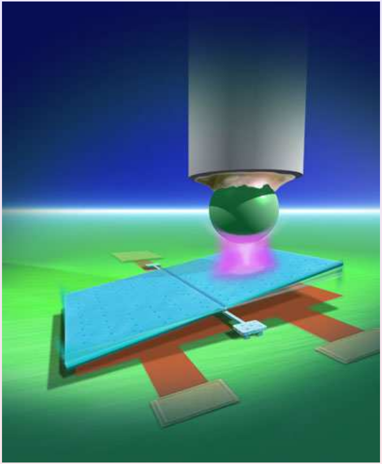
The Mathematics of the Casimir Effect

Thomas Prellberg

School of Mathematical Sciences
Queen Mary, University of London

Annual Lectures
February 19, 2007



Casimir forces: still surprising after 60 years

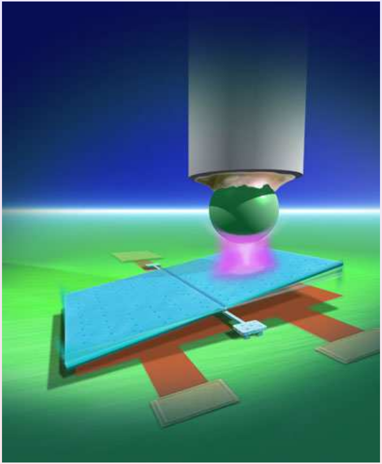
Physics Today, February 2007

The Casimir effect heats up

AIP News Update, February 7, 2007

Scientists devise test for string theory

EE Times, February 6, 2007



Casimir forces: still surprising after 60 years

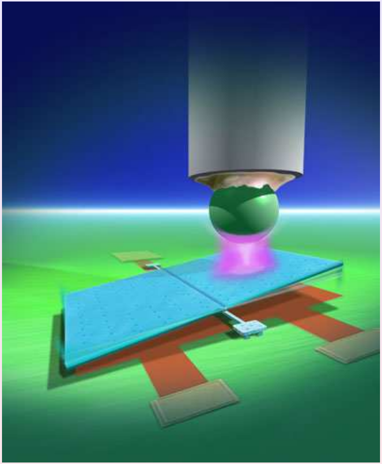
Physics Today, February 2007

The Casimir effect heats up

AIP News Update, February 7, 2007

Scientists devise test for string theory

EE Times, February 6, 2007



Casimir forces: still surprising after 60 years

Physics Today, February 2007

The Casimir effect heats up

AIP News Update, February 7, 2007

Scientists devise test for string theory

EE Times, February 6, 2007

Topic Outline

Outline

- student of Ehrenfest, worked with Pauli and Bohr
- *retarded Van-der-Waals-forces*

$$E = \frac{23}{4\pi}\hbar c \frac{\alpha}{R^7}$$

Casimir and Polder, 1948

- *Force between cavity walls*

$$F = -\frac{\pi^2 \hbar c}{240}\frac{A}{d^4}$$

Casimir, 1948

Hendrik Brugt Gerhard Casimir, 1909 - 2000

- student of Ehrenfest, worked with Pauli and Bohr
- *retarded Van-der-Waals-forces*

$$E = \frac{23}{4\pi}\hbar c \frac{\alpha}{R^7}$$

Casimir and Polder, 1948

- *Force between cavity walls*

$$F = -\frac{\pi^2 \hbar c}{240}\frac{A}{d^4}$$

Casimir, 1948

Hendrik Brugt Gerhard Casimir, 1909 - 2000

- student of Ehrenfest, worked with Pauli and Bohr
- *retarded Van-der-Waals-forces*

$$E = \frac{23}{4\pi}\hbar c \frac{\alpha}{R^7}$$

Casimir and Polder, 1948

- *Force between cavity walls*

$$F = -\frac{\pi^2 \hbar c}{240}\frac{A}{d^4}$$

Casimir, 1948

Hendrik Brugt Gerhard Casimir, 1909 - 2000

The Electromagnetic Field

- The electromagnetic field, described by the *Maxwell Equations*, satisfies the *wave equation*

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{x},t) = 0$$

- Fourier-transformation ($\vec{x} \leftrightarrow \vec{k}$) gives

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\vec{A}(\vec{k},t) = 0 \quad \text{with } \omega = c|\vec{k}|$$

  which, for each $\vec{k}$, describes a **harmonic oscillator**
- Quantising harmonic oscillators is easy...

The Electromagnetic Field

- The electromagnetic field, described by the *Maxwell Equations*, satisfies the *wave equation*

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{x},t) = 0$$

- Fourier-transformation ($\vec{x} \leftrightarrow \vec{k}$) gives

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\vec{A}(\vec{k},t) = 0 \quad \text{with } \omega = c|\vec{k}|$$

  which, for each $\vec{k}$, describes a **harmonic oscillator**
- Quantising harmonic oscillators is easy...

The Electromagnetic Field

- The electromagnetic field, described by the *Maxwell Equations*, satisfies the *wave equation*

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{x},t) = 0$$

- Fourier-transformation ($\vec{x} \leftrightarrow \vec{k}$) gives

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\vec{A}(\vec{k},t) = 0 \quad \text{with } \omega = c|\vec{k}|$$

  which, for each $\vec{k}$, describes a **harmonic oscillator**

- Quantising harmonic oscillators is easy...

Quantisation of the Field

- Each harmonic oscillator can be in a discrete state of energy

$$E_m(\vec{k}) = \left(m + \frac{1}{2}\right)\hbar\omega \quad \text{with } \omega = c|\vec{k}|$$

- Interpretation: m photons with energy $\hbar\omega$ and momentum $\hbar\vec{k}$
- In particular, the ground state energy $\frac{1}{2}\hbar\omega$ is non-zero!
- This leads to a **zero-point energy** density of the field

$$\frac{E}{V} = 2\int E_0(\vec{k})\,\frac{d^3k}{(2\pi)^3}$$

(factor 2 due to polarisation of the field)

- **Caveat:** this quantity is infinite...

Quantisation of the Field

- Each harmonic oscillator can be in a discrete state of energy

$$E_m(\vec{k}) = \left(m + \frac{1}{2}\right)\hbar\omega \quad \text{with } \omega = c|\vec{k}|$$

- Interpretation: m photons with energy $\hbar\omega$ and momentum $\hbar\vec{k}$
- In particular, the ground state energy $\frac{1}{2}\hbar\omega$ is non-zero!
- This leads to a **zero-point energy** density of the field

$$\frac{E}{V} = 2\int E_0(\vec{k})\,\frac{d^3k}{(2\pi)^3}$$

(factor 2 due to polarisation of the field)

- **Caveat:** this quantity is infinite...

Quantisation of the Field

- Each harmonic oscillator can be in a discrete state of energy

$$E_m(\vec{k}) = \left(m + \frac{1}{2}\right)\hbar\omega \quad \text{with } \omega = c|\vec{k}|$$

- Interpretation: m photons with energy $\hbar\omega$ and momentum $\hbar\vec{k}$
- In particular, the ground state energy $\frac{1}{2}\hbar\omega$ is non-zero!
- This leads to a **zero-point energy** density of the field

$$\frac{E}{V} = 2\int E_0(\vec{k})\,\frac{d^3k}{(2\pi)^3}$$

(factor 2 due to polarisation of the field)

- **Caveat:** this quantity is infinite...

Quantisation of the Field

- Each harmonic oscillator can be in a discrete state of energy

$$E_m(\vec{k}) = \left(m + \frac{1}{2}\right)\hbar\omega \quad \text{with } \omega = c|\vec{k}|$$

- Interpretation: m photons with energy $\hbar\omega$ and momentum $\hbar\vec{k}$
- In particular, the ground state energy $\frac{1}{2}\hbar\omega$ is non-zero!
- This leads to a **zero-point energy** density of the field

$$\frac{E}{V} = 2\int E_0(\vec{k})\,\frac{d^3k}{(2\pi)^3}$$

(factor 2 due to polarisation of the field)

- **Caveat:** this quantity is infinite...

Quantisation of the Field

- Each harmonic oscillator can be in a discrete state of energy

$$E_m(\vec{k}) = \left(m + \frac{1}{2}\right)\hbar\omega \quad \text{with } \omega = c|\vec{k}|$$

- Interpretation: m photons with energy $\hbar\omega$ and momentum $\hbar\vec{k}$
- In particular, the ground state energy $\frac{1}{2}\hbar\omega$ is non-zero!
- This leads to a **zero-point energy** density of the field

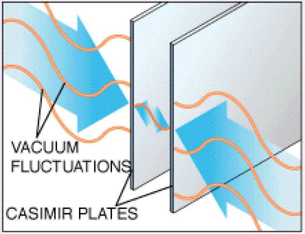
$$\frac{E}{V} = 2\int E_0(\vec{k})\,\frac{d^3k}{(2\pi)^3}$$

(factor 2 due to polarisation of the field)

- **Caveat:** this quantity is infinite...

Making (Physical) Sense of Infinity

The zero-point energy shifts due to a restricted geometry



- In the presence of the boundary

$$E_{discrete} = \sum_n E_{0,n}$$

is a sum over discrete energies $E_{0,n} = \frac{1}{2}\hbar\omega_n$

- In the absence of a boundary

$$E = 2V \int E_0(\vec{k}) \frac{d^3k}{(2\pi)^3}$$

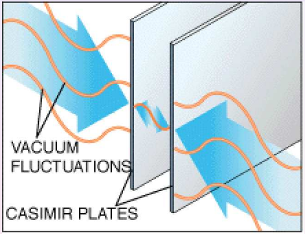
The difference of the infinite zero-point energies is finite!

$$\Delta E = E_{discrete} - E = -\frac{\pi^2\hbar c}{720}\frac{L^2}{d^3}$$

for a box of size $L \times L \times d$ with $d \ll L$

Making (Physical) Sense of Infinity

The zero-point energy shifts due to a restricted geometry



- In the presence of the boundary

$$E_{discrete} = \sum_n E_{0,n}$$

is a sum over discrete energies $E_{0,n} = \frac{1}{2}\hbar\omega_n$

- In the absence of a boundary

$$E = 2V \int E_0(\vec{k}) \frac{d^3k}{(2\pi)^3}$$

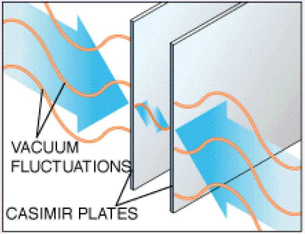
The difference of the infinite zero-point energies is finite!

$$\Delta E = E_{discrete} - E = -\frac{\pi^2\hbar c}{720}\frac{L^2}{d^3}$$

for a box of size $L \times L \times d$ with $d \ll L$

Making (Physical) Sense of Infinity

The zero-point energy shifts due to a restricted geometry



- In the presence of the boundary

$$E_{discrete} = \sum_n E_{0,n}$$

is a sum over discrete energies $E_{0,n} = \frac{1}{2}\hbar\omega_n$

- In the absence of a boundary

$$E = 2V \int E_0(\vec{k}) \frac{d^3k}{(2\pi)^3}$$

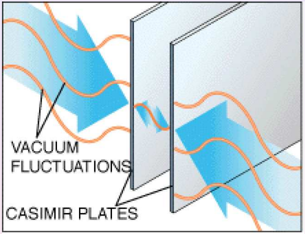
The difference of the infinite zero-point energies is finite!

$$\Delta E = E_{discrete} - E = -\frac{\pi^2 \hbar c}{720} \frac{L^2}{d^3}$$

for a box of size $L \times L \times d$ with $d \ll L$

Making (Physical) Sense of Infinity

The zero-point energy shifts due to a restricted geometry



- In the presence of the boundary

$$E_{discrete} = \sum_n E_{0,n}$$

is a sum over discrete energies $E_{0,n} = \frac{1}{2}\hbar\omega_n$

- In the absence of a boundary

$$E = 2V \int E_0(\vec{k}) \frac{d^3k}{(2\pi)^3}$$

The difference of the infinite zero-point energies is finite!

$$\Delta E = E_{discrete} - E = -\frac{\pi^2 \hbar c}{720} \frac{L^2}{d^3}$$

for a box of size $L \times L \times d$ with $d \ll L$

- Wolfgang Pauli's initial reaction: 'absolute nonsense'

- Experimental verification
  - Sparnaay (1958): 'not inconsistent with'
  - van Blokland and Overbeek (1978): experimental accuracy of 50%
  - Lamoreaux (1997): experimental accuracy of 5%

- Theoretical extensions
  - Geometry dependence
  - Dynamical Casimir effect
  - Real media: non-zero temperature, finite conductivity, roughness, . . .

We are done with the physics. Let's look at some mathematics!

- Wolfgang Pauli's initial reaction: 'absolute nonsense'

- Experimental verification
  - Sparnaay (1958): 'not inconsistent with'
  - van Blokland and Overbeek (1978): experimental accuracy of 50%
  - Lamoreaux (1997): experimental accuracy of 5%

- Theoretical extensions
  - Geometry dependence
  - Dynamical Casimir effect
  - Real media: non-zero temperature, finite conductivity, roughness, . . .

We are done with the physics. Let's look at some mathematics!

- Wolfgang Pauli's initial reaction: 'absolute nonsense'

- Experimental verification
  - Sparnaay (1958): 'not inconsistent with'
  - van Blokland and Overbeek (1978): experimental accuracy of 50%
  - Lamoreaux (1997): experimental accuracy of 5%

- Theoretical extensions
  - Geometry dependence
  - Dynamical Casimir effect
  - Real media: non-zero temperature, finite conductivity, roughness, ...

We are done with the physics. Let's look at some mathematics!

- Wolfgang Pauli's initial reaction: 'absolute nonsense'

- Experimental verification
  - Sparnaay (1958): 'not inconsistent with'
  - van Blokland and Overbeek (1978): experimental accuracy of 50%
  - Lamoreaux (1997): experimental accuracy of 5%

- Theoretical extensions
  - Geometry dependence
  - Dynamical Casimir effect
  - Real media: non-zero temperature, finite conductivity, roughness, ...

We are done with the physics. Let's look at some mathematics!

Outline

Spectral Theory

- Consider $-\Delta$ for a compact manifold Ω with a smooth boundary $\partial\Omega$
- On a suitable function space, this operator is self-adjoint and positive with pure point spectrum
- One finds formally

$$E_{discrete} = \frac{1}{2}\hbar c \,\mathrm{Trace}(-\Delta)^{1/2}$$

This would be a different talk — let's keep it simple for today

- Choose

$$\Omega = [0, L] \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2}$$

with Dirichlet boundary conditions $f(0) = f(L) = 0$.

Spectral Theory

- Consider $-\Delta$ for a compact manifold Ω with a smooth boundary $\partial\Omega$
- On a suitable function space, this operator is self-adjoint and positive with pure point spectrum
- One finds formally

$$E_{discrete} = \frac{1}{2}\hbar c \, \mathrm{Trace}(-\Delta)^{1/2}$$

This would be a different talk — let's keep it simple for today

- Choose

$$\Omega = [0, L] \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2}$$

with Dirichlet boundary conditions $f(0) = f(L) = 0$.

Spectral Theory

- Consider $-\Delta$ for a compact manifold Ω with a smooth boundary $\partial\Omega$
- On a suitable function space, this operator is self-adjoint and positive with pure point spectrum
- One finds formally

$$E_{discrete} = \frac{1}{2}\hbar c \, \mathrm{Trace}(-\Delta)^{1/2}$$

This would be a different talk — let's keep it simple for today

- Choose

$$\Omega = [0, L] \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2}$$

with Dirichlet boundary conditions $f(0) = f(L) = 0$.

Spectral Theory

- Consider $-\Delta$ for a compact manifold Ω with a smooth boundary $\partial\Omega$
- On a suitable function space, this operator is self-adjoint and positive with pure point spectrum
- One finds formally

$$E_{discrete} = \frac{1}{2}\hbar c \,\mathrm{Trace}(-\Delta)^{1/2}$$

This would be a different talk — let's keep it simple for today

- Choose

$$\Omega = [0, L] \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2}$$

with Dirichlet boundary conditions $f(0) = f(L) = 0$.

Spectral Theory

- Consider $-\Delta$ for a compact manifold Ω with a smooth boundary $\partial\Omega$
- On a suitable function space, this operator is self-adjoint and positive with pure point spectrum
- One finds formally

$$E_{discrete} = \frac{1}{2}\hbar c \, \mathrm{Trace}(-\Delta)^{1/2}$$

This would be a different talk — let's keep it simple for today

- Choose

$$\Omega = [0, L] \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2}$$

with Dirichlet boundary conditions $f(0) = f(L) = 0$.

Spectral Theory

- Consider $-\Delta$ for a compact manifold Ω with a smooth boundary $\partial\Omega$
- On a suitable function space, this operator is self-adjoint and positive with pure point spectrum
- One finds formally

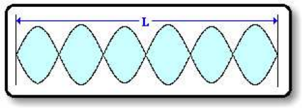
$$E_{discrete} = \frac{1}{2}\hbar c \,\mathrm{Trace}(-\Delta)^{1/2}$$

This would be a different talk — let's keep it simple for today

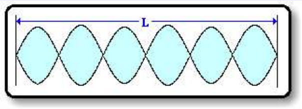
- Choose

$$\Omega = [0, L] \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2}$$

with Dirichlet boundary conditions $f(0) = f(L) = 0$.



Casimir Effect in One Dimension



- The solutions are standing waves with wavelength λ satisfying

$$n\frac{\lambda}{2} = L$$

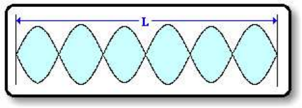
- We therefore find

$$E_{0,n} = \frac{1}{2}\hbar c \frac{n\pi}{L}$$

- The zero-point energies are given by

$$E_{discrete} = \frac{\pi}{2L}\hbar c \sum_{n=0}^{\infty} n \quad \text{and} \quad E = \frac{\pi}{2L}\hbar c \int_0^{\infty} t\, dt$$

Casimir Effect in One Dimension



- The solutions are standing waves with wavelength λ satisfying

$$n\frac{\lambda}{2} = L$$

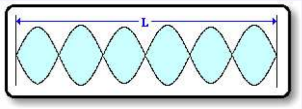
- We therefore find

$$E_{0,n} = \frac{1}{2}\hbar c \frac{n\pi}{L}$$

- The zero-point energies are given by

$$E_{discrete} = \frac{\pi}{2L}\hbar c \sum_{n=0}^{\infty} n \qquad and \qquad E = \frac{\pi}{2L}\hbar c \int_0^{\infty} t\, dt$$

Casimir Effect in One Dimension



- The solutions are standing waves with wavelength λ satisfying

$$n\frac{\lambda}{2} = L$$

- We therefore find

$$E_{0,n} = \frac{1}{2}\hbar c \frac{n\pi}{L}$$

- The zero-point energies are given by

$$E_{discrete} = \frac{\pi}{2L}\hbar c \sum_{n=0}^{\infty} n \qquad and \qquad E = \frac{\pi}{2L}\hbar c \int_0^{\infty} t \, dt$$

The Mathematical Problem

- We need to make sense of

$$\Delta E = E_{discrete} - E = \frac{\pi}{2L}\hbar c \left(\sum_{n=0}^{\infty} n - \int_0^{\infty} t \, dt \right)$$

- More generally, consider

$$\Delta(f) = \sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t) \, dt$$

The Mathematical Problem

- We need to make sense of

$$\Delta E = E_{discrete} - E = \frac{\pi}{2L}\hbar c \left(\sum_{n=0}^{\infty} n - \int_0^{\infty} t \, dt \right)$$

- More generally, consider

$$\Delta(f) = \sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t) \, dt$$

Divergent Series

On the Whole, Divergent Series are the Works of the Devil and it's a Shame that one dares base any Demonstration upon them. You can get whatever result you want when you use them, and they have given rise to so many Disasters and so many Paradoxes. Can anything more horrible be conceived than to have the following oozing out at you:

$$0 = 1 - 2^n + 3^n - 4^n + etc.$$

where n is an integer number?

Niels Henrik Abel

Summing Divergent Series

- Some divergent series can be summed in a sensible way ...

$$S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

  - Cesaro summation: let $S_N = \sum_{n=0}^{N} (-1)^n$ and compute

$$S = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N = \frac{1}{2}$$

  - Abel summation:

$$S = \lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}$$

  - Borel summation, Euler summation, ...: again $S = \frac{1}{2}$

- ... but some (such as $\sum_{n=0}^{\infty} n$) cannot

Summing Divergent Series

- Some divergent series can be summed in a sensible way ...

$$S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

  - Cesaro summation: let $S_N = \sum_{n=0}^{N} (-1)^n$ and compute

$$S = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N = \frac{1}{2}$$

  - Abel summation:

$$S = \lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}$$

  - Borel summation, Euler summation, ...: again $S = \frac{1}{2}$

- ... but some (such as $\sum_{n=0}^{\infty} n$) cannot

Summing Divergent Series

- Some divergent series can be summed in a sensible way ...

$$S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

  - Cesaro summation: let $S_N = \sum_{n=0}^{N} (-1)^n$ and compute

$$S = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N = \frac{1}{2}$$

  - Abel summation:

$$S = \lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}$$

  - Borel summation, Euler summation, ...: again $S = \frac{1}{2}$

- ... but some (such as $\sum_{n=0}^{\infty} n$) cannot

Summing Divergent Series

- Some divergent series can be summed in a sensible way ...

$$S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

  - Cesaro summation: let $S_N = \sum_{n=0}^{N} (-1)^n$ and compute

$$S = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N = \frac{1}{2}$$

  - Abel summation:

$$S = \lim_{x\to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}$$

  - Borel summation, Euler summation, ...: again $S = \frac{1}{2}$

- ... but some (such as $\sum_{n=0}^{\infty} n$) cannot

Summing Divergent Series

- Some divergent series can be summed in a sensible way ...

$$S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

  - Cesaro summation: let $S_N = \sum_{n=0}^{N} (-1)^n$ and compute

$$S = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N = \frac{1}{2}$$

  - Abel summation:

$$S = \lim_{x\to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}$$

  - Borel summation, Euler summation, ...: again $S = \frac{1}{2}$

- ... but some (such as $\sum_{n=0}^{\infty} n$) cannot

Regularising Divergent Series

If a divergent series cannot be summed, physicists like to remove infinity

- Regularisation of $\sum_{n=0}^{\infty} f(n)$ (in particular, $f(n) = n$)
  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s - 1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^{\infty} t\, e^{-st}\, dt = \frac{1}{s^2}$: divergent terms cancel
  - Zeta function regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, n^{-s}$

    in particular, $\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1)\,, \qquad \zeta(-1) = -\frac{1}{12}$

Digression

Regularising Divergent Series

If a divergent series cannot be summed, physicists like to remove infinity

- Regularisation of $\sum_{n=0}^{\infty} f(n)$ (in particular, $f(n) = n$)
  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s-1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^{\infty} t\, e^{-st}\, dt = \frac{1}{s^2}$: divergent terms cancel
  - Zeta function regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, n^{-s}$

    in particular, $\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1)\,, \qquad \zeta(-1) = -\frac{1}{12}$

Digression

Regularising Divergent Series

If a divergent series cannot be summed, physicists like to remove infinity

- Regularisation of $\sum_{n=0}^{\infty} f(n)$ (in particular, $f(n) = n$)
  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s-1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^{\infty} t\, e^{-st} dt = \frac{1}{s^2}$: divergent terms cancel
  - Zeta function regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, n^{-s}$

    in particular, $\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1)\,, \qquad \zeta(-1) = -\frac{1}{12}$

Digression

Regularising Divergent Series

If a divergent series cannot be summed, physicists like to remove infinity

- Regularisation of $\sum_{n=0}^{\infty} f(n)$ (in particular, $f(n) = n$)
  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s - 1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^{\infty} t\, e^{-st} dt = \frac{1}{s^2}$: divergent terms cancel
  - Zeta function regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, n^{-s}$

    in particular, $\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1)$, $\qquad \zeta(-1) = -\frac{1}{12}$

Digression

Regularising Divergent Series

If a divergent series cannot be summed, physicists like to remove infinity

- Regularisation of $\sum_{n=0}^{\infty} f(n)$ (in particular, $f(n) = n$)
  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s-1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^\infty t\, e^{-st} dt = \frac{1}{s^2}$: divergent terms cancel
  - Zeta function regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, n^{-s}$

    in particular, $\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1)$, $\quad \zeta(-1) = -\frac{1}{12}$

Digression

Regularising Divergent Series

If a divergent series cannot be summed, physicists like to remove infinity

- Regularisation of $\sum_{n=0}^{\infty} f(n)$ (in particular, $f(n) = n$)
  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s - 1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^\infty t\, e^{-st} dt = \frac{1}{s^2}$: divergent terms cancel
  - Zeta function regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, n^{-s}$

    in particular, $\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1)$, $\qquad \zeta(-1) = -\frac{1}{12}$

Digression

Regularising Divergent Series

If a divergent series cannot be summed, physicists like to remove infinity

- Regularisation of $\sum_{n=0}^{\infty} f(n)$ (in particular, $f(n) = n$)
  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s-1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^\infty t\, e^{-st} dt = \frac{1}{s^2}$: divergent terms cancel
  - Zeta function regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, n^{-s}$

    in particular, $\sum_{n=0}^{\infty} n\, n^{-s} = \zeta(s-1)\,, \qquad \zeta(-1) = -\frac{1}{12}$

Digression

Regularising Divergent Series

- Regularisation result should be independent of the method used
- In particular, for a reasonable class of *cutoff functions*

$$g(t;s) \quad \text{with} \quad \lim_{t\to\infty} g(t;s) = 0 \quad \text{and} \quad \lim_{s\to 0^+} g(t;s) = 1$$

replacing $f(t)$ by $f(t)g(t;s)$ should give the same result for $s \to 0^+$

- We need to study

$$\lim_{s\to 0^+} \Delta(fg) = \lim_{s\to 0^+} \left(\sum_{n=0}^{\infty} f(n)g(n;s) - \int_0^{\infty} f(t)g(t;s)\,dt \right)$$

Two mathematically sound approaches are

- *Euler-Maclaurin Formula*
- *Abel-Plana Formula*

Regularising Divergent Series

- Regularisation result should be independent of the method used
- In particular, for a reasonable class of *cutoff functions*

$$g(t;s) \quad \text{with} \quad \lim_{t\to\infty} g(t;s) = 0 \quad \text{and} \quad \lim_{s\to 0^+} g(t;s) = 1$$

replacing $f(t)$ by $f(t)g(t;s)$ should give the same result for $s \longrightarrow 0^+$

- We need to study

$$\lim_{s\to 0^+} \Delta(fg) = \lim_{s\to 0^+} \left(\sum_{n=0}^{\infty} f(n)g(n;s) - \int_0^{\infty} f(t)g(t;s)\,dt \right)$$

Two mathematically sound approaches are

- *Euler-Maclaurin Formula*
- *Abel-Plana Formula*

Regularising Divergent Series

- Regularisation result should be independent of the method used
- In particular, for a reasonable class of *cutoff functions*

$$g(t;s) \quad \text{with} \quad \lim_{t\to\infty} g(t;s) = 0 \quad \text{and} \quad \lim_{s\to 0^+} g(t;s) = 1$$

  replacing $f(t)$ by $f(t)g(t;s)$ should give the same result for $s \to 0^+$
- We need to study

$$\lim_{s\to 0^+} \Delta(fg) = \lim_{s\to 0^+} \left(\sum_{n=0}^{\infty} f(n)g(n;s) - \int_0^{\infty} f(t)g(t;s)\,dt \right)$$

Two mathematically sound approaches are

- *Euler-Maclaurin Formula*
- *Abel-Plana Formula*

Regularising Divergent Series

- Regularisation result should be independent of the method used
- In particular, for a reasonable class of *cutoff functions*

$$g(t;s) \quad \text{with} \quad \lim_{t\to\infty} g(t;s) = 0 \quad \text{and} \quad \lim_{s\to 0^+} g(t;s) = 1$$

replacing $f(t)$ by $f(t)g(t;s)$ should give the same result for $s \to 0^+$

- We need to study

$$\lim_{s\to 0^+} \Delta(fg) = \lim_{s\to 0^+} \left(\sum_{n=0}^{\infty} f(n)g(n;s) - \int_0^{\infty} f(t)g(t;s)\,dt \right)$$

Two mathematically sound approaches are

- *Euler-Maclaurin Formula*
- *Abel-Plana Formula*

The Euler-Maclaurin Formula

Leonhard Euler, 1707 - 1783

Colin Maclaurin, 1698 - 1746

The Euler-Maclaurin Formula

A formal derivation (Hardy, *Divergent Series*, 1949)

- Denoting $Df(x) = f'(x)$, the Taylor series can be written as

$$f(x+n) = e^{nD} f(x)$$

- It follows that

$$\sum_{n=0}^{N-1} f(x+n) = \frac{e^{ND}-1}{e^{D}-1} f(x) = \frac{1}{e^{D}-1}\left(f(x+N) - f(x)\right)$$

$$= \left(D^{-1} + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1}\right)\left(f(x+N) - f(x)\right)$$

$$\sum_{n=0}^{\infty} f(x+n) - \int_0^{\infty} f(x+t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1} f(x)$$

The Euler-Maclaurin Formula

A formal derivation (Hardy, *Divergent Series*, 1949)

- Denoting $Df(x) = f'(x)$, the Taylor series can be written as

$$f(x+n) = e^{nD} f(x)$$

- It follows that

$$\sum_{n=0}^{N-1} f(x+n) = \frac{e^{ND}-1}{e^{D}-1} f(x) = \frac{1}{e^{D}-1}\left(f(x+N)-f(x)\right)$$

$$= \left(D^{-1} + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1}\right)\left(f(x+N)-f(x)\right)$$

$$\sum_{n=0}^{\infty} f(x+n) - \int_0^{\infty} f(x+t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1} f(x)$$

The Euler-Maclaurin Formula

A formal derivation (Hardy, *Divergent Series*, 1949)

- Denoting $Df(x) = f'(x)$, the Taylor series can be written as

$$f(x+n) = e^{nD} f(x)$$

- It follows that

$$\sum_{n=0}^{N-1} f(x+n) = \frac{e^{ND}-1}{e^{D}-1} f(x) = \frac{1}{e^{D}-1}\left(f(x+N)-f(x)\right)$$

$$= \left(D^{-1} + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1}\right)\left(f(x+N)-f(x)\right)$$

▸ Skip precise statement

$$\sum_{n=0}^{\infty} f(x+n) - \int_0^{\infty} f(x+t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1} f(x)$$

The Euler-Maclaurin Formula

A formal derivation (Hardy, *Divergent Series*, 1949)

- Denoting $Df(x) = f'(x)$, the Taylor series can be written as

$$f(x+n) = e^{nD} f(x)$$

- It follows that

$$\sum_{n=0}^{N-1} f(x+n) = \frac{e^{ND}-1}{e^{D}-1} f(x) = \frac{1}{e^{D}-1} \left(f(x+N) - f(x)\right)$$

$$= \left(D^{-1} + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1}\right) \left(f(x+N) - f(x)\right)$$

Skip precise statement

$$\sum_{n=0}^{\infty} f(x+n) - \int_0^{\infty} f(x+t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1} f(x)$$

The Euler-Maclaurin Formula

A formal derivation (Hardy, *Divergent Series*, 1949)

- Denoting $Df(x) = f'(x)$, the Taylor series can be written as

$$f(x+n) = e^{nD} f(x)$$

- It follows that

$$\sum_{n=0}^{N-1} f(x+n) = \frac{e^{ND}-1}{e^{D}-1} f(x) = \frac{1}{e^{D}-1}\left(f(x+N)-f(x)\right)$$

$$= \left(D^{-1} + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1}\right)\left(f(x+N)-f(x)\right)$$

Skip precise statement

$$\sum_{n=0}^{\infty} f(x+n) - \int_0^{\infty} f(x+t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1} f(x)$$

Theorem (Euler-Maclaurin Formula)

If $f \in C^{2m}[0, N]$ *then*

$$\sum_{n=0}^{N} f(n) - \int_0^N f(t)\,dt = \frac{1}{2}\left(f(0) + f(N)\right) + \sum_{k=1}^{m-1} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(N) - f^{(2k-1)}(0)\right) + R_m$$

where

$$R_m = \int_0^N \frac{B_{2m} - B_{2m}(t - \lfloor t \rfloor)}{(2m)!} f^{(2m)}(t)\,dt$$

Here, $B_n(x)$ *are Bernoulli polynomials and* $B_n = B_n(0)$ *are Bernoulli numbers*

Applying the Euler-Maclaurin Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0)$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ we find $RHS = -\frac{B_1}{1!} f(0) - \frac{B_2}{2!} f'(0)$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

Applying the Euler-Maclaurin Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0)$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ we find $RHS = -\frac{B_1}{1!} f(0) - \frac{B_2}{2!} f'(0)$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

Applying the Euler-Maclaurin Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0)$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ we find $RHS = -\frac{B_1}{1!} f(0) - \frac{B_2}{2!} f'(0)$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

Applying the Euler-Maclaurin Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0)$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ we find $RHS = -\frac{B_1}{1!} f(0) - \frac{B_2}{2!} f'(0)$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

The Abel-Plana Formula

Niels Henrik Abel, 1802 - 1829

Giovanni Antonio Amedeo Plana, 1781 - 1864

The Abel-Plana Formula

> *. . . a remarkable summation formula of Plana . . .*

Germund Dahlquist, 1997

> *The only two places I have ever seen this formula are in Hardy's book and in the writings of the "massive photon" people — who also got it from Hardy.*

Jonathan P Dowling, 1989

The only other applications I am aware of, albeit for convergent series, are

- q-Gamma function asymptotics (Adri B Olde Daalhuis, 1994)
- uniform asymptotics for $\prod_{k=0}^{\infty}(1-q^{n+k})^{-1}$ (myself, 1995)

The Abel-Plana Formula

. . . a remarkable summation formula of Plana . . .

Germund Dahlquist, 1997

The only two places I have ever seen this formula are in Hardy's book and in the writings of the "massive photon" people — who also got it from Hardy.

Jonathan P Dowling, 1989

The only other applications I am aware of, albeit for convergent series, are

- q-Gamma function asymptotics (Adri B Olde Daalhuis, 1994)
- uniform asymptotics for $\prod_{k=0}^{\infty}(1-q^{n+k})^{-1}$ (myself, 1995)

The Abel-Plana Formula

> *. . . a remarkable summation formula of Plana . . .*

Germund Dahlquist, 1997

> *The only two places I have ever seen this formula are in Hardy's book and in the writings of the "massive photon" people — who also got it from Hardy.*

Jonathan P Dowling, 1989

The only other applications I am aware of, albeit for convergent series, are

- q-Gamma function asymptotics (Adri B Olde Daalhuis, 1994)
- uniform asymptotics for $\prod_{k=0}^{\infty}(1-q^{n+k})^{-1}$ (myself, 1995)

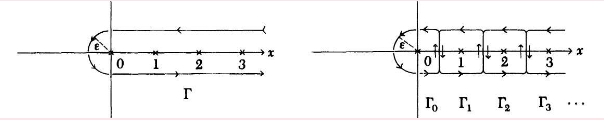
The Abel-Plana Formula

- Use Cauchy's integral formula $f(\zeta) = \frac{1}{2\pi i} \oint_{\Gamma_\zeta} \frac{f(z)}{z-\zeta}\, dz$ together with

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z-n}$$

to get

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{\Gamma_n} \frac{f(z)}{z-n}\, dz = \frac{1}{2i} \int_{\Gamma} \cot(\pi z) f(z)\, dz$$



The Abel-Plana Formula

- Rotate the upper and lower arm of Γ by $\pm\pi/2$ to get

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy)) \coth(\pi y)\, dy$$

- A similar trick gives

$$\int_0^{\infty} f(t)\, dt = \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy))\, dy$$

Taking the difference gives the elegant result

Skip precise statement

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\, dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy$$

The Abel-Plana Formula

- Rotate the upper and lower arm of Γ by $\pm\pi/2$ to get

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy)) \coth(\pi y)\, dy$$

- A similar trick gives

$$\int_0^{\infty} f(t)\, dt = \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy))\, dy$$

Taking the difference gives the elegant result

Skip precise statement

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\, dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy$$

The Abel-Plana Formula

- Rotate the upper and lower arm of Γ by $\pm\pi/2$ to get

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy)) \coth(\pi y)\, dy$$

- A similar trick gives

$$\int_0^{\infty} f(t)\, dt = \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy))\, dy$$

Taking the difference gives the elegant result

Skip precise statement

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\, dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy$$

Theorem (Abel-Plana Formula)

Let $f : \mathbb{C} \to \mathbb{C}$ satisfy the following conditions

(a) $f(z)$ *is analytic for* $\Re(z) \geq 0$ *(though not necessarily at infinity)*

(b) $\lim_{|y| \to \infty} |f(x+iy)| e^{-2\pi|y|} = 0$ *uniformly in* x *in every finite interval*

(c) $\int_{-\infty}^{\infty} |f(x+iy) - f(x-iy)| e^{-2\pi|y|}\, dy$ *exists for every* $x \geq 0$ *and tends to zero for* $x \to \infty$

(d) $\int_0^\infty f(t)\, dt$ *is convergent, and* $\lim_{n\to\infty} f(n) = 0$

Then

$$\sum_{n=0}^{\infty} f(n) - \int_0^\infty f(t)\, dt = \frac{1}{2} f(0) + i \int_0^\infty \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy$$

Applying the Abel-Plana Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = \frac{1}{2}f(0) + i\int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\,dy$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ one finds $RHS = -2\int_0^{\infty} \frac{y\,dy}{e^{2\pi y} - 1}$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

Applying the Abel-Plana Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\,dy$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ one finds $RHS = -2 \int_0^{\infty} \frac{y\,dy}{e^{2\pi y} - 1}$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

Applying the Abel-Plana Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\,dy$$

- Introducing suitable cutoff functions $g(t; s)$ one can justify applying this to divergent series
- For $f(t) = t$ one finds $RHS = -2 \int_0^{\infty} \frac{y\,dy}{e^{2\pi y} - 1}$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

Applying the Abel-Plana Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\,dy$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ one finds $RHS = -2 \int_0^{\infty} \frac{y\,dy}{e^{2\pi y} - 1}$

$$\sum_{n=0}^{\infty} n - \int_0^{\infty} t\,dt = -\frac{1}{12}$$

Outline

Conclusion

- Mathematical question posed in theoretical physics
- Some really nice, old formulæ from classical analysis
- The result has been verified in the laboratory
- The motivation for this talk:
  - Jonathan P Dowling "The Mathematics of the Casimir Effect" Math Mag **62** (1989) 324

Doing mathematics and physics together can be more stimulating than doing either one separately, not to mention it's downright fun.

Conclusion

- Mathematical question posed in theoretical physics
- Some really nice, old formulæ from classical analysis
- The result has been verified in the laboratory
- The motivation for this talk:
  - Jonathan P Dowling "The Mathematics of the Casimir Effect" Math Mag **62** (1989) 324

Doing mathematics and physics together can be more stimulating than doing either one separately, not to mention it's downright fun.

Conclusion

- Mathematical question posed in theoretical physics
- Some really nice, old formulæ from classical analysis
- The result has been verified in the laboratory
- The motivation for this talk:
  - Jonathan P Dowling "The Mathematics of the Casimir Effect" Math Mag **62** (1989) 324

Doing mathematics and physics together can be more stimulating than doing either one separately, not to mention it's downright fun.

Conclusion

- Mathematical question posed in theoretical physics
- Some really nice, old formulæ from classical analysis
- The result has been verified in the laboratory
- The motivation for this talk:
  - Jonathan P Dowling "The Mathematics of the Casimir Effect" Math Mag **62** (1989) 324

Doing mathematics and physics together can be more stimulating than doing either one separately, not to mention it's downright fun.

Conclusion

- Mathematical question posed in theoretical physics
- Some really nice, old formulæ from classical analysis
- The result has been verified in the laboratory
- The motivation for this talk:
    - Jonathan P Dowling "The Mathematics of the Casimir Effect" Math Mag **62** (1989) 324

Doing mathematics and physics together can be more stimulating than doing either one separately, not to mention it's downright fun.

I had a feeling once about Mathematics - that I saw it all. . . . I saw a quantity passing through infinity and changing its sign from plus to minus. I saw exactly why it happened and why the tergiversation was inevitable but it was after dinner and I let it go.

Sir Winston Spencer Churchill, 1874 - 1965

The End

- Define for an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ the zeta function

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

- If the zeta function has an analytic extension up to 0 then define the regularised infinite sum by

$$\sum_{n=1}^{\infty} \log \lambda_n = -\zeta_\lambda'(0)$$

- Alternatively, the regularised infinite product is given by

$$\prod_{n=1}^{\infty} \lambda_n = e^{-\zeta_\lambda'(0)}$$

- Define for an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ the zeta function

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

- If the zeta function has an analytic extension up to 0 then define the regularised infinite sum by

$$\sum_{n=1}^{\infty} \log \lambda_n = -\zeta'_\lambda(0)$$

- Alternatively, the regularised infinite product is given by

$$\prod_{n=1}^{\infty} \lambda_n = e^{-\zeta'_\lambda(0)}$$

- Define for an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ the zeta function

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

- If the zeta function has an analytic extension up to 0 then define the regularised infinite sum by

$$\sum_{n=1}^{\infty} \log \lambda_n = -\zeta_\lambda'(0)$$

- Alternatively, the regularised infinite product is given by

$$\prod_{n=1}^{\infty} \lambda_n = e^{-\zeta_\lambda'(0)}$$

- Let $\lambda_n = p_n$ be the n-th prime so that

$$\prod_p p = e^{-\zeta_p'(0)} \qquad \textit{where} \qquad \zeta_p(s) = \sum_p p^{-s}$$

- Using $e^x = \prod_{n=1}^{\infty} (1 - x^n)^{-\frac{\mu(n)}{n}}$, one gets

$$e^{\zeta_p(s)} = \prod_p e^{p^{-s}} = \prod_p \prod_{n=1}^{\infty} (1 - p^{-ns})^{-\frac{\mu(n)}{n}}$$

- Observing that $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$, one gets

$$e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$$

(Edmund Landau and Arnold Walfisz, 1920)

- Let $\lambda_n = p_n$ be the n-th prime so that

$$\prod_p p = e^{-\zeta_p'(0)} \qquad \textit{where} \qquad \zeta_p(s) = \sum_p p^{-s}$$

- Using $e^x = \prod_{n=1}^{\infty} (1 - x^n)^{-\frac{\mu(n)}{n}}$, one gets

$$e^{\zeta_p(s)} = \prod_p e^{p^{-s}} = \prod_p \prod_{n=1}^{\infty} \left(1 - p^{-ns}\right)^{-\frac{\mu(n)}{n}}$$

- Observing that $\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$, one gets

$$e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$$

(Edmund Landau and Arnold Walfisz, 1920)

- Let $\lambda_n = p_n$ be the n-th prime so that

$$\prod_p p = e^{-\zeta_p'(0)} \qquad \textit{where} \qquad \zeta_p(s) = \sum_p p^{-s}$$

- Using $e^x = \prod_{n=1}^{\infty} (1-x^n)^{-\frac{\mu(n)}{n}}$, one gets

$$e^{\zeta_p(s)} = \prod_p e^{p^{-s}} = \prod_p \prod_{n=1}^{\infty} \left(1-p^{-ns}\right)^{-\frac{\mu(n)}{n}}$$

- Observing that $\zeta(s) = \prod_p \left(1-p^{-s}\right)^{-1}$, one gets

$$e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$$

(Edmund Landau and Arnold Walfisz, 1920)

- From $e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$ one gets the *divergent* expression

$$\zeta_p'(s) = \sum_{n=1}^{\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)}$$

- At $s = 0$, this simplifies to

$$\zeta_p'(0) = \frac{1}{\zeta(0)} \frac{\zeta'(0)}{\zeta(0)} = -2\log(2\pi)$$

- This calculation "à la Euler" can be made rigorous, so that

$$\prod_p p = 4\pi^2$$

(Elvira Muñoz Garcia and Ricardo Pérez-Marco, preprint 2003)

- Corollary: there are infinitely many primes

Back to main talk

- From $e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$ one gets the *divergent* expression

$$\zeta_p'(s) = \sum_{n=1}^{\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)}$$

- At $s = 0$, this simplifies to

$$\zeta_p'(0) = \frac{1}{\zeta(0)} \frac{\zeta'(0)}{\zeta(0)} = -2\log(2\pi)$$

- This calculation "à la Euler" can be made rigorous, so that

$$\prod_p p = 4\pi^2$$

(Elvira Muñoz Garcia and Ricardo Pérez-Marco, preprint 2003)

- Corollary: there are infinitely many primes

Back to main talk

- From $e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$ one gets the *divergent* expression

$$\zeta_p'(s) = \sum_{n=1}^{\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)}$$

- At $s = 0$, this simplifies to

$$\zeta_p'(0) = \frac{1}{\zeta(0)} \frac{\zeta'(0)}{\zeta(0)} = -2\log(2\pi)$$

- This calculation “à la Euler” can be made rigorous, so that

$$\prod_p p = 4\pi^2$$

(Elvira Muñoz Garcia and Ricardo Pérez-Marco, preprint 2003)

- Corollary: there are infinitely many primes

Back to main talk

- From $e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$ one gets the *divergent* expression

$$\zeta_p'(s) = \sum_{n=1}^{\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)}$$

- At $s = 0$, this simplifies to

$$\zeta_p'(0) = \frac{1}{\zeta(0)} \frac{\zeta'(0)}{\zeta(0)} = -2\log(2\pi)$$

- This calculation "à la Euler" can be made rigorous, so that

$$\prod_p p = 4\pi^2$$

(Elvira Muñoz Garcia and Ricardo Pérez-Marco, preprint 2003)

- Corollary: there are infinitely many primes

Back to main talk