C_k/C_1 = C'_k/C'_1$ and $C_l/C_1 = (C'_l/C'_1)^*$; we will show that in this case, one of the two ratios is real. So we can suppose $k \neq 1 \neq l$. We consider the vector $\Phi = \frac{1}{\sqrt{3}}(\psi_1 + \psi_k + \psi_l)$,

$$\Phi' = \frac{\alpha}{\sqrt{3}}\big(U\psi_1 + U\psi_k + U\psi_l\big)$$

where $\alpha \in \mathbb{C}$ satisfies $|\alpha| = 1$. The conservation of probability $|\langle\Phi|\psi\rangle|^2 = |\langle\Phi'|\psi'\rangle|^2$ gives $|C_1 + C_k + C_l|^2 = |C'_1 + C'_k + C'_l|^2$. Since $|C_1|^2 = |C'_1|^2$, we can divide the left hand side by $|C_1|^2$ and the right one by $|C'_1|^2$. We find

$$\left|1 + \frac{C_k}{C_1} + \frac{C_l}{C_1}\right|^2 = \left|1 + \frac{C'_k}{C'_1} + \frac{C'_l}{C'_1}\right|^2.$$

Using the assumption $C_k/C_1 = C'_k/C'_1$ and $C_l/C_1 = (C'_l/C'_1)^*$, we are in a case of an equation of the form $|u+v|^2 = |u+v^*|^2$ with $u, v \in \mathbb{C}$. If we write $u = a+bi$ and $v = x+iy$, we find $b + y = \pm(b-y)$, so that it leaves the choice $y = 0$ or $b = 0$ which corresponds to $(C_k/C_1) \in \mathbb{R}$ or $(C_l/C_1) \in \mathbb{R}$. So the coefficients C'_k $(k \neq 1)$ in the expansion (4.10)-131 must satisfy

$$C_k/C_1 = C'_k/C'_1 \quad \forall k \tag{4.16a}$$

xor

$$C_k/C_1 = (C'_k/C'_1)^* \quad \forall k. \tag{4.16b}$$

[1]We will show later that for a given T, the choice must be the same for all the ψ.

Note that the phase of C_1 is not yet fixed. We naturally choice $C_1 = C_1'$ or $C_1 = {C_1'}^*$ following the case. We define $U\colon \mathscr{H} \to \mathscr{H}$ by

$$U\left(\sum_k C_k\psi_k\right) = \sum_k C_k U\psi_k \qquad \text{if (4.16a)-132,} \tag{4.17a}$$

xor

$$U\left(\sum_k C_k\psi_k\right) = \sum_k C_k^* U\psi_k \qquad \text{if (4.16b)-132.} \tag{4.17b}$$

One can explicitly check that it preserves the probability because $|\langle\psi|\psi_k\rangle|^2 = |C_k|^2$ while $|\langle U\psi|U\psi_k\rangle|$ is equal to $|C_k|^2$ or $|C_k^*|^2$ (which are the same) following the case (4.17a)-133 or (4.17b)-133.

Now we have to prove that the choice (4.17a)-133 or (4.17b)-133 is fixed by the data of T and must be the same for all the $\psi \in \mathscr{H}$. For, let us consider two vectors $\phi = \sum A_k\psi_k$, $\varphi = \sum B_k\psi_k$ and suppose that

$$U\phi = \sum_k A_k U\psi_k \text{ but } U\varphi = \sum_k B_k^* U\psi_k.$$

In order to see that it is impossible, looks at the conservation of probability $|\sum_k A_k B_k^*|^2 = |\sum_k A_k B_k|^2$, then

$$\sum_{kl}\left(B_l^* B_k A_l A_k^* - B_l^* B_k A_l^* A_k\right) = \sum_{kl} B_l^* B_k \operatorname{Im}(A_l A_k^*) = 0. \tag{4.18}$$

Since $A_l A^* l \in \mathbb{R}$, we can regroup each term (k, l) with the corresponding term (l, k). We get

$$0 = \sum_{kl} \operatorname{Im}(A_l A_k^*)(B_l^* B_k - B_k^* B_l) = \sum_{kl} \operatorname{Im}(A_k^* A_l) \operatorname{Im}(B_k^* B_l). \tag{4.19}$$

We can find a vector $\sum_k C_k\psi_k$ such that

$$\sum_{kl} \operatorname{Im}(C_k^* C_l) \operatorname{Im}(A_k^* A_l) \neq 0 \tag{4.20a}$$

and

$$\sum_{kl} \operatorname{Im}(C_k^* C_l) \operatorname{Im}(B_k^* B_l) \neq 0. \tag{4.20b}$$

In order to see how to find such a vector, let us show that there always exists a choice (i, j) such that $B_i^* B_j$ is not real. Let us say $B_1 = x + iy$ and $B_k = a_k + bi$. If $y \neq 0$, the condition $\operatorname{Im}(B_1^* B_k) = 0$ gives $B_k = \frac{b_k}{y} B_1$. It is always possible to find a sequence (b_k) which gives 1 as norm for $\sum B_k\psi_k$; the problem is not there. The problem is that $B_k/B_1 \in \mathbb{R}$, so that the choice (4.17)-133 is not a true choice. For the same reason, all the $B_i^* B_k$ can't be pure imaginary.

Now we can find the vector which satisfy (4.20)-133. There are several cases. If there is a pair (k, l) such that $A_k^* A_l$ and $B_k^* B_l$ are both complex, we can take all C_i's zero for $k \neq i \neq l$ and choose C_k and C_l in such a way that $C_k^* C_l$ is not real. If there is a pair (k, l) with $A_k^* A_l$ complex and $B_k^* B_l$ real, we consider a pair (m, n) such that $B_m^* B_n$ is complex. If $A_m^* A_n$ is complex, we take all the C_i's zero except C_m and C_n such that $\operatorname{Im}(C_m^* C_n) \neq 0$. If $A_m^* A_n$ is real, we take all the C_i's zero except C_k, C_l, C_m, C_n which we choose in such a way that $\mathrm{Im}(C_m^* C_n) \neq 0$ and $\mathrm{Im}(C_k^* C_k) \neq 0$.

Equation (4.20a)-133 makes that the same choice must be made for $\sum A_k\psi_k$ and $\sum C_k\psi_k$ (if it was not the case, we would have an equation of the form of (4.19)-133). For the same reason, the same choice must be made for $\sum B_k\psi_k$ and $\sum C_k\psi_k$. So we conclude that the data of T fixes the choice between (4.17a)-133 and (4.17b)-133 and that this choice must be the same for all the vectors of $\mathscr{H}$.

We have to show that the possibility (4.17a)-133 makes U linear and unitary while the possibility (4.17b)-133 makes U antilinear and antiunitary. For we consider $\psi = \sum_k A_k\psi_k$ and $\phi = \sum_k B_k\psi_k$. If (4.17a)-133 works,

$$\begin{aligned} U(\alpha\psi + \beta\phi) &= U\Big(\sum_k (\alpha A_k + \beta B_k)\psi_k\Big) \\ &= \sum_k (\alpha A_k + \beta B_k) U\psi_k \\ &= \alpha U\psi + \beta U\phi, \end{aligned} \tag{4.21}$$

and

$$\langle U\psi|U\phi\rangle = \sum_{kl} A_k^* B_l \langle U\psi_k|U\psi_l\rangle = \sum_k A_k^* B_k, \tag{4.22}$$

so that $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$. Thus in this case U is linear and unitary. In the case where (4.17a)-133 works, the computations are almost the same:

$$\begin{aligned} U(\alpha\psi + \beta\phi) &= U\Big(\sum_k (\alpha A_k + \beta B_k)\psi_k\Big) \\ &= \sum_k (\alpha^* A_k^* + \beta^* B_k^*) U\psi_k \\ &= \alpha^* U\psi + \beta^* U\phi, \end{aligned} \tag{4.23}$$

and

$$\langle U\psi|U\phi\rangle = \sum_{kl} A_k B_l^* \langle U\psi_k|U\psi_l\rangle = \sum_k A_k B_k^*, \tag{4.24}$$

so that $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle^*$. In this case, U is antilinear and antiunitary.

□

4.1.3 Projective representations

If $T_1(\mathcal{R}_n) = \mathcal{R}'_n$ and $\psi_n \in \mathcal{R}_n$, then $U(T_1)\psi_n \in \mathcal{R}'_n$. If $T_2(\mathcal{R}') = \mathcal{R}''$, then $U(T_2)U(T_1)\psi_n \in \mathcal{R}''_n$. But $U(T_2T_1)\psi_n$ also belongs to $\mathcal{R}''_n$. Then there exists a $\phi_n(T_2, T_1) \in \mathbb{R}$ such that

$$U(T_2)U(T_1)\psi_n = e^{i\phi_n(T_2,T_1)} U(T_2T_1)\psi_n.$$

Note that for any $\psi \in \mathscr{H}$, there exists a $\lambda \in \mathbb{R}$ such that $\|\lambda\psi\| = 1$. Since a real can be sent out the $U(T)$'s, for *any* $\psi \in \mathscr{H}$, there exists a ϕ which only depends on $\psi/\|\psi\|$ such that

$$U(T_2)U(T_1)\psi = e^{i\phi(T_2,T_1)} U(T_2T_1)\psi \tag{4.25}$$

Proposition 4.3.
The ϕ doesn't depend at all on the ψ:

$$U(T_2)U(T_1) = e^{i\phi(T_2,T_1)} U(T_2T_1). \tag{4.26}$$

Proof. Let us consider a ψ_A and a ψ_B which are not proportional each other. One has a $\phi_{AB}(T_2, T_1)$ such that

$$\begin{aligned} e^{i\phi_{AB}(T_2,T_1)}U(T_2T_1)(\psi_A+\psi_B) &= U(T_2)U(T_1)(\psi_A+\psi_B) \\ &= e^{i\phi_A(T_2,T_1)}U(T_2T_1)\psi_A \\ &\quad + e^{i\phi_B(T_2,T_1)}U(T_2T_1)\psi_B. \end{aligned} \tag{4.27}$$

Now, we apply $U(T_2T_1)^{-1}$ to both sides. If it is unitary, the $e^{i\phi}$ get out without problems; else is get out as $e^{-i\phi}$:

$$e^{\pm i\phi_{AB}}(\psi_A+\psi_B) = e^{\pm i\phi_A}\psi_A + e^{\pm i\phi_B}\psi_B. \tag{4.28}$$

Since ψ_A and ψ_B are linearly independent, the only solution is $e^{i\phi_{AB}} = e^{i\phi_A} = e^{i\phi_B}$.

□

Since the operators $U(T)$ must only fulfil

$$U(T_2)U(T_1) = e^{i\phi(T_2,T_1)}U(T_2,T_1), \tag{4.29}$$

these form a **projective representation** of the symmetry group on the physical Hilbert space $\mathscr{H}$.

Remark 4.4. In order to have some physical relevance, this demonstration supposes that a state $\psi_A+\psi_B$ exists in nature. If one can divide the particles in several "incompatibles" classes labeled by a, b such that $\psi_a+\psi_b$ doesn't exist, then equation (4.29)-135 is false and one has to write

$$U(T_2)U(T_1)\psi_a = e^{i\phi_a(T_2,T_1)}U(T_2T_1)\psi_a$$

because we can't show that $\phi_a = \phi_b$ from the simple fact that $\psi_a+\psi_b$ doesn't exist !

For example, physicists think that there are no superposition of state of integer and semi-integer spin.

Remark 4.5. If the group satisfies some requirements, one can choose $\phi = 0$. From now we suppose that we are in this case: we work with "true" representations.

4.1.4 Representations and power expansions

Let G be an arc connected Lie group whose elements are denoted by $T(\theta)$ with θ, a continuous family of parameters (from a local chart). The multiplication law is given by a function $f\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$:

$$T(\theta')T(\theta) = T\big(f(\theta',\theta)\big). \tag{4.30}$$

If $\theta = 0$ is the coordinate of the identity,

$$f(0,\theta) = f(\theta,0) = \theta. \tag{4.31}$$

We suppose that G acts on the rays of a Hilbert space $\mathscr{H}$, so that there are represented on $\mathscr{H}$ by unitary operators $U\big(T(\theta)\big)$. We denote by W the group of transformations of $\mathscr{H}$; roughly speaking,

$$W = U(G).$$

Now, we are going to cheat a little. We know that there exists a normal neighbourhood of e in W. In simple words, the map $\exp\colon \mathcal{W}\to W$ is a diffeomorphism between the elements of $\mathcal{W}$

"close" to 0 and the ones of W close to e. By *close to*, we mean that the components of θ are small enough. If $\{it_a\}$ is a basis of $\mathcal{W}$, we define

$$U(T(\theta)) = e^{i\theta^a t_a}. \tag{4.32}$$

In other words, one considers the exponential map for a neighbourhood of identity.

The cheat is the fact that $U(T(\theta))$ is actually defined by Wigner's theorem from the data of the group G. So equation (4.32)-136 should be seen as a requirement in the choice of the basis $\{t_a\}$.

Remark 4.6. The i in the exponential in (4.32)-136 and in the definition of the basis $\{it_a\}$ is a convention in order the t_a's to be hermitian. Indeed, the Lie algebra of a group of unitary matrices is made of *anti*hermitian matrices.

With all that,

$$U(T(\theta)) = \mathbb{1} + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \ldots \tag{4.33}$$

where t_{bc} is defined (among other requirements) to absorb the "intuitive" minus sign in the third term.

Now we are going to explore some consequences of equation (4.30)-135. Equation (4.31)-135 makes the expansion of f as

$$f^a(\theta', \theta) = \theta^a + {\theta'}^a + f^a_{bc}{\theta'}^b\theta^c + \ldots \tag{4.34}$$

From expansions (4.34)-136 and (4.33)-136 of f and $U(T(\theta))$, "group structure" equation (4.30)-135 gives (at order two):

$$t_{bc} = -t_b t_c - i f^a_{bc} t_a \tag{4.35}$$

and nothing for the first order. Then, providing that one knows the group structure (the f), one knows the second order of the representation from the first one. From equation (4.32)-136, one finds the value of t_{ab}:

$$e^{i\theta^a t_a} = 1 + i\theta^a t_a + \frac{1}{2}(i)^2(\theta^a t_a)(\theta^b t_b),$$

up to constant coefficients, one can choose t_{ab} to be symmetric with respect to a and b:

$$t_{ab} = \frac{1}{2}(t_a t_b + t_b t_a).$$

Taking this convention and computing $t_{bc} - t_{cb}$ from (4.35)-136, we find

$$[t_a, t_b] = iC^c_{ab} t_c \tag{4.36}$$

with $C^c_{ab} = f^c_{ab} - f^c_{ba}$.

On the other hand, one knows that if a group is abelian, its algebra is also abelian; we can see it here by considering that if G is abelian, $f(\theta,\theta') = f(\theta',\theta)$, then f^c_{ab} is symmetric and $[t_a, t_b] = 0$. We can say more about f Since the t_a commute, equations (4.30)-135 and (4.32)-136 make that

$$e^{if(\theta,\theta')^a t_a} = e^{i\theta^a t_a} e^{i{\theta'}^b t_b} = e^{i(\theta^a + {\theta'}^a)t_a}, \tag{4.37}$$

so that

$$f(\theta,\theta') = \theta + \theta'.$$

4.2 The symmetry group of nature

4.2.1 Spin and double covering

Some of literature carry an ambiguity in the choice of the right space-time symmetry group in the quantum field theory. A very good and deep discussion about the choice of the space-time symmetry group of nature is given in the book [8] which will be used here. An other enlightening review can be found in [36].

From a relativistic point of view, the group is the Poincaré group of all the maps $\mathbb{R}^4 \to \mathbb{R}^4$ which leaves invariant the quantity $s^2 = -t^2 + x^2 + y^2 + z^2$. At this point we can already make an important remark: the so defined quantity s is in fact *not* a relativistic invariant. Indeed if I follow a (spatially) closed path, I will measure $\Delta t \neq 0$ and $\Delta x = \Delta y = \Delta z = 0$ because in *my* frame, my displacement is zero. A guy who keeps at my starting point will measure (between the beginning and the end of my travel) $\Delta' t \neq 0$ and also $\Delta x = \Delta y = \Delta z = 0$. If $s = s'$, then $\Delta t = \Delta t'$.

So the relativistic invariance is only local: $ds^2 = ds'^2$, and as far as relativity is concerned, one can work with infinitesimal transformations only. In this case, the distinction between the *groups* $L^{\uparrow}_{+}$ and $\mathrm{SL}(2,\mathbb{C})$ is no relevant. Intuitively, we choose $L^{\uparrow}_{+}$ to be the space-time symmetry group. As we will see the difference will reveal to be crucial in relativistic field theory because $L^{\uparrow}_{+}$ has no half-integer spin representations.

This group naturally splits into two parts: the translations and the rotations (and boost). As far as I know, the translation part makes no difficulties. For the other one, there are some difficulties to find the *minimal* group of symmetry. First, one often want to separate the space-time inversions P and T from the remaining: the group then becomes the homogeneous orthochrone Lorentz group $L^{\uparrow}_{+}$. An other often presented group is $\ldots\mathrm{SL}(2,\mathbb{C})$. This is our choice here. The physical reason of this choice is all but immediate. As we will see during the following pages, an elementary particle is an irreducible representation of the symmetry group.

For massive particles, the relevant subgroup of $\mathrm{SL}(2,\mathbb{C})$ reveals to be $SU(2)$. If we had chosen the most intuitive $L^{\uparrow}_{+}$, we would have found $\mathrm{SO}(3)$. There is an important difference between $SU(2)$ and $\mathrm{SO}(3) = SU(2)/\mathbb{Z}_2$: the first one admits representations of any integer and half-integer spin while the second only posses the integer spin representations (cf. page 85).

Let us now be more precise about the relation between $L^{\uparrow}_{+}$ and $\mathrm{SL}(2,\mathbb{C})$. A know result is

$$L^{\uparrow}_{+} = \frac{\mathrm{SL}(2,\mathbb{C})}{\mathbb{Z}_2}.$$

Let $\mathrm{Spin}\colon \mathrm{SL}(2,\mathbb{C}) \to L^{\uparrow}_{+}$ be the surjective homomorphism with kernel $\pm\mathbb{1}_{2\times 2}$ giving this relation. We will not give a complete proof, but we will explain how $\mathrm{SL}(2,\mathbb{C})$ acts by isometries on $\mathbb{R}^4$. First, we remark that there exists a bijection between $\mathbb{R}^4$ and the 2×2 complex hermitian matrices:

$$v = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \tag{4.38}$$

If $\lambda \in \mathrm{SL}(2,\mathbb{C})$, the matrix $\lambda v \lambda^{\dagger}$ is also hermitian and $\|v\|^2 = \det v$. Thus

$$\begin{aligned} \Lambda(\lambda)\colon \mathbb{R}^4 &\to \mathbb{R}^4 \\ v &\mapsto \lambda v \lambda^{\dagger} \end{aligned} \tag{4.39}$$

is a Lorentz transformation if and only if $|\det\lambda| = 1$. Moreover,

$$\Lambda(\lambda\lambda') = \Lambda(\lambda)\Lambda(\lambda').$$

If $\lambda' = e^{\phi}\lambda$, then $\Lambda(\lambda') = \Lambda(\lambda)$, thus it is natural to impose $\det v = 1$ and to consider $\mathrm{SL}(2,\mathbb{C})$ instead of $L(2,\mathbb{C})$ to fit $L_+^{\uparrow}$. Now, $\Lambda(\lambda) = \Lambda(-\lambda)$, and we wish to consider $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

I think the problem is the following: as far as the action of the "nature group" on the space-time is concerned, it is sufficient to consider $L_+^{\uparrow}$. But the group which acts on the state space is wider: it must be $SL(2,\mathbb{C})$.

From now, when we say "Poincaré group", we mean $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{R}^4$ while "Lorentz" means $\mathrm{SL}(2,\mathbb{C})$ acting on $\mathbb{R}^4$ by $\Lambda(\lambda)v = \lambda v\lambda^{\dagger}$.

Let us continue the discussion of page 84. A know result is the fact that the map Spin restricts to a surjective homomorphism $\mathrm{Spin}\colon SU(2) \to \mathrm{SO}(3)$ with kernel $\pm\mathbb{1}$ giving the relation $\mathrm{SO}(3) = SU(2)/\mathbb{Z}_2$. If one considers a representation $\rho\colon \mathrm{SO}(3) \to GL(V)$, then $\tilde{\rho} = \rho \circ \mathrm{Spin}$ is a representation of $SU(2)$ on V. So every representation of SO(3) comes from a representation of $SU(2)$.

As far as the transformation rule of a (quantum mechanical) wave function under a rotation $R \in \mathrm{SO}(3)$ is concerned, one can see (it is done in [8]) that the try

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to T(R)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

doesn't works if $T(R)$ is a representation of SO(3) on $\mathbb{C}^2$. If one allows T to be a representation of $SU(2)$, then our choice —for an electron— should naturally be the spin one half representation $T = D^{(1/2)}$. Let us do it. The remaining problem is the following. Let's consider that in a certain frame, an electron is described by the wave function $\begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}$, the question is to know the wave function observed by a guy which use another frame linked to the first frame by $R \in \mathrm{SO}(3)$. We always have exactly two elements in SU(2) projected to R by Spin; namely $\mathrm{Spin}(\pm g) = R$; so how to choose between

$$D^{(1/2)}(g)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad\text{and}\quad D^{(1/2)}(-g)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\ ?$$

The trick is to remark that a change of frame is not the mathematical process described by a single element R of SO(3), but a physical *continuous* process which begins at the identity and stops at R. In other word, we have to ask ourself *how to go from a frame to another* ? Taking as example the rotations around the x axis, we can look at two different path in SO(3) from $\mathbb{1}$ to $\mathbb{1}$ given by the same expression

$$R_1(t) = R_2(t) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos t & \sin t\\ 0 & -\sin t & \cos t\end{pmatrix},$$

but considering $t\colon 0 \to 2\pi$ for R_1 and $t\colon 0 \to 4\pi$ for R_2. The covering map $\mathrm{Spin}\colon SU(2) \to \mathrm{SO}(3)$ allows us to lift any path in SO(3) to a path in $SU(2)$ in an unique way providing a starting point. In other words, if $\mathrm{Spin}(g) = R$,

$$\exists!\, \tilde{R}(t) \in SU(2) \text{ such that } \mathrm{Spin}\circ\tilde{R} = R \text{ and } \tilde{R}(0) = \mathbb{1},$$
$$\exists!\, \tilde{R}(t) \in SU(2) \text{ such that } \mathrm{Spin}\circ\tilde{R} = R \text{ and } \tilde{R}(0) = -\mathbb{1}.$$

The question is now: how to choose the right path among these two ? The answer comes from the homotopy of SO(3): the path R_1 and R_2 belongs to two different classes.

Considering the "change of frame" as a continuous process, the initial point is naturally chosen to be $\mathbb{1}$. With this choice, the lift of R_1 and R_2 are given by

$$g_1(t) = g_2(t) = \begin{pmatrix} \cos\frac{t}{2} & -i\sin\frac{t}{2} \\ -i\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}$$

with $t\colon 0 \to 2\pi$ for g_1 and $t\colon 0 \to 4\pi$ for g_2. In $SU(2)$, the ending point of g_1 is $-\mathbb{1}$ while the one of g_2 is $\mathbb{1}$.

It is still possible to say a lot of interesting thinks about the space-time symmetry group of nature; let's just conclude saying that $SU(2)$ is more adapted to the rotations of non zero spin than SO(3). (it is not intuitive !)

4.2.2 How to implement the Poincaré group

We are not making physics here, but differential geometry and group theory; so we will not discuss the physical relevance of the Poincaré group from a "speed of light" point of view. We consider the **Poincaré group** as the group of all the affine isometries of metric $\eta = diag(-1,1,1,1)$ and the **Lorentz group** as the subgroup of rotations and boost.

A Poincaré transformation of $\mathbb{R}^4$ is given by (Λ, a) with Λ a 4×4 matrix and $a \in \mathbb{R}^4$, a translation vector. The composition of (Λ, a) with (Λ', a') is given by $(\Lambda'\Lambda, \Lambda' a + a')$, the inverse is $(\Lambda^{-1}, -\Lambda^{-1}a)$, the neutral is $(\mathbb{1}, 0)$, and $(\det\Lambda)^2 = 1$.

The axiom (iv) at page 129 gives us a group of transformation of the rays in $\mathscr{H}$ parametrised by (Λ, a) such that

$$T(\Lambda', a')T(\Lambda, a) = T(\Lambda'\Lambda, \Lambda' a + a'), \tag{4.40}$$

$T(\Lambda, a)\colon \operatorname{Ray}\mathscr{H} \to \operatorname{Ray}\mathscr{H}$. Then Wigner's theorem defines a representation of the Poincaré group on $\mathscr{H}$ by unitary matrices :

$$\psi \to U(\Lambda, a)\psi.$$

Remark 4.7. Wigner only ensure existence of *projective* representations. Here we suppose that our symmetry group (maybe slightly different that Poincaré) is such that any projective representations can be turn into a classic representation. We will therefore use the composition law

$$U(\Lambda', a')U(\Lambda) = U(\Lambda'\Lambda, \Lambda' a + a') \tag{4.41}$$

instead of $U(\Lambda', a')U(\Lambda, a') = e^{i\phi(\Lambda, a, \Lambda', a')}U(\Lambda'\Lambda, \Lambda' a + a')$.

By axiom, the (connected) Poincaré group acts on rays of $\mathscr{H}$, and we have the representation U which form a group acting on $\mathscr{H}$. The Lie algebra acts also :

$$u\psi = \frac{d}{dt}\Big[U(t)\Big]_{t=0}\psi := \frac{d}{dt}\Big[U(t)\psi\Big]_{t=0}. \tag{4.42}$$

This definition is natural because $\mathscr{H}$ is a vector space: it can be identified with its tangent space: $U(t)\psi$ is a path in $\mathscr{H}$ and its derivative at $t = 0$ is still a well defined element in $\mathscr{H}$. Now recall that the operators U are unitary, so that the corresponding operators u are hermitian (therefore diagonalisable).

Let us consider an abelian subgroup A of Poincaré with Lie algebra $\mathfrak{a}$. One can find a basis of $\mathscr{H}$ made of common eigenvectors of a basis of $\mathfrak{a}$. In other words, one can find a basis of $\mathscr{H}$ which simultaneously diagonalises all $\mathfrak{a}$. If $\{a_i\}$ is a basis of $\mathfrak{a}$, one can find a basis $\{|\psi_\lambda\rangle\}$ (here λ labels a basis of $\mathscr{H}$: it might take continuous values) such that

$$a_i|\psi_\lambda\rangle = \lambda_i|\psi_\lambda\rangle. \tag{4.43}$$

4.2.3 Momentum operator

Of course, there exists an abelian subgroup of Poincaré: the pure translations, $A = \{U(\mathbb{1}, a)\}$. A basis of the Lie algebra is given by four vectors labeled as P^μ and defined by

$$P^\mu = \frac{d}{dt}\Big[U(\mathbb{1}, te^\mu)\Big]_{t=0}$$

where e^μ is the unit vector following the direction μ (for $\mu = 0$, $e^0 = (1,0,0,0)$). One can consider a basis which diagonalises the P^μ's:

$$P^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle \tag{4.44}$$

where by definition,

$$P^\mu|p,\sigma\rangle = \frac{d}{dt}\Big[U(\mathbb{1}, te^\mu)|p,\sigma\rangle\Big]_{t=0}. \tag{4.45}$$

Remark 4.8. Be careful on a point: we don't say anything about the symbol "p" in the ket. The only property is that it labels a Hilbert space $\mathscr{H}$. But nothing is already imposed to $\mathscr{H}$: it must just carry a representation of the Poincaré group on its rays. In particular, it is *a priori* false to say that p is a "momentum 4-vector" and that p^μ is a component of p. Naturally, our notations are adapted to think that ! Maybe it is a pedagogical mistake; I don't know.

This remark can be disturbing: why is generally $|p,\sigma\rangle$ called "a state of momentum p" ? Since $U(\mathbb{1}, a)$ is unitary, P^μ is hermitian; the p^μ are eigenvalues for an hermitian operator, so by axiom (ii) (page 129) they are candidate to be physical values. But equation (4.45)-140 shows that P^μ is what a physicist should call an "infinitesimal translation", so that Noether suggests us to interpret the eigenvalue as momentum. We are safe !

The parameters σ are not yet defined neither. It will come later. For the moment, we include into the definition of a **one particle state** that σ takes discrete values.

Since $U(\mathbb{1}, a) = e^{a_\mu P^\mu}$,

$$U(\mathbb{1}, a)|p,\sigma\rangle = e^{ia_\mu p^\mu}|p,\sigma\rangle.$$

Now we are interested in the determination of $U(\Lambda, a)|p,\sigma\rangle$.

Proposition 4.9.
The operators P^μ are subject to the "transformation law"

$$U(\Lambda, a)P^\mu U(\Lambda, a)^{-1} = \Lambda^\mu_\nu P^\nu. \tag{4.46}$$

Proof. Since operators $U(\Lambda, a)$ are linear, they can be putted in the derivative which defines P^μ. Using the composition law (4.41)-139 we find :

$$\begin{aligned} U(\Lambda, a)P^\mu U(\Lambda, a)^{-1} &= \frac{d}{dt}\Big[U(\Lambda, a)U(\mathbb{1}, te^\mu)U(\Lambda, a)^{-1}\Big]_{t=0} \\ &= \frac{d}{dt}\Big[U(\mathbb{1}, t\Lambda e^\mu)\Big]_{t=0}. \end{aligned} \tag{4.47}$$

The Λ can be putted out of derivative; let us see it for a sum of two terms (here it is four) :

$$\begin{aligned} \frac{d}{dt}\Big[U(\mathbb{1}, t(e^\mu + e^\nu))\Big]_{t=0} &= \frac{d}{dt}\Big[U(\mathbb{1}, te^\mu)U(\mathbb{1}, te^\nu)\Big]_{t=0} \\ &= \frac{d}{dt}\Big[U(\mathbb{1}, te^\mu)U(\mathbb{1}, 0)\Big]_{t=0} + \frac{d}{dt}\Big[U(\mathbb{1}, 0)U(\mathbb{1}, te^\nu)\Big]_{t=0} \\ &= P^\mu + P^\nu. \end{aligned} \tag{4.48}$$

Thus
$$\frac{d}{dt}\Big[U(\mathbb{1},\Lambda^{\mu}_{\nu}e^{\nu})\Big]_{t=0}=\Lambda^{\mu}_{\nu}\frac{d}{dt}\Big[U(\mathbb{1},te^{\nu})\Big]_{t=0}=\Lambda^{\mu}_{\nu}P^{\nu}. \tag{4.49}$$

□

4.2.4 Pure Lorentz transformation

Now we consider a pure Lorentz transformation $U(\Lambda)\equiv U(\Lambda,0)$, and we want to look at $U(\Lambda)|p,\sigma\rangle$. In order to see its decomposition into others $|k,\sigma'\rangle$, we apply a P^{μ} :

$$\begin{aligned}P^{\mu}U(\Lambda)|p,\sigma\rangle&=U(\Lambda)\Big(U(\Lambda)^{-1}P^{\mu}U(\Lambda)\Big)|p,\sigma\rangle\\&=U(\Lambda)(\Lambda^{-1})^{\mu}_{\nu}P^{\nu}|p,\sigma\rangle\\&=(\Lambda^{-1})^{\mu}_{\nu}p^{\nu}U(\Lambda)|p,\sigma\rangle.\end{aligned} \tag{4.50}$$

Thus the vector $U(\Lambda)|p,\sigma\rangle\in\mathscr{H}$ has $(\Lambda^{-1})^{\mu}_{\nu}p^{\nu}$ as eigenvalue for P^{μ}. If the p^{μ}'s are seen as components of a 4-vector p, one can write

$$P^{\mu}U(\Lambda)|p,\sigma\rangle=(\Lambda p)^{\mu}U(\Lambda)|p,\sigma\rangle;$$

thus we naturally write
$$U(\Lambda)|p,\sigma\rangle=\sum_{\sigma'}C_{\sigma'\sigma}(\Lambda,p)|\Lambda p,\sigma'\rangle. \tag{4.51}$$

Note that we had not yet given anything about the nature of the p in the ket $|p,\sigma\rangle$ so we can *define* the product Λp by the fact that the ket $|\Lambda p,\sigma\rangle$ has eigenvalue $(\Lambda^{-1})^{\mu}_{\nu}p^{\nu}$ for the operator P^{μ}. So it is one of the $|p',\sigma'\rangle$.

4.2.5 Rebuilding of a basis for $\mathscr{H}$

From general considerations about the Lorentz group (many physicists had written very better books than me about) anyone knows that the only functions of the p^{μ}'s which are invariant under all the Lorentz transformations are $p^2=\eta_{\mu}nup^{\mu}p^{\nu}$ and the sign of p^0 when $p^2<0$.

For any value of p^2 and sign of p^0, one consider a "standard vector" k. For example :

$$k=(1,0,0,1)\qquad\text{for } p^2=0, \tag{4.52a}$$
$$k=(1,0,0,0)\qquad\text{for } p^2<0,\ p^0>0, \tag{4.52b}$$
$$k=(-1,0,0,0)\qquad\text{for } p^2<0,\ p^0<0. \tag{4.52c}$$

With this convention, p can be written as $p=L(p)k$ for a suitable Lorentz transformation $L(p)$. The vector $U(L(p))|k,\sigma\rangle$ has eigenvalue $L(p)k$ for the operator P, thus it is a linear combination of some $|p,\sigma'\rangle$.

Now we will cheat and redefine our basis of the Hilbert space $\mathscr{H}$. First, we consider a fixed k; in other words, we build the state space for a given particle which has given momentum p. The basis vectors must be eigenvectors for the fours operators P^{μ}. As far as we say no more, any eigenvalue is possible. Thus our basis must be labelled by at least an element p of $\mathbb{R}^4$ with only one constraint: the value of p^2 (plus eventually the sign of p^0). So we define the $|k,\sigma\rangle$ to be such that

$$P^{\mu}|k,\sigma\rangle=k^{\mu}|k,\sigma\rangle.$$

Since we know that with this definition of $|k,\sigma\rangle$, the eigenvalue of $U(L(p))|k,\sigma\rangle$ for P^μ is p^μ, we *define* $|p,\sigma\rangle$ as

$$|p,\sigma\rangle = N(p)U(L(p))|k,\sigma\rangle. \tag{4.53}$$

where $N(p)$ is a normalization to be discussed later. With this construction, we have an eigenvector for any possible eigenvalue for P^μ. We have to show that these vectors are linearly independent.

The set of the $|p,\sigma\rangle$ with different p is free in $\mathscr{H}$ because they are eigenvectors for different eigenvalue of an hermitian operator [2] . There are no reason to think that the set of operators P^μ is complete; in other words, it remains not clear that there exist only one way to diagonalise the all the P^μ. The function of the extra label σ is to label different linearly independent vectors with same eigenvalue for P.

From now, we are interested in $|k,\sigma\rangle$ and $N(p)$.

4.2.6 Little group

We have :

$$\begin{aligned} U(\Lambda)|p,\sigma\rangle &= N(p)U(\Lambda L(p))|k,\sigma\rangle \\ &= N(p)U(L(\Lambda p))U\big(L(\Lambda p)^{-1}\Lambda L(p)\big)|k,\sigma\rangle, \end{aligned} \tag{4.54}$$

So we will try to understand the operation $L(\Lambda p)^{-1}\Lambda L(p)$. First remark that

$$U(L(\Lambda p)^{-1})|\Lambda p,\sigma\rangle = N(\Lambda p)|k,\sigma\rangle,$$

and then compute :

$$\begin{aligned} U(L(\Lambda p)^{-1}\Lambda L(p))N(p)|k,0\rangle &= U(L(\Lambda p)^{-1}\Lambda)|p,\sigma\rangle \\ &= U(L(\Lambda p)^{-1})\sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)|\Lambda p,\sigma'\rangle \\ &= \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)N(\Lambda p)|k,\sigma'\rangle. \end{aligned} \tag{4.55}$$

The **little group** is the subgroup of the Lorentz transformations which leaves the chosen standard vector k invariant: $Wk = k$. For any W in the little group,

$$U(W)|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W)|k,\sigma'\rangle$$

With this definition, the D's form a representation of the little group. Indeed for any V, W in the little group,

$$\begin{aligned} \sum_{\sigma'} D_{\sigma'\sigma}(VW)|k,\sigma'\rangle &= U(VW)|k,\sigma\rangle \\ &= U(V)\sum_{\sigma''} D_{\sigma''\sigma}(W)|k,\sigma''\rangle \\ &= \sum_{\sigma'\sigma''} D_{\sigma'\sigma''}(V)D_{\sigma''\sigma}(W)|k,\sigma'\rangle. \end{aligned} \tag{4.56}$$

[2] I did not checked that it is sufficient

Since we want the $|p,\sigma\rangle$ with different p and σ to form a basis of $\mathscr{H}$, they are linearly independent, then we can get rid of the sum over the σ' and keep the equation

$$D_{\sigma'\sigma}(VW) = \sum_{\sigma''} D_{\sigma'\sigma''}(VW) D_{\sigma''\sigma}(VW);$$

if we adopt a more "matricial" notation,

$$D(VW) = D(V)D(W). \tag{4.57}$$

We are now able to perform a step in the study of the vector $U(\Lambda)|p,\sigma\rangle$. We naturally define $W(\Lambda,p) = L(\Lambda p)^{-1}\Lambda L(p)$. This belongs to the little group[3]. Then,

$$\begin{aligned} U(\Lambda)|p,\sigma\rangle &= N(p)U(L(\Lambda p))U(W(\Lambda,p))|k,\sigma\rangle \\ &= N(p)\sum_{\sigma'} D_{\sigma'\sigma}(W)U(L(\Lambda p))|k,\sigma'\rangle \\ &= \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|\Lambda p,\sigma'\rangle. \end{aligned} \tag{4.58}$$

But we have no constraint on the D's: it must just form a representation of the little group. Consequently, we are at a point in which our axioms are no more sufficient to continue the building of quantum field theory: we will get as many theories as representations of the little group.

The physical interpretation is the following : each type of particle has its own representation. When we consider a Hilbert space on which $U(\Lambda)$ acts via one given representation of the little group, we consider the Hilbert space which describes the corresponding particle. Note that the little group depends on the choice of k, and therefore depends on the particle which is studied (massive or not).

In this sense, a particle is a representation of the Poincaré group [4] . In particular, the nature of the index σ can change from the one representation to the other.

Remark 4.10. As far as normalization is concerned, we will pose

$$N(p) = \sqrt{k^0/p^0}.$$

There are some good reasons to take it; but it is irrelevant from our group point of view of the theory.

4.2.7 Positive mass

This is the easy case. The choice of standard momentum is $k = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$. One could believe that the little group is SO(3). It would be the case if we had chosen $L_+^\uparrow$ instead of $\mathrm{SL}(2,\mathbb{C})$ –see point 4.2.1. In our hermitian representation of $\mathbb{R}^4$, $k = \mathbb{1}$. Then a matrix of $\mathrm{SL}(2,\mathbb{C})$ which leaves it invariant fulfills

$$\lambda k \lambda^\dagger = \lambda\lambda^\dagger = \mathbb{1},$$

this is $\lambda \in SU(2)$. By the way, note that $\mathrm{SO}(3) = SU(2)/\mathbb{Z}_2$.

[3] Pay attention that $L(p)$ depends implicitly on the choice of k.

[4] I think that the irreducibility of a representation is related to *elementary* particles.

The celebrated "law of transformation" of a massive particle of spin j (integer or half integer) under the Lorentz transformation Λ is

$$U(\Lambda)|p,\sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))|\Lambda p,\sigma'\rangle \tag{4.59}$$

where σ runs from $-j$ to j by step of 1.

4.2.8 Null mass

In the case of a null mass, the standard vector is $k = (1,0,0,1)$ and an element of the little group fulfils $Wk = k$. As the little group is part of the Lorentz group, this is an isometry, so

$$\langle Wt|Wk\rangle = \langle t|k\rangle \tag{4.60a}$$
$$\langle Wt|Wt\rangle = \langle t|t\rangle, \tag{4.60b}$$

for any $t \in \mathbb{R}^4$. Taking in particular $t = (1,0,0,0)$,

$$(Wt)^\mu k_\mu = t^\mu k_\mu = -1 \tag{4.61a}$$
$$(Wt)^\mu (Wt)_\mu = t^\mu t_\mu = -1. \tag{4.61b}$$

If we write $Wt = (a,b,c,d)$, the first relation gives $d = a-1$, so that $Wt = (1+\xi,\alpha,\beta,\xi)$, while the second one gives $\xi = (\alpha^2+\beta^2)/2$. The conclusion is that W acts on t as a certain Lorentz transformation $S(\alpha,\beta)$:

$$Wt = \begin{pmatrix} 1+\xi \\ \alpha \\ \beta \\ \xi \end{pmatrix} = \begin{pmatrix} 1+\xi & -\xi & \alpha & \beta \\ \alpha & -\alpha & 1 & 0 \\ \beta & -\beta & 0 & 1 \\ \xi & (1+\xi) & \alpha & \beta. \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.62}$$

Be careful: it doesn't means that $W = S$, but $Wt = St$. However it is an information: $S(\alpha,\beta)^{-1}W$ is a Lorentz transformation which leaves t invariant. Then it is a spatial rotation. More precisely, since W and S conserve $(1,0,0,1)$, it is a rotation around the z axis: $S(\alpha,\beta)^{-1}W = R(\theta)$, and

$$W(\theta,\alpha,\beta) = S(\alpha,\beta)R(\theta) \tag{4.63}$$

is the most general element of the non massive little group.

This chapter actually don't deals with *quantum* field theory in the sense that our wave functions aren't operators which acting on a Fock space. So this is just relativistic field theory.

4.3 Connections

4.3.1 Gauge potentials

Let us consider a **section** σ_α of P over $\mathcal{U}_\alpha$. It is a map $\sigma_\alpha\colon \mathcal{U}_\alpha \to P$ such that $\pi\circ\sigma_\alpha = \text{id}$. A **connection** on P is a 1-form $\omega\colon T_pP \to \mathcal{G} \in \Omega^1(P)$ which satisfies the following two conditions:

$$\omega_p(Y^*_p) = Y, \tag{4.64a}$$
$$\omega(dR_g\xi) = g^{-1}\omega(\xi)g. \tag{4.64b}$$

The **gauge potential** of ω with respect of the local section σ_α is the 1-form on $\mathcal{U}_\alpha$ given by

$$A_\alpha(x)(v) = (\sigma_\alpha^*\omega)_x(v). \tag{4.65}$$

We will not always explicitly write the dependence of A_α in x. Now we consider another section $\sigma_\beta\colon\mathcal{U}_\beta \to P$ which is related on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ to σ_α by $\sigma_\beta(x) = \sigma_\alpha(x)\cdot g_{\alpha\beta}(x)$ for a well defined map $g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta \to G$.

Proposition 4.11.
The gauge potentials A_α and A_β are related by

$$A_\beta = g^{-1}A_\alpha g - g^{-1}dg. \tag{4.66}$$

Proof. By definition, for $v \in T_x\mathcal{U}_\alpha$,

$$A_\beta(v) = (\sigma_\beta^*\omega)_x(v) = \omega_{\sigma_\alpha(x)\cdot g_{\alpha\beta}(x)}\big((d\sigma_\beta)_x(v)\big).$$

We begin by computing $d\sigma_\beta(v)$. Let us take a path $v(t)$ in $\mathcal{U}_\alpha$ such that $v(0) = x$ and $v'(0) = v$. We have :

$$\begin{aligned}
(d\sigma_\beta)_x(v) &= \frac{d}{dt}\sigma_\beta(v(t))\Big|_{t=0}\\
&= \frac{d}{dt}\sigma_\alpha(v(t))\cdot g_{\alpha\beta}(v(t))\Big|_{t=0}\\
&= \frac{d}{dt}\Big[\sigma_\alpha(v(t))\cdot g_{\alpha\beta}(x)\Big]_{t=0} + \frac{d}{dt}\Big[\sigma_\alpha(x)\cdot g_{\alpha\beta}(v(t))\Big]_{t=0}\\
&= dR_{g_{\alpha\beta}(x)}d\sigma_\alpha(v) + \frac{d}{dt}\Big[\sigma_\alpha(x)\cdot g_{\alpha\beta}(x)e^{-ts}\Big]_{t=0}\\
&= dR_{g_{\alpha\beta}(x)}d\sigma_\alpha(v) + s^*_{\sigma_\alpha(x)\cdot g_{\alpha\beta}(x)}
\end{aligned} \tag{4.67}$$

where s is defined by the requirement that $g_{\alpha\beta}(x)^{-1}g_{\alpha\beta}(v(t))$ can be replaced in the derivative by e^{-ts}, so that we can replace $g_{\alpha\beta}(v(t))$ by $g_{\alpha\beta}(x)e^{-ts}$. As far as the derivatives are concerned, $e^{-ts} = g_{\alpha\beta}(x)^{-1}g_{\alpha\beta}(v(t))$, then

$$s = -\frac{d}{dt}g_{\alpha\beta}(x)^{-1}g_{\alpha\beta}(v(t))\Big|_{t=0} = -g_{\alpha\beta}(x)^{-1}dg_{\alpha\beta}(v),$$

this equality being a notation. Now, properties (4.64a)-144 and (4.64b)-144 make that

$$A_\beta(v) = g_{\alpha\beta}(x)^{-1}\omega_{\sigma_\alpha(x)}(d\sigma_\alpha(v))g_{\alpha\beta}(x) + s.$$

The thesis is just the same, with "reduced" notations (see section 4.6.2). . □

An explicit form for this transformation law is :

$$A_\beta(v) = \frac{d}{dt}\Big[g^{-1}e^{tA_\alpha(v)}g\Big]_{t=0} - \frac{d}{dt}\Big[g^{-1}g_{\alpha\beta}(v(t))\Big]_{t=0}, \tag{4.68}$$

where $g := g_{\alpha\beta}(x)$.

4.3.2 Covariant derivative

When we have a connection on a principal bundle, we can define a covariant derivative on any associated bundle. Let us quickly review it. An associated bundle is the semi-product $E = P \times_\rho V$ where V is a vector space on which acts the representation ρ of G. We denote the canonical projection by $\pi_p\colon E \to M$. The classes are taken with respect to the equivalence relation $(p, v) \sim (p \cdot g, \rho(g^{-1})v)$.

A **section** of E is a map $\psi\colon M \to E$ such that $\pi \circ \psi = \mathrm{id}$. We denote by $\Gamma(E)$ the set of all the sections of E. A section of E defines (and is defined by) an equivariant function $\hat{\psi}\colon P \to V$ such that

$$\psi(\pi(\xi)) = [\xi, \hat{\psi}(\xi)], \tag{4.69a}$$
$$\hat{\psi}(\xi \cdot g) = \rho(g^{-1})\hat{\psi}(\xi). \tag{4.69b}$$

For a section $\psi \in \Gamma(E)$, we define $\psi_{(\alpha)}\colon \mathcal{U}_\alpha \to V$ by

$$\psi_{(\alpha)}(x) = \hat{\psi}(\sigma(x)).$$

We saw in (1.161)-59 that a covariant derivative on E is given by

$$(D_X\psi)_{(\alpha)}(x) = X_x\psi_{(\alpha)} - \rho_*\Big((\sigma_\alpha^*\omega)_x(X_x)\Big)\psi_{(\alpha)}(x). \tag{4.70}$$

Since $(d\psi)(X) = X(\psi)$, we can rewrite this formula in a simpler manner by forgetting the index α and the mention of X :

$$D\psi = d\psi - (\rho_* A_\alpha)\psi.$$

If we note $(\rho_* A_\alpha)\psi$ by $A_\alpha\psi$, we have

$$D\psi = d\psi - A\psi. \tag{4.71}$$

One has to understand that equation as a “notational trick” for (4.70)-146. By the way, remark that $(\rho_* A_\alpha)$ is the only “reasonable” way for A to act on ψ.

4.4 Gauge transformation

A **gauge transformation** of a G-principal bundle is a diffeomorphism $\varphi\colon P \to P$ which satisfies

$$\pi \circ \varphi = \pi, \tag{4.72a}$$
$$\varphi(\xi \cdot g) = \varphi(\xi) \cdot g. \tag{4.72b}$$

In local coordinates, it can be expressed in terms of a function $\tilde{\varphi}_\alpha\colon \mathcal{U}_\alpha \to G$ by the requirement that

$$\varphi(\sigma_\alpha(x)) = \sigma_\alpha(x) \cdot \tilde{\varphi}_\alpha(x). \tag{4.73}$$

We have shown in proposition 1.59 that, if ω is a connection 1-form on P, the form $\varphi{\cdot}\omega := \varphi^*\omega$ is still a connection 1-form on P. Of course, with the same section σ_α than before, we can define a gauge potential $(\varphi \cdot A)_\alpha$ for this new connection. We will see how it is related to A_α. The reader can guess the result (it will be the same as proposition 4.11). We show it.

Proposition 4.12.

$$(\varphi \cdot A) = \tilde{\varphi}^{-1} A \tilde{\varphi} - \tilde{\varphi}^{-1} d\tilde{\varphi}. \tag{4.74}$$

Proof. Let us consider $x \in \mathcal{U}_\alpha$, and $v \in T_x\mathcal{U}_\alpha$, the vector which is tangent to the curve $v(t) \in \mathcal{U}_\alpha$. We compute

$$\sigma_\alpha^*(\varphi^*\omega)_x(v) = \omega_{(\varphi\circ\sigma_\alpha)(x)}((d\varphi \circ d\sigma_\alpha)(v)),$$

but equation (4.73)-146 makes

$$\begin{aligned}(d\varphi \circ d\sigma_\alpha)(v) &= \frac{d}{dt}\varphi(\sigma_\alpha(v(t)))\Big|_{t=0}\\ &= \frac{d}{dt}\sigma_\alpha(v(t)) \cdot \tilde{\varphi}_\alpha(v(t))\Big|_{t=0}.\end{aligned} \tag{4.75}$$

Now, we are in the same situation as in equation (4.67)-145. □

If $\psi\colon M \to E$ is a section of E, the gauge transformation $\varphi\colon P \to P$ acts on ψ by

$$\widehat{\varphi\cdot\psi}(\xi) = \hat{\psi}(\varphi^{-1}(\xi)). \tag{4.76}$$

On the other hand, φ acts on the covariant derivative (and the potential) : $\varphi \cdot D$ is the covariant derivative of the connection $\varphi\cdot\omega$. Of course, we define

$$(\varphi\cdot D)\psi = d\psi - (\varphi\cdot A)\psi. \tag{4.77}$$

Lemma 4.13.
If $\varphi\colon P \to P$ is a gauge transformation, then

(i) φ^{-1} is also a gauge transformation and $(\widetilde{\varphi^{-1}})_\alpha(x) = \tilde{\varphi}_\alpha(x)^{-1}$,

(ii) $(\varphi\cdot\psi)_{(\alpha)}(x) = \rho(\tilde{\varphi}_x^{-1})\psi_{(\alpha)}(x)$.

Proof. The first part is clear while the second is a computation :

$$(\varphi\cdot\psi)_{(\alpha)} = \widehat{\varphi\cdot\psi}(\sigma_\alpha(x)) = \hat{\psi}(\varphi^{-1}(\sigma_\alpha(x))) = \hat{\psi}(\sigma_\alpha(x)\cdot\tilde{\varphi}_\alpha(x)^{-1}) = \rho(\tilde{\varphi}_\alpha(x))\psi_{(\alpha)}(x). \tag{4.78}$$

□

Now, we will proof the main theorem: the one which explains why the covariant derivative is “covariant”.

Theorem 4.14.
The covariant derivative D fulfils a “covariant” transformation rule under gauge transformations:

$$(\varphi\cdot D)(\varphi^{-1}\cdot\psi) = \varphi^{-1}(D\psi). \tag{4.79}$$

Remark 4.15. Let us use more intuitive notations: we write (4.74)-146 under the form $A' = g^{-1}Ag - g^{-1}dg$. If we have two sections ψ and ψ', they are necessarily related by a gauge transformation: $\psi' = g^{-1}\psi$. Then, the theorem tells us that the equation $D\psi = d\psi - A\psi$ becomes $D'\psi' = g^{-1}D\psi$ “under a gauge transformation”. This is: $D\psi$ transforms under a gauge transformation as $d\psi$ transforms under a constant linear transformation. This is the reason why D is a *covariant* derivative. The physicist way to write (4.79)-147 is

$$D'\psi' = g^{-1}D\psi \tag{4.80}$$

Proof of theorem 4.14. First, we look at $(\varphi \cdot A)\psi_\alpha$. Using all the notational tricks used to give a sens to $A\psi$, we write :

$$[(\varphi \cdot A)_X \psi]_{(\alpha)}(x) = (\varphi \cdot A)_X \psi_{(\alpha)}(x) = \rho_*(\varphi \cdot A(X))\psi_{(\alpha)}(x).$$

But we know that $\varphi \cdot A = \tilde{\varphi}^{-1} A \tilde{\varphi} - \tilde{\varphi}^{-1} d\tilde{\varphi}$, then

$$\begin{aligned}(\varphi \cdot A)_X \psi_{(\alpha)}(x) &= \rho_*(\tilde{\varphi}^{-1} A(X)\tilde{\varphi})\psi_{(\alpha)}(x)\\ &\quad - \rho_*(\tilde{\varphi}^{-1} d\tilde{\varphi}(X))\psi_{(\alpha)}(x)\\ &= \frac{d}{dt}\Big[\rho(\tilde{\varphi}^{-1} e^{tA(X)}\tilde{\varphi})\psi_{(\alpha)}(x)\Big]_{t=0}\\ &\quad - \frac{d}{dt}\Big[\rho(\tilde{\varphi}^{-1}\tilde{\varphi}(X_t))\psi_{(\alpha)}(x)\Big]_{t=0}\end{aligned} \tag{4.81}$$

Now, we have to write this equation with $\varphi^{-1} \cdot \psi$ instead of ψ. Using lemma 4.13, we find :

$$\begin{aligned}(\varphi \cdot A)_X(\varphi^{-1} \cdot \psi)_{(\alpha)}(x) &= \frac{d}{dt}\Big[\rho(\tilde{\varphi}^{-1} e^{tA(X)}\tilde{\varphi}\tilde{\varphi}^{-1})\psi_{(\alpha)}(x)\Big]_{t=0}\\ &\quad - \frac{d}{dt}\Big[\rho(\tilde{\varphi}^{-1}\tilde{\varphi}(X_t)\tilde{\varphi}^{-1})\psi_{(\alpha)}(x)\Big]_{t=0}\end{aligned} \tag{4.82}$$

After simplification, the first term is a term of the thesis: $\tilde{\varphi}(x)^{-1}(A\psi)_\alpha(x)$ and we let the second one as it is. Now, we turn our attention to the second term of (4.79)-147; the same argument gives:

$$\begin{aligned}d(\varphi^{-1}\psi_{(\alpha)})_x X &= \frac{d}{dt}\Big[(\varphi^{-1} \cdot \psi)_{(\alpha)}(X_t)\Big]_{t=0}\\ &= \frac{d}{dt}\Big[\rho(\tilde{\varphi}(X_t)^{-1})\psi_{(\alpha)}(X_t)\Big]_{t=0}\\ &= \frac{d}{dt}\Big[\rho(\tilde{\varphi}(X_t)^{-1})\psi_{(\alpha)}(x)\Big]_{t=0} + \frac{d}{dt}\Big[\rho(\tilde{\varphi}^{-1})\psi_{(\alpha)}(X_t)\Big]_{t=0}.\end{aligned} \tag{4.83}$$

The second term is $\tilde{\varphi}^{-1} d\psi_\alpha(X)$. In definitive, we need to prove that the two exceeding terms cancel each other :

$$\frac{d}{dt}\Big[\rho(\tilde{\varphi}^{-1}\tilde{\varphi}(X_t)\tilde{\varphi}^{-1})\psi_{(\alpha)}(x)\Big]_{t=0} + \frac{d}{dt}\Big[\rho(\tilde{\varphi}(X_t)^{-1})\psi_{(\alpha)}(x)\Big]_{t=0} \tag{4.84}$$

must be zero.

One can find a $g(t) \in G$ such that $\tilde{\varphi}(X_t) = \tilde{\varphi}g(t)$, $g(0) = e$. Then, what we have in the ρ of these two terms is respectively $g(t)\tilde{\varphi}^{-1}$ and $g(t)^{-1}\tilde{\varphi}^{-1}$. As far as the derivative are concerned, $g(t)$ can be written as e^{tZ} for a certain $Z \in \mathcal{G}$. So we see that $g(t)^{-1} = e^{-tZ}$ and the derivative will come with the right sign to makes the sum zero. □

Remark 4.16. If we naively make the computation with the notations of remark 4.15, we replace $\psi' = g^{-1}\psi$ and $A' = g^{-1}Ag - g^{-1}dg$ in

$$D'\psi' = d\psi' - A'\psi',$$

using some intuitive "Leibnitz formulas", we find : $D'\psi' = dg^{-1}\psi + g^{-1}d\psi + g^{-1}A\psi + g^{-1}dgg^{-1}\psi$. It is exactly $g^{-1}d\psi + g^{-1}A\psi$ with two additional terms: $dg^{-1}\psi$ and $g^{-1}dgg^{-1}\psi$. One sees that these are precisely the two terms of the expression (4.84)-148. We will give a sens to this "naive" computation in section 4.6.2.

4.5 A bite of physics

4.5.1 Example: electromagnetism

Let us consider the electromagnetism as the simplest example of a gauge invariant physical theory. We first discuss the theory of free electromagnetic field (this is: without taking into account the interactions with particles) from Maxwell's equations, see [31, 37]. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are subject to following relations :

$$\nabla \cdot \mathbf{E} = \rho, \tag{4.85a}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{4.85b}$$
$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \tag{4.85c}$$
$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{j}. \tag{4.85d}$$

Comparing (4.85a)-149 and (4.85b)-149, we see that Maxwell's theory does not incorporate magnetic monopoles. Suppose that we can use the Poincaré lemma. Equation (4.85b)-149 gives a vector field $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$, so that (4.85c)-149 becomes $\nabla \times (\mathbf{E} + \partial_t \mathbf{A}) = 0$ which gives a scalar field ϕ such that $-\nabla \cdot \phi = \mathbf{E} + \partial_t \mathbf{A}$.

Now the equations (4.85a)-149–(4.85d)-149 are equations for the potentials $\mathbf{A}$ and ϕ, and we find back the "physical" field by

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4.86a}$$
$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}. \tag{4.86b}$$

One can easily see that there are several choice of potentials which describe the same electromagnetic field. Indeed, if

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda, \tag{4.87a}$$
$$\phi' = \phi - \partial_t \lambda, \tag{4.87b}$$

the electromagnetic field given (via (4.86)-149) by $\{\phi', \mathbf{A}'\}$ is the same as the one given by $\{\phi, \mathbf{A}\}$

The Maxwell's equations can be written in a more "covariant" way by defining

$$F = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ \cdot & 0 & -B_z & \cdot \\ \cdot & \cdot & 0 & -B_x \\ \cdot & -B_y & \cdot & 0 \end{pmatrix}, \tag{4.88}$$

$F^{\mu\nu} = -F^{\nu\mu}$ and

$$J = \begin{pmatrix} c\rho & j_x & j_y & j_z \end{pmatrix}.$$

We also define $\star F^{\alpha\beta} = \frac{1}{2} e^{\alpha\beta\lambda\mu} F_{\lambda\mu}$. With all that, Maxwell's equations read :

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \mu_0 J^\nu, \\ \partial_\alpha \star F^{\alpha\beta} &= 0. \end{aligned} \tag{4.89}$$

If we define

$$A = \begin{pmatrix} \frac{\phi}{c} & -A_x & -A_y & -A_z \end{pmatrix}, \tag{4.90}$$

the physical fields are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The **gauge invariance** of this theory is the fact that

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} \tag{4.91a}$$

when

$$A'_\mu(x) = A_\mu(x) + \partial_\mu f(x) \tag{4.91b}$$

for any scalar *function* f (to be compared with (4.87)-149).

This is: in the picture of the world in which we see the A as fundamental field of physics, several (as much as you have functions in $C^\infty(\mathbb{R}^4)$) fields A, A',... describe the *same* physical situation because the fields **E** and **B** which acts on the particle are the same for A and A'.

Now, we turn our attentions to the interacting field theory of electromagnetism. As far as we know, the electron makes interactions with the electromagnetic field via a term $\overline{\psi}A_\mu\psi$ in the Lagrangian. The free Lagrangian for an electron is

$$\mathcal{L} = \overline{\psi}(\gamma^\mu\partial_\mu + m)\psi. \tag{4.92}$$

The easiest way to include a $\overline{\psi}A\psi$ term is to change ∂_μ to $\partial_\mu + A_\mu$. But we want to preserve the powerful gauge invariance of classical electrodynamics, then we want the new Lagrangian to keep unchanged if we do

$$A_\mu \to A'_\mu = A_\mu - i\partial_\mu\phi. \tag{4.93}$$

In order to achieve it, we remark that the ψ must be transformed *simultaneously* into

$$\psi'(x) = e^{i\phi(x)}\psi(x). \tag{4.94}$$

The conclusion is that if one want to write down a Lagrangian for QED, one must find a Lagrangian which remains unchanged under certain transformation $A \to A'$ and $\psi \to \psi'$. In other words the set $\{\psi, A\}$ of fields which describe the world of an electron in an electromagnetic field is not well defined from data of the physical situation alone: it is defined up to a certain invariance which is naturally called a **gauge invariance**.

Remark 4.17. In the physics books, the matter is presented in a slightly different way. We observe that the Lagrangian (4.92)-150 is invariant under

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) \tag{4.95}$$

for any *constant* α. One can see that the associated conserved current (Noether) is closely related to the electric current. The idea (of Yang-Mills) is to develop this symmetry. Since the symmetry (4.95)-150 depends only on a constant, we say it a **global** symmetry; we will simultaneously add a new field A_μ and upgrade (4.95)-150 to a **local** symmetry:

$$\psi(x) \to \psi'(x) = e^{i\phi(x)}\psi(x). \tag{4.96}$$

Then, we deduce the transformation law of A_μ.

Because of the form of (4.94)-150, we say that the electromagnetism is a $U(1)$-gauge theory. The fact that this is an abelian group have a deep physical meaning and many consequences.

4.5.2 Little more general, slightly more formal

The aim of this text is to interpret the field A as a gauge potential for a connection. But equation (4.93)-150 is not exactly the expected one which is (4.74)-146. The point is that equation (4.93)-150 concerns a theory in which the gauge transformation of the field was a simple multiplication by a scalar field, so that simplifications as $e^{-i\phi(x)}A_\mu(x)e^{i\phi(x)} = A_\mu(x)$ are allowed.

Now, we consider a vector space V, a manifold M and a function $\psi\colon M \to V$ which "equation of motion" is

$$L^i(\partial_i + m_i)\psi = 0$$

Where we imply an unit matrix behind ∂ and m; the indices i, j are the (local) coordinates in M and a, b, the coordinates in V. Let G be a matrix group which acts on V. If ψ is a solution, $\Lambda^{-1}\psi$ is also a solution as far as Λ is a constant –does not depend on $x \in M$– matrix of G. In other words, $L^i(\partial_i + m_i)\psi_a = 0$ for all a implies $L^i(\partial_i + m_i)((\Lambda^{-1})^b_a\psi_b) = 0$.

The function, $\psi'(x) = \Lambda(x)^{-1}\psi(x)$ is no more a solution. If we want it to be solution of the same equation as ψ, we have to change the equation and consider

$$L^i(\partial_i + A_i + m_i)\psi = 0.$$

This equation is preserved under the *simultaneous* change

$$\begin{cases} \psi'_a = (\Lambda^{-1})^b_a\psi_b \\ (A'_i)^a_b = (\Lambda^{-1})^c_b(A_i)^d_c(\Lambda^a_d) - (\partial_i\Lambda^{-1})^d_b\Lambda^a_d. \end{cases} \tag{4.97}$$

The second line show that the formalism in which A is a connection is the good one to write down covariant equations. This has to be compared with (4.66)-145. Logically, a theory which includes an invariance under transformations as (4.97)-151 is called a G-gauge theory.

4.5.3 A "final" formalism

Now, we work with fields which are sections of some fiber bundle build over M, the physical space. More precisely, let G be a matrix group. We search for a theory which is "locally invariant under G". In order to achieve it, we consider a G-principal bundle P over M and the associated bundle $E = P \times_\rho V$ for a certain vector space V, and a representation ρ of G on V. Typically, V is $\mathbb{C}$ or the vector space on which the spinor representation acts.

The physical fields are sections $\psi\colon M \to E$. If we choose some reference sections $\sigma_\alpha\colon M \to P$, they can be expressed by $\psi_{(\alpha)}(x) = \hat{\psi}(\sigma_\alpha(x))$. We translate the idea of a local invariance under G by requiring an invariance under

$$\psi'_{(\alpha)}(x) = \rho(g(x))\psi_{(\alpha)}(x)$$

for every $g\colon M \to G$. By (ii) of lemma 4.13, we see that $\psi'_{(\alpha)}(x) = (\varphi^{-1} \cdot \psi)_{(\alpha)}(x)$, where $\varphi\colon P \to P$ is the gauge transformation given by

$$\varphi(\sigma_\alpha(x)) = \sigma_\alpha(x) \cdot g(x).$$

We want ψ and ψ' to "describe the same physics". From a mathematical point of view, we want ψ and ψ' to *satisfy the same equation*. It is clear that equation $d\psi = 0$ will not work.

The trick is to consider any connection ω on P and the gauge potential A of ω. In this case the equation

$$(d - A)\psi = 0 \qquad \text{or} \qquad D\psi = 0 \tag{4.98}$$

is preserved under

$$\begin{aligned} A &\to \varphi \cdot A, \\ \psi &\to \varphi^{-1} \cdot \psi. \end{aligned}$$

Theorem 4.14 powa !

In this sense, we say that equation (4.98)-151 is gauge invariant, and is thus taken by physicists to build some theories when they need a "local G-covariance". This gives rise to the famous Yang-Mills theories.

In this picture the matter field ψ and the bosonic field A are both defined from a $U(1)$-principal bundle. The sense of "ψ transforms as ... under a $U(1)$ transformation" is the sense of the transformation of a section of an associated bundle; the sense of "A transforms as ... under a $U(1)$ transformation" is the one of the transformation of the gauge potential of a connection on a $U(1)$-principal bundle.

Remark 4.18. The mathematics of equation (4.98)-151 only requires a $\mathcal{G}$-valued connection on P. There are several physical constraints on the choice of the connection. These give rise to interaction terms between the gauge bosons. We will not discuss it at all. This a matter of books about quantum field theories.

The most used Yang-Mills groups in physics are $U(1)$ for the QED, $SU(2)$ for the weak interactions and $SU(3)$ for chromodynamic.

4.6 Curvature

4.6.1 Intuitive setting

From the $\mathcal{G}$-valued connection 1-form ω on P, we may define its **curvature 2-form** :

$$\Omega = d\omega + \omega \wedge \omega. \tag{4.99}$$

As before, we can see Ω as a 2-form on M instead of P. For this, we just need some sections $\sigma_\alpha : \mathcal{U}_\alpha \to P$ and define

$$F_\alpha = \sigma_\alpha^* \Omega. \tag{4.100}$$

This F is called the **Yang-Mills field strength**. The question is now to see how does it transform under a change of chart ? What is $F_\beta = \sigma_\beta^* \Omega$ in terms of F_α ?

Theorem 4.19.

$$F_\beta = g^{-1} F_\alpha g. \tag{4.101}$$

Naive proof. Let us accept $F_\beta = dA_\beta + A_\beta \wedge A_\beta$. With proposition 4.11, we can perform a simple computation with all the intuitive "Leibnitz rules" :

$$dA_\beta = -g^{-1} dg\, g^{-1} \wedge A_\alpha g + g^{-1} dA_\alpha g + g^{-1} A_\alpha \wedge dg - g^{-1} dg\, g^{-1} \wedge dg,$$

and

$$A_\beta \wedge A_\beta = g^{-1} A_\alpha g \wedge g^{-1} A_\alpha g + g^{-1} A_\alpha g \wedge g^{-1} dg + g^{-1} dg \wedge g^{-1} A_\alpha g + g^{-1} dg \wedge g^{-1} dg.$$

The sum is obviously the announced result. $\square$

This proof seems too beautiful to be false[5]. We will now try to give a sense to this computation. A complete proof of the theorem is reported until page 156.

[5] More precisely, it is as beautiful as we want it to be true.

First, note that we can't try to find a relation like $d(g\omega) = dg \wedge \omega + g\, d\omega$. Pose $A_x = g(x)\omega_x$:

$$A_x(v) = \frac{d}{dt} g(x) e^{t\omega_x(v)} \Big|_{t=0}.$$

Using

$$(d\alpha)(v, w) = v(\alpha(w)) - w(\alpha(v)) - \alpha([v, w]),$$

we are led to write

$$w(A(v)) = d(A(v))w = \frac{d}{du} A_{w_u}(v)\Big|_{u=0} = \frac{d}{du}\frac{d}{dt}\Big[g(w_u)e^{t\omega_{w_u}(v)}\Big]_{t=0}\Big|_{u=0}. \tag{4.102}$$

But at $t = u = 0$, the expression in the bracket is $g(x)$, and not e. Then the whole expression is not an element of $\mathcal{G}$. In other words, the problem is that for $g\colon M \to G$, we have $dg_x\colon T_xM \to T_{g(x)}G \neq T_eG$.

Now, remark that in our matter, the problem will not arise because in the expressions $A_\beta = g^{-1}A_\alpha g + g^{-1}dg$, each term has a g and a g^{-1}.

Lemma 4.20.

$$d(g^{-1})_x(v) = -g(x)^{-1} dg(v) g(x)^{-1}. \tag{4.103}$$

Proof. Let v_t be a path which defines the vector v, and define $Y \in \mathcal{G}$ such that as far as the derivative are concerned, we have $g(v_t) = g(x)e^{tY}$. Then,

$$g(g^{-1})(v) = \frac{d}{dt}\Big[g(v_t)^{-1}\Big]_{t=0} = \frac{d}{dt}\Big[e^{-tY} g(x)^{-1}\Big]_{t=0}.$$

But on the other hand,

$$g^{-1}dg(v)g^{-1} = \frac{d}{dt}\Big[g(x)^{-1}g(v_t)g(x)^{-1}\Big]_{t=0} = \frac{d}{dt}\Big[e^{tY} g(x)^{-1}\Big]_{t=0},$$

thus $d(g^{-1})_x(v) = -g(x)^{-1}dg(v)g(x)^{-1}$, as we want. $\square$

4.6.2 A digression: $T_Y\mathcal{G}$ and $\mathcal{G}$

We define two product: $G \times \mathcal{G} \to TG$ and $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$. If $g \in G$ and $X \in \mathcal{G}$, we put

$$gX = \frac{d}{dt} g e^{tX}\Big|_{t=0}, \tag{4.104a}$$

and if X, $Y \in \mathcal{G}$,

$$XY = \frac{d}{dt}\frac{d}{du}\Big[e^{tX}e^{uY}\Big]_{\substack{u=0\\t=0}}. \tag{4.104b}$$

We naturally define the product of a $\mathcal{G}$-valued 1-form A by an element $g \in G$ by $(gA)v = gA(v)$.

Note that gX does not belong to $\mathcal{G}$ but to T_gG. Fortunately, in the expressions which we will meet, there will always be a g^{-1} to save the situation.

Let us now see a great consequence of the second definition.

Proposition 4.21.
The formula

$$XY - YX = [X, Y]. \tag{4.105}$$

links the formal product inside the Lie algebra and the Lie bracket.

In order to get a real proof (given from page 155) of this, we have to give some precisions about derivatives as (4.104b)-153. We consider the expression

$$\frac{d}{du}\left(\frac{d}{dt}c_u(t)\Big|_{t=0}\right)_{u=0},$$

which will be more simply written as :

$$\frac{d}{du}\frac{d}{dt}\Big[c_u(t)\Big]_{\substack{t=0\\u=0}} \tag{4.106}$$

with $c_u(t) \in G$ for all u, t; $c_u(0) = e$ for all u and $c_0'(0) = Y \in \mathcal{G}$ where the prime stands for the derivative with respect of t. So $\frac{d}{dt}c_u(t)\big|_{t=0} \in \mathcal{G}$ for each u and (4.106)-154 belongs to $T_Y\mathcal{G}$. But we know that $\mathcal{G}$ is a vector space, then $T_Y\mathcal{G} \simeq \mathcal{G}$, the isomorphism being given by the following idea: if $\{\partial_i\}$ is a basis of $\mathcal{G}$ and $\{e_i\}$ the corresponding basis of $T_Y\mathcal{G}$, we define the action of $A^i e_i \in T_Y\mathcal{G}$ on $f\colon G \to \mathbb{R}$ by $(A^i e_i)f := A^i\partial_i f$.

Lemma 4.22.
Seen as an equality in $\mathcal{G}$, for $f\colon G \to \mathbb{R}$ we have :

$$\frac{d}{du}\frac{d}{dt}\Big[c_u(t)\Big]_{\substack{t=0\\u=0}} f = \frac{d}{du}\frac{d}{dt}\Big[f(c_u(t))\Big]_{\substack{t=0\\u=0}}. \tag{4.107}$$

Proof. Let us consider $X_u = X_u^i\partial_i = c_u'(0)$ and $X_0 = Y$. We naturally have

$$X_u f = \frac{d}{dt}f(c_u(t))\Big|_{t=0}, \qquad \text{and} \qquad \frac{d}{du}X_u\Big|_{u=0} \in T_Y\mathcal{G}. \tag{4.108}$$

Now, we consider a function $h\colon \mathcal{G} \to \mathbb{R}$ and compute :

$$\frac{d}{du}\Big[X_u\Big]_{u=0} h = \frac{d}{du}\Big[h(X_u)\Big]_{u=0} = \frac{d}{du}h\Big(\frac{d}{dt}\Big[c_u(t)\Big]_{t=0}\Big)\Big|_{u=0}.$$

If $\{\partial_i\}$ is a basis of $\mathcal{G}$ and $\{e_i\}$, the corresponding one of $T_Y\mathcal{G}$, thus

$$\frac{d}{du}\Big[X_u\Big]_{u=0} h = \frac{\partial h}{\partial e_i}\Big|_Y \frac{d}{du}\frac{d}{dt}\Big[c_u^i(t)\Big]_{\substack{t=0\\u=0}}. \tag{4.109}$$

So, we can write

$$\frac{d}{du}\Big[X_u\Big]_{u=0} = \frac{d}{du}\frac{d}{dt}\Big[c_u^i(t)\Big]_{\substack{t=0\\u=0}} \frac{\partial}{\partial e_i}\Big|_Y,$$

and, as element of $\mathcal{G}$, we consider

$$\frac{d}{du}\Big[X_u\Big]_{u=0} = \frac{d}{du}\frac{d}{dt}\Big[c_u^i(t)\Big]_{\substack{t=0\\u=0}} \partial_i|_e.$$

Now, we can compute the action of $\frac{d}{du}X_u\big|_{u=0}$ on a function $f\colon G\to\mathbb{R}$ as

$$\begin{aligned}\frac{d}{du}\Big[X_u\Big]_{u=0}f &= \frac{d}{du}\frac{d}{dt}\Big[c_u^i(t)\Big]_{\substack{t=0\\u=0}}\frac{\partial f}{\partial x^i}\Big|_e\\ &= \frac{d}{du}\Big[\frac{\partial f}{\partial x^i}\Big|_e\frac{d}{dt}c_u^i(t)\Big|_{t=0}\Big]_{u=0}\\ &= \frac{d}{du}\Big[\frac{d}{dt}f(c_u(t))\Big|_{t=0}\Big]_{u=0}.\end{aligned}\tag{4.110}$$

□

Proof of proposition 4.21. From this, we can precise our definition of XY when X, $Y\in\mathcal{G}$. The product XY acts on $f\colon G\to\mathbb{R}$ by

$$(XY)f=\frac{d}{dt}\frac{d}{du}\Big[f(e^{tX}e^{uY})\Big]_{\substack{u=0\\t=0}}.$$

We can get a more geometric interpretation of this. We know how to build a left invariant vector field $\tilde{Y}$ from any $Y\in\mathcal{G}$: for each $g\in G$ we just have to define

$$\tilde{Y}_g(f)=\frac{d}{ds}\Big[f(gY(s))\Big]_{s=0}.$$

First remark: $\tilde{Y}_g$ is precisely the object that previously named "gY". In order to construct the basis blocks of the formula $XY-YX=[X,Y]$, we turn our attention to $\tilde{X}_e\tilde{Y}$. It is clear that $\tilde{Y}(f)$ is a function from G to $\mathbb{R}$, so we can apply $\tilde{X}_e$ on it. If X_t is a path which gives the vector $\tilde{X}_e$ (for example: $X_t=e^{tX}$), we have

$$\tilde{X}_e(\tilde{Y}(f))=\frac{d}{dt}\Big[\tilde{Y}(f)_{X_t}\Big]_{t=0}=\frac{d}{du}\frac{d}{dt}\Big[f(X_tY(u))\Big]_{\substack{t=0\\u=0}}=\frac{d}{du}\frac{d}{dt}\Big[f(e^{tX}e^{uY})\Big]_{\substack{t=0\\u=0}}.\tag{4.111}$$

Thus we have: $XY=\tilde{X}_e\tilde{Y}$, but it is clear that $[\tilde{X},\tilde{Y}]_e=\tilde{X}_e\tilde{Y}-\tilde{Y}_e\tilde{X}$. The claim reads now: $[\tilde{X},\tilde{Y}]_e=[X,Y]$. We can actually take it as de *definition* of $[X,Y]$. It is done in [3]. The link with the definition in terms of successive derivations of $\mathbf{Ad}_g(x)=gxg^{-1}$ is not trivial but it can be done. □

Now, we can give a powerful definition of the wedge for two $\mathcal{G}$-valued 1-forms. If A, $B\in\Omega^1(M,\mathcal{G})$ and v, $w\in\mathfrak{X}(M)$, we define

$$(A\wedge B)(v,w)=A(v)B(w)-A(w)B(v).\tag{4.112}$$

For A^2, we find back the usual definition :

$$(A\wedge A)(v,w)=A(v)A(w)-A(w)A(v)=[A(v),A(w)].$$

One can see that for any section $\sigma_\alpha\colon\mathcal{U}_\alpha\to P$, we have

$$\sigma_\alpha^*(A\wedge A)=(\sigma_\alpha^*A)\wedge(\sigma_\alpha^*A).\tag{4.113}$$

Lemma 4.23.
If A and B are two $\mathcal{G}$-valued 1-forms, one can make "simplifications" as

$$(Ag)\wedge(g^{-1}B)=A\wedge B.\tag{4.114}$$

Proof. We just have to prove that for $A, B \in \mathcal{G}$, $(Ag)(g^{-1}B) = AB$ with definitions (4.104a)-153 and (4.104b)-153. Remark that $Ag = \frac{d}{ds}\Big[e^{sA}g\Big]_{s=0}$, so

$$e^{tAg} = \exp(t\,\frac{d}{ds}e^{sA}g\Big|_{s=0}) = \exp(\frac{d}{ds}e^{stA}g\Big|_{s=0}) = e^{tA}g.$$

Therefore

$$(Ag)(g^{-1}B) = \frac{d}{dt}\frac{d}{du}\Big[e^{tAg}e^{ug^{-1}B}\Big]_{\substack{u=0\\t=0}} = \frac{d}{dt}\frac{d}{du}\Big[e^{tA}gg^{-1}e^{uB}\Big]_{\substack{u=0\\t=0}} = AB.$$

□

Lemma 4.24.

$$F_\beta = dA_\beta + A_\beta^2. \tag{4.115}$$

Proof. This is a direct consequence of (4.113)-155 and $[\sigma_\beta^*, d] = 0$. □

Now, we can prove the theorem.

Ultimate proof of theorem 4.19. First we compute $d(g^{-1}A_\alpha g)$. In order to do this, remark that the 1-form $g^{-1}A_\alpha g$ is explicitly given on $v \in \mathfrak{X}(M)$ by

$$(g^{-1}A_\alpha g)(v)_x = \frac{d}{dt}\Big[g(x)^{-1}e^{tA(v)_x}g(x)\Big]_{t=0}.$$

For all $x \in M$, this expression is an element of $\mathcal{G}$; then we can say that $(g^{-1}A_\alpha g)(v)$ is a map $(g^{-1}A_\alpha g)(v)\colon M \to \mathcal{G}$. So it is unambiguous to write $w((g^{-1}A_\alpha g)(v)) \in \mathcal{G}$ for $w \in T_xM$.

We will use the formula

$$d(g^{-1}A_\alpha g)(v,w) = v(g^{-1}A_\alpha g)(w) - w(g^{-1}A_\alpha g)(v) - (g^{-1}A_\alpha g)([v,w]).$$

As $w((g^{-1}A_\alpha g)(v)) = d((g^{-1}A_\alpha g)(v))w$, we have

$$\begin{aligned}
w((g^{-1}A_\alpha g)(v)) &= \frac{d}{du}(g^{-1}A_\alpha g)(v)_{w_u}\Big|_{u=0}\\
&= \frac{d}{du}\frac{d}{dt}\Big[g(w_u)^{-1}e^{tA(v)_{w_u}}g(w_u)\Big]_{t=0}\Big|_{u=0}\\
&= \frac{d}{dt}\frac{d}{du}\Big[g(w_u)^{-1}\Big]_{u=0}e^{tA(v)_x}g(x)\Big|_{t=0}\\
&\quad + \frac{d}{dt}g(x)^{-1}\frac{d}{du}\Big[e^{tA(v)_{w_u}}\Big]_{u=0}g(x)\Big|_{t=0}\\
&\quad + \frac{d}{dt}g(x)^{-1}e^{tA(v)_x}\frac{d}{du}\Big[g(w_u)\Big]_{u=0}\Big|_{t=0}\\
&= d(g^{-1})(w)A(v)_xg(x)\\
&\quad + g(x)^{-1}w_x(A(v))g(x)\\
&\quad + g(x)^{-1}A(v)_xdg(w).
\end{aligned} \tag{4.116}$$

On the other hand, one easily finds that

$$(g^{-1}A_\alpha g)([v,w]) = g(x)^{-1}A([v,w])g(x).$$

Using lemma 4.20, we have

$$\begin{aligned}
d(g^{-1}A_\alpha g)_x(v,w) = &-g(x)^{-1}dg(v)g(x)^{-1}A(w)_x g(x) + g(x)^{-1}v(A(w))g(x)\\
&+ g(x)^{-1}A(w)_x dg(v)_x\\
&+ g(x)^{-1}dg(w)_x g(x)^{-1}A(v)_x g(x) - g(x)^{-1}w(A(v))g(x)\\
&- g(x)^{-1}A(v)_x dg(w)\\
&- g(x)^{-1}A([v,w])g(x).
\end{aligned} \tag{4.117}$$

We can regroup the terms two by two in order to form dA_α and some wedge; with simpler notations, we can write :

$$d(g^{-1}A_\alpha g) = -(g^{-1}dg\, g) \wedge (A_\alpha g) - (g^{-1}A) \wedge dg + (g^{-1}dAg). \tag{4.118}$$

We compute $d(g^{-1}dg)$ in the same way; the result is

$$(g^{-1}dg)(v)_x = \frac{d}{dt}\Big[g(x)^{-1}g(v_t)\Big]_{t=0} \in \mathcal{G}.$$

For v, $w \in \mathfrak{X}(M)$, we have :

$$\begin{aligned}
w\big((g^{-1}dg)(v)\big) &= \frac{d}{du}(g^{-1}dg)(v)_{w_u}\Big|_{u=0}\\
&= \frac{d}{du}\frac{d}{dt}\Big[g(w_u)^{-1}g(v_{w_u}(t))\Big]_{\substack{t=0\\u=0}}\\
&= \frac{d}{dt}\frac{d}{du}\Big[g(w_u)^{-1}g(v_t)\Big]_{\substack{u=0\\t=0}} + \frac{d}{dt}\frac{d}{du}\Big[g(x)^{-1}g(w_u(t))\Big]_{\substack{u=0\\t=0}}\\
&= d(g^{-1})(w)dg(v) + \frac{d}{du}\Big[g(x)^{-1}dg(v_{w_u})\Big]_{u=0}\\
&= -g^{-1}dg(w)g^{-1}dg(v) + g(x)^{-1}w(dg(v))
\end{aligned} \tag{4.119}$$

where w_u is a path such that $w_0' = w_x$ and $v_{w_u}(t)$ is, with respect of t, a path which gives the vector v_{w_u}. On the another hand, we have

$$(g^{-1}dg)([v,w]) = g^{-1}dg([v,w]).$$

Remark that the term $g(x)^{-1}w(dg(v))$ of $w((g^{-1}dg)(v))$ together with the same with $v \leftrightarrow w$ and $(g^{-1}dg)([v,w])$ which comes from $(g^{-1}dg)([v,w])$ will give $g(x)^{-1}(d^2g)(v,w) = 0$ when we will compute $d(g^{-1}dg)$. Finally,

$$d(g^{-1}dg) = -(g^{-1}dg\, g^{-1} \wedge dg). \tag{4.120}$$

The equations (4.118)-157 and (4.120)-157 allow us to write :

$$\begin{aligned}
(dA_\beta) &= d(g^{-1}A_\alpha g) + d(g^{-1}dg)\\
&= -(g^{-1}dg\, g^{-1}) \wedge (A_\alpha g) - (g^{-1}A_\alpha) \wedge dg\\
&\quad + (g^{-1}dA_\alpha g) - (g^{-1}dg\, g^{-1}) \wedge dg.
\end{aligned} \tag{4.121}$$

Notice that the term $(g^{-1}dA_\alpha g)$ corresponds to the first one in $F_\beta = g^{-1}(dA_\beta + A_\beta \wedge A_\beta)g$.

For anyone who had understood the whole computations up to here, it is clear that

$$\begin{aligned}[A_\beta(v), A_\beta(w)] = \frac{d}{dt}\frac{d}{du}\Big[e^{tA_\beta(v)}e^{tA_\beta(w)}\Big]_{\substack{u=0\\t=0}} \\ -\frac{d}{dt}\frac{d}{du}\Big[e^{tA_\beta(w)}e^{tA_\beta(v)}\Big]_{\substack{u=0\\t=0}},\end{aligned} \tag{4.122}$$

so that

$$\begin{aligned}A_\beta \wedge A_\beta = g^{-1}A_\alpha g \wedge g^{-1}A_\alpha g + g^{-1}A_\alpha g \wedge g^{-1}dg \\ + g^{-1}dg \wedge g^{-1}A_\alpha g + g^{-1}dg \wedge g^{-1}dg.\end{aligned} \tag{4.123}$$

Lemma 4.23 allows us to write it under the form

$$\begin{aligned}A_\beta \wedge A_\beta = g^{-1}A_\alpha g \wedge g^{-1}A_\alpha g + g^{-1}A_\alpha g \wedge g^{-1}dg \\ + g^{-1}dg \wedge g^{-1}A_\alpha g + g^{-1}dg \wedge g^{-1}dg.\end{aligned} \tag{4.124}$$

Here the term $(g^{-1}A_\alpha \wedge A_\alpha g)$ corresponds to the second one in $F_\beta = g^{-1}(dA_\beta + A_\beta \wedge A_\beta)g$. The sum of (4.121)-157 and (4.124)-158 is

$$F_\beta = g^{-1}F_\alpha g.$$

□

4.6.3 The electromagnetic field F

Now, we are able to interpret the field F introduced in equation (4.88)-149. We follow [32]. From now, we use the usual Minkowski metric $g = diag(-,+,+,+)$. From the vector given by (4.90)-149, we define a (local) potential 1-form

$$A = A_\mu dx^\mu = -\phi dt + A_x dx + A_y dy + A_z dz.$$

The field strength is $F = dA$. We easily find that

$$\begin{aligned}F = (dt \wedge dx)(\partial_x \phi + \partial_t A_x) + \ldots \\ + (dx \wedge dy)(-\partial_z A_x + \partial_x A_y) + \ldots\end{aligned} \tag{4.125}$$

But the fields $\mathbf{B}$ and $\mathbf{E}$ are defined from $\mathbf{A}$ and ϕ by (4.86)-149, so

$$\begin{aligned}F = -E_x(dt \wedge dx) - E_y(dt \wedge dy) - E_z(dt \wedge dz) \\ + B_x(dy \wedge dz) + B_y(dz \wedge dx) + B_z(dx \wedge dy).\end{aligned} \tag{4.126}$$

We naturally have $dF = d^2A = 0$. But conversely, $dF = 0$ ensures the existence of a 1-form A such that $F = dA$. If we define[6] $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$, equations (4.85b)-149 and (4.85c)-149 are obviously satisfied. So in the connection formalism, the equations "without sources" are written by

$$dF = 0. \tag{4.127}$$

In order to write the two others, we introduce the current 1-form :

$$j = j_\mu dx^\mu = -\rho dt + j_x dx + j_y dy + j_z dz.$$

[6] *i.e.* we consider F as the main physical field while $\mathbf{E}$ and $\mathbf{B}$ are "derived" fields.

One sees that

$$\begin{aligned}\delta F := \star d \star F = &-dt(\nabla\cdot\mathbf{E})\\ &+dx(-\partial_t\mathbf{E}_x+(\nabla\times\mathbf{B})_x)\\ &+dy(-\partial_t\mathbf{E}_y+(\nabla\times\mathbf{B})_y)\\ &+dz(-\partial_t\mathbf{E}_z+(\nabla\times\mathbf{B})_z),\end{aligned} \tag{4.128}$$

so that equation $\delta F = j$ gives equations (4.85a)-149 and (4.85d)-149. Now, the complete set of Maxwell's equations is :

$$dF = 0 \tag{4.129a}$$

$$\delta F = j \tag{4.129b}$$

with

$$j = -\rho dt + j_x dx + j_y dy + j_z dz, \tag{4.130a}$$

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{4.130b}$$

$$\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A} \tag{4.130c}$$

where A is a 1-form such that $F = dA$ whose existence is given by (4.129a)-159.

4.7 Inclusion of the Lorentz group

Up to now we had seen how to express the *gauge* invariance of a physical theory. In particle physics, a really funny field theory must be invariant under the Lorentz group; it is rather clear that, from the bundle point of view, this feature will be implemented by a Lorentz-principal bundle and some associated bundles. A spinor will be a section of an associated bundle for spin one half representation of the Lorentz group on $\mathbb{C}^4$. In order to describe non-zero spin particle interacting with an electromagnetic field (represented by a connection on a $U(1)$-principal bundle), we have to build a correct $\mathrm{SL}(2,\mathbb{C})\times U(1)$-principal bundle. We are going to use the ideas of 4.2.1.

A **space-time** is a differentiable **pseudo-Riemannian** 4-dimensional manifold. The pseudo-Riemannian structure is a 2-form $g \in \Omega^2(M)$ for which we can find at each point $x \in M$ a basis $b = (\mathbf{b}_0, \ldots, \mathbf{b}_3)$ which fulfils

$$g_x(\mathbf{b}_i, \mathbf{b}_j) = \eta_{ij}.$$

When we use an adapted coordinates, the metric reads $g = \eta_{ij}dx^i \otimes dx^j$.

One says that M is **time orientable** if one can find a vector field $T \in \mathfrak{X}(M)$ such that $g_x(T_x, T_x) > 0$ for all $x \in M$. A **time orientation** is a choice of such a vector field. A vector $v \in T_xM$ is **future directed** if $g_x(T_x, v) > 0$.

The Lorentz group L acts on the orthogonal basis of each T_xM, but you may note that L don't acts on M; it's just when the metric is flat that one can identify the whole manifold with a tangent space and consider that L is the space-times isometry group. In the case of a curved metric, the Lorentz group have to be introduced pointwise and the building of a frame bundle is natural.

Now, we are mainly interested in the frame related each other by a transformation of $L_+^\uparrow$. An arising question is to know if one can make a choice of some basis of each T_xM in such a manner that

(i) pointwise, the chosen frames are related by a transformation of $L_+^\uparrow$,

(ii) the choice is globally well defined.

The first point is trivial to fulfil from the definition of a space-time. For the second, it turns out that a good choice can be performed if and only if there exists a vector field $V \in \mathfrak{X}(M)$ such that $g_x(V_x, V_x) > 0$ for all $x \in M$. We suppose that it is the case[7].

So our first principal bundle attempt to describe the space-time symmetry is the $L_+^\uparrow$-principal bundle of orthonormal oriented frame on M :

$$\begin{tikzcd} L_+^\uparrow \arrow[r, rightsquigarrow] & L(M) \arrow[d, "p_L"] \\ & M \end{tikzcd} \tag{4.131}$$

The notion of "**relativistic invariance**" has to be understood in the sense of associated bundle to this one. The next step is to recall ourself (see subsection 4.2.1) that the physical fields doesn't transform under representation of the group $L_+^\uparrow$ but rather under representations of $\mathrm{SL}(2,\mathbb{C})$. So we build a $\mathrm{SL}(2,\mathbb{C})$-principal bundle

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{C}) \arrow[r, rightsquigarrow] & S(M) \arrow[d, "p_S"] \\ & M \end{tikzcd}$$

In order this bundle to "fit" as close as possible the bundle (4.131)-160, we impose the existence of a map $\lambda\colon S(M) \to L(M)$ such that

(i) $p_B(\lambda(\xi)) = p_S(\xi)$ for all $\xi \in S(M)$ and

(ii) $\lambda(\xi \cdot g) = \lambda(\xi) \cdot \mathrm{Spin}(g)$ for all $g \in \mathrm{SL}(2,\mathbb{C})$.

You can recognize the definition of a **spin structure**. Notice that the existence of a spin structure on a given manifold is a non trivial issue.

Now a physical field is given by a section of the associated bundle $E = S(M) \times_\rho V$ where ρ is a representations of $\mathrm{SL}(2,\mathbb{C})$ on V. For an electron, it is $V = \mathbb{C}^4$ and $\rho = D^{(1/2,0)} \oplus D^{(0,1/2)}$. That describes a *free* electron is the sense that it doesn't interacts with a gauge field. So in order to write down the formalism in which lives a non zero spin particle, we have to build a $U(1) \times \mathrm{SL}(2,\mathbb{C})$-principal bundle. For this, we follow the procedure given in section 1.15

4.8 Interactions

4.8.1 Spin zero

The general framework is the following :

$$\begin{tikzcd} U(1) \arrow[r, rightsquigarrow] & P \arrow[d, "\pi"'] & & E = P \times_\rho V \\ & M & \mathcal{U}_\alpha \arrow[l] \arrow[ul, "\sigma_\alpha"'] \arrow[ur, "\phi"'] & \end{tikzcd}$$

[7] That condition is rather restrictive because we cannot, for example, find an everywhere non zero vector field on the sphere S^n with n even.

a $U(1)$-principal bundle over a manifold M (as far as topological subtleties are concerned, we suppose $M = \mathbb{R}^4$) and a section ϕ of an associated bundle for a representation ρ of $U(1)$ on V. We consider M with the Lorentzian metric but, since we are intended to treat with scalar (spin zero) fields, we still don't include the Lorentz (or $\mathrm{SL}(2,\mathbb{C})$) group in the picture. We also consider local sections $\sigma_\alpha\colon \mathcal{U}_\alpha \to P$, a connection ω on P and Ω its curvature. We define $A_\alpha = \sigma_\alpha^*\omega$.

Now we particularize ourself to the target space $V = \mathbb{C}$ on which we put the scalar product

$$\langle z_1, z_2\rangle = \frac{1}{2}(z_1\overline{z}_2 + z_2\overline{z}_1), \tag{4.132}$$

and the representation $\rho_n\colon U(1) \to GL(\mathbb{C})$,

$$\rho_n(g)z = g \cdot z = g^n z$$

where we identify $U(1)$ to the unit circle in $\mathbb{C}$ in order to compute the product. A property of the product (4.132)-161 is to make ρ_n an isometry: for all $g \in U(1)$, $z_1, z_2 \in \mathbb{C}$,

$$\langle \rho_n(g)z_1, \rho_n(g)z_2\rangle = \langle z_1, z_2\rangle.$$

Our first aim is to write the covariant derivative of ϕ with respect to the connection ω. For this we work on the section ϕ under the form $\phi_{(\alpha)}\colon M \to V$ and we use formula (4.70)-146 :

$$(D_X\phi)_{(\alpha)}(x) = X_x\phi_{(\alpha)} - \rho_*\big((\sigma_\alpha^*\omega)_x X_x\big)\phi_{(\alpha)}(x). \tag{4.133}$$

Let us study this formula. We know that $(\sigma_\alpha^*\omega)_x = A_\alpha(x) : T_x\mathcal{U}_\alpha \xrightarrow{\sigma} T_{\sigma_\alpha(x)}P \xrightarrow{\omega} u(1)$. Thus $A_\alpha(x)X_x$ is given by a path in $U(1)$; it is this path which is taken by ρ_*. Therefore (we forget some dependences in x)

$$\begin{aligned}
\rho_*\big(A_\alpha(x)X_x\big)\phi_{(\alpha)}(x) &= \frac{d}{dt}\Big[\rho_n\big((A_\alpha X)(t)\big)\phi_{(\alpha)}(x)\Big]_{t=0}\\
&= \frac{d}{dt}\Big[(A_\alpha X)(t)^n\Big]_{t=0}\phi_{(\alpha)}(x)\\
&= n\frac{d}{dt}\Big[(A_\alpha X)(t)\Big]_{t=0}\phi_{(\alpha)}(x)\\
&= nA_\alpha(X)\phi_{(\alpha)}(x).
\end{aligned} \tag{4.134}$$

Thus the covariant derivative is given by

$$(D_X\phi)_{(\alpha)}(x) = X_x\phi_{(\alpha)} - nA_\alpha(x)(X_x)\phi_{(\alpha)}(x). \tag{4.135}$$

One can guess an electromagnetic coupling for a particle of electric charge n. If this reveals to be physically relevant, it shows that the "electromagnetic identity card" of a particle is given by a representation of $U(1)$. This has to be seen in relation to the discussion on page 143 where the "type of particle" was closely related to representations of the Lorentz group. It is a remarkable piece of quantum field theory: the properties of a particle are encoded in representations of some symmetry groups.

Now we are going to prove that $\|D\phi\|^2$ is a gauche invariant quantity. The first step is to give a sense to this norm. We consider X_i ($i = 0,1,2,3$), an orthonormal basis of T_xM and we naturally denote $D_i = D_{X_i}$, $\partial_i = X_i$ and $A_{\alpha i} = A_\alpha(\partial_i)$. Remark that

$$A_\alpha(x)X_x = (\sigma_\alpha^*)_x X_x = \omega(d\sigma_\alpha X_x) = \omega\frac{d}{dt}\Big[\sigma_\alpha(X(t))\Big]_{t=0} \in u(1), \tag{4.136}$$

so this is given by a path in $U(1)$ which can be taken by ρ. Let $c(t)$ be this path, then

$$A_\alpha \phi_{(\alpha)}(x) = \frac{d}{dt}\Big[e^{ic(t)}\phi_{(\alpha)}(x)\Big]_{t=0},$$

so that under the conjugation, $\overline{A_\alpha \phi_{(\alpha)}(x)} = -A_\alpha \overline{\phi}_{(\alpha)}(x)$. Now our definition of $\|D\phi\|^2$ is a composition of the norm on V and the one on T_xM :

$$\|D\phi\|^2 = \eta^{ij}\langle D_i\phi_{(\alpha)}, D_j\phi_{(\alpha)}\rangle \tag{4.137}$$

Using the notation in which the upper indices are contractions with η^{ij}, we have

$$\|D\phi\|^2 = \Big((\partial_i\phi_{(\alpha)})(x) - nA_{\alpha i}\phi_{(\alpha)}(x)\Big)\Big((\partial^i\overline{\phi}_{(\alpha)})(x) + nA^i_\alpha\overline{\phi}_{(\alpha)}(x)\Big).$$

Gauge transformation law

A gauge transformation φ is given by an equivariant function $\tilde{\varphi}_\alpha\colon \mathcal{U}_\alpha \to U(1)$ which can be written under the form

$$\tilde{\varphi}_\alpha(x) = e^{i\Lambda(x)}$$

for a certain function $\Lambda\colon \mathcal{U}_\alpha \to \mathbb{R}$. From the general formula (ii) of lemma 4.13,

$$(\varphi\cdot\phi)_{(\alpha)}(x) = \rho_n(e^{-i\Lambda(x)})\phi_{(\alpha)}(x) = e^{-ni\Lambda(x)}\phi_{(\alpha)}(x). \tag{4.138}$$

The transformation of the gauche field A is given by equation (4.74)-146. Let us see the meaning of the term $d\tilde{\varphi}$. For $v \in T_x\mathcal{U}_\alpha$,

$$(d\tilde{\varphi}_\alpha)_x v = \frac{d}{dt}\Big[\tilde{\varphi}_\alpha(v(t))\Big]_{t=0} = \frac{d}{dt}\Big[e^{i\Lambda(v(t))}\Big]_{t=0} = i\frac{d}{dt}\Big[\Lambda(v(t))\Big]_{t=0} e^{i\Lambda(v(0))} = i(d\Lambda)_x v e^{i\Lambda(x)}. \tag{4.139}$$

Thus $\tilde{\varphi}_\alpha^{-1}(x)(d\tilde{\varphi}_\alpha)_x = i(d\Lambda)_x$. Since $U(1)$ is abelian, $\tilde{\varphi}^{-1}A\tilde{\varphi} = A$. Finally,

$$(\varphi\cdot A)_\alpha(x) = A_\alpha(x) + i(d\Lambda)_x. \tag{4.140}$$

Now we are able to prove the invariance of $\|D\phi\|^2$. First,

$$(\varphi\cdot A)_{i\alpha}(x) = (\varphi\cdot A)_\alpha(\partial_i) = A_{i_\alpha}(x) + i(\partial_i\Lambda)(x); \tag{4.141}$$

second,

$$\partial_i\left(e^{-ni\Lambda(x)}\phi_{(\alpha)}(x)\right) = -ni(\partial_i\Lambda)(x)\phi_{(\alpha)}(x) + e^{-in\Lambda(x)}(\partial_i\phi_{(\alpha)})(x). \tag{4.142}$$

With these two results,

$$\partial_i(\varphi\cdot\phi)_{(\alpha)}(x) + n(\varphi\cdot A)_{\alpha i}(\varphi\cdot\phi)_{(\alpha)}(x) = e^{-in\Lambda(x)}(nA_{\alpha i}(x) + \partial_i\phi_{(\alpha)}(x)). \tag{4.143}$$

The Yang-Mills **field strength** is given by $F_{(\alpha)} = \sigma_\alpha^*\Omega$ (cf. page 56). Since $U(1)$ is abelian, $dF_{(\alpha)} = 0$, so that the second pair of Maxwell's equations is complete without any Lagrangian assumptions.

The full Yang-Mills action is written as

$$S(\omega,\phi) = \int_M \left[-\frac{1}{4}F_{(\alpha)ij}F_{(\alpha)}{}^{ij} + \frac{1}{2}\|D\phi\|^2 + \frac{1}{2}m\phi_{(\alpha)}\overline{\phi_{(\alpha)}}\right].$$

The Euler-Lagrange equations are

$$(\partial_i - inA_{\alpha i})(\partial^i - inA^i_\alpha)\phi_\alpha + m^2\phi_\alpha = 0 \tag{4.144a}$$
$$\partial_i {F_{(\alpha)}}^{ij} = 0. \tag{4.144b}$$

So the Yang-Mills Lagrangian only gives the first pair of Maxwell's equations while the second one is given by the geometric nature of fields.

As explained in [38], the topology of the physical space has deep implications on the physics of Yang-Mills equations. The absence of magnetic monopoles for example is ultimately linked to the (simple) connectedness of $\mathbb{R}^4$. When one consider the $U(1)$ Yang-Mills on a sphere, some topological charges appear and magnetic monopoles naturally arise.

4.8.2 Non zero spin formalism

The formalism for a non zero spin particle in an electromagnetic field is described in section 1.15. We consider the spinor bundle

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{C}) \arrow[r, squiggly] & S(M) \arrow[d, "p_S"] \\ & M \end{tikzcd}$$

with the spinor connection on $S(M)$, and ρ_1, a representation of $\mathrm{SL}(2,\mathbb{C})$ on V. For an electron, it is $V = \mathbb{C}^4$ and $\rho_1 = D^{(1/2,0)} \oplus D^{(0,1/2)}$, so for $g_1 \in \mathrm{SL}(2,\mathbb{C})$,

$$\rho_1(g_1)\begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} = \begin{pmatrix} g_1 & \\ & (\overline{g_1}^t)^{-1} \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix}. \tag{4.145}$$

On the other hand, we consider the principal bundle

$$\begin{tikzcd} U(1) \arrow[r, squiggly] & P \arrow[d, "p_U"] \\ & M \end{tikzcd}$$

with a connection ω_2 which describes the electromagnetic field. As representation $\rho_2 \colon U(1) \to GL(\mathbb{C}^4)$ we choose the multiplication coordinate by coordinate :

$$\rho_2(g_2)\begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} = \begin{pmatrix} g_2 z_1 \\ \vdots \\ g_2 z_4 \end{pmatrix}. \tag{4.146}$$

The physical picture of the electron is now the principal bundle

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{C}) \times U(1) \arrow[r, squiggly] & S(M) \circ P \arrow[d, "p"] \\ & M, \end{tikzcd}$$

and the field is a section of the associated bundle $(S(M) \circ P) \times_\rho \mathbb{C}^4$.

.1 Alternative formalism for the quantum mechanics

We can a little reformulate the axioms of the quantum mechanics. Since we are in a Hilbert space $\mathscr{H}$ we can speak about orthogonal projections; if $\phi \in \mathcal{R}$, we can consider the projection on the space spanned by ϕ:

$$P_\phi e_k = \frac{\langle \phi, e_k \rangle}{\|\phi\|}\phi$$

where $\{e_i\}$ is a basis of $\mathscr{H}$. It is pretty clear that

$$\mathrm{Tr}(P_\phi P_\psi) = \frac{|\langle \psi, \phi \rangle|^2}{\|\psi\|\|\phi\|}. \tag{.147}$$

If $\psi \in \mathcal{R}$ and $\phi \in \mathcal{R}'$ are unimodular, then

$$P(\mathcal{R} \to \mathcal{R}') = \mathrm{Tr}(P_\phi P_\psi), \tag{.148}$$

so we can express the axioms in terms of projections instead of rays. For notational convenience, we put

$$\mathscr{H}_1 = \{\psi \in \mathscr{H} \text{ st } \|\psi\| = 1\}. \tag{.149}$$

We denote by $\mathscr{S}$ the space of the projections into one dimensional subspaces of $\mathscr{H}$ (in other words $\mathscr{S}$ is the space of physical states) and for P, $Q \in \mathscr{S}$, the transition probability is $P \cdot Q = \mathrm{Tr}(PQ)$. Now a **quantum symmetry** is a map $T\colon \mathscr{S} \to \mathscr{S}'$ such that $(TP) \cdot (TQ) = P \cdot Q$.

One can prove the following :

Theorem .25.
If $T\colon \mathscr{S} \to \mathscr{S}'$ is a quantum symmetry, then there exists an operator $U\colon \mathscr{H} \to \mathscr{H}'$ such that

(i) $P_{U\phi} = TP_\phi$,

(ii) $U(\xi + \eta) = U(\xi) + U(\eta)$,

(iii) $\langle U\xi, U\eta \rangle = \kappa(\langle \xi, \eta \rangle)$

where P_ψ is the projection onto the one dimensional space spanned by ψ and $\kappa\colon \mathbb{C} \to \mathbb{C}$ fulfils $\kappa(\lambda) = \lambda$ or $\kappa(\lambda) = \overline{\lambda}$ and

(iv) $U(\lambda\xi) = \kappa(\lambda)\xi$.

Here is why this implies Wigner's theorem as given by theorem 4.2. Let us consider some $\varphi_i \in \mathscr{H}$ such that $\|\varphi_i\| = 1$ and P_{φ_i}, the corresponding projections. Let

$$\Delta(P_1, P_2, P_3) = \langle \varphi_1, \varphi_2 \rangle \langle \varphi_2, \varphi_3 \rangle \langle \varphi_3, \varphi_1 \rangle.$$

It is clear that this expression doesn't depend on the choice of φ_i in its ray. We have

$$\begin{aligned}
\Delta(TP_1; TP_2, TP_3) &= \Delta(P_{U\varphi_1}, P_{U\varphi_2}, P_{U\varphi_3}) \\
&= \langle U\varphi_1, U\varphi_2 \rangle \langle U\varphi_2, U\varphi_3 \rangle \langle U\varphi_3, U\varphi_1 \rangle \\
&= \kappa(\langle U\varphi_1, U\varphi_2 \rangle)\kappa(\langle U\varphi_2, U\varphi_3 \rangle)\kappa(\langle U\varphi_3, U\varphi_1 \rangle) \\
&= \kappa\big(\Delta(P_1, P_2, P_3)\big).
\end{aligned} \tag{.150}$$

We can see from this that the choice of $\kappa(\lambda) = \lambda$ or $\kappa(\lambda) = \overline{\lambda}$ is determined by the data of T if $\dim \mathscr{H} \geqslant 2$. In the case where $\dim \mathscr{H} = 1$, Δ is always equals to 1 and the equality (.150)-164 don't gives any informations. In the case $\dim \mathscr{H} \geqslant 2$, we can choice φ_1 and φ_2 such that $\langle \varphi_1, \varphi_2 \rangle$ takes any value $z \in \mathbb{C}$ with $\|z\| \leqslant 1$. Taking $\varphi_3 = \varphi_1 + \varphi_2$, we find

$$\Delta(P_1, P_2, P_3) = z(1 + \overline{z})^2.$$

which is easily non real for a suitable choice of $z \in \mathbb{C}$. Let us suppose that we have an operator U which satisfies the theorem .25. If $\kappa(\lambda) = \lambda$, then

$$U(z\psi + z'\phi) = U(z\psi) + U(z'\phi) = zU(\psi) + zU(\phi) \tag{.151}$$

and

$$\langle U\psi, U\phi \rangle = \kappa(\langle \psi, \phi \rangle) = \langle \psi, \phi \rangle, \tag{.152}$$

so that U is linear. If $\kappa(\lambda) = \overline{z}$, then

$$U(z\psi) = \overline{z} U\psi \tag{.153}$$

and

$$\langle U\xi, U\eta \rangle = \kappa(\langle \xi, \eta \rangle) = \overline{\langle \xi, \eta \rangle}. \tag{.154}$$

.2 Statement of some results

This appendix is devoted to the statement of some results which are used in the text, but whose demonstration should be out of our purpose.

Theorem .26.
Let G be a Lie group and H a subgroup (with no special other structures) of G. If H is a closed subset of G then there exists an unique analytic structure on H such that H is a topological Lie subgroup of G.

This comes from [3], chapter 2, theorem 2.3.

Theorem .27.
Let G be a Lie group, H a closed subgroup of G and G/H the space of left cosets $[g] = \{gh | h \in H\}$ with the natural topology. Then G/H has an unique analytic structure with the property that G is a Lie transformation group of G/H.

This comes from [3], chapter 2, theorem 4.2.

Lemma .28.
Let G be a connected Lie group with Lie algebra $\mathcal{G}$ and let φ be an analytic homomorphism of G into a Lie group X with Lie algebra $\mathcal{X}$. Then

(i) The kernel $\varphi^{-1}(e)$ is a topological Lie subgroup of G. Its Lie algebra is the kernel of $d\varphi_e$.

(ii) The image $\varphi(G)$ is a Lie subgroup of $\mathcal{X}$ with Lie algebra $d\varphi(\mathcal{G}) \subset \mathcal{X}$.

This comes from [3], chapter 2, lemma 5.1.

Lemma .29.
Let G and H be two Lie group, whose Lie algebra are $\mathcal{G}$ and $\mathcal{H}$. If $\theta\colon G \to H$ is a surjective map, then we have $\mathcal{H} \simeq \mathcal{G}/Ker\, d\theta_e$.

Theorem .30.
Let us consider $Ad\colon SU(2) \to GL(3)$, $Ad(U)X = UXU^{-1}$. We have the following properties:

(i) *Ad is a linear homomorphism,*

(ii) *it takes his values in $\mathrm{SO}(3)$; then we can write $Ad\colon SU(2) \to \mathrm{SO}(3)$,*

(iii) *it is surjective,*

(iv) *$Ker\, Ad = \mathbb{Z}_2$,*

(v) *all these properties show that*
$$\mathrm{SO}(3) = \frac{SU(2)}{\mathbb{Z}_2}.$$

Corollary .31.
An useful formula:
$$\mathrm{Ad}(e^X) = e^{\mathrm{ad}\, X}.$$

Corollary .32.
Another useful corollary of lemma 1.20 is the particular case $\phi = \boldsymbol{Ad}(e^X)$:
$$e^X e^Y e^{-X} = e^{Ad(e^Y)X}.$$

Definition .33.
If (a_k) is a sequence in $\mathbb{R}$, its ***upper limit*** *is the real number*
$$\limsup_{n\to\infty} a_n = \lim_{l\to\infty} \sup\{a_k : k \geqslant l\}.$$

Lemma .34.
If ω is a k-form (not specially a symplectic one), and ∇ a torsion free connection, one has
$$(d\omega)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \ldots, \hat{X}_i, \ldots X_k). \tag{.155}$$

Remark .35. The link between d and ∇ comes from the fact that in the left hand side of (.155)-166 appears some commutators $[X_i, X_j]$, but since the connection is torsion-free,
$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i$$

The main consequence of this lemma is that $\nabla\omega = 0$ implies $d\omega = 0$.

Proposition .36.
Consider a function $f\colon X \times E \to \overline{\mathbb{R}}$ and $z_0 \in E$ such that

- *for all $z \in E$, the function $x \to (x, z)$ is integrable,*

- *for (almost) all* $x \in X$*, the function* $z \to f(x, z)$ *is continuous at* z_0*,*
- *there exists a function* $g \geqslant 0$ *such that for all* $z \in E$*,* $|f(x, z)| \leqslant g(x)$ *almost everywhere in* X*.*

Then the function $h\colon E \to \mathbb{R}$ *defined by* $h(z) = \int_X f(x, z)$ *is continuous at* z_0*.*

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List of symbols

Algebra

(α, β) Inner product on the dual $\mathfrak{h}^*$ of a Cartan algebra, page 69

$\mathfrak{A}^\times$ The set of invertible elements of the algebra $\mathfrak{A}$; for example $\mathbb{R}^\times = \mathbb{R}\backslash\{0\}$, page 103

χ A representation of $\Gamma(p, q)$, page 101

$\tilde{\rho}\colon (\mathbb{R}^2)^{\mathbb{C}} \to \mathrm{End}(\Lambda W)$ Spinor representation, page 111

$\tilde{\rho}\colon (\mathbb{R}^{1+3})^{\mathbb{C}} \to \mathrm{End}(\Lambda W)$ Spinor representation, page 96

$N : \Gamma(p, q) \to \Gamma(p, q)$ Spin norm, page 101

$\mathbb{H}$ quaternionic algebra, page 111

$\Gamma(E)$ Space of sections of the vector bundle E, page 30

γ_i Abstract definition of Dirac matrices, page 98

γ_i Explicit form of gamma matrices, page 100

$\mathrm{Ind}_{\mathfrak{A}}^{\mathrm{Cl}(V)}(E_1)$ Induced Clifford module, page 113

ΛW Space of spinor representation, page 96

$\Lambda W^\pm$ Spinor space, page 111

$\Omega(M, V)$ V valued 1-forms, page 17

ω_i^j Connection form, page 51

$\mathrm{Ray}\,\mathscr{H}$ Rays in a Hilbert space, page 129

$\mathcal{Q}$ A subgroup of $\mathcal{G}$, page 104

A_α Gauge potentials, page 145

d_ω Exterior covariant derivative associated with the connection form ω, page 57

$F_{\mu\nu}$ Electromagnetic field strength, page 149

J_μ Electromagnetic 4-current, page 149

$U(n, \theta)$ Rotation operator on functions, page 84

$W, \underline{W}$ Totally isotropic subspace, page 96

$x^{\perp}$ Space orthogonal to x, page 102

Differential geometry

$(\theta_\alpha)_i^j$ Matrix associated with a connection, page 47

$[\omega \wedge \eta]$ Combination of the wedge and the bracket in the case of Lie algebra-valued forms, page 31

$\mathrm{Ad}(P)$ Adjoint bundle of the principal bundle P, page 46

$\mathcal{D}$ Dirac operator, page 121

$\gamma\colon \mathfrak{X}(M) \to \operatorname{End}\Gamma(\mathcal{S})$ A key ingredient for Dirac operator, page 121

$\tilde{\nabla}\colon \mathfrak{X}(M) \times \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ Covariant derivative for the spinor connection, page 119

$\Omega(M, E)$ the set of E-valued differential forms, page 30

Functional analysis

Δf Laplace operator, page 50

∇f Gradient of the function f, page 50

$\nabla \cdot X$ divergence of the vector field X, page 50

Lie groups and algebras

$\mathrm{Cl}(2)$ Clifford algebra of $\mathbb{R}^2$, page 111

$\mathrm{Cl}(p, q)$ Clifford algebra of $\mathbb{R}^{1,3}$, page 92

$\mathrm{Cl}(p, q)^{\pm}$ Grading of Clifford algebra, page 101

$\Gamma(p, q)$ Clifford group, page 101

$\mathfrak{Lie}(\Gamma(p, q)^{+})$ Algèbre de $\Gamma(p, q)^{+}$, page 109

$\mathrm{Spin}(p, q)$ Spin group of $\mathbb{R}^{1,3}$, page 101

$\mathfrak{spin}(p, q)$ Lie algebra of the group $\mathrm{Spin}(p, q)$, page 109

$\mathrm{Spin}(V)$ The spin group, page 108

$\mathrm{Spin}^c(V)$ A group related to Spin, page 108

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