The *dot product* (scalar product)

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

is a scalar

The *cross product* (vector product) $\mathbf{a} \times \mathbf{b}$ is a vector with magnitude $|\mathbf{a}||\mathbf{b}| \cos\theta$ and a direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in a right-handed sense.

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3$$

The *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is a scalar

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a}$$

The *vector triple product* is a vector

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

The Kronecker Delta is *symmetric* $\delta_{ij} = \delta_{ji}$ and $\delta_{ij}a_j = a_i$

The Alternating Tensor:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i,\ j \text{ or } k \text{ are equal,} \\ 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (3,2,1) \text{ or } (2,1,3) \end{cases}$$

The Alternating Tensor is *antisymmetric*:

$$\epsilon_{ijk} = -\epsilon_{jik}$$

The Alternating Tensor is invariant under cyclic permutations of the indices:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon kij$$

The vector product:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}\, a_j b_k$$

The relation between δ_{ij} and ϵ_{ijk}:

$$\epsilon_{ijk}\, \epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

In all of the below formulae we are considering the vector $\boldsymbol{F} = (F_1, F_2, F_3)$

Basic Vector Differentiation

If $\boldsymbol{F} = \boldsymbol{F}(t)$ then

$$\frac{d\boldsymbol{F}}{dt} = \left(\frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt}\right)$$

The unit tangent to the curve $\mathbf{x} = \boldsymbol{\psi}(t)$ is given by

$$\frac{d\mathbf{x}/dt}{|d\mathbf{x}/dt|}$$

Grad, Div and Curl

The *gradient* of a scalar field $f(x, y, z)$ $(= f(x_1, x_2, x_3))$ is given by

$$\text{grad} f = \boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

$\boldsymbol{\nabla} f$ is the vector field with a direction perpendicular to the isosurfaces of f with a magnitude equal to the rate of change of f in that direction.

The *directional derivative* of f in the direction of a unit vector $\hat{\boldsymbol{u}}$ is $(\boldsymbol{\nabla} f) \cdot \hat{\boldsymbol{u}}$

$\boldsymbol{\nabla}$ pronounced *del* or *nabla* is a *vector differential operator*. It is possible to study the 'algebra of $\boldsymbol{\nabla}$'.

The *divergence* of a vector field $\boldsymbol{F}$ is given by

$$\text{div}\, \boldsymbol{F} = \boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

A vector field $\boldsymbol{F}$ is *solenoidal* if $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$ everywhere.

The *curl* of a vector field $\boldsymbol{F}$ is given by

$$\begin{aligned}
\text{curl}\, \boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathbf{e}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{e}_3 \\
&= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) \mathbf{e}_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) \mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) \mathbf{e}_3 \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}
\end{aligned}$$

A vector field $\boldsymbol{F}$ is *irrotational* if $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$ everywhere.

$(\boldsymbol{F}\cdot\boldsymbol{\nabla})$ is a vector differential operator which can act on a scalar or a vector

$$(\boldsymbol{F}\cdot\boldsymbol{\nabla})\, f = F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z}$$

$$(\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G} = ((\boldsymbol{F}\cdot\boldsymbol{\nabla})\, G_1\,,\, (\boldsymbol{F}\cdot\boldsymbol{\nabla})\, G_2\,,\, (\boldsymbol{F}\cdot\boldsymbol{\nabla})\, G_3)$$

The *Laplacian* operator $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ can act on a scalar or a vector.

Grad, Div and Curl and suffix notation

In suffix notation

$$\mathbf{r} = (x, y, z) = x_i$$

$$\text{grad} f = (\boldsymbol{\nabla} f)_i = \frac{\partial f}{\partial x_i}$$

$$(\boldsymbol{\nabla})_i = \frac{\partial}{\partial x_i}$$

$$\text{div}\,\boldsymbol{F} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_j}{\partial x_j}$$

$$(\text{curl}\,\boldsymbol{F})_i = (\boldsymbol{\nabla}\times\boldsymbol{F})_i = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}$$

$$(\boldsymbol{F}\cdot\boldsymbol{\nabla}) = F_j\frac{\partial}{\partial x_j}$$

Note: Here you cannot move the $\dfrac{\partial}{\partial x_j}$ around as it acts on everything that follows it.

If $\boldsymbol{F}$ and $\boldsymbol{G}$ are vector fields and φ and ψ are scalar fields then

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\varphi) = \nabla^2\varphi$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi) = \mathbf{0}$$

$$\boldsymbol{\nabla}(\varphi\psi) = \varphi\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\varphi$$

$$\boldsymbol{\nabla}\cdot(\varphi\boldsymbol{F}) = \varphi\boldsymbol{\nabla}\cdot\boldsymbol{F} + \boldsymbol{F}\cdot\boldsymbol{\nabla}\varphi$$

$$\boldsymbol{\nabla}\times(\varphi\boldsymbol{F}) = \varphi\boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla}\varphi\times\boldsymbol{F}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

$$\boldsymbol{\nabla}(\boldsymbol{F}\cdot\boldsymbol{G}) = \boldsymbol{F}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) + \boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) + (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G})$$

$$\boldsymbol{\nabla}\times(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{F}(\boldsymbol{\nabla}\cdot\boldsymbol{G}) - \boldsymbol{G}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G}$$

An alternative definition of *divergence* is given by

$$\nabla\cdot\boldsymbol{F} = \lim_{\delta V\to 0}\frac{1}{\delta V}\oiint_{\delta S}\boldsymbol{F}\cdot\boldsymbol{n}\,dS,$$

where δV is a small volume bounded by a surface δS which has outward-pointing normal $\boldsymbol{n}$.

An alternative definition of *curl* is given by

$$\boldsymbol{n}\cdot\nabla\times\boldsymbol{F} = \lim_{\delta S\to 0}\frac{1}{\delta S}\oint_{\delta C}\boldsymbol{F}\cdot d\boldsymbol{r},$$

where δS is a small open surface bounded by a curve δC which is oriented in a right-handed sense.

Physical Interpretation of divergence and curl

The divergence of a vector field gives a measure of how much expansion and contraction there is in the field.

The curl of a vector field gives a measure of how much rotation or twist there is in the field.

The Divergence and Stokes' Theorems

The *divergence theorem* states that

$$\iiint_V \nabla\cdot\boldsymbol{F} = \oiint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS,$$

where S is the closed surface enclosing the volume V and $\boldsymbol{n}$ is the outward-pointing normal from the surface.

Stokes' theorem states that

$$\iint_S \nabla\times\boldsymbol{F}\cdot\boldsymbol{n}\,dS = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r},$$

where C is the closed curve enclosing the open surface S and $\boldsymbol{n}$ is the normal from the surface.

Conservative Vector fields, line integrals and exact differentials

The following 5 statements are equivalent in a simply-connected domain:

(i) $\nabla\times\boldsymbol{F} = 0$ at each point in the domain.

(ii) $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ around every closed curve in the region.

(iii) $\int_P^Q \boldsymbol{F}\cdot d\boldsymbol{r}$ is independent of the path of integration from P to Q.

(iv) $\boldsymbol{F}\cdot d\boldsymbol{r}$ is an exact differential.

(v) $\boldsymbol{F} = \nabla\phi$ for some scalar ϕ which is single-valued in the region.

If $\nabla\cdot\boldsymbol{F} = 0$ then $\boldsymbol{F} = \nabla\times\boldsymbol{A}$ for some $\boldsymbol{A}$. (This *vector potential* $\boldsymbol{A}$ is not unique.)