COURSE 7

MOTT INSULATORS, SPIN LIQUIDS AND QUANTUM DISORDERED SUPERCONDUCTIVITY

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Abstract

These introductory lecture notes describe recent results on novel Mott insulating phases which are "descendents" of superconductors – obtained by "quantum disordering". After a brief overview of quantum magnetism, attention is focussed on the spin – liquid phase of the two-leg Hubbard ladder and the *nodal liquid* – a descendent of the $d_{x^2-y^2}$ superconductor. These notes are self-contained and an effort has been made to keep them accessible.

1 Introduction

At the foundation of the quantum theory of metals is the theory of the non-interacting electron gas, in which the electrons move through the material interacting only with the periodic potential of the ions, and not with one another. Surprisingly, the properties of most metals are quite well described by simply ignoring the strong Coulomb repulsion between electrons, essentially because Pauli exlusion severely limits the phase space for electron collisions [1]. But in some cases electron interactions can have dramatic effects leading to a complete breakdown of the metallic state, even when the conduction band is only partially occupied. In the simplest such Mott insulator [2] there is only one electron per crystalline unit cell, and so a half-filled metallic conduction band would be expected.

With the discovery of the cuprate superconductors in 1986 [3], there has been a resurgence of interest in Mott insulators. There are two broad classes of Mott insulators, distinguished by the presence or absence of magnetic order [4, 5]. More commonly spin rotational invariance is spontaneously broken, and long-range magnetic order, typically antiferromagnetic, is realized [43]. There are then low energy spin excitations, the spin one magnons. Alternatively, in a spin-liquid [4] Mott insulator there are no broken symmetries. Typically, the magnetic order is short-ranged and there is a gap to all spin excitations: a spin-gap.

In the cuprates the Mott insulator is antiferromagnetically ordered [7,8], but upon doping with holes the antiferromagnetism is rapidly destroyed, and above a certain level superconductivity occurs with $d_{x^2-y^2}$ pairing symmetry. But at intermediate doping levels between the magnetic and d-wave superconducting phases, there are experimental signs of a spin gap opening below a crossover temperature $T^*(x)$ (see Fig. 1). The ultimate nature of the underlying quantum ground state in this portion of the phase diagram – commonly called the *pseudo-gap* regime – is an intriguing puzzle. More generally, the apparent connection between a spin-gap and superconductivity has been a source of motivation to search for Mott insulators of the spin-liquid variety.

Generally, spin liquids are more common in low dimensions where quantum fluctuations can suppress magnetism. Quasi-one-dimensional ladder materials [9, 10] are promising in this regard and have received extensive attention, particularly the two-leg ladder [11]. The Mott insulating spin-liquid phase of the two-leg ladder can be understood by mapping to an appropriate spin-model – the Heisenberg antiferromagnet. Spin-liquid behavior results from the formation of singlet bond formation across the rungs of the ladder [12, 13].

Almost without exception, theoretical studies of spin-liquids *start* by mapping to an appropriate spin-model, and the charge degrees of freedom are thereafter ignored. This represents an enormous simplification, since spin models are so much easier to analyze that the underlying interacting electron model. This approach to quantum magnetism has yielded tremendous progress in the past decade [5]. But is the simplification to a spin-model always legitimate? A central goal of these lectures is to analyze a novel two-dimensional spin-liquid phase – called a nodal liquid [14, 15] – which probably *cannot* be described in terms of a spin model. Although the nodal liquid is a Mott insulator with a charge gap and has no broken symmetries, it possesses gapless *Fermionic* degrees of freedom which carry spin.

The standard route to the spin-liquid invokes quantum fluctuations to suppress the *magnetic* order of a quantum spin-model [4]. The proximity of antiferromagnetism to d-wave superconductivity in the cuprates suggests an alternate route. Indeed, as we shall see, the nodal liquid phase results when a d-wave *superconductor* is "quantum disordered". The gapless Fermionic excitations in the nodal liquid are descendents of the low energy quasiparticles of the d-wave superconductor.

The spin-liquid phase of the two-leg ladder gives us a simpler example of a quantum disordered superconductor. To see this, we will revisit the two-leg ladder, employing a model of interacting *electrons* [16], rather than truncating to a spin-model. Retaining the charge degrees of freedom will enable us to show that the Mott-insulating phase of the two-leg

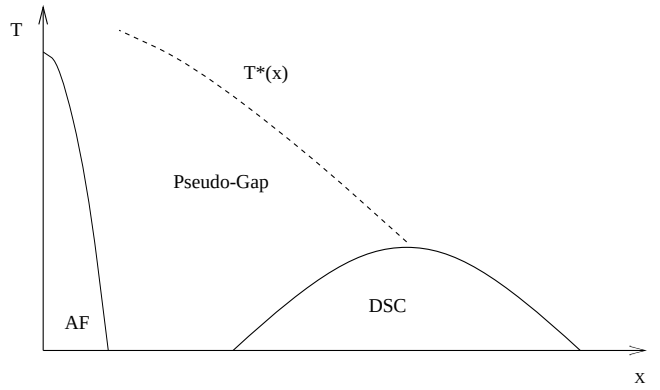

Fig. 1. Schematic phase diagram of a high-temperature superconductor as a function of doping x and temperature T.

ladder actually exhibits pairing, with an approximate d-wave symmetry. Moreover, upon doping, the two-leg ladder exhibits quasi-long-range superconducting (d-wave) pairing correlations. This behavior is reminiscent of that seen in the underdoped cuprate superconductors.

These notes are organized as follows. In Section 2 a simple tight binding model of interacting electrons is introduced and it's symmetry properties are discussed. Section 3 specializes to the Mott insulating state at half-filling, focussing on the magnetic properties employing the Heisenberg antiferromagnet spin-model. In Section 4 the method of Bosonization is briefly reviewed for the case of a one-dimensionless spinless electron gas. Section 5 is devoted to an analysis of the Mott insulating state of the two-leg Hubbard ladder, employing a weak coupling perturbative renormalization group approach. The remaining sections focus on the two-dimensional d-wave superconductor, and the nodal liquid phase which descends from it upon quantum disordering. Specifically, Section 6 briefly reviews BCS theory for a d-wave superconductor focussing on the gapless quasiparticles. An effective field theory for this state, including quantum phase fluctuations, is obtained in Section 7. Section 8 implements a duality transformation of this effective field theory, which enables a convenient description of the nodal liquid phase in Section 9.

2 Models and metals

2.1 Noninteracting electrons

In metals the highest lying band of Bloch states is only partially occupied, and there are low energy electronic excitations which consist of

exciting electrons from just below the Fermi energy into unoccupied states. These excitations can be thermally excited and contribute to thermodynamic properties such as the specific heat, as well as to electrical conduction [1]. Tight binding models give a particularly simple description of the conduction band. In the simplest case the states in the conduction band are built up from a single atomic orbital on each of the ions in the solid. The conduction electrons are presumed to be moving through the solid, tunnelling between ions. We denote the creation and annihilation operators for an electron with spin $\alpha =\uparrow,\downarrow$ on the ion at position $\boldsymbol{x}$ by $c^\dagger_\alpha(\boldsymbol{x})$ and $c_\alpha(\boldsymbol{x})$. These operators satisfy the canonical Fermionic anti-commutation relations,

$$[c_\alpha(\boldsymbol{x}), c^\dagger_\beta(\boldsymbol{x}')]_- = \delta_{\alpha\beta}\delta_{\boldsymbol{x},\boldsymbol{x}'}. \tag{2.1}$$

If the orbitals in question form a simple Bravais lattice with, say, cubic symmetry, then the appropriate tight binding Hamiltonian is,

$$H_0 = -t \sum_{\langle \boldsymbol{x}\boldsymbol{x}'\rangle} \left[c^\dagger_\alpha(\boldsymbol{x})c_\alpha(\boldsymbol{x}') + \text{h.c.}\right] - \mu \sum_{\boldsymbol{x}} n(\boldsymbol{x}), \tag{2.2}$$

where the first summation is over near neighbor sites. Here t is the tunnelling rate between neighboring ions and for simplicity we have ignored further neighbor tunnelling. The electron density $n(\boldsymbol{x}) = c^\dagger_\alpha(\boldsymbol{x})c_\alpha(\boldsymbol{x})$ can be adjusted by tuning the chemical potential, μ.

In the Cuprate superconductors Copper and Oxygen atoms form two dimensional sheets [7], with the Copper atoms sitting at the sites of a square lattice and the Oxygen atoms sitting on the bonds, as depicted schematically in Figure 2. In the simplest one-band models the sites of the tight binding model are taken as the Copper atoms, and $c^\dagger(\boldsymbol{x})$ *removes* an electron (adds a hole) from a Copper 3d orbital. In most of the materials the 3d shell is almost filled with roughly one hole per Copper atom, so that the tight binding model is close to half-filling with $\langle n(\boldsymbol{x})\rangle \approx 1$.

The tight binding Hamiltonian is invariant under translations by an arbitrary real space lattice vector, $\boldsymbol{R}$,

$$c_\alpha(\boldsymbol{x}) \to c_\alpha(\boldsymbol{x} + \boldsymbol{R}). \tag{2.3}$$

This discrete symmetry implies the conservation of crystal momentum, up to a reciprocal lattice vector, $\boldsymbol{G}$, with $\exp(i\boldsymbol{G}\cdot\boldsymbol{R}) = 1$. Being quadratic, the Hamiltonian can be diagonalized by transforming to (crystal) momentum space by defining,

$$c_\alpha(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} c_{\mathbf{k}\alpha} \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{x}}. \tag{2.4}$$

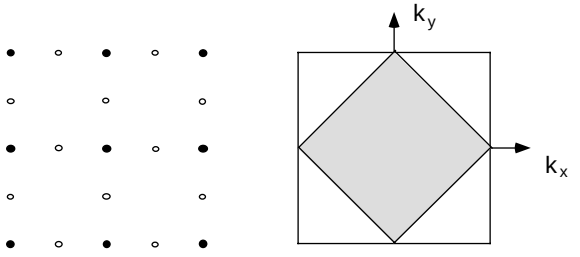

Fig. 2. Schematic illustration of a single Copper-Oxygen plane, consisting of a square lattice of Copper atoms (solid points) and Oxygen atoms (open circles). Two-dimensional Brillouin zone for the 2d square lattice tight binding model with near neighbor hopping is shown at right. At half-filling all states in the Fermi sea (shaded) are occupied.

Here V denotes the "volume" of the system, equal to the total number of sites N with the lattice spacing set to unity, and the sum is over crystal momentum within the first Brillouin zone compatible with periodic boundary conditions. The momentum space creation and anihillation operators also satisfy canonical Fermion anticommutation relations:

$$[c_{\mathbf{k}\alpha}, c^\dagger_{\mathbf{k}'\beta}]_- = \delta_{\alpha\beta}\delta_{\mathbf{k}\mathbf{k}'}. \tag{2.5}$$

In momentum space the Hamiltonian takes the standard diagonal form,

$$H_0 = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}, \tag{2.6}$$

invariant under the discrete translation symmetry: $c_{\mathbf{k}\alpha} \to \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{R}} c_{\mathbf{k}\alpha}$. For a 2d square lattice with near-neighbor hopping, the energy is simply

$$\epsilon_{\mathbf{k}} = -2t[\cos k_x + \cos k_y] - \mu. \tag{2.7}$$

The ground state consists of filling those states in momentum space with $\epsilon_{\mathbf{k}}$ negative, leaving the positive energy states unoccupied. The Fermi surface, separating the occupied from empty states, is determined by the condition $\epsilon_{\mathbf{k}} = 0$. For the 2d square lattice at half-filling with energy dispersion equation (2.7) (at $\mu = 0$), the Fermi surface is a diamond, as shown in Figure 2.

Particle/hole excitations above the ground state consist of removing an electron from within the full Fermi sea, and placing it in an unoccupied positive energy state. In most metals the width of the conduction band (proportional to t) is of order an electron volt (roughly 10^4 K) so that even at room temperature only "low energy" particle/hole states confined within

close proximity to the Fermi surface are thermally excited. In addition to being thermally active, these low energy particle/hole excitations can be excited by an electric field, and lead to metallic electrical conduction.

In the band theory of solids, insulators occur whenever the highest lying energy band is *fully* occupied. Excited states then involve promoting electrons into the next available band which typically requires a very large energy (electron volts). Not surprisingly, such band insulators are very poor conductors of electricity. By constrast, in Mott insulators the highest band is only partially occupied, yet conduction is blocked by strong electron interactions.

Before addressing the complications of electron interactions, it is instructive to briefly consider the symmetries of the above Hamiltonian, and the associated conserved quantities. There are only two *continous* symmetries, associated with conservation of charge and spin. The Hamiltonian is invariant under the global $U(1)$ charge symmetry,

$$c_\alpha(\boldsymbol{x}) \to \mathrm{e}^{i\theta_0} c_\alpha(\boldsymbol{x}), \tag{2.8}$$

for arbitrary (constant) angle θ_0. Conservation of spin is due to the global $SU(2)$ symmetry, $c_\alpha(\boldsymbol{x}) \to U_{\alpha\beta} c_\beta(\boldsymbol{x})$, with $U = \exp(i\boldsymbol{\theta}\cdot\boldsymbol{\sigma})$ and Pauli matrices $\boldsymbol{\sigma}_{\alpha\beta}$. The Hamiltonian is invariant under this transformation, $H_0 \to H_0$, for *arbitrary* spin rotations $\boldsymbol{\theta}$. Here and below we ignore spin-orbit effects which (usually weakly) break the continuous spin rotational symmetry.

There are also a number of discrete symmetries. The Hamiltonian is real, $H_0^* = H_0$, a signature of time reversal invariance (for models with spin-independent interactions). For a square lattice the Hamiltonian is also invariant under reflection (or parity) symmetry, $c_\alpha(\boldsymbol{x}) \to c_\alpha(-\boldsymbol{x})$. This implies that $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$. On the square lattice, a discrete particle/hole transformation is implemented by

$$c_\alpha(\boldsymbol{x}) \xrightarrow{p/h} \mathrm{e}^{i\boldsymbol{\pi}\cdot\boldsymbol{x}} c_\alpha^\dagger(\boldsymbol{x}), \tag{2.9}$$

with $\boldsymbol{\pi} = (\pi, \pi)$. At half-filling when $\mu = 0$, H_0 is invariant under this symmetry, but with further neighbor hopping terms the kinetic energy will generally *not* be particle/hole symmetric. In momentum space the particle/hole transformation is implemented *via* $c_{\mathbf{k}\alpha} \to c^\dagger_{\boldsymbol{\pi}-\mathbf{k}\alpha}$ and invariance of the kinetic energy implies that $\epsilon_{\mathbf{k}} = -\epsilon_{\mathbf{k}+\boldsymbol{\pi}}$.

2.2 Interaction effects

Spin-independent density interactions can be included by adding an additional term to the Hamiltonian:

$$H_1 = \frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{x}'} v(\boldsymbol{x}-\boldsymbol{x}') n(\boldsymbol{x}) n(\boldsymbol{x}'). \tag{2.10}$$

For Coulomb interactions $v(x) \sim e^2/|x|$ is long-ranged. For simplicity the long-ranged interactions are often ignored. In the Hubbard model [2,5] only the *on-site* repulsive interaction is retained,

$$H_u = u \sum_{\boldsymbol{x}} n_\uparrow(\boldsymbol{x}) n_\downarrow(\boldsymbol{x}), \tag{2.11}$$

with $n_\alpha = c_\alpha^\dagger c_\alpha$. This can be re-cast into a manifestly spin-rotationally invariant form:

$$H_u = \frac{u}{2} \sum_{\boldsymbol{x}} n(\boldsymbol{x})[n(\boldsymbol{x}) - 1]. \tag{2.12}$$

Despite the deceptive simplicity of these effective models, they are exceedingly difficult to analyze. Even the Hubbard Hamiltonian, $H = H_0 + H_u$, which is parameterized by just two energy scales, t and u, is largely intractable [4], except in one-dimension. Since the typical interaction scale u is comparable to the kinetic energy t there is no small parameter. Moreover, one is typically interested in phenomena occuring on temperature scales which are *much* smaller than both u and t.

In most metals, the low energy properties are quite well described by simply ignoring the (strong!) interactions. This surprising fact can be understood (to some degree) from Landau's Fermi-liquid theory [1], and more recent renormalization group arguments [17]. The key point is that the phase space available for collisions between excited particles and holes *vanishes* with their energy. In metals the phase space is evidently so restrictive that the surviving interactions do not change the *qualitative* behavior of the low energy particle/hole excitations. Indeed, the quasiparticle excitations within Landau's Fermi liquid theory have the same quantum numbers as the electron (charge e spin $1/2$ and momentum), but move with a "renormalized" velocity. But some materials such as the Cuprates are not metallic, even when band structure considerations would suggest a partially occupied conduction band. In these Mott insulators one must invoke electron interactions.

3 Mott insulators and quantum magnetism

The Hubbard model at half-filling is perhaps the simplest example of a Mott insulator. To see this, consider the behavior as the ratio u/t is varied. As discussed above, for $u/t = 0$ the model is diagonalized in momentum space, and exhibits a Fermi surface. But at half-filling the model is also soluble when $u/t = \infty$. Since the onsite Hubbard energy takes the form, $u(n-1)^2/2$, in this limit the ground state consists simply of one electron on each site. The electrons are frozen and immobile, since doubly occupied

and unoccupied sites cost an energy proportional to u. The state is clearly insulating – a Mott insulator.

In this large u limit it is very costly in energy to add an electron, and the state exhibits a *charge gap* of order u. But there are many low energy *spin* excitations, which consist of flipping the spin of an electron on a given site. For infinite u this spin-one excitation costs no energy at all, and indeed the ground state is highly degenerate since the spins of each of the N localized electrons can be either up or down. For large but finite u/t one still expects a charge gap, but the huge spin degeneracy will be lifted.

The fate of the spin degrees of freedom in the Mott insulator is enormously interesting. Broadly speaking, Mott insulators come in two classes, distinguished by the presence or absence of spontaneously broken symmetries. Often the spin rotational invariance is spontaneously broken and the ground state is magnetic, but $SU(2)$ invariant spin structures which break translational symmetries are also possible. In the second class, usually referred to as *spin liquid* states there are *no* broken symmetries.

3.1 Spin models and quantum magnetism

Traditionally, spin physics in the Mott insulating states have been analyzed by studying simple spin models. These focus on the electron spin operators:

$$\boldsymbol{S}(\boldsymbol{x}) = \frac{1}{2} c^\dagger_\alpha(\boldsymbol{x}) \boldsymbol{\sigma}_{\alpha\beta} c_\beta(\boldsymbol{x}), \tag{3.1}$$

where $\boldsymbol{\sigma}$ is a vector of Pauli matrices. These spin operators satisfy standard angular momentum commutation relations:

$$[S_\mu(\boldsymbol{x}), S_\nu(\boldsymbol{x}')] = i\delta_{\boldsymbol{x}\boldsymbol{x}'}\epsilon_{\mu\nu\lambda} S_\lambda. \tag{3.2}$$

They also satisfy,

$$\boldsymbol{S}^2(\boldsymbol{x}) = \frac{3}{4} n(\boldsymbol{x})[2 - n(\boldsymbol{x})]. \tag{3.3}$$

Within the restricted sector of the full Hilbert space with exactly one electron per site, these operators are bone fide spin 1/2 operators satisfying $\boldsymbol{S}^2 = s(s+1)$ with $s = 1/2$. Their matrix elements in the restricted Hilbert space are identical to the Pauli matrices: $\boldsymbol{\sigma}/2$.

The simplest spin model consists of a (square) lattice of spin 1/2 operators coupled *via* a near neighbor exchange interaction, J:

$$H = J \sum_{\langle \boldsymbol{x}\boldsymbol{x}' \rangle} \boldsymbol{S}(\boldsymbol{x}) \cdot \boldsymbol{S}(\boldsymbol{x}'). \tag{3.4}$$

This spin model can be obtained from the half-filled Hubbard model [5], by working perturbatively in small t/u. For $t/u = 0$ the spins are decoupled,

but an antiferromagnetic exchange interaction $J = 4t^2/u$ is generated at second order in t. Specifically, the matrix elements of the spin Hamiltonian in the restricted Hilbert space are obtained by using second order perturbation theory in t. The intermediate virtual states are doubly occupied, giving an energy denominator u.

Mapping the Hubbard model to a spin model represents an *enormous* simplification. The complications due to the Fermi statistics of the underlying electrons have been subsumed into an exchange interaction. The spin operators are essentially *bosonic*, commuting at different sites. It should be emphasized that at higher order in t/u multi-spin exchange interactions will be generated, also between further separated spins. If t/u is of order one, then it is by no means obvious that it is legitimate to truncate to a spin model at all.

A central focus of quantum magnetism during the past decade has been exploring the possible ground states and low energy excitations of such spin models [4, 5]. The above $s = 1/2$ square lattice Heisenberg antiferromagnet is, of course, only one member of a huge class of such models. These models can be generalized to larger spin s, to different lattices and/or dimensionalities, to include competing or frustrating interactions, to include multi-spin interactions, to "spins" in different groups such as $SU(N)$, etc. Not surprisingly, there is an almost equally rich set of possible ground states.

The main focus of these notes is the 2d "nodal liquid", a spin-liquid phase obtained by quantum disordering a d-wave superconductor. As we shall see in Section 9, in the nodal liquid the spin excitations are carried by *Fermionic* degrees of freedom and cannot be described by (Bosonic) spin operators. In truncating to the restricted Hilbert space with one electron per site, one has effectively "thrown out the baby with the bath water". The nodal liquid phase probably *requires* retaining the charge degrees of freedom.

But spin models are *much* simpler than interacting electron models, relevant to many if not most Mott insulators (as well as other localized spin systems) and extremely rich and interesting in their own right. So I would like to briefly summarize some of the possible ground states, focussing on spin 1/2 models on bi-partite lattices [5, 43]. Consider first those ground states with spontaneously broken symmetries. Most common is the breaking of spin-rotational invariance. If the spin operators are treated as *classical* fixed length vectors, which is valid in the large spin limit ($s \to \infty$), the ground state of the near neighbor square lattice antiferromagnet is the Neel state (up on one sublattice, down on the other) which breaks the $SU(2)$ symmetry. For finite s the Neel state is *not* the exact ground state, but the ground state is still antiferromagnetically ordered, even for $s = 1/2$. Quantum fluctuations play a role in reducing the sub-lattice magnetization, but

(for the 2d square lattice) do not drive it to zero. The low energy excitations are *gapless* spin-waves (*i.e.* magnons), as expected when a continuous symmetry is spontaneously broken.

For some spin models the ground state is spin rotationally invariant but spontaneously breaks (discrete) translational symmetry. The classic example is the Majumdar – Ghosh Hamiltonian [5],

$$H_{\rm MG} = J \sum_x \left[\boldsymbol{S}(x) \cdot \boldsymbol{S}(x+1) + \frac{1}{2} \boldsymbol{S}(x) \cdot \boldsymbol{S}(x+2) \right], \tag{3.5}$$

which describes a one dimensional $s = 1/2$ Heisenberg antiferromagnetic spin chain with a second neighbor exchange interaction. The exact ground state of this model is a two-fold degenerate "spin-Peierls" state:

$$|G\rangle_{\pm} = \prod_x [|\uparrow_{2x}\rangle|\downarrow_{2x\pm1}\rangle - |\downarrow_{2x}\rangle|\uparrow_{2x\pm1}\rangle]. \tag{3.6}$$

This state consists of a product of "singlet bonds" formed from neighboring pairs of spins, and breaks invariance under translations by one lattice spacing. Since the singlet bonds are rotationally invariant, the $SU(2)$ symmetry remains unbroken. The second neighbor interaction has effectively suppressed the tendency towards antiferromagnetic order.

3.2 Spin liquids

Spin liquid ground states in which *no* symmetries are broken generally occur more readily in low dimensions where quantum fluctuations are more effective at destroying magnetic order. The one-dimensional $s = 1/2$ chain with near neighbor antiferromagnetic exchange exhibits power law magnetic correlations at the antiferromagnetic wave vector π [43]. Although "almost" magnetically ordered the $SU(2)$ symmetry is *not* broken in the ground state, which thus technically qualifies as a spin liquid. More dramatic is the behavior of the $s = 1/2$ antiferromagnetic two-leg ladder, shown in Figure 3. This model exhibits a featureless spin-rotationally invariant ground state with exponentially decaying spin correlation fuctions and a non-zero energy *gap* for all spin excitations [12,13]. The physics can be best understood in the limit in which the exchange interaction across the rungs of the ladder greatly exceeds the intra-leg exchange: $J_\perp \gg J$. When $J = 0$ the ground state consists of singlet bonds formed across the rungs of the ladder, with triplet excitations separated by an energy gap of order $J_\perp$. Perturbing in small J will cause these singlet bonds to "resonate", but one expects the spin gap to survive at least for $J \ll J_\perp$. It turns out that the ground state evolves adiabatically and smoothly with increasing J, and in fact the spin-liquid survives for arbitrarily large $J_\perp/J$.

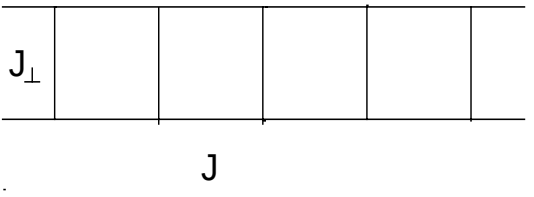

Fig. 3. Heisenberg spin model on a two-leg ladder. Spin 1/2 operators sit on the sites of the ladder, interacting *via* an antiferromagnetic exchange J along the ladder and $J_\perp$ across the rungs.

There has been an enormous amount of theoretical effort expended searching for *two-dimensional* spin 1/2 models which exhibit spin-liquid ground states analogous to the two-leg ladder – but with little success. The original motivation soon after the discovery of superconductivity in the Cuprates was based on Anderson's ideas [41] that a Mott insulating spin-liquid exhibits "pre-formed" Cooper pairing. Doping the Mott insulator would give the Cooper pairs room to move and to condense into a superconducting state, presumed to have s-wave pairing symmetry. But it soon became clear that the undoped Mott insulator in the Cuprates is *not* a spin-liquid, but actually antiferromagnetically ordered. Moreover, recent experiments have established that the pairing symmetry in the superconducting phase is d-wave rather than s-wave [19, 20].

However, recent theoretical work [16] (see Sect. 5 below) has established that the pairing in the spin-liquid phase of the two-leg ladder actually has (approximate) d-wave symmetry. Moreover, doping this Mott insulator does indeed give the pairs room to move [21, 22], and they form a one-dimensional d-wave "superconductor" (with quasi-long-ranged pairing correlations). The nodal liquid phase [14, 15] discussed extensively below is a two-dimensional analog of this spin-liquid phase. Indeed, we shall explicitly construct the nodal liquid by quantum disordering a two-dimensional d-wave superconductor. As we shall see, the resulting 2d nodal liquid posesses gapless Fermionic excitations, which are descendents of the d-wave quasiparticles. These Fermions carry spin but no charge. The nodal liquid presumably cannot be the ground state of any (Bosonic) spin-model. To describe the nodal liquid one must employ the underlying interacting electron model which retains the charge degrees of freedom.

Recent experiment has focussed attention on the underdoped regime of the Cuprate materials [8], occuring between the antiferromagnetic and superconducting phases (see Fig. 1). In this pseudo-gap regime insulating behavior is seen at low temperatures, and there are indications for a spin gap – behavior reminiscent of a Mott insulating spin-liquid. We have

suggested [14] that this strange phase can perhaps be understood in terms of a doped nodal liquid.

Before discussing further the 2d nodal liquid, it is instructive to revisit the spin liquid phase of the two-leg ladder and analyze it directly with a model of interacting electrons. Specifically, we consider weak interactions (small u/t), a limit in which truncation to a spin model is *not* possible. This analysis is greatly aided by "Bosonization" – a powerful method which enables an interacting electron model in one dimension to be re-formulated in terms of collective Bosonic degrees of freedom. See references [23–27] as well as Fradkin's book [4] for useful reviews of Bosonization. First, in Section 4 we briefly review Bosonization for the simplest case of a spinless one-dimensional electron gas, before turning to the two-leg ladder in Section 5.

4 Bosonization primer

Consider the Hamiltonian for non-interacting spinless electrons hopping on a 1d lattice,

$$H = -t \sum_x c^\dagger(x) c(x+1) + \text{h.c.} \tag{4.1}$$

with hopping strength t. One can diagonalize this Hamiltonian by Fourier transforming to momentum space as in equation (2.4), giving

$$H = \sum_k \epsilon_k c_k^\dagger c_k, \tag{4.2}$$

with energy dispersion $\epsilon_k = -t\cos(k)$ for momentum $|k| < \pi$, as shown in Figure 4. In the ground state all of the negative energy states with momentum $|k| \leq k_{\rm F}$ are occupied. At half-filling the Fermi wavevector $k_{\rm F} = \pi/2$. An effective low energy theory for these excitations can be obtained by focussing on momenta close to $\pm k_{\rm F}$ and defining *continuum* Fermi fields:

$$\psi_{R}(q) = c_{k_{\rm F}+q}; \qquad \psi_{L}(q) = c_{-k_{\rm F}+q}. \tag{4.3}$$

Here the subscripts R/L refer to the right/left Fermi points, and q is assumed to be smaller than a momentum cutoff, $|q| < \Lambda$ with $\Lambda \ll k_{\rm F}$. One can then linearize the dispersion about the Fermi points, writing $\epsilon_{\pm k_{\rm F}+q} = \pm v_{\rm F} q$ with $v_{\rm F}$ the Fermi velocity. It is convenient to transform back to real space, defining fields

$$\psi_{P}(x) = \frac{1}{\sqrt{V}} \sum_{|q|<\Lambda} \mathrm{e}^{iqx} \psi_{P}(q), \tag{4.4}$$

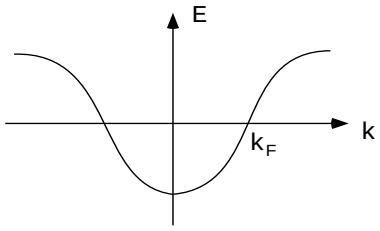

Fig. 4. Schematic energy dispersion for the one-dimensional electron gas. The negative energy states are occupied, with momentum $|k| < k_F$. The dispersion can be linearized around $\pm k_F$, leading to a continuum Dirac Fermion theory.

(with $P = R, L$) which vary slowly on the scale of the lattice spacing. This is equivalent to expanding the lattice electron operators in terms of continuum fields,

$$c(x) \sim \psi_R(x) e^{ik_F x} + \psi_L(x) e^{-ik_F x}. \tag{4.5}$$

After linearization, the effective low energy Hamiltonian takes the form, $H = \int dx \mathcal{H}$, with Hamiltonian density,

$$\mathcal{H} = -v_F[\psi_R^\dagger i\partial_x \psi_R - \psi_L^\dagger i\partial_x \psi_L]. \tag{4.6}$$

describing a one-dimensional relativistic Dirac particle. The associated Lagrangian density is simply

$$\mathcal{L} = \psi_R^\dagger i\partial_t \psi_R + \psi_L^\dagger i\partial_t \psi_L - \mathcal{H}. \tag{4.7}$$

Consider a particle/hole excitation about the right Fermi point, where an electron is removed from a state with $k < k_F$ and placed into an unoccupied state with $k + q > k_F$. For small momentum change q, the energy of this excitation is $\omega_q = v_F q$. Together with the negative momentum excitations about the left Fermi point, this linear dispersion relation is identical to that for phonons in one-dimension. The method of Bosonization exploits this similarity by introducing a phonon displacement field, θ, to decribe this linearly dispersing density wave [23, 25]. We follow the heuristic development of Haldane [27], which reveals the important physics, dispensing with mathematical rigor. To this end, consider a Jordan-Wigner transformation [4] which replaces the electron operator, $c(x)$, by a (hard-core) boson operator,

$$c(x) = \mathcal{O}(x) b(x) \equiv \exp\left[i\pi \sum_{x' < x} n(x') \right] b(x). \tag{4.8}$$

where $n(x) = c^\dagger(x)c(x)$ is the number operator. One can easily verify that the Bose operators commute at different sites. Moreover, the lattice Hamiltonian equation (4.1) can be re-expressed in terms of these Bosons, and takes the identical form with $c's$ replaced by $b's$. This transformation, exchanging Fermions for Bosons, is a special feature of one-dimension. The Boson operators can be (approximately) decomposed in terms of an amplitude and a phase,

$$b(x) \to \sqrt{\rho}\mathrm{e}^{i\varphi}. \tag{4.9}$$

We now imagine passing to the continuum limit, focussing on scales long compared to the lattice spacing. In this limit we decompose the total density as, $\rho(x) = \rho_0 + \tilde{\rho}$, where the mean density, $\rho_0 = k_\mathrm{F}/\pi$, and $\tilde{\rho}$ is an operator measuring fluctuations in the density. As usual, the density and phase are canonically conjugate quantum variables, taken to satisfy

$$[\varphi(x), \tilde{\rho}(x')] = i\delta(x - x'). \tag{4.10}$$

Now we introduce a phonon-like displacement field, $\theta(x)$, *via* $\tilde{\rho}(x) = \partial_x\theta(x)/\pi$. The full density takes the form: $\pi\rho(x) = k_\mathrm{F} + \partial_x\theta$. The above commutation relations are satisfied if one takes,

$$[\varphi(x), \theta(x')] = -i\pi\Theta(x' - x). \tag{4.11}$$

Here $\Theta(x)$ denotes the heavyside step function, not to be confused with the displacement field θ. Notice that $\partial_x\varphi/\pi$ is the momentum conjugate to θ.

The effective (Bosonized) Hamiltonian density which describes the 1d density wave takes the form:

$$\mathcal{H} = \frac{v}{2\pi}[g(\partial_x\varphi)^2 + g^{-1}(\partial_x\theta)^2]. \tag{4.12}$$

This Hamiltonian describes a wave propagating at velocity v, as can be readily verified upon using the commutation relations to obtain the equations of motion, $\partial_t^2\theta = v^2\partial_x^2\theta$, and similarly for φ. Clearly one should equate v with the Fermi velocity, v_F. The additional dimensionless parameter, g, can be determined as follows. A small variation in density, $\tilde{\rho}$, will lead to a change in energy, $E = \tilde{\rho}^2/2\kappa$, where $\kappa = \partial\rho/\partial\mu$ is the compressibility. Since $\partial_x\theta = \pi\tilde{\rho}$, one deduces from $\mathcal{H}$ that $\kappa = g/\pi v$. But for a non-interacting electron gas, $\pi v\kappa = 1$, so that $g = 1$. In the presence of (short-ranged) interactions between the (spinless) electrons, one can argue that the above Hamiltonian density remains valid, but with renormalized values of both g and v. This Hamiltonian would then describe a (spinless) Luttinger liquid [27, 28], rather than the free electron gas.

The power of Bosonization relies on the ability to re-express the electron operator $c(x)$ in terms of the Boson fields. Clearly $c(x)$ must remove a unit

charge (e) at x, and satisfy Fermion anticommutation relations. Consider first the *Bose* operator, $b \sim \exp(i\varphi)$, which removes unit charge. To see this, note that one can write,

$$\mathrm{e}^{i\varphi(x)} = \mathrm{e}^{i\pi \int_{-\infty}^{x} \mathrm{d}x' P(x')}, \tag{4.13}$$

where $P = \partial_x \varphi/\pi$ is the momentum conjugate to θ. Since the momentum operator is the generator of translations (in θ), this creates a kink in θ of height π centered at position x – which corresponds to a localized unit of charge since the density $\tilde{\rho} = \partial_x \theta/\pi$. To construct the (Fermionic) electron operator requires multiplying this Bose operator by a Jordan-Wigner "string":

$$\mathcal{O}(x) = \mathrm{e}^{i\pi \sum_{x'<x} n(x')} \to \mathrm{e}^{i\pi \int^{x} \rho(x')} = \mathrm{e}^{i(k_{\mathrm{F}} x + \theta)}. \tag{4.14}$$

Since this string operator carries momentum k_{F}, the resulting Fermionic operator $\mathcal{O}\mathrm{e}^{i\varphi}$ should be identified with the right moving continuum Fermi field, ψ_R. We have thereby identified the correct Bosonized form for the (continuum) electron operators:

$$\psi_P(x) = \mathrm{e}^{i\phi_P(x)}; \qquad \phi_P = \varphi + P\theta, \tag{4.15}$$

with $P = R/L = \pm$. From equation (4.10) the *chiral* Boson fields ϕ_P can be shown to satisfy the so-called Kac-Moody commutation realtions:

$$[\phi_P(x), \phi_P(x')] = \quad iP\pi \quad \mathrm{sgn}(x - x'), \tag{4.16}$$

$$[\phi_R(x), \phi_L(x')] = \quad i\pi. \tag{4.17}$$

These commutation relations can be used to show that ψ_R and ψ_L anticommute.

It is instructive to re-express the Bosonized Hamiltonian density in terms of the *chiral* boson fields,

$$\mathcal{H} = \pi v_{\mathrm{F}} [n_R^2 + n_L^2], \tag{4.18}$$

where we have defined right and left moving densities

$$n_P = P \frac{1}{2\pi} \partial_x \phi_P, \tag{4.19}$$

which sum to give the total density, $n_R + n_L = \tilde{\rho}$. These chiral densities can be expressed in terms of the chiral electron operators as,

$$n_P =: \psi_P^\dagger \psi_P :\equiv \psi_P^\dagger \psi_P - \langle \psi_P^\dagger \psi_P \rangle. \tag{4.20}$$

Notice that the Bosonized Hamiltonian decouples into right and left moving sectors.

An advantage of Bosonization is the ease with which electron interactions can be incorporated. Consider a (short-range) density-density interaction added to the original lattice Hamiltonian. Using equation (4.5) this can be decomposed into the continuum Dirac fields, and will be quartic and spatially local. Due to momentum conservation, only three terms are possible: Two chiral terms of the form $(\psi_P^\dagger \psi_P)^2$ with $P = R/L$, and a right/left mixing term of the form, $\psi_R^\dagger \psi_R \psi_L^\dagger \psi_L$. Under Bosonization the chiral terms are proportional to $(\partial_x \phi_P)^2$, and can be seen to simply shift the Fermi velocity in equation (4.18). The right/left mixing term also Bosonizes into a quadratic form proportional to $(\partial_x \theta)^2 - (\partial_x \varphi)^2$. When added to the Hamiltonian in equation (4.12), this term can be absorbed by shifting *both* the Fermi velcocity *and* the dimensionless Luttinger parameter, g, which is then no longer equal to one. For repulsive interactions $g < 1$, whereas $g > 1$ with attractive interactions. This innocuous looking shift in g has profound effects on the nature of the electron correlation functions. In fact, it leads to new chiral operators which have fractional charge, ge. The resulting one-dimensional phase is usually called a "Luttinger liquid" [27]. For electrons with spin or for 1d models with multiple bands, the quartic Fermion operators can have even more dramatic consequences, for example opening up energy gaps as we shall see in Section 5.

The Lagrangian density in the Bosonized representation takes the form of a free scalar field,

$$\mathcal{L} = \frac{g}{2} \kappa_\mu (\partial_\mu \varphi)^2, \tag{4.21}$$

with $g = 1$ for the free Fermion gas, and $g \neq 1$ in the interacting Luttinger liquid. The Greek index μ runs over time and the spatial coordinate, $\mu = 0, 1 = t, x$. Here $\kappa_0 = 1/\pi v$ and $\kappa_1 = -v^2 \kappa_0$. When re-expressed in terms of θ the Lagrangian takes the *identical* form, except with $g \to 1/g$ for the Luttinger liquid. Changing from the φ to the θ representation can be viewed as a *duality* transformation. In Section 8 we will consider an analogous duality transformation in *two* spatial dimensions.

5 2 Leg Hubbard ladder

5.1 Bonding and antibonding bands

We now consider electrons hopping on a two-leg ladder as shown in Figure 5. The kinetic energy takes the form,

$$H_0 = -t \sum_{\langle \boldsymbol{x} \boldsymbol{x}' \rangle} \left[c_\alpha^\dagger(\boldsymbol{x}) c_\alpha(\boldsymbol{x}') + \text{h.c.} \right] - \mu \sum_{\boldsymbol{x}} n(\boldsymbol{x}), \tag{5.1}$$

where $n(\boldsymbol{x}) = c_\alpha^\dagger(\boldsymbol{x}) c_\alpha(\boldsymbol{x})$, and the summation is taken over near neighbors on the two-leg ladder, with $y = 1, 2$. Due to a parity symmetry under

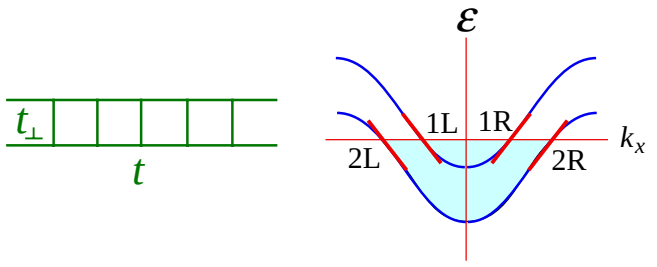

Fig. 5. A two-leg ladder and its band structure. In the low-energy limit, the energy dispersion is linearized near the Fermi points. The two resulting relativistic Dirac Fermions are distinguished by pseudospin indices $i = 1, 2$ for the anti-bonding and bonding bands, respectively.

interchange of the two legs of the ladder, it is convenient to consider even and odd parity bonding and anti-bonding operators:

$$b_\alpha(x) = \frac{1}{\sqrt{2}}[c_\alpha(x, y = 1) + c_\alpha(x, y = 2)], \tag{5.2}$$

$$a_\alpha(x) = \frac{1}{\sqrt{2}}[c_\alpha(x, y = 1) - c_\alpha(x, y = 2)], \tag{5.3}$$

which depend only on the coordinate x along the ladder. The Hamiltonian splits into even and odd contributions, $H_0 = H_a(a) + H_b(b)$. Each is a one-dimensional tight binding model which can be readily diagonalized by transforming to momentum space,

$$b(x) = \frac{1}{\sqrt{N}} \sum_k b_k \mathrm{e}^{ikx}, \tag{5.4}$$

and similarly for the anti-bonding operator. Here N denotes the number of sites *along* the ladder. The diagonal form is

$$H_0 = \sum_k [\epsilon_k^a a_{k\alpha}^\dagger a_{k\alpha} + \epsilon_k^b b_{k\alpha}^\dagger b_{k\alpha}], \tag{5.5}$$

which describes *two* one-dimensional bands with dispersion $\epsilon_k^{a/b} = -2t \cos k \pm t - \mu$. These are sketched in Figure 5.

Focussing on the case at half-filling with one electron per site ($\mu = 0$), both bands intersect the Fermi energy, $\epsilon_\mathrm{F} = 0$. There are four Fermi *points* at $\pm k_{\mathrm{F}1}$ and $\pm k_{\mathrm{F}2}$, for the antibonding and bonding bands, respectively. Gapless particle/hole excitations exist at each of the four Fermi points.

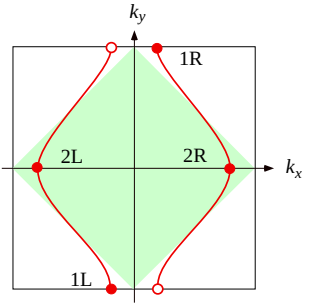

Fig. 6. Fermi points for the two-leg ladder plotted in the two-dimensional Brillouin zone, with the antibonding band (denoted 1) at $k_y = \pi$ and the bonding band (2) at $k_y = 0$. The shaded region represents the Fermi sea for a *two-dimensional* square lattice model at half-filling.

Due to particle/hole symmetry present with near neighbor hopping, $\epsilon^a_k + \epsilon^b_{k+\pi} = 0$, which implies that $k_{\mathrm{F}1} + k_{\mathrm{F}2} = \pi$. Moreover, the Fermi velocity in each band is the same, hereafter denoted as v. It is instructive to plot these Fermi points in *two-dimensional* momentum space, taking transverse momentum $k_y = 0, \pi$ for the two bands, as shown in Figure 6. The four Fermi points can be viewed as constant k_y slices through a two-dimensional Fermi surface.

As we shall see, with even weak electron interactions present the gapless Fermi points are unstable, and a gap opens in the spectrum. Of interest are the properties of the resulting Mott insulator. As discussed in Section 3, for *strong* interactions mapping to a spin model is possible, and the electron spins across the rungs of the ladder are effectively locked into singlets:

$$|RS\rangle = \frac{1}{\sqrt{2}}[c^\dagger_\uparrow(1)c^\dagger_\downarrow(2) - c^\dagger_\downarrow(1)c^\dagger_\uparrow(2)]|0\rangle, \tag{5.6}$$

where $y = 1, 2$ refers to the two legs of the ladder, and we have suppressed the rung position x. The state $|0\rangle$ denotes a rung with no electrons.

It is extremely instructive to re-express this rung-singlet state in terms of the bonding and anti-bonding operators. One finds,

$$|RS\rangle = \frac{1}{\sqrt{2}}[b^\dagger_\uparrow b^\dagger_\downarrow - a^\dagger_\uparrow a^\dagger_\downarrow]|0\rangle, \tag{5.7}$$

a linear combination of adding a singlet (Cooper) pair into the bonding and antibonding orbitals. This paired form is suggestive of superconductivity. Indeed, when viewed in momentum space, the ground state of a superconductor is a product of singlet pairs with zero center of mass momentum at different points around the Fermi surface. In an s-wave superconductor, the

pairs are all added with the *same* sign, but if the pairs are formed with a relative angular momentum (*e.g.* d-wave) sign changes are expected. But notice the most important relative *minus* sign in the rung singlet state! The spin-liquid phase of the two-leg ladder is evidently related to a paired superconductor with *non-zero* angular momentum. Since pairing in the bonding band at $k_y = 0$ has a positive sign and pairing in the anti-bonding band at $k_y = \pi$ is negative, in the two-dimensional Brillouin zone (see Fig. 6) the sign is proportional to $k_x^2 - k_y^2$, consistent with a so-called $d_{x^2-y^2}$ pairing symmetry.

If the interactions are weak, it is legitimate to focus on electronic states near the Fermi points. As in Section 4, the electron operators can be conveniently decomposed into continuum fields near the Fermi points which vary slowly on the scale of the lattice. Denoting $c_1 = a$ and $c_2 = b$, the bonding and antibonding operators are expanded as,

$$c_{i\alpha} \sim \psi_{Ri\alpha} \mathrm{e}^{ik_{Fi}x} + \psi_{Li\alpha} \mathrm{e}^{-ik_{Fi}x}, \tag{5.8}$$

with $i = 1, 2$. Upon linearizing the spectrum around the four Fermi points the kinetic energy takes the form, $H_0 = \int \mathrm{d}x \mathcal{H}_0$, with Hamiltonian density,

$$\mathcal{H}_0 = -v \sum_{i,\alpha} [\psi^\dagger_{Ri\alpha} i\partial_x \psi_{Ri\alpha} - \psi^\dagger_{Li\alpha} i\partial_x \psi_{Li\alpha}]. \tag{5.9}$$

This Hamiltonian describes massless Dirac Fermions, with four flavors labelled by band and spin indices. Implicit in this theory is a momentum cutoff, Λ, whose inverse exceeds the lattice spacing. Only modes with momentum $|k| < \Lambda$ are included in these continuum fields. Since the spectrum is massless, this simple theory is "critical" and scale invariant behavior is expected. This can be seen by considering the (Euclidian) action, written as a space-time integral of the Lagrangian density,

$$S = \int \mathrm{d}\tau \mathrm{d}x \mathcal{L}_0, \tag{5.10}$$

$$\mathcal{L}_0 = \sum_{P\alpha} \psi^\dagger_{Pi\alpha} \partial_\tau \psi_{Pi\alpha} + \mathcal{H}_0, \tag{5.11}$$

with $P = R/L$, and τ denoting imaginary time. The partition function, $Z = Tr \exp(-\beta H_0)$, can be expressed as a (coherent state Grassman) path integral [29],

$$Z = \int [\mathrm{D}\psi][\mathrm{D}\bar{\psi}] e^{-S(\bar{\psi},\psi)}. \tag{5.12}$$

A simple renormalization group can be implemented [17,30] by first integrating out fields $\psi(k, \omega)$ with momentum k lying in the interval $\Lambda/b < |k| < \Lambda$,

with rescaling parameter $b > 1$. Since modes with different momentum and frequency are not coupled, the action takes the same form after this integration, except with a smaller momentum cutoff, Λ/b. The renormalization group transformation is completed by a rescaling procedure which returns the cutoff to it's original value:

$$x \to bx; \quad \tau \to b\tau; \quad \psi \to b^{-1/2}\psi. \tag{5.13}$$

The field rescaling has been chosen to leave the action invariant. This simple theory is at a renormalization group *fixed point*.

5.2 Interactions

Electron-electron interactions scatter right-moving electrons into left-moving electrons and *vice-versa*. We consider general finite-ranged spin-independent interactions, but assume that the typical interaction strength, u, is weak – much smaller than the bandwidth. We focus on the effects of the interactions to *leading* non-vanishing order in u. In this limit it is legitimate to keep only those pieces of the interactions which scatter the low energy Dirac Fermions. A general four Fermion interaction on the two-leg ladder (such as the Hubbard u) can be readily decomposed in terms of the continuum Dirac fields. It is instructive to see how these quartic terms in $\psi(x)$ transform under the rescaling transformation equation (5.13). A simple quartic term with no spatial gradients is seen to be invariant, so that these operators are "marginal" under the renormalization group. The corresponding interaction strengths will "flow" under the renormalization group transformation due to non-linear interaction effects. On the other hand, a quartic term involving gradients such as $u_2(\psi^\dagger \partial_x \psi)^2$, would rapidly scale to zero under rescaling: $u_2 \to u_2/b^2$, and can thus be ignored. Moreover, four-Fermion interactions which are chiral, say only scattering right movers, do *not* renormalize to lowest order in u and can thus also be neglected [21,22]. A As discussed in Section 4, these terms simply lead to small shifts in the Fermi velocity. All of the remaining four-Fermion interactions can be conveniently expressed in terms of currents, defined as

$$J_{ij} = \psi^\dagger_{i\alpha}\psi_{j\alpha}, \qquad \boldsymbol{J}_{ij} = \frac{1}{2}\,\psi^\dagger_{i\alpha}\boldsymbol{\sigma}_{\alpha\beta}\psi_{j\beta}; \tag{5.14}$$

$$I_{ij} = \psi_{i\alpha}\epsilon_{\alpha\beta}\psi_{j\beta}, \qquad \boldsymbol{I}_{ij} = \frac{1}{2}\,\psi_{i\alpha}(\epsilon\boldsymbol{\sigma})_{\alpha\beta}\psi_{j\beta}, \tag{5.15}$$

where the R, L subscript has been suppressed. Both J and I are invariant under global $SU(2)$ spin rotations, whereas $\boldsymbol{J}$ and $\boldsymbol{I}$ rotate as $SU(2)$ vectors. Due to Fermi statistics, some of the currents are (anti-)symmetric

$$I_{ij} = I_{ji} \qquad \boldsymbol{I}_{ij} = -\boldsymbol{I}_{ji}, \tag{5.16}$$

so that $\boldsymbol{I}_{ii} = 0$ (no sum on i).

The full set of marginal momentum-conserving four-Fermion interactions can be written

$$\begin{aligned}\mathcal{H}_I^{(1)} &= b_{ij}^{\rho} J_{Rij} J_{Lij} - b_{ij}^{\sigma} \boldsymbol{J}_{Rij} \cdot \boldsymbol{J}_{Lij}, \\ &\quad + f_{ij}^{\rho} J_{Rii} J_{Ljj} - f_{ij}^{\sigma} \boldsymbol{J}_{Rii} \cdot \boldsymbol{J}_{Ljj}. \end{aligned} \tag{5.17}$$

Here f_{ij} and b_{ij} denote the forward and backward (Cooper) scattering amplitudes, respectively, between bands i and j. Summation on $i, j = 1, 2$ is implied. To avoid double counting, we set $f_{ii} = 0$ (no sum on i). Hermiticity implies $b_{12} = b_{21}$ and parity symmetry ($R \leftrightarrow L$) gives $f_{12} = f_{21}$, so that there are generally eight independent couplings $b_{11}^{\rho,\sigma}$, $b_{22}^{\rho,\sigma}$, $b_{12}^{\rho,\sigma}$, and $f_{12}^{\rho,\sigma}$. At half-filling with particle/hole symmetry $b_{11} = b_{22}$. Additional momentum non-conserving Umklapp interactions of the form

$$\mathcal{H}_I^{(2)} = u_{ij}^{\rho} I_{Rij}^{\dagger} I_{L\hat{i}\hat{j}} - u_{ij}^{\sigma} \boldsymbol{I}_{Rij}^{\dagger} \cdot \boldsymbol{I}_{L\hat{i}\hat{j}} + \text{h.c.} \tag{5.18}$$

are also allowed, (here $\hat{1} = 2, \hat{2} = 1$). Because the currents $(\boldsymbol{I}_{ij}), I_{ij}$ are (anti-)symmetric, one can always choose $u_{12} = u_{21}$ for convenience. We also take $u_{ii}^{\sigma} = 0$ since $\boldsymbol{I}_{ii} = 0$. With particle/hole symmetry there are thus just three independent Umklapp vertices, u_{11}^{ρ}, u_{12}^{ρ}, and u_{12}^{σ}. Together with the six forward and backward vertices, nine independent couplings are required to describe the most general set of marginal non-chiral four-Fermion interactions for a two-leg ladder with particle/hole symmetry at half-filling.

The renormalization group transformation described above can be implemented by working perturbatively for small interaction parameters [16,21]. Upon systematically integrating out high-energy modes away from the Fermi points and then rescaling the spatial coordinate and Fermi fields, a set of renormalization group (RG) transformations can be derived for the interaction strengths. Denoting the nine interaction strengths as g_i, and setting the rescaling parameter $b = 1 + d\ell$ with $d\ell$ infinitesimal, the leading order differential RG flow equations take the general form,

$$\partial_{\ell} g_i = A_{ijk} g_j g_k, \tag{5.19}$$

valid up to order g^3. The matrix of coefficients A_{ijk} is given explicitly in reference [16].

These nine coupled non-linear differential equations are quite complicated, but can be integrated numerically starting with initial values appropriate to a lattice interaction (such as the Hubbard interaction). This integration reveals that some of the couplings remain small, while others tend to increase, sometimes after a sign change, and then eventually diverge. Quite surprisingly, though, the ratios of the growing couplings tend

to approach fixed constants, which are *independent* of the initial coupling strengths, at least over a wide range in the nine dimensional parameter space. These constants can be determined by inserting the Ansatz,

$$g_i(\ell) = \frac{g_{i0}}{(\ell_d - \ell)}, \tag{5.20}$$

into the RG flow equations, to obtain nine *algebraic* equations quadratic in the constants g_{i0}. There are various distinct solutions of these algebraic equations, or rays in the nine-dimensional space, which correspond to different possible phases. But for generic *repulsive* interactions between the electrons on the two-leg ladder, a numerical integration reveals that the flows are essentially always attracted to one particular ray [16]. This is the spin-liquid phase of interest, which we refer to as a d-Mott phase. In the d-Mott phase, two of the nine coupling constants, b^{ρ}_{11} and f^{σ}_{12}, remain small, while the other seven grow large with fixed ratios:

$$b^{\rho}_{12} = \frac{1}{4} b^{\sigma}_{12} \;\; = f^{\rho}_{12} = -\frac{1}{4} b^{\sigma}_{11} = \tag{5.21}$$

$$2u^{\rho}_{11} = 2u^{\rho}_{12} = \frac{1}{2} u^{\sigma}_{12} = g > 0. \tag{5.22}$$

Once the ratio's are fixed, there is a single remaining coupling contant, denoted g, which measures the distance from the origin along a very special direction (or "ray") in the nine dimensional space of couplings. The RG equations reveal that as the flows scale towards strong coupling, they are *attracted* to this special direction. If the initial bare interaction parameters are sufficiently weak, the RG flows have sufficient "time" to renormalize onto this special "ray", before scaling out of the regime of perturbative validity. In this case, the low energy physics, on the scale of energy gaps which open in the spectrum, is *universal*, depending only on the properties of the physics along this special ray, and independent of the precise values of the bare interaction strengths.

5.3 Bosonization

To determine the properties of the resulting d-Mott phase, it is extremely helpful to Bosonize the theory. As discussed in Section 4 the (continuum) electron fields can expressed in terms of Boson fields:

$$\psi_{Pi\alpha} = \kappa_{i\alpha} \mathrm{e}^{i\phi_{Pi\alpha}}; \qquad \phi_{Pi\alpha} = \varphi_{i\alpha} + P\theta_{i\alpha}, \tag{5.23}$$

with $P = R/L = \pm$. The displacement field $\theta_{i\alpha}$ and phase field $\varphi_{i\alpha}$ satisfy the commutation relations

$$[\varphi_{i\alpha}(x), \theta_{j\beta}(x')] = -i\pi\delta_{ij}\delta_{\alpha\beta}\Theta(x' - x). \tag{5.24}$$

Klein factors, satisfying

$$\{\kappa_{i\alpha}, \kappa_{j\beta}\} = 2\delta_{ij}\delta_{\alpha\beta}, \tag{5.25}$$

have been introduced so that the Fermionic operators in different bands or with different spins anticommute with one another. When the Hamiltonian is Bosonized, the Klein factors only enter in the combination, $\Gamma = \kappa_{1\uparrow}\kappa_{1\downarrow}\kappa_{2\uparrow}\kappa_{2\downarrow}$. Since $\Gamma^2 = 1$, one can take $\Gamma = \pm 1$. Hereafter, we will put $\Gamma = 1$.

The Bosonized form for the kinetic energy equation (5.9) is

$$\mathcal{H}_0 = \frac{v}{2\pi}\sum_{i,\alpha}[(\partial_x\theta_{i\alpha})^2 + (\partial_x\varphi_{i\alpha})^2], \tag{5.26}$$

which describes density waves propagating in band i and with spin α.

This expression can be conveniently separated into charge and spin modes, by defining

$$\theta_{i\rho} = (\theta_{i\uparrow} + \theta_{i\downarrow})/\sqrt{2} \tag{5.27}$$

$$\theta_{i\sigma} = (\theta_{i\uparrow} - \theta_{i\downarrow})/\sqrt{2}, \tag{5.28}$$

and similarly for φ. The $\sqrt{2}$ ensures that these new fields satisfy the same commutators, equation (5.24). It is also convenient to combine the fields in the two bands into a $\pm$ combination, by defining

$$\theta_{\mu\pm} = (\theta_{1\mu} \pm \theta_{2\mu})/\sqrt{2}, \tag{5.29}$$

where $\mu = \rho, \sigma$, and similarly for φ.

The Hamiltonian density $\mathcal{H}_0$ can now be re-expressed in a charge/spin and flavor decoupled form,

$$\mathcal{H}_0 = \frac{v}{2\pi}\sum_{\mu,\pm}[(\partial_x\theta_{\mu\pm})^2 + (\partial_x\varphi_{\mu\pm})^2]. \tag{5.30}$$

The fields $\theta_{\rho+}$ and $\varphi_{\rho+}$ describe the total charge and current fluctuations, since under Bosonization, $\psi^\dagger_{Pi\alpha}\psi_{Pi\alpha} = 2\partial_x\theta_{\rho+}/\pi$ and $vP\psi^\dagger_{Pi\alpha}\psi_{Pi\alpha} = 2\partial_x\varphi_{\rho+}/\pi$.

While it is possible to Bosonize the interaction Hamiltonians in full generality [16], we do not reproduce it here. In addition to terms quadratic in gradients of the Boson fields (as in $\mathcal{H}_0$), the Bosonized interaction consists of terms bi-linear in $\cos 2\theta$ and $\cos 2\varphi$. More specifically, of the eight non-chiral Boson fields ($\theta_{\mu\pm}$ and $\varphi_{\mu\pm}$) only five enter as arguments of cosine terms. In the momentum conserving terms these are $\theta_{\sigma\pm}$, $\varphi_{\rho-}$ and $\varphi_{\sigma-}$. The Umklapp terms also involve the overall charge displacement field, *via* $\cos 2\theta_{\rho+}$. This can be understood by considering how the Boson fields transform under a spatial translation, $x \to x + x_0$. The chiral electron operators

transform as $\psi_{Pi} \to \psi_{Pi} e^{ipk_{Fi}x_0}$, which is equivalent to $\theta_{i\alpha} \to \theta_{i\alpha} + k_{Fi}x_0$. Three of the charge/spin and flavor fields are thus invariant under spatial translations, whereas $\theta_{\rho+} \to \theta_{\rho+} + \pi x_0$. The momentum conserving terms are invariant under spatial translations, so *cannot* depend on $\cos 2\theta_{\rho+}$.

The full interacting theory is invariant under spatially constant shifts of the remaining three Boson fields – $\varphi_{\rho+}, \varphi_{\sigma+}$ and $\theta_{\rho-}$. For the first two of these, the conservation law responsible for this symmetry is readily apparent. Specifically, the operators $\exp(iaQ)$ and $\exp(iaS_z)$, with Q the total electric charge and S_z the total z-component of spin, generate "translations" proportional to a in the two fields $\varphi_{\rho+}$ and $\varphi_{\sigma+}$. To see this, we note that $Q = \int \mathrm{d}x \rho(x)$ with $\rho(x) = 2\partial_x \theta_{\rho+}/\pi$ the momentum conjugate to $\varphi_{\rho+}$, whereas S_z can be expressed as an integral of the momentum conjugate to $\varphi_{\sigma+}$. Since the total charge is conserved, $[Q, H] = 0$, the full Hamiltonian must therefore be invariant under $\varphi_{\rho+} \to \varphi_{\rho+} + a$ for arbitrary constant a, precluding a cosine term for this field. Similarly, conservation of S_z implies invariance under $\varphi_{\sigma+} \to \varphi_{\sigma+} + a$.

The five Boson fields entering as arguments of various cosine terms will tend to be pinned at the minima of these potentials. Two of these 5 fields, $\theta_{\sigma-}$ and $\varphi_{\sigma-}$, are dual to one another so that the uncertainty principle precludes pinning both fields. Since there are various competing terms in the potential seen by these 5 fields, minimization for a given set of bare interaction strengths is generally complicated. However, along the special ray in the nine dimensional space of interaction parameters the nine independent coupling constants can be replaced by a *single* parameter g. The resulting Bosonized theory is found to reduce to a very simple and highly symmetrical form when expressed in terms of a new set of Boson fields, defined by

$$\begin{aligned} (\theta,\varphi)_1 &= (\theta,\varphi)_{\rho+}, & (\theta,\varphi)_2 &= (\theta,\varphi)_{\sigma+}, \\ (\theta,\varphi)_3 &= (\theta,\varphi)_{\sigma-}, & (\theta,\varphi)_4 &= (\varphi,\theta)_{\rho-}. \end{aligned} \tag{5.31}$$

The first three are simply the charge/spin and flavor fields defined earlier. However, in the fourth pair of fields, θ and φ have been interchanged.

In terms of these new fields, the full interacting Hamiltonian density along the special ray takes an exceedingly simple form: $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$, with

$$\mathcal{H}_0 = \frac{v}{2\pi} \sum_a [(\partial_x \theta_a)^2 + (\partial_x \varphi_a)^2], \tag{5.32}$$

$$\begin{aligned} \mathcal{H}_I &= \frac{g}{2\pi^2} \sum_a [(\partial_x \theta_a)^2 - (\partial_x \varphi_a)^2] \\ &\quad -4g \sum_{a \neq b} \cos 2\theta_a \cos 2\theta_b. \end{aligned} \tag{5.33}$$

5.4 *d*-Mott phase

We now briefly discuss some of the general physical properties of the *d*-Mott phase which follow from this Hamiltonian. Ground state properties can be inferred by employing semi-classical considerations. Since the fields φ_a enter quadratically, they can be integrated out when the partition function is expressed as a path integral over Boson fields. This leaves an effective action in terms of the four fields θ_a. Since the single coupling constant g is marginally relevant and flowing off to strong coupling, these fields will be pinned in the minima of the cosine potentials. Specifically, there are two sets of semiclassical ground states with all $\theta_a = n_a\pi$ or all $\theta_a = (n_a + 1/2)\pi$, where n_a are integers. It can be shown [16] that these different solutions actually correspond to the *same* physical state, so that the ground state is unique. Excitations will be separated from the ground state by a finite energy gap, since the fields are harmonically confined, and instanton excitations connecting different minima are also costly in energy.

Consider first those fields which are pinned by momentum conserving interaction terms. Since both $\theta_{\sigma\pm}$ fields are pinned, so are the spin-fields in each band, $\theta_{i\sigma}$ $(i = 1, 2)$. Since $\partial_x\theta_{i\sigma}$ is proportional to the z-component of spin in band i, a pinning of these fields implies that the spin in each band vanishes, and excitations with non-zero spin are expected to cost finite energy: the spin gap. This can equivalently be interpreted as singlet pairing of electron pairs in each band. It is instructive to consider the pair field operator in band i:

$$\Delta_i = \psi_{Ri\uparrow}\psi_{Li\downarrow} = \kappa_{i\uparrow}\kappa_{i\downarrow}\mathrm{e}^{i\sqrt{2}(\varphi_{i\rho}+\theta_{i\sigma})}. \tag{5.34}$$

With $\theta_{i\sigma} \approx 0$, $\varphi_{i\rho}$ can be interpreted as the phase of the pair field in band i. The relative phase of the pair field in the two bands follows by considering the product

$$\Delta_1\Delta_2^\dagger = -\Gamma\mathrm{e}^{i2\theta_{\sigma-}}\mathrm{e}^{i2\varphi_{\rho-}}, \tag{5.35}$$

with $\Gamma = \kappa_{1\uparrow}\kappa_{1\downarrow}\kappa_{2\uparrow}\kappa_{2\downarrow} = 1$. Since $\theta_4 = \varphi_{\rho-}$ the relative phase is also pinned by the cosine potential, with a sign change in the relative pair field, $\Delta_1\Delta_2^\dagger < 0$, corresponding to an approximate *d*-wave symmetry.

To discuss the physics of the remaining overall charge mode $(\theta_{\rho+})$, it is convenient to first imagine "turning off" the Umklapp interactions. After pinning the other three fields to the minima of the cosine potentials, the pair field operator in band i becomes

$$\Delta_i \sim (-1)^i\mathrm{e}^{i\varphi_{\rho+}}, \tag{5.36}$$

so that $\varphi_{\rho+}$ is the phase of the pair field. In the absence of Umklapp scattering, the Lagrangian for this phase field is simply,

$$\mathcal{L} = \frac{1}{2}\kappa_\mu(\partial_\mu\varphi_{\rho+})^2. \tag{5.37}$$

Being in one-spatial dimension, these gapless *phase* fluctuations lead to power law decay of the pair field spatial correlation function, $\Delta^*(x)\Delta(0) \sim 1/x^\eta$. A true superconductor (for $d > 1$) exhibits (off-diagonal) long-ranged order, and this correlation function would not decay to zero even as $x \to \infty$. But in one-dimension a "superconductor" can at best exhibit power law decay, since true off-diagonal long-ranged order is not possible [5]. Thus, in the absence of Umklapp scattering the 2-leg ladder would be a one-dimensional d-wave "superconductor".

But what is the effect of the momentum non-conserving Umklapp interactions? Once the other three fields are pinned in the minima of the cosine potentials in the above Hamiltonian equation (5.33), the Umklapp scattering terms take the simple form,

$$\mathcal{H}_u = -12g \cos 2\theta_{\rho+}. \tag{5.38}$$

This term tends to pin the field $\theta_{\rho+}$. The pair field phase, $\varphi_{\rho+}$, being the *conjugate* field will fluctuate wildly. These quantum flucutations will destroy the power-law 1d "superconducting" phase, leading to an exponentially decaying pair-field correlation function. What is the fate of this one-dimensional "quantum disordered d-wave superconductor"?

To see this, one simply has to consider the "dual" representation in terms of the $\theta_{\rho+}$ field, rather than $\varphi_{\rho+}$. A *lattice* version of this duality transformation is carried out in detail in the Appendix. Alternatively, one can obtain the dual theory directly from the Bosonized Hamiltonian equation (5.32). The appropriate Lagrangian dual to equation (5.37) above, is simply

$$\mathcal{L} = \frac{1}{2}\kappa_\mu (\partial_\mu \theta_{\rho+})^2, \tag{5.39}$$

which describes gapless *density* waves. These density flucutations will be pinned by the Umklapp terms in H_u, leading to a Mott insulator with a gap to charge excitations. Since there is also a spin-gap this phase is equivalent to the spin-liquid, discussed at strong coupling in terms of the Heisenberg model in Section 3. But we now see that this spin-liquid phase exhibits superconducitng d-wave pairing correlations, despite being an insulator. The spin-liquid phase can thus be described as a quantum disordered one-dimensional d-wave "superconductor".

The Euclidian action associated with the phase Lagrangian in equation (5.37) is equivalent to the effective Hamiltonian in the low temperature phase of the classical 2d xy model, (with imaginary time playing the role of a second spatial coordinate). The 2d xy model can be disordered by introducing vortices into the phase of the order parameter [31]. For this it is convenient to go to a dual representation [32]. As shown explicitly in the Appendix, the dual represention is equivalent to the $\theta_{\rho+}$ representation, with the strength of the Umklapp term playing the role of a vortex

fugacity. In Section 8, we will quantum disorder a *two-dimensional* d-wave superconductor, and it will be extremely convenient to consider a duality transformation – a three dimensional version of the 2d $\theta \leftrightarrow \varphi$ duality discussed here. The resulting nodal liquid phase will be particularly simple to analyze in the dual representation.

5.5 Symmetry and doping

Due to the highly symmetric form of the Hamiltonian in equations (5.32) and (5.33), it is possible to make considerable further progress in analyzing it's properties. Indeed, as shown in reference [16], under a re-Fermionization procedure this Hamiltonian is equivalent to the $SO(8)$ Gross-Neveu model [33], which has been studied extensively by particle field theorists. The $SO(8)$ Gross-Neveu model posesses a remarkable symmetry known as triality [34], which can be used to equate the energies of various excited states. In particular, the energy of the lowest excited state with the quantum numbers of an electron (charge e and $s = 1/2$) is *equal* to the energy of the lowest lying spinless charge $2e$ exited state (a Cooper pair). This beautifully demonstrates *pairing* in the insulating d-Mott phase: the energy to add two electrons of opposite spin far apart is twice as large as the energy to add them into a Cooper pair bound state. It turns out, moreover, that the Gross-Neveu model is *integrable* [35] so it is possible to fully enumerate the energies and quantum numbers of *all* the low energy excited states [16] (grouped into $SO(8)$ multiplets) and compute exactly various correlation functions [36].

We finally briefly mention the effects of doping the d-Mott phase away from half-filling. This can be achieved by adding a chemical potential term to the Hamiltonian in equations (5.32) and (5.33), with $H_\mu = H - \mu Q$, where Q is the *total* electric charge:

$$Q = \frac{2}{\pi} \int \partial_x \theta_{\rho+}. \tag{5.40}$$

Since the field $\theta_{\rho+}$ is pinned in the cosine potential by the Umklapp interaction terms, H_u, for small μ the density will stay fixed at half-filling. Eventually, μ will pass through the Mott charge gap and the density will change. This occurs *via* π instantons in $\theta_{\rho+}$, connecting adjacent minima of the cosine potential. Each instanton carries charge $2e$, but no spin, so can be intepreted as a Cooper pair. In this doped phase, the Umklapp scattering terms will no longer we able to freeze the charge fluctuations, and one expects gapless excitations in the density and pair field phase, $\varphi_{\rho+}$. This doped phase will exhibit power-law d-wave superconducting correlations [21].

6 *d*-Wave superconductivity

We now turn to the case of a two-dimensional superconductor which exhibits a particular type of d-wave pairing (denoted $d_{x^2-y^2}$) appropriate to the Cuprates. Our ultimate goal is to quantum disorder this state to obtain a description of the "nodal liquid". There are two main distinctions between the 2d d-wave superconductor and it's one-dimensional counterpart considered above. Firstly, a 2d superconductor exhibits *true* (off-diagonal) long-ranged order at $T = 0$. But more importantly, due to sign changes in the pair wave function, the $d_{x^2-y^2}$ superconductor exhibits *gapless* quasiparticle excitations. We first briefly review BCS theory which gives one a powerful framework to describe d-wave pairing and the gapless quasiparticles. In Section 7 below we incorporate *quantum flucutations* of the order parameter phase to obtain a complete *effective* low-energy theory of the $d_{x^-y^2}$ phase. In Section 8 a dual represention is derived, and used to quantum disorder the superconductor in Section 9.

6.1 BCS theory re-visited

It is instructive to briefly review BCS theory [37], focussing on the symmetries of the pair wave function and the superconducting order parameter. In particular, it is important to emphasize the important distinction between the wave function for the center of mass of the Cooper pair (often ignored) and the wavefunction for the relative coordinate.

Consider a Hamiltonian expressed as a sum of kinetic energy and interaction terms, $H = H_0 + H_{\rm int}$, with H_0 given in equation (2.2). We consider a rather general form for the electron interactions:

$$H_{\rm int} = \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k'q}} v_{\mathbf{q}}(\mathbf{k},\mathbf{k'}) c^\dagger_{\mathbf{k+q}\alpha} c^\dagger_{-\mathbf{k}+\mathbf{q}\beta} c_{-\mathbf{k'}+\mathbf{q}\beta} c_{\mathbf{k'}+\mathbf{q}\alpha}, \tag{6.1}$$

which is invariant under global charge $U(1)$ and spin $SU(2)$ symmetries. For simplicity Umklapp interaction terms have been ignored, so that the crystal momentum is conserved. The interaction term describes a two electron scattering process with $2\mathbf{q}$ the total conserved momentum of the pair. For a density-density interaction in real space, such as the Coulomb interaction, $v_{\mathbf{q}}(\mathbf{k},\mathbf{k'}) = v(|\mathbf{k}-\mathbf{k'}|)$, so is independent of $\mathbf{q}$.

Superconductivity within BCS theory requires an attractive interaction (in the appropriate angular momentum channel) between electrons. But the bare Coulomb interaction is of course strongly repulsive. In traditional low temperature superconductors, phonons are believed to drive the pairing, inducing a retarded attractive interaction at low energies below the deBye energy. Superconductivity in the high temperature Cuprates is probably of electronic origin. In this case, retardation leading to an attractive

interaction at low energies would be due to virtual interactions *via* high energy electron states well away from E_F. These processes can be studied *via* a renormalization group procedure [17], which consists of "integrating out" high energy electron states, and seeing how the remaining interactions between those electrons near the Fermi energy are modified. This is precisely what we implemented in detail for the two-leg ladder in Section 5. One thereby arrives at an effective low energy theory involving electron states within a small energy range of width 2Λ around E_F, scattering off one another with an *effective* (or renormalized) interaction potential. In the following, we view $v_{\mathbf{q}}(\mathbf{k},\mathbf{k}')$ as an effective low energy interaction. For the two-leg ladder the renormalized potential is given by putting the nine coupling contants equal to their values along the special ray. Upon Bosonization, the effective potential is given explicitly in equation (5.33). More generally, the form of the renormalized potential will be constrained by the original symmetries of the Hamiltonian. Specifically, time reveral and parity symmetries imply that $v_{\mathbf{q}}(\mathbf{k},\mathbf{k}')$ is real, and odd in it's arguments: $v_{\mathbf{q}}(\mathbf{k},\mathbf{k}') = v_{-\mathbf{q}}(-\mathbf{k},-\mathbf{k}')$. Hermiticity implies $v_{\mathbf{q}}(\mathbf{k},\mathbf{k}') = v_{\mathbf{q}}(\mathbf{k}',\mathbf{k})$. The summation over momentum is now understood to be constrained, involving only electron operators with energy in a shell of width 2Λ about E_F.

BCS theory can be implemented by considering the operator,

$$P^{\alpha\beta}_{\mathbf{k}}(\mathbf{q}) = c_{-\mathbf{k}+\mathbf{q}\alpha}c_{\mathbf{k}+\mathbf{q}\beta}, \tag{6.2}$$

which destroys a pair of electrons, with total momentum $2\mathbf{q}$. For $\mathbf{k}$ near the Fermi surface, and $|\mathbf{q}| \ll k_F$, $[P_{\mathbf{k}}(\mathbf{q}), P^{\dagger}_{\mathbf{k}}(\mathbf{q}')] = 0$ for $\mathbf{q} \neq \mathbf{q}'$, so that the pair operator resembles a boson operator, $b(\mathbf{q})$. By analogy with Bose condensation, in the superconducting phase one expects a non-zero expectation value for the pair operator: $\langle P\rangle \neq 0$. The pair operators entering into $H_{\rm int}$ are expressed as $P = \langle P\rangle + \delta P$, and the fluctuations $\delta P = P - \langle P\rangle$ are presumed to be small. Upon ignoring terms quadratic in δP, $H_{\rm int}$ can be written (dropping additive constants),

$$H_1 = \frac{1}{2V}\sum_{\mathbf{k},\mathbf{q}}[c^{\dagger}_{\mathbf{k}+\mathbf{q}\alpha}c^{\dagger}_{-\mathbf{k}+\mathbf{q}\beta}\Delta^{\beta\alpha}_{\mathbf{k}}(\mathbf{q}) + \text{h.c.}], \tag{6.3}$$

where we have introduced the (complex) superconducting order parameter (or "gap"), Δ, defined as,

$$\Delta^{\alpha\beta}_{\mathbf{k}}(\mathbf{q}) = \sum_{\mathbf{k}'} v_{\mathbf{q}}(\mathbf{k},\mathbf{k}')\langle c_{-\mathbf{k}'+\mathbf{q}\alpha}c_{\mathbf{k}'+\mathbf{q}\beta}\rangle. \tag{6.4}$$

BCS is a self-consistent mean field theory: the full mean field (or quasiparticle) Hamiltonian, $H_{qp} = H_0 + H_1$, which depends on Δ, is employed to compute the expectation value $\langle c_{-\mathbf{k}'+\mathbf{q}\alpha}c_{\mathbf{k}'+\mathbf{q}\beta}\rangle$. Upon insertion in equation (6.4) one obtains a self-consistent equation which determines Δ – the

celebrated BCS gap-equation. Notice that H_{qp} is bi-linear in electron operators and hence tractable, although it does involve "anomalous" terms involving pairs of creation or annihilation operators.

Before carrying through this procedure, it is instructive to consider the form for the pair wavefunction which follows from a non-zero expectation value of the pair operator $\langle P \rangle \neq 0$. Consider removing a pair of electrons, at positions $\boldsymbol{R} \pm \boldsymbol{r}/2$, with $\boldsymbol{R}$ the center of mass position and $\boldsymbol{r}$ the relative coordinate. The pair wave function can be defined as,

$$\Phi^{\alpha\beta}(\boldsymbol{R}, \boldsymbol{r}) = \langle c_\alpha(\boldsymbol{R} - \boldsymbol{r}/2) c_\beta(\boldsymbol{R} + \boldsymbol{r}/2) \rangle, \tag{6.5}$$

which depends on the *spin* of the electrons as well as the (center of mass and relative) positions. Upon transforming the electron operators into momentum space, one finds that

$$\Phi^{\alpha\beta}(\boldsymbol{R}, \boldsymbol{r}) = \sum_{\boldsymbol{Q}} \mathrm{e}^{i\boldsymbol{Q}\cdot\boldsymbol{R}} \Phi^{\alpha\beta}(\boldsymbol{Q}, \boldsymbol{r}), \tag{6.6}$$

with $\boldsymbol{Q}$ the center of mass momentum and

$$\Phi^{\alpha\beta}(\boldsymbol{Q}, \boldsymbol{r}) = \frac{1}{N} \sum_{\mathbf{k}} \langle P^{\alpha\beta}_{\mathbf{k}}(\boldsymbol{Q}/2) \rangle \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{r}}. \tag{6.7}$$

Notice that the wavefunction in the *relative* coordinate, involves a Fourier transform with respect to the relative pair momentum, $\mathbf{k}$.

It is also instructive to define a *spatially varying* superconducting order parameter by Fourier transforming the gap function, $\Delta_{\mathbf{k}}(\mathbf{q})$:

$$\Delta^{\alpha\beta}_{\mathbf{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{Q}} \mathrm{e}^{i\boldsymbol{Q}\cdot\boldsymbol{x}} \Delta_{\mathbf{k}}(\boldsymbol{Q}/2). \tag{6.8}$$

In the superconducting phase one can often ignore the spatial dependence of the complex order parameter $\Delta_{\mathbf{k}}(\boldsymbol{x})$, and indeed in BCS theory this $\boldsymbol{x}$ dependence is dropped. However, if one wishes to include the effects of quantum fluctuations (to quantum disorder the superconductor) it is necessary to consider a spatially varying order parameter as discussed in Section 7 below.

By analogy with Bose condensation, one expects the Cooper pairs to be condensed into a state of *zero* momentum, $\boldsymbol{Q} = 0$. This requires

$$\langle P^{\alpha\beta}_{\mathbf{k}}(\mathbf{q}) \rangle = \delta_{\mathbf{q},\mathbf{0}} \langle c_{-\mathbf{k}\alpha} c_{\mathbf{k}\beta} \rangle, \tag{6.9}$$

which gives a *relative* pair wavefunction, $\Phi(\boldsymbol{r}) \equiv \Phi(\boldsymbol{Q} = 0, \boldsymbol{r})$ of the form,

$$\Phi^{\alpha\beta}(\boldsymbol{r}) = \frac{1}{N} \sum_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{r}} \Phi^{\alpha\beta}_{\mathbf{k}}; \qquad \Phi^{\alpha\beta}_{\mathbf{k}} = \langle c_{-\mathbf{k}\alpha} c_{\mathbf{k}\beta} \rangle. \tag{6.10}$$

Due to the electron anticommutation relations one has $\Phi^{\alpha\beta}_{\mathbf{k}} = -\Phi^{\beta\alpha}_{-\mathbf{k}}$, which implies that the pair wavefunction is *antisymmetric* under exchange of the two electrons: $\Phi^{\alpha\beta}(\boldsymbol{r}) = -\Phi^{\beta\alpha}(-\boldsymbol{r})$.

When the Cooper pairs are condensed into a state with zero momentum, the superconducting order parameter becomes spatially uniform: $\Delta^{\alpha\beta}_{\mathbf{k}}(\boldsymbol{x}) \equiv \Delta^{\alpha\beta}_{\mathbf{k}}$, as seen from equation (6.4). The mean field Hamiltonian then takes a rather simpler form:

$$H_1 = \frac{1}{2}\sum_{\mathbf{k}}[c^\dagger_{\mathbf{k}\alpha}c^\dagger_{-\mathbf{k}\beta}\Delta^{\beta\alpha}_{\mathbf{k}} + \text{h.c.}], \tag{6.11}$$

whereas the self-consistentcy condition becomes,

$$\Delta^{\alpha\beta}_{\mathbf{k}} = \frac{1}{V}\sum_{\mathbf{k}'} v_0(\mathbf{k},\mathbf{k}')\langle c_{-\mathbf{k}'\alpha}c_{\mathbf{k}'\beta}\rangle \cdot \tag{6.12}$$

Since the full model has a conserved $SU(2)$ spin symmetry, the relative pair wavefunction can be expressed as the product of an orbital and a spin wavefunction: $\Phi^{\alpha\beta}_{\mathbf{k}} = \phi_{\alpha\beta}\Phi_{\mathbf{k}}$. The spin piece can be chosen as an eigenfunction of the total spin of the pair, that is a singlet with $S = 0$ or a triplet with $S = 1$. In conventional low temperature superconductors and in the Cuprates the Cooper pairs are singlets with,

$$\phi_{\alpha\beta} = \delta_{\alpha\downarrow}\delta_{\beta\uparrow} - \delta_{\alpha\uparrow}\delta_{\beta\downarrow}, \tag{6.13}$$

in which case the orbital wavefunction is symmetric: $\Phi_{\mathbf{k}} = \Phi_{-\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\rangle$. (In the superfluid phases of $3-$He on the other hand, the Cooper pairs have $S = 1$.) The superconducting order parameter is then also a singlet; $\Delta^{\alpha\beta}_{\mathbf{k}} \equiv \phi_{\alpha\beta}\Delta_{\mathbf{k}}$, with $\Delta_{\mathbf{k}} = \Delta_{-\mathbf{k}}$ satisfying

$$\Delta_{\mathbf{k}} = \frac{1}{V}\sum_{\mathbf{k}'} v_0(\mathbf{k},\mathbf{k}')\langle c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}\rangle \cdot \tag{6.14}$$

For singlet pairing, the final mean field (quasiparticle) Hamiltonian becomes, $H_{qp} = H_0 + H_1$ with,

$$H_1 = \sum_{\mathbf{k}}[\Delta_{\mathbf{k}}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} + \Delta^*_{\mathbf{k}}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}]. \tag{6.15}$$

To complete the self-consistency requires diagonalizing the quasiparticle Hamiltonian. This is usually done in a way which masks the spin rotational invariance [37]. We prefer to keep the spin rotational invariance explicit, by defining a new set of Fermion operators, for $k_y > 0$:

$$\chi_{1\alpha}(\mathbf{k}) = c_{\mathbf{k}\alpha}; \qquad \chi_{2\alpha}(\mathbf{k}) = i\sigma^y_{\alpha\beta}c^\dagger_{-\mathbf{k}\beta}, \tag{6.16}$$

which satisfy canonical Fermion anti-commutation relations:

$$[\chi_{a\alpha}(\mathbf{k}), \chi^\dagger_{b\beta}(\mathbf{k}')]_- = \delta_{ab}\delta_{\alpha\beta}\delta_{\mathbf{k}\mathbf{k}'}. \tag{6.17}$$

The first index $a, b = 1, 2$ acts in the particle/hole subspace. The σ^y in the definition of $\chi_{2\alpha}$ has been introduced so that these new operators transform like $SU(2)$ spinors under spin rotations: $\chi_{a\alpha} \to U_{\alpha\beta}\chi_{a\beta}$, with $U = \exp(i\boldsymbol{\theta}\cdot\boldsymbol{\sigma})$ a global spin rotation.

In these variables, the quasiparticle Hamiltonian becomes

$$H_{qp} = {\sum_{\mathbf{k}}}' \chi^\dagger(\mathbf{k})[\tau^z\epsilon_{\mathbf{k}} + \tau^+\Delta_{\mathbf{k}} + \tau^-\Delta^*_{\mathbf{k}}]\chi(\mathbf{k}), \tag{6.18}$$

where the prime on the summation denotes over k_y positive, only, and we have introduced a vector of Pauli matrices, $\vec{\tau}_{ab}$ acting in the particle/hole subspace. Also, we are employing the notation $\tau^\pm = (\tau^x \pm i\tau^y)/2$. To evaluate the self-consistency condition equation (6.14) we need the anomalous average of two electron fields (the orbital piece of the relative pair wavefunction), which is re-expressed as,

$$\Phi_{\mathbf{k}} \equiv \frac{1}{2}\sum_{\pm}\langle c_{\mp\mathbf{k}\downarrow}c_{\pm\mathbf{k}\uparrow}\rangle = \frac{1}{2}\langle\chi^\dagger(\mathbf{k})\tau^+\chi(\mathbf{k})\rangle\cdot \tag{6.19}$$

Diagonalization is now achieved by performing an $SU(2)$ rotation in the particle/hole subspace, by defining rotated Fermion fields: $\chi(\mathbf{k}) \equiv U(\mathbf{k})\tilde{\chi}(\mathbf{k})$, with $U(\mathbf{k}) = \mathrm{e}^{-i\boldsymbol{\theta}_{\mathbf{k}}\cdot\boldsymbol{\tau}}$. Assuming for simplicity that $\Delta_{\mathbf{k}}$ is real, the appropriate rotation is around the y-axis by an angle $\theta_{\mathbf{k}}$, $U(\mathbf{k}) = \mathrm{e}^{-i\theta_{\mathbf{k}}\tau^y/2}$, with

$$\sin(\theta_{\mathbf{k}}) = \frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}}; \qquad E_{\mathbf{k}} = \sqrt{\epsilon^2_{\mathbf{k}} + \Delta^2_{\mathbf{k}}}. \tag{6.20}$$

In terms of the rotated Fermion fields, $\tilde{\chi}$, the quasiparticle Hamiltonian is diagonal,

$$H_{qp} = {\sum_{\mathbf{k}}}' E_{\mathbf{k}}\tilde{\chi}^\dagger(\mathbf{k})\tau^z\tilde{\chi}(\mathbf{k}), \tag{6.21}$$

with $E_{\mathbf{k}}$ the quasiparticle energy. Finally, we define a set of rotated electron operators *via*

$$\tilde{\chi}_{1\alpha}(\mathbf{k}) = a_{\mathbf{k}\alpha}; \qquad \tilde{\chi}_{2\alpha}(\mathbf{k}) = i\sigma^y_{\alpha\beta}a^\dagger_{-\mathbf{k}\beta}, \tag{6.22}$$

and the quasiparticle Hamiltonian can be re-expressed in standard form,

$$H_{qp} = \sum_{\mathbf{k}} E_{\mathbf{k}}a^\dagger_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}, \tag{6.23}$$

where we have dropped an additive constant. Notice that the quasiparticle energy $E_{\mathbf{k}} \geq 0$ for all momentum. On the Fermi surface, $\epsilon_{\mathbf{k}} = 0$ and the quasiparticle energy is given by $|\Delta_{\mathbf{k}}|$ – the energy gap.

To complete the self-consistentcy, the anomalous electron average (or relative orbital pair wavefunction from Eq. (6.19)) is expressed in terms of the quasiparticle operators. Upon using the fact that $[U^{\dagger}\tau^{+}U]_{\rm diag} = \sin(\theta)\tau^{z}/2$ one obtains,

$$\Phi_{\mathbf{k}} = \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}[\langle a^{\dagger}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}\rangle - 1], \tag{6.24}$$

which reduces to $\Phi_{\mathbf{k}} = -\Delta_{\mathbf{k}}/2E_{\mathbf{k}}$ at zero temperature. At finite temperature the number of quasiparticles is simply a Fermi function: $\langle a^{\dagger}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}\rangle = 2f(E_{\mathbf{k}})$, with $f(E) = [\exp(\beta E) + 1]^{-1}$. One thereby obtains the celebrated BCS gap equation:

$$\Delta_{\mathbf{k}} = -\frac{1}{V}\sum_{\mathbf{k}'} v_0(\mathbf{k},\mathbf{k}')\frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}[1 - 2f(E_{\mathbf{k}'})]. \tag{6.25}$$

6.2 d-wave symmetry

In a system with rotational invariance the orbital piece of the pair wavefunction, proportional to $\Delta_{\mathbf{k}}$ from equation (6.24), can be chosen as an eigenstate of angular momentum, a spherical harmonic $Y_{\rm lm}$ in three dimensions. The simplest case is s-wave, with $\Delta_{\mathbf{k}}$ a constant over the (spherical) Fermi surface. Real materials of course do not share the full continuous rotational symmetry of free space. Nevertheless, a superconductor in which $\Delta_{\mathbf{k}}$ is everywhere positive over the Fermi surface is (loosely) referred to as having s-wave pairing – a property of all conventional low temperature superconductors. Since $|\Delta_{\mathbf{k}}|$ is the quasiparticle energy on the Fermi surface, there are no low energy electronic excitations in an *s*-wave superconductor – the Fermi surface is *fully* gapped. Within BCS theory the magnitude of the (zero temperature) energy gap is related to the superconducting transition temperature: $2|\Delta| \approx 3.5 k_B T_c$. The presence of an energy gap leads to thermally activated behavior for various low temperature properties, such as the electronic specific heat and the magnetic penetration length.

It is clear from the self-consistent gap equation (Eq. (6.25)) that a purely repulsive effective interaction, $v_0(\mathbf{k},\mathbf{k}') > 0$, precludes s-wave pairing within BCS theory (since $1 - 2f(E_{\mathbf{k}}) \geq 0$). In conventional superconductors, phonons are believed to drive s-wave pairing [37], generating an effective attractive interaction at low energies.

Recent experiment [19,20] has established that in the high temperature superconductors the orbital pairing symmetry is a particular form of *d*-wave, usually denoted as $d_{x^2-y^2}$. Here x and y refer to the directions along the crystalline axis of a single Cu-O sheet, within which the Cu atoms form a

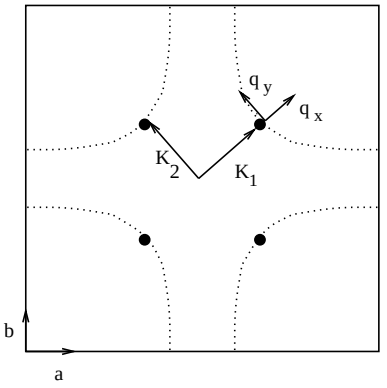

Fig. 7. In the $d_{x^2-y^2}$ superconductor the quasiparticle energy vanishes at four points ($\pm\mathbf{K}_1$ and $\pm\mathbf{K}_2$) in the Brillouin zone. The dotted line represents the Fermi surface. The wavevector $\mathbf{q}$ is rotated with respect to the a and b axis of the square lattice.

square lattice. In terms of the corresponding two dimensional momentum, $\mathbf{k} = (k_x, k_y)$, the angular dependence of the gap function in this state is $\Delta_{\mathbf{k}} \sim k_x^2 - k_y^2$, and from equation (6.24) the orbital piece of the relative pair wave function has the same d-wave symmetry.

A novel feature of the $d_{x^2-y^2}$ state is that the gap function $\Delta_{\mathbf{k}}$ *vanishes* along lines in $k-space$ with $k_x = \pm k_y$, corresponding to *nodes* in the relative pair wave function. These lines intersect the (two-dimensional) Fermi surface at four *points* in momentum space. Near these four points (or "nodes") in momentum space there are electronic excitations with arbitrary low energy, in striking constast to the fully gapped s-wave case. These low energy quasiparticle excitations dominate the physics of the $d_{x^2-y^2}$ superconductor at temperatures well below T_c, leading to power law temperature corrections in such quantities as the electronic specific heat and the magnetic penetration length.

6.3 Continuum description of gapless quasiparticles

It is convenient to obtain a *continuum* description of the gapless d-wave quasiparticles, analogous to the Dirac theory description of the low energy properties of the 1d free Fermions employed in Section 4. A continuum form can be obtained directly from the general quasiparticle Hamiltonian equation (6.18) by specializing to $d_{x^2-y^2}$ symmetry and then focussing on those momenta close to the four nodes where the quasiparticle energy $E_{\mathbf{k}} = 0$ (see Fig. 7). For a model with particle/hole symmetry $\epsilon_{\mathbf{k}} = -\epsilon_{\mathbf{k}+\pi}$, which together with parity symmetry implies that the four nodes occur at the

special wavevectors $\pm\mathbf{K}_j$, with $\mathbf{K}_1 = (\pi/2, \pi/2)$ and $\mathbf{K}_2 = (-\pi/2, \pi/2)$. It is convenient to introduce *two* continuum fields Ψ_j, one for each pair of nodes, expanded around $\pm\mathbf{K}_1, \pm\mathbf{K}_2$:

$$\Psi_{ja\alpha}(\mathbf{q}) = \chi_{a\alpha}(\mathbf{K}_j + \mathbf{q}). \tag{6.26}$$

Here, the wavevectors $\mathbf{q}$ are assumed to be small, within a circle of radius Λ around the origin. With this definition, the particle/hole transformation is extremely simple,

$$\Psi \to \Psi^\dagger. \tag{6.27}$$

For this reason it is convenient to *always* define the continuum fields Ψ around $\pm\mathbf{K}_j$, and account for deviations of the node momenta from these values by a particle/hole symmetry-breaking parameter λ.

Once we have restricted attention to the momenta near the nodes, it is legitimate to linearize in the quasiparticle Hamiltonian. The resulting theory is more conveniently written in coordinates perpendicular and parallel to the Fermi surface, so we perform the rotation *via* $x \to (x-y)/\sqrt{2}$ and $y \to (x+y)/\sqrt{2}$, correspondingly transforming the momenta q_x and q_y (see Fig. 7). Linearizing near the nodes, we put $\epsilon_{\mathbf{K}_1+\mathbf{q}} = v_{\rm F} q_x$ where $v_{\rm F}$ is the Fermi velocity and

$$\Delta_{\mathbf{K}_1+\mathbf{q}} = \tilde{\Delta} q_y + O(q^2), \tag{6.28}$$

where $\tilde{\Delta}$ has dimensions of a velocity. An identical linearization is possible around the second pair of nodes, except with $q_x \leftrightarrow q_y$. It is finally convenient to Fourier transform back into real space by defining,

$$\Psi_j(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}} \Psi_j(\mathbf{q}), \tag{6.29}$$

where the momentum summation is for $q < \Lambda$. The continuum fields $\Psi(\mathbf{x})$ describe long lengthscale variations of the quasiparticles, on scales greater than Λ^{-1}. We thereby arrive at a compact form for the continuum quasiparticle Hamiltonian in a $d_{x^2-y^2}$ superconductor: $\mathcal{H}_{qp} = \mathcal{H}_\Psi + \mathcal{H}_\lambda$ with

$$\begin{aligned} \mathcal{H}_\Psi &= \Psi_1^\dagger [v_{\rm F} \tau^z i\partial_x + (\tilde{\Delta}\tau^+ + \tilde{\Delta}^* \tau^-) i\partial_y] \Psi_1 \\ &\quad + (1 \leftrightarrow 2; x \leftrightarrow y), \end{aligned} \tag{6.30}$$

and the particle/hole symmetry breaking term,

$$\mathcal{H}_\lambda = \lambda \Psi_j^\dagger \tau^z \Psi_j. \tag{6.31}$$

The quasiparticle Hamiltonian takes the form of (four) Dirac equations in $2+1$ space-time dimensions, and can be readily diagonalized. For the first pair of nodes one obtains the relativistic dispersion,

$$E_1(\mathbf{q}) = \sqrt{(v_{\mathrm{F}} q_x + \lambda)^2 + |\tilde{\Delta}|^2 q_y^2}, \tag{6.32}$$

and a similar expression is obtained for E_2 except with q_x and q_y interchanged. As usual in Dirac theory, the negative energy single particle states with energy $-E_j(\mathbf{q})$ are filled but positive energy holes states can be created. As expected, the quasiparticle energy vanishes at the nodes ($\mathbf{q} = 0$ with particle/hole symmetry $\lambda = 0$), so the "relativistic" particle is massless. Notice that non-zero λ indeed shifts the positions of the nodes.

In this continuum description $\tilde{\Delta}$ serves as a complex superconducting order parameter for the $d_{x^2-y^2}$ state. Indeed, when $\tilde{\Delta} = 0$ one recovers the metallic Fermi surface and the quasiparticle Hamiltonian describes gapless excitations for all q_y. Below we will include quantum fluctuations by allowing $\tilde{\Delta}$ to depend on space and time. Before doing so, it is convenient to see how $\tilde{\Delta}$ transforms under a particle/hole transformation. From the transformation properties of the electron fields one deduces that the gap transforms as, $\Delta_{\mathbf{k}} \to -\Delta^*_{-\mathbf{k}+\boldsymbol{\pi}}$, which is equivalent to complex conjugation for the (linearized) order parameter,

$$\tilde{\Delta} \to \tilde{\Delta}^*. \tag{6.33}$$

Together with equation (6.27) this implies that the quasiparticle Hamiltonian in equation (6.30) is indeed particle/hole symmetric: $\mathcal{H}_\Psi \to \mathcal{H}_\Psi$.

7 Effective field theory

7.1 Quasiparticles and phase flucutations

Our goal in this section is to obtain a complete low-energy effective theory for the $d_{x^2-y^2}$ superconductor. This task is complicated by the existence of *additional* gapless excitations, besides the quasiparticles. Specifically, since the global $U(1)$ charge conservation symmetry ($c_\alpha \to \mathrm{e}^{i\theta_0} c_\alpha$) is spontaneously broken by the existence of a non-zero order parameter, $\tilde{\Delta} \neq 0$, gapless Goldstone modes are expected. (In a three-dimensional superconductor these modes are actually gapped, due to the presence of long-ranged Coulomb interactions, but would be gapless for a thin 2d film.) These modes propogate in the *phase* of the complex order parameter. Thus a correct low energy theory for the $d_{x^2-y^2}$ state requires consideration of a *spatially* varying order parameter, $\tilde{\Delta}(\mathbf{x})$. Generally, both the magnitude and the phase

of the complex order parameter can vary, but we will focus exclusively on the phase fluctuations, writing

$$\tilde{\Delta}(\mathbf{x}) = |\Delta| e^{i\varphi(\mathbf{x})}, \tag{7.1}$$

with $|\Delta|$ a (real) *constant*. Since amplitude fluctuations are costly in energy this should suffice in the superconducting phase, and will also allow us to describe the nodal liquid in which superconductivity is destroyed by *phase* fluctuations. The desired low energy effective theory can be obtained from symmetry considerations, and the form of the continuum quasiparticle Hamiltonian. A more microscopic approach, discussed briefly below, would entail integrating out high energy degrees of freedom in a functional integral representation.

7.1.1 Symmetry considerations

Since the BCS gap equation has a degenerate manifold of solutions for arbitrary phase φ, the energy should only depend on *gradients* of $\varphi(\mathbf{x})$. The appropriate Lagrangian which describes the *fluctuations* of the phase of the superconducing order parameter can thus be developed as a gradient expansion, with lowest order terms of the form,

$$\mathcal{L}_\varphi = \frac{1}{2}\kappa_\mu (\partial_\mu \varphi)^2, \tag{7.2}$$

where the Greek index μ runs over time and two spatial coordinates: $\mu = 0, 1, 2 = t, x, y$. Here κ_0 is equal to the compressibility of the condensate (ignoring for the moment long-ranged Coulomb forces) and $\kappa_j = -v_c^2 \kappa_0$ (for $j = 1, 2 = x, y$) with v_c the superfluid sound velocity. This form is largely dictated by symmetry. Parity and four-fold rotational symmetry *determine* the form of the spatial gradient terms. The stiffness coefficients, κ_μ, can be estimated as follows. The pair compressibility κ_0 should be roughly one half the electron compressibility – at least for weak interactions. If the pairing is electronic in origin, the Fermi velocity sets the scale for v_c.

In general a Berry's phase term [4] *linear* in $\partial_t \varphi$ is allowed,

$$\mathcal{L}_{\text{Berry}} = n_0 \partial_t \varphi, \tag{7.3}$$

where n_0 is a two-dimensional number density. For a model with particle/hole symmetry which must be invariant under

$$\varphi \to -\varphi, \tag{7.4}$$

(which follows from the particle/hole transformation properties of the order parameter $\tilde{\Delta} \sim e^{i\varphi} \to \tilde{\Delta}^*$) it naively appears that the number density n_0 must vanish. However, this is not the case [15]. To see this it is necessary to

return to the lattice where the term in the (Euclidian) action which follows from $\mathcal{L}_{\rm Berry}$ is,

$$S_{\rm Berry} = in_0 \int_0^{\beta\hbar} {\rm d}\tau \sum_i \partial_\tau \varphi_i, \tag{7.5}$$

where i labels the sites of a square lattice with lattice spacing set to one and $\beta = 1/k_B T$. The partition function is expressed as a functional integral of $\exp(-S)$ over configurations $\varphi_i(\tau)$, with β periodic boundary conditions on the fields $\mathrm{e}^{i\varphi}$. This implies the boundary conditions,

$$\varphi_i(\beta) = \varphi_i(0) + 2\pi N_i, \tag{7.6}$$

with *integer* winding numbers N_i. We thus see that the Berry's phase term contributes a multiplicative piece to the partition function (in each winding sector) of the form;

$$\exp(-S_{\rm Berry}) = \mathrm{e}^{i2\pi n_0 N_{\rm W}}, \tag{7.7}$$

with integer $N_{\rm W} = \sum_i N_i$. Under the particle/hole transformation equation (7.4), the winding numbers change sign, $N_{\rm W} \to -N_{\rm W}$. The Berry's phase term is thus invariant under the particle/hole transformation *provided* n_0 is integer or half-integer.

The appropriate value for n_0 can be readily determined by obtaining the lattice Hamiltonian associated with the Lagrangian density $\mathcal{L}_\varphi + \mathcal{L}_{\rm Berry}$. The first contribution can be conveniently regularized on the lattice as,

$$L_\varphi = -t \sum_{<i,j>} \cos(\varphi_i - \varphi_j) - \frac{1}{u} \sum_i (\partial_t \varphi_i)^2. \tag{7.8}$$

Upon inclusion of the Berry's phase term this gives the lattice Hamiltonian,

$$H_\varphi = -t \sum_{<i,j>} \cos(\varphi_i - \varphi_j) + u \sum_i (n_i - n_0)^2. \tag{7.9}$$

Here n_i denotes a Cooper-pair number *operator*, canonically conjugate to the phase fields:

$$[\varphi_i, n_j] = i\delta_{ij}. \tag{7.10}$$

The first term in H_φ describes the hopping of charge $2e$ (spinless) Cooper pairs between neighboring sites of the lattice, and the second term is an onsite repulsive interaction. The parameter n_0 plays the role of an "off-set" charge and determines the average number of Cooper pairs per site. For the Hubbard model at half-filling with one electron per site, the number of Cooper pairs clearly equals *one-half* the number of lattice sites. This is

especially apparent in the limit of very large attractive Hubbard interaction when the electrons pair into on-site singlets, but is expected to be more generally valid. Thus, it is clear that one should take $n_0 = 1/2$. Tuning away from particle/hole symmetry with a chemical potantial μ, corresponds to changing n_0 away from one-half.

In the superconducting phase one expects that the winding numbers will all vanish, since the phase field φ is essentially constant in space and time, and the Berry's term plays no role. But when the superconductor is "quantum disordered", the phase field fluctuates wildly with signifigant winding, and inclusion of the Berry's phase term is expected to be important (but see Sect. 8 below).

It remains to couple these phase fluctuations to the gapless quasiparticles. Since the order parameter $\tilde{\Delta}$ directly enters the quasiparticle Hamiltonian equation (6.30), one can readily guess the appropriate coupling. We should simply replace $\tilde{\Delta} \to v_\Delta e^{i\varphi}$ with v_Δ real. Since φ varies spatially, some care is required. In the quasiparticle Hamiltonian we let,

$$\tilde{\Delta}\tau^+ i\partial_y \to v_\Delta \tau^+ e^{i\varphi/2}(i\partial_y)e^{i\varphi/2}, \tag{7.11}$$

and similarly for the τ^- term. This "symmetric" form leads to an hermitian Hamiltonian, physical currents, and respects the symmetries of the problem. A careful derivation of equation (7.11) is given below. With this prescription, the quasiparticle Hamiltonian becomes

$$\begin{aligned} \mathcal{H}_{qp} &= \sum_{s=\pm} \Psi_1^\dagger [v_F \tau^z i\partial_x + v_\Delta \tau^s e^{is\varphi/2}(i\partial_y)e^{is\varphi/2}]\Psi_1 \\ &\quad + (1 \leftrightarrow 2; x \leftrightarrow y). \end{aligned} \tag{7.12}$$

Since φ can also fluctuate with time, it will convenient to consider the time dependence *via* a Lagrangian formulation. The Lagrangian density is

$$\mathcal{L}_{qp} = \Psi_j^\dagger i\partial_t \Psi_j - \mathcal{H}_{qp}. \tag{7.13}$$

The full low-energy effective Lagrangian in the d-wave superconductor is obtained by adding the two contributions: $\mathcal{L}_\varphi + \mathcal{L}_{qp}$.

7.1.2 Microscopic approach

To illustrate how one might try to "derive" this effective theory from a more microscopic starting point, we briefly consider a simple model Hamiltonian,

$$H = H_0 - V \sum_{\langle \vec{x}\vec{x}' \rangle} c_\alpha^\dagger(\vec{x}) c_\beta^\dagger(\vec{x}') c_\beta(\vec{x}') c_\alpha(\vec{x}), \tag{7.14}$$

where H_0 is the usual kinetic energy describing hopping on a 2d square lattice and we have added an *attractive* near-neighbor interaction with strength

V. To derive the effective field theory, it is convenient to express the partition function $Z = Tr\ \mathrm{e}^{-\beta H}$, as an *imaginary time* coherent state path integral [29],

$$Z = \int [\mathcal{D}c][\mathcal{D}\overline{c}]\mathrm{e}^{-S}, \tag{7.15}$$

where c and $\overline{c}$ are Grassman fields and the Euclidean action is simply

$$S = \int \mathrm{d}\tau \left\{ \sum_{\vec{x}} \overline{c}_\alpha(\vec{x})\partial_\tau c_\alpha(\vec{x}) + H[\overline{c}, c] \right\}. \tag{7.16}$$

We consider here only $T = 0$, for which the τ integration domain is infinite. The possibility of a d-wave superconducting phase can be entertained by decoupling the above action using a Hubbard-Stratonovich transformation:

$$Z = \int [\mathcal{D}c][\mathcal{D}\overline{c}][\mathcal{D}\Delta][\mathcal{D}\Delta^*]\mathrm{e}^{-S_1}, \tag{7.17}$$

with $S_1 = \int \mathrm{d}\tau[\sum_{\vec{x}} \overline{c}_\alpha(\vec{x})\partial_\tau c_\alpha(\vec{x}) + H_{\mathrm{eff}}]$. The effective Hamiltonian can be decomposed into $H_{\mathrm{eff}} = H_0 + H_{\mathrm{int}} + H_\Delta$, with

$$H_{\mathrm{int}} = \sum_{\langle \vec{x}\vec{x}' \rangle} \left[\Delta^{\alpha\beta}_{\vec{x}\vec{x}'} \overline{c}_\alpha(\vec{x})\overline{c}_\beta(\vec{x}') + \mathrm{h.c.} \right], \tag{7.18}$$

$$H_\Delta = \frac{1}{V} \sum_{\langle \vec{x}\vec{x}' \rangle} |\Delta^{\alpha\beta}_{\vec{x}\vec{x}'}|^2. \tag{7.19}$$

Equations (7.18- 7.19) form a basis for studying the original electron model. At this stage BCS mean field theory could be implemented by integrating out the electron degrees of freedom to obtain an effective action only depending on Δ, $S_{\mathrm{eff}}(\Delta)$. Minimizing this action with respect to Δ would give the gap equation. One could imagine including fluctuations by expanding about the saddle point solution. But for a d-wave superconductor this procedure is problematic, since integrating out *gapless* quasiparticles will generate *singular* long-ranged interactions in $S_{\mathrm{eff}}(\Delta)$. It is preferable to *retain* the gapless quasiparticles in the effective theory, and only integrate out the *high frequency* electron modes which are well away from the nodes. In this way, the dynamics and interactions generated for the order parameter Δ will be *local*.

Rather than trying to implement this procedure, we content ourselves with arguing that the "symmetric" prescription adopted above indeed gives the correct form for the phase-quasiparticle coupling term. To this end we focus on singlet pairing, defining

$$\Delta^{\alpha\beta}_{\vec{x}\vec{x}'} = \Delta(\vec{x}, \vec{x}')(\delta_{\alpha\uparrow}\delta_{\beta\downarrow} - \delta_{\alpha\downarrow}\delta_{\beta\uparrow}). \tag{7.20}$$

The triplet pieces of Δ are presumed to be massive, so that they can be safely integrated out. Since Δ lives on the bonds, it is convenient to associate two such fields with each site on the square lattice, *i.e.*

$$\Delta_1(\vec{x}) \equiv \Delta(\vec{x}, \vec{x}+\hat{e}_1), \tag{7.21}$$
$$\Delta_2(\vec{x}) \equiv \Delta(\vec{x}, \vec{x}+\hat{e}_2), \tag{7.22}$$

where $\hat{e}_1, \hat{e}_2$ are unit vectors along the a and b axes of the square lattice, respectively. The interaction Hamiltonian becomes,

$$H_{\rm int} = \sum_{j,\vec{x}} \left\{ \Delta_j(\vec{x}) \left[c^\dagger_\uparrow(\vec{x}) c^\dagger_\downarrow(\vec{x}+\hat{e}_j) - \uparrow\leftrightarrow\downarrow \right] + \text{h.c.} \right\}, \tag{7.23}$$

where the sum includes all lattice sites and $j = 1, 2$. The magnitudes of Δ_1 and Δ_2, as well as their relative sign, are determined by the effective action generated upon integrating out the high-energy modes. For a d-wave superconductor the effective action will be minimized for $\Delta_1 = -\Delta_2 = \Delta_0 e^{i\varphi}$, up to massive modes. We can now take the continuum limit. For agreement with Section 6, we define $v_\Delta = 2\sqrt{2}\Delta_0$, or $\Delta_1 = -\Delta_2 = \tilde{\Delta}/2\sqrt{2}$. In addition, we take the continuum limit of the electron fields, using the decompositions

$$c^\dagger_\uparrow \sim \Psi^\dagger_{111} i^{x+y} - \Psi_{122}(-i)^{x+y} + \Psi^\dagger_{211}(-i)^{x-y} - \Psi_{222} i^{x-y},$$
$$c^\dagger_\downarrow \sim \Psi^\dagger_{112} i^{x+y} + \Psi_{121}(-i)^{x+y} + \Psi^\dagger_{212}(-i)^{x-y} + \Psi_{221} i^{x-y},$$

and the hermitian conjugates of these equations. Inserting these into equation (7.23), gradient-expanding the Ψ fields, and rotating 45 degrees to $x-y$ coordinates along the (π, π) and $(-\pi, \pi)$ directions, one obtains $H_{\rm int} = \int d^2x \mathcal{H}_{\rm int}$, with

$$\mathcal{H}_{\rm int} = \left[\frac{\tilde{\Delta}}{2} \left(\Psi^\dagger_1 \tau^+ i\partial_y \Psi_1 - (i\partial_y \Psi^\dagger_1)\tau^+ \Psi_1 \right) + \text{h.c.} \right] + (1 \leftrightarrow 2, x \leftrightarrow y). \tag{7.24}$$

This form is identical to the $\tilde{\Delta}$ term in equation (6.30) when the order parameter $\tilde{\Delta}$ is constant, but the symmetric placement of derivatives is important in the presence of phase gradients. In particular, now let $\tilde{\Delta} = v_\Delta e^{i\varphi}$ and integrate by parts to transfer the derivative in the second term from the $\Psi^\dagger$ to the $\tilde{\Delta}\Psi$ combination. Upon using the operator identity

$$\frac{1}{2}\left(e^{i\varphi} i\partial_y + i\partial_y e^{i\varphi} \right) = e^{i\varphi/2} i\partial_y e^{i\varphi/2}, \tag{7.25}$$

this becomes identical to the symmetrized form of the phase-quasiparticle interaction hypothesized in equation (7.12).

7.2 Nodons

Treatment of quantum phase fluctuations is complicated by the coupling between the quasiparticle Fermion operators, Ψ, and exponentials of the phase φ, as seen explicitly in $\mathcal{H}_{qp}$ in equation (7.12). The form of the coupling is determined by the electric charge carried by Ψ, which is uncertain – being built from electron *and* hole operators. To isolate the uncertain charge of Ψ it is extremely convenient to perform a change of variables [14], defining a new set of fermion fields ψ_j *via*

$$\psi_j = \exp(-i\varphi\tau^z/2)\Psi_j. \tag{7.26}$$

In the superconducting phase, and in the absence of quantum flucutations of the order-paramater phase, one can set $\varphi = 0$, and these new fermions are simply the d-wave quasiparticles. However, when the field φ is dynamical and fluctuates strongly this change of variables is non-trivial. In particular, the new fermion fields ψ are electrically *neutral*, invariant under a global $U(1)$ charge transformation (since $\varphi \to \varphi + 2\theta_0$ under the $U(1)$ charge transformation in Eq. (2.8)). As we shall see, when the d-wave superconductivity is quantum disordered, these new fields will play a fundamental role, describing low energy gapless excitations, centered at the former nodes. For this reason, we refer to these fermions as *nodons*. For completeness, we quote the symmetry properties of the nodon field under a particle/hole transformation. Since $\varphi \to -\varphi$, one has simply

$$\psi \to \psi^\dagger. \tag{7.27}$$

The full Lagrangian in the d-wave superconductor, $\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_{qp}$, can be conveniently re-expressed in terms of these nodon fields since $\mathcal{L}_{qp} = \mathcal{L}_\psi + \mathcal{L}_{\rm int} + \mathcal{L}_\lambda$ with a free nodon piece,

$$\begin{aligned} \mathcal{L}_\psi &= \psi_1^\dagger[i\partial_{\rm t} - v_{\rm F}\tau^z i\partial_x - v_\Delta\tau^x i\partial_y]\psi_1 \\ &\quad +(1 \leftrightarrow 2, x \leftrightarrow y), \end{aligned} \tag{7.28}$$

interacting with the phase of the order-parameter:

$$\mathcal{L}_{\rm int} = \partial_\mu\varphi J_\mu. \tag{7.29}$$

Here the electrical 3-current J_μ is given by

$$J_0 = \frac{1}{2}\psi_j^\dagger\tau^z\psi_j, \tag{7.30}$$

$$J_j = \frac{v_{\rm F}}{2}\psi_j^\dagger\psi_j. \tag{7.31}$$

Because the transformation in equation (7.26) is local, identical expressions hold for these currents in terms of the quasiparticle fields, Ψ. The form of the particle/hole asymmetry term remains the same in terms of the nodon fields:

$$\mathcal{L}_\lambda = \lambda \psi_j^\dagger \tau^z \psi_j. \tag{7.32}$$

It is instructive to re-express the components of the currents J_μ back in terms of the original electron operators. One finds

$$J_0 = \frac{1}{2}\left(c^\dagger_{\mathbf{K}_j} c_{\mathbf{K}_j} + c^\dagger_{-\mathbf{K}_j} c_{-\mathbf{K}_j} \right), \tag{7.33}$$

(with an implicit spin summation) which corresponds physically to the total electron density living at the nodes, in units of the Cooper pair charge. Similarly,

$$J_j = \frac{v_\mathrm{F}}{2}\left(c^\dagger_{\boldsymbol{K}_j} c_{\boldsymbol{K}_j} - c^\dagger_{-\boldsymbol{K}_j} c_{-\boldsymbol{K}_j} \right) \tag{7.34}$$

corresponds to the *current* carried by the electrons at the nodes. Thus, J_μ can be correctly interpreted as the quasiparticles three-current.

To complete the description of a quantum mechanically fluctuating order parameter phase interacting with the gapless fermionic excitations at the nodes, we minimally couple to an external electromagnetic field, A_μ. Since the nodon fermions are neutral, the only coupling is to the order-parameter phase, *via* the substitution $\partial_\mu \varphi \to \partial_\mu \varphi - 2A_\mu$. Here we have set the electron charge $e = 1$, with a factor of 2 appropriate for Cooper pairs. The final Lagrangian then takes the form $\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_\mathrm{int} + \mathcal{L}_\lambda$, with

$$\mathcal{L}_\varphi = \frac{1}{2}\kappa_\mu (\partial_\mu \varphi - 2A_\mu)^2, \tag{7.35}$$

$$\mathcal{L}_\mathrm{int} = (\partial_\mu \varphi - 2A_\mu) J_\mu, \tag{7.36}$$

and $\mathcal{L}_\psi$ still given by equation (7.28). Here we have dropped the Berry's phase term, which is not expected to play an important role in the superconducting phase. Long-ranged Coulomb interactions could be readily incorporated at this stage by treating A_0 as a dynamical field and adding a term to the Lagrangian of the form, $\mathcal{L}_\mathrm{coul} = (1/2)(\partial_j A_0)^2$. The spatial components of the electromagnetic field, A_j, have been included to keep track of the current operator.

7.2.1 Symmetries and conservation laws

If the full effective Lagrangian $\mathcal{L}$ is to correctly describe the low energy physics it must exhibit the same symmetries as the original electron Hamiltonian – the most important being charge and spin conservation. Since the

ψ operators are electrically neutral the full $U(1)$ charge transformation is implemented by $\varphi \to \varphi + 2\theta_0$ for constant θ_0, and $\mathcal{L}$ is indeed invariant. Moreover, the Lagrangian is invariant under $\psi_\alpha \to U_{\alpha\beta}\psi_\beta$ for arbitrary (global) $SU(2)$ spin rotations $U = \exp(i\boldsymbol{\theta}\cdot\boldsymbol{\sigma})$. Since the Cooper pairs are in spin singlets, *all* of the spin is carried by the nodons.

As usual, associated with each continuous symmetry is a conserved "charge" which satisfies a continuity equation (Noether's theorem). Since the Lagrangian only depends on gradients of φ, the Euler-Lagrange equation of motion reduces to the continuity equation,

$$\partial_\mu J^{\rm tot}_\mu = 0, \tag{7.37}$$

where the total electric 3-current is given by $J^{\rm tot}_\mu = \partial\mathcal{L}/\partial(\partial_\mu\varphi) = -\partial\mathcal{L}/\partial A_\mu$. This gives,

$$J^{\rm tot}_\mu = \kappa_\mu(\partial_\mu\varphi - A_\mu) + J_\mu, \tag{7.38}$$

where the first term is the Cooper pair 3-current and the second the quasiparticles current.

The analogous conserved *spin* currents can be obtained by considering infinitesimal spin rotations,

$$U = 1 + i\boldsymbol{\theta}(\mathbf{x},t)\cdot\boldsymbol{\sigma}, \tag{7.39}$$

for *slowly* varying $\boldsymbol{\theta}(\mathbf{x},t)$. Under this spin rotation the Lagrangian transforms as,

$$\mathcal{L} \to \mathcal{L} + \partial_\mu\boldsymbol{\theta}\cdot\boldsymbol{j}_\mu, \tag{7.40}$$

with $\boldsymbol{j}_\mu$ given below. After an integration by parts, invariance of the action S under global spin rotations implies continuity equations $\partial_\mu\boldsymbol{j}_\mu = 0$ for *each* of the three *spin* polarizations, $\boldsymbol{j}$. The space-time components of the conserved spin currents are given explicitly by,

$$\boldsymbol{j}_0 = \frac{1}{2}\psi_1^\dagger\boldsymbol{\sigma}\psi_1 + (1\to 2), \tag{7.41}$$

$$\boldsymbol{j}_x = \frac{1}{2}v_{\rm F}\psi_1^\dagger\boldsymbol{\sigma}\tau^z\psi_1 + \frac{1}{2}v_\Delta\psi_2^\dagger\boldsymbol{\sigma}\tau^x\psi_2, \tag{7.42}$$

and $\boldsymbol{j}_y$ the same as $\boldsymbol{j}_x$ except with $\psi_1 \leftrightarrow \psi_2$. Notice that in contrast to the electrical current, the spin current operator has a contribution which is proportional to the velocity tangential to the Fermi surface, v_Δ, which is anomalous when re-expressed in terms of the original electron operators.

Surprisingly, the effective Lagrangian exhibits *additional* continuous symmetries, *not* present in the original Hamiltonian. Firstly, $\mathcal{L}$ is invariant under *separate* $SU(2)$ spin rotations on the two pairs of nodes, ψ_j for

$j = 1, 2$. Moreover, the Lagrangian is also invariant under two additional $U(1)$ transformations $\psi_j \to \mathrm{e}^{i\theta_j}\psi_j$ for arbitrary constant phases, θ_j. These latter symmetries imply two new conserved "charges", $\psi_j^\dagger\psi_j$ (no sum on j). We refer to these conserved quantities as "nodon charges". The associated conserved nodon 3-currents take the same form as the spin currents above, except replacing $\boldsymbol{\sigma}/2$ by the identity. As seen from equation (7.31), the conserved nodon charges are proportional to the quasiparticle electrical *current*, since $J_j = (v_\mathrm{F}/2)\psi_j^\dagger\psi_j$.

It is possible to add to $\mathcal{L}$ additional interaction terms which are consistent with the original $U(1)$ and $SU(2)$ symmetries, but do *not* conserve the "nodon charge". Specifically, anomalous quartic interaction terms of the form ψ^4 arise from Umklapp scattering processes in the original electron Hamiltonian and clearly change the nodon charge. However, such interactions are unimportant at low energies due to severe phase space restrictions. To see this, consider how the action, $S = \int \mathrm{d}^2x\mathrm{d}t\mathcal{L}$ transforms under a renormalization group (RG) rescaling transformation,

$$x_\mu \to bx_\mu; \qquad \psi \to b^{-1}\psi; \qquad \varphi \to b^{-1/2}\varphi, \tag{7.43}$$

with rescaling parameter $b > 1$. By construction, this leaves the quadratic pieces S_ψ and S_φ invariant, but interaction terms such as $u\psi^4$ scale to zero under the RG ($b \to \infty$) since $u \to u/b$. It is the $T = 0$ "fixed point" theory described by the quadratic terms which exhibits the additional symmetries. Incidentally, the coupling term $\mathcal{L}_\mathrm{int}$ above also scales to zero (as $b^{-1/2}$) under the renormalization group. In the resulting quadratic theory the quasiparticles and phase fluctuations actually decouple.

7.2.2 Superfluid stiffness

The above effective theory is particularly convenient for examining very low temperatures properties of the $d_{x^2-y^2}$ state. Of interest are charge response functions such as the electrical conductivity and the superfluid stiffness (measureable *via* the penetration length). The spin excitations (carried by the quasiparticles) can also be probed *via* resonance techniques, such as NMR and ESR. Impurity scattering can be readily incorporated by coupling a random potential to the electron density (which can be re-expressed as a nodon bi-linear).

For illustrative purposes we briefly consider the quasiparticle contribution to the low temperature superfluid stiffness and extract the famous T-linear dependence. For a Galilean invariant system of mass m bosons the superfluid stiffness K_s equals the superfluid density divided by m. But more generally K_s can be extracted rather directly by considering the response of the system to a *transverse* vector potential [38]. We set $A_0 = 0$ and decompose the *static* vector potential A_j into longitudinal and transverse

pieces:

$$A_j = A_{\ell,j} + A_{t,j}, \tag{7.44}$$

with $\partial_j A_{t,j} = 0$ and $\epsilon_{ij} A_{\ell,j} = 0$. The superfluid stiffness is then given by,

$$K_{\rm s} = \frac{1}{V} \frac{\partial^2 F}{\partial A_{t,x}^2}, \tag{7.45}$$

where $F = -k_B T \ln Z$ is the Free energy and $V \to \infty$ is the area of the 2d system. Here $A_{t,x}$ can be taken spatially constant.

To extract F the partition function can be written as an imaginary time coherent state path integral [29],

$$Z = \int [\mathcal{D}\varphi][\mathcal{D}\psi][\mathcal{D}\overline{\psi}] \exp(-S_{\rm E}), \tag{7.46}$$

with Euclidian action $S_{\rm E} = \int {\rm d}^2x {\rm d}\tau \mathcal{L}_{\rm E}$. The longitudinal vector potential, which can be expressed as a gradient of a scalar field $A_{\ell,j} = \partial_j \Lambda$, can be eliminated entirely by shifting $\varphi \to \varphi + \Lambda$. Moreover, the crossterm between $\partial_j \varphi$ and $A_{t,j}$ vanishes since A_t is divergenceless. The Gaussian integral over φ can then be readily perfomed and simply generates an irrelevant interaction term ($J \sim (\psi^\dagger \psi)^2$) which can be ignored. One thereby arrives at an effective action depending only on ψ and A_j with associated Hamiltonian density of the form: $\mathcal{H}_{\rm eff} = \mathcal{H}_\psi + \mathcal{H}_{\rm A}$, with $\mathcal{H}_\psi$ the free nodon Hamiltonian and

$$\mathcal{H}_{\rm A} = \frac{1}{2} K_{\rm s}^0 A_{t,j}^2 + A_{t,j} J_j. \tag{7.47}$$

Here $K_{\rm s}^0 = \kappa_0 v_{\rm c}^2$ is the superfluid stiffness from the Cooper pairs, and $J_j = (v_{\rm F}/2) \psi_j^\dagger \psi_j$. Notice that the (transverse) vector potential acts as an *effective* chemical potential for the "nodon charge" density, $\rho_{\rm n} = \psi_j^\dagger \psi_j$. Thus, the superfluid stiffness can be expressed in terms of the nodon "compressibility" as

$$K_{\rm s} = K_{\rm s}^0 - (v_{\rm F}/2)^2 \kappa_{\rm n}, \tag{7.48}$$

where $\kappa_{\rm n} = \partial \rho_{\rm n} / \partial \mu_{\rm n}$ and $\mu_{\rm n} = (v_{\rm F}/2) A_{t,x}$ is the nodon "chemical potential".

The nodon compressibility can be extracted by diagonalizing the Hamiltonian, H_ψ. From the first pair of nodes one obtains the free Fermion form,

$$H_\psi = \sum_{\mathbf{q}} E_1(\mathbf{q}) [a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + b_{\mathbf{q}}^\dagger b_{\mathbf{q}}], \tag{7.49}$$

where $E_1(\mathbf{q})$ is given in equation (6.32) and we have suppressed the spin index. Here a and b are particle and hole operators, respectively. The nodon charge is simply,

$$\rho_{\rm n} = \frac{1}{V}\sum_{\mathbf{q}}[\langle a^\dagger_{\mathbf{q}} a_{\mathbf{q}}\rangle - \langle b^\dagger_{\mathbf{q}} b_{\mathbf{q}}\rangle], \tag{7.50}$$

where the averages are taken with $\mathcal{H}_\psi - \mu_{\rm n}\rho_{\rm n}$. At finite temperatures one obtains

$$\rho_{\rm n} = 2\int \frac{{\rm d}\mathbf{q}}{(2\pi)^2}[f(E_1(\mathbf{q}) - \mu_{\rm n}) - f(E_1(\mathbf{q}) + \mu_{\rm n})], \tag{7.51}$$

where $f(E)$ are Fermi functions, and the factor of 2 is from the spin sum. Finally, upon differentiating with respect to $\mu_{\rm n}$ and performing the momentum integral one extracts the desired result for the low temperature superfluid stiffness:

$$K_{\rm s}(T) = K^0_{\rm s} - c\frac{v_{\rm F}}{v_\Delta}k_B T, \tag{7.52}$$

with the dimensionless constant $c = (\ln 2/2\pi)$.

8 Vortices

8.1 hc/2e versus hc/e vortices

Having successively incorporated phase fluctuations into the effective low energy description of the $d_{x^2-y^2}$ state, we now turn to a more interesting task – quantum disordering the superconductivity to obtain the nodal liquid phase, a novel Mott insulator. The superconductivity is presumed to be destroyed by strong *quantum* fluctuations of the order parameter phase φ driven by vortex excitations. In two-dimensions vortices are simply whorls of current swirling around a core region. But in a superconductor the circulation of such vortices is *quantized*, since upon encircling the core the phase φ can only change by integer multiples of 2π. Inside the core of a vortex the *magnitude* of the complex order parameter $|\tilde{\Delta}|$ vanishes, but is essentially constant outside. In the superconducting phase, the size of the core is set by the coherence length - roughly 10 Å in the Cuprate materials. Such vortices are thus tiny "point-like" objects, with a truly microscopic size in the Cuprate materials.

The "elementary" vortex has a phase winding of $\pm 2\pi$. When a superconductor is placed in an external magnetic field, the currents circulating around the core of a vortex tend to screen out the magnetic field, except within a region of the penetration length, λ, from the vortex core. (In the cuprate materials λ is in the range of a thousand angströms.) In addition

to the circulation, the total magnetic flux near a vortex is *quantized* – in units of the flux quantum $hc/2e$. An "elementary" vortex quantizes precisely $hc/2e$ of magnetic flux, and will thus henceforth be referred to as an $hc/2e$ vortex. As we shall argue [15], to obtain the nodal liquid phase it will be necessary to "liberate" double-strength hc/e vortices, keeping the $hc/2e$ vortices "confined".

Generally, the position of these "point-like" vortices can change with time, and their dynamics requires a quantum mechanical description. Thus a collection of many vortices can be viewed as a many body system of "point-like" particles. Since positive $(+1)$ and negative (-1) circulation vortices can annihilate – and disappear (just as for real elementary particles like electrons and positrons), they behave as "relativistic" particles. There is a conserved vortex "charge" in this process, namely the total circulation, and an associated current. Since the Cooper pairs are Bosons, one anticipates that the "dual particles" – the vortices – are also Bosonic forming a relativistic Boson system, and this is indeed the case [39].

However, in the superconducting phase at zero temperature there are *no* vortices present – this phase constitutes a "vacuum" of vortices. More precisely, due to quantum fluctuations vortices are present as short-lived "virtual" fluctuations, popping out of the "vacuum" in the form of small tightly bound (neutral) pairs. For the low energy properties of the superconductor these fluctuations can be largely ignored. But what happens if these virtual pairs unbind into a proliferation of free mobile vortices? Vortex motion is very effective at scrambling the phase φ of the superconducting order, so that mobile vortices will in fact destroy the superconductivity. Since the vortices are Bosonic, once they are free and mobile they will "Bose condense", at least at zero temperature. One thereby obtains a non-superconducting insulating state, with the "vortex-condensate" serving as an appropriate order parameter. As we shall see, it will be extremely convenient to pass to a "dual" representation [39, 40] in which the vortices are the basic "particles" – rather than the Cooper pairs.

Consider first unbinding and condensing the "elementary" $hc/2e$ vortices [15]. When a Cooper pair is taken around such a vortex it's wave function acquires a $\pm 2\pi$ phase change. Likewise, when an $hc/2e$ vortex is taken around a Cooper pair, the *vortex* wavefunction acquires the 2π phase change. Thus, $hc/2e$ vortices "see" Cooper pairs as a source of "dual flux", each carrying one unit. (This notion can be made precise by performing a duality transformation – see below and the Appendix.) For a Hubbard model of electrons at half-filling, on average there is one-half of a Cooper pair per site, as seen explicitly in the effective lattice Cooper pair Hamiltonian, equation (7.9), which has offset charge $n_0 = 1/2$. Thus, these elementary vortices "see" a dual "magnetic field", with one-half of a dual flux-quantum per plaquette. When the $hc/2e$ vortices unbind and condense, they will

quantize this dual flux, in precisely the same way that the condensation of Cooper pairs in a real superconductor will quantize an applied magnetic field – forming an Abrikosov flux-lattice (if Type II). The analog of the Abrikosov flux-lattice for the $hc/2e$ *vortex condensate* is an ordered lattice of *Cooper pairs*. In this "crystal" state at half-filling, the Cooper pairs will preferentially sit on one of the two equivalent sub-lattices of the square lattice. This state can be described as a commensurate charge-density-wave with ordering wavevector ($\mathbf{Q} = \pi, \pi$), which spontaneously breaks the discrete symmetry under translation by one lattice spacing. Such ordering implies a considerable degree of double occupancy for the electrons, and thus seems most reasonable for a Hubbard type model with an attractive on-site interaction (negative u). In the Cuprate materials there is a strong on-site *repulsion*, and moreover there is no evidence for "charge-ordering" near $\mathbf{Q}$. Thus, for a description of the pseudo-gap regime in the Cuprate materials, we can rule out the $hc/2e$ vortex-condensate on phenomenological grounds.

Instead, we consider the possibility of unbinding and condensing double-strength hc/e vortices, keeping the elementary $hc/2e$ vortices confined [15]. When an hc/e vortex is taken around a Cooper pair it acquires a 4π phase change. A 2π phase change corresponds to taking such an hc/e vortex around "half" of a Cooper pair – which has charge e. Thus, a condensation of hc/e vortices should correspond to a "crystal" of such charge e objects. But at half-filling with charge e per lattice site, this should correspond to a state *without* charge ordering or translational symmetry breaking. As we shall see, for a $d_{x^2-y^2}$ superconductor the resulting hc/e "vortex-condensate" gives a description of the nodal liquid phase.

This procedure – keeping the elementary $hc/2e$ vortices confined and only liberating the hc/e vortices – is responsible for the remarkable properties of the nodal liquid [15]. To see why, consider first the Berry's phase term in equation (7.3). With only hc/e vortices present, the Cooper pair phase, φ, only winds by integer multiples of 4π – *not* 2π. At half-filling (with $n_0 = 1/2$) the Berry's phase term will *not* contribute to the partition function (see Eq. (7.7)) and can thus be dropped entirely in the description of the nodal liquid. This can be implemented by defining a new phase field:

$$\phi = \varphi/2, \tag{8.1}$$

and only allowing vortices in $\phi(x)$ with circulation 2π times an integer. This restriction precludes $hc/2e$ vortices, and guarantees that the field

$$b = \mathrm{e}^{i\phi}, \tag{8.2}$$

is *single-valued*. As an operator, b creates a spinless excitation with charge e. When re-written in terms of ϕ, the effective Lagrangian for a d-wave superconductor with quantum phase fluctuations (from Eqs. (7.35, 7.36))

becomes $\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_{\rm int} + \mathcal{L}_\psi$ with

$$\mathcal{L}_\phi + \mathcal{L}_{\rm int} = \frac{1}{2}\kappa_\mu(\partial_\mu\phi - A_\mu + \kappa_\mu^{-1}J_\mu)^2, \tag{8.3}$$

and $\mathcal{L}_\psi$ given in equation (7.28). The Berry's phase term has been dropped, since it plays no role when $\exp(i\phi)$ is a single valued field. Here, we have absorbed a factor of two into κ_μ and also completed the square with the nodon current, J_μ, dropping order J_μ^2 terms which are irrelevant as discussed after equation (7.43). Notice that the coefficient of A_μ is one – as expected for a charge e operator $\exp(i\phi)$. By precluding $hc/2e$ vortices, we see the emergence of a new bosonic field, $\exp(i\phi)$, with exotic quantum numbers – charge e but spin zero – which will be referred to as a "holon". This is the first hint of spin-charge separation [41–43] in the nodal liquid.

As we shall see, another remarkable consequence of precluding $hc/2e$ vortices, is that the charge neutral spin one-half nodons survive under hc/e vortex condensation into the nodal liquid. To see why this is *not* the case if elementary $hc/2e$ vortices are condensed [15] (as in the charge-density-wave), it is very instructive to consider the transformation which relates the nodons to the d-wave quasiparticles, equation (7.26), which can be written in terms of the new field ϕ $(= \varphi/2)$ as:

$$\psi = \exp(-i\tau_z\phi)\Psi. \tag{8.4}$$

In the presence of vortices, the nodon field ψ only remains single-valued if $hc/2e$ vortices are excluded (so that $\exp(\pm i\phi)$ is single valued). Indeed, when a nodon is taken around an $hc/2e$ vortex, it's wavefunction *changes sign*, since ϕ winds by π. This implies a very strong and long-ranged "statistical" interaction between nodons and $hc/2e$ vortices. If $hc/2e$ vortices proliferate and condense, it will clearly be very difficult for the nodons to propogate coherently. In fact, we have argued recently [15] that in this case the nodons are bound (actually "confined") to the holons, leaving only the electron in the spectrum of the charge-density-wave.

8.2 Duality

We now consider implementing the procedure of unbinding and condensing hc/e vortices in the $d_{x^2-y^2}$ superconductor. To this end, it is extremely convenient to pass to the "dual" representation [39, 40] in which the vortices are the basic "particles", rather than the Cooper pairs. The most straightforward way to incorporate hc/e vortices is by placing the (single-valued) field $\exp(i\phi)$ on the *sites* of a lattice [39], so that vortices can exist in the *plaquettes*. A lattice duality transformation can be implemented in which the phase ϕ is replaced by a dual field, θ, which is the phase of a vortex complex field, $\Phi \sim \mathrm{e}^{i\theta}$. In a Hamiltonian description, Φ and $\Phi^\dagger$ can

be viewed as vortex quantum field operators – which destroy and create hc/e vortices. On a $2+1$-dimensional Euclidian space-time lattice, the appropirate model corresponding to the phase Lagrangian equation (8.3) is essentially a classical 3d-xy model with an effective gauge field:

$$A_\mu^{\rm eff} = A_\mu - \kappa_\mu^{-1} J_\mu. \tag{8.5}$$

The lattice duality transformation for the 3d-xy model with gauge field is implemented in some detail in the Appendix. An alternative method which we sketch below, involves implementing the duality transformation directly in the continuum [40].

To this end we introduce a vortex 3-current, j_μ^v, which satisfies,

$$j_\mu^v = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\phi. \tag{8.6}$$

In the presence of hc/e vortices, ϕ is multi-valued, $\partial_\mu\phi$ is not curl-free, and j_μ^v is non-vanishing. Even in the dual vortex representation the total electrical charge must be conserved. This can be achieved by expressing the *total* electrical 3-current (in units of the electron charge e) as a curl,

$$J_\mu^{\rm tot} = \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda, \tag{8.7}$$

where we have introduced a "fictitious" dynamical gauge field, a_μ. (In the Appendix the electrical 3-current is expressed as a lattice curl of a_μ.) Upon combining equation (7.38) with (8.6) and (8.7), one can eliminate the phase field, ϕ, and relate a_μ to the vortices:

$$j_\mu^v = \epsilon_{\mu\nu\lambda}\partial_\nu[\kappa_\lambda^{-1}\epsilon_{\lambda\alpha\beta}\partial_\alpha a_\beta + A_\lambda - \kappa_\lambda^{-1} J_\lambda], \tag{8.8}$$

where J_μ is the quasiparticle 3-current defined earlier in equations (7.30-7.31).

In this continuum approach to duality, a dual description is obtained by constructing a Lagrangian, $\mathcal{L}_{\rm D}$, depending on a_μ, J_μ and j_μ^v, whose equation of motion, obtained by differentiating the action with respect to a_μ, leads to the above equation. It is convenient to first express the vortex 3-current in terms of a complex field, Φ, which can be viewed as an hc/e vortex destruction operator. The dual Lagrangian is constructed to have an an associated $U(1)$ invariance under $\Phi \to {\rm e}^{i\alpha}\Phi$, which guarantees that j_μ^v is indeed conserved. When an hc/e vortex is taken around a Cooper pair it aquires a 4π phase change (2π around a charge e "holon"). In the dual representation the vortex wavefunction Φ should acquire a 4π phase change (or 2π for a "holon"). This can be achieved by minimally coupling derivatives af Φ to the "fictitious" vector potential a_μ.

The appropriate dual Lagrangian can be conveniently decomposed as $\mathcal{L}_{\rm D} = \mathcal{L}_\psi + \mathcal{L}_v + \mathcal{L}_a$, where $\mathcal{L}_\psi$ is given in equation (7.28). The vortex piece

has the Ginzburg-Landau form [44],

$$\mathcal{L}_v = \frac{\kappa_\mu}{2}|(\partial_\mu - ia_\mu)\Phi|^2 - V_\Phi(|\Phi|), \tag{8.9}$$

as constructed explicitly with lattice duality in the Appendix. The vortex 3-current, following from $j^v_\mu = -\partial\mathcal{L}_v/\partial a_\mu$, is

$$j^v_\mu = \kappa_\mu \mathrm{Im}[\Phi^*(\partial_\mu - ia_\mu)\Phi]. \tag{8.10}$$

For small $|\Phi|$ (appropriate close to a second order transition) one can expand the potential as, $V_\Phi(X) = r_\Phi X^2 + u_\Phi X^4$. The remaining piece of the dual Lagrangian is

$$\mathcal{L}_a = \frac{1}{2\kappa_0}(\mathrm{e}_j^2 - b^2) + a_\mu \epsilon_{\mu\nu\lambda}\partial_\nu(A_\lambda - \kappa_\lambda^{-1} J_\lambda), \tag{8.11}$$

with dual "magnetic" and "electric" fields: $b = \epsilon_{ij}\partial_i a_j$ and $e_j = v_c^{-1}(\partial_j a_0 - \partial_0 a_j)$. It can be verified that the dual Lagrangian has the desired property that equation (8.8) follows from the equation of motion $\delta S_\mathrm{D}/\delta a_\mu = 0$.

9 Nodal liquid phase

In this section we employ the dual representation of the $d_{x^2-y^2}$ superconductor to analyze the quantum disordered phase - the *nodal liquid*. The dual representation comprises a complex vortex field, which is minimally coupled to a gauge field, as well as a set of neutral nodon fermions. Without the nodons and in imaginary time, the dual Lagrangian is formally equivalent to a classical three-dimensional superconductor at finite temperature, coupled to a fluctuating electromagnetic field. To disorder the d-wave superconductor, we must order the dual "superconductor" – that is, condense the hc/e vortices. The nature of the resulting phase will depend sensitively on doping, since upon doping, the dual "superconductor" starts seeing an applied "magnetic field". Below, we first consider the simpler case of half-filling. We then turn to the doped case, where two scenarios are possible depending on whether the dual "superconductor" is Type I or Type II [44].

9.1 Half-filling

Specialize first to the case of electrons at half-filling, with particle-hole symmetry. In the dual representation, the "magnetic field", b, is equal to the deviation of the total electron density from half-filling. Thus at half-filling $\langle b\rangle = 0$ and the dual Ginzburg-Landau theory is in zero applied field. The quantum disordered phase corresponds to condensing the hc/e vortices, setting $\langle\Phi\rangle = \Phi_0 \neq 0$. In this dual Meissner phase the vortex Lagrangian

becomes

$$\mathcal{L}_v = \frac{1}{2}\kappa_\mu \Phi_o^2 (a_\mu^{\rm t})^2, \tag{9.1}$$

where $a^{\rm t}$ represents the transverse piece of a_μ. It is then possible to integrate out the field a_μ which now enters *quadratically* in the Lagrangian. Equivalently, a_μ can be eliminated using the equation of motion which follows from $\delta S_{\rm D}/\delta a_\mu = 0$. The full Lagrangian in the nodal liquid phase is then

$$\mathcal{L}_{nl} = \mathcal{L}_\psi + A_\mu I_\mu + \frac{\epsilon_0}{2}E_j^2 - \frac{B^2}{2\mu_0} + O\left[(\partial J)^2\right], \tag{9.2}$$

where we have introduced the physical magnetic and electric fields: $B = \epsilon_{ij}\partial_i A_j$ and $E_j = \partial_j A_0 - \partial_t A_j$. The last two terms describe a dielectric, with magnetic permeability $\mu_0 = \kappa_0 \Phi_0^2$ and dielectric constant $\epsilon_0 = (\mu_0 v_{\rm c}^2)^{-1}$, with the sound velocity entering, rather than the speed of light. The external electromagnetic field is coupled to the 3-current I_μ, which can be expressed as a bi-linear of the nodon fermions as,

$$I_\mu = \frac{\epsilon_0}{\kappa_0^2 v_{\rm c}^2}[\kappa_\nu \partial_\nu^2 J_\mu - \kappa_\mu \partial_\mu (\partial_\nu J_\nu)]. \tag{9.3}$$

Notice that this 3-current is automatically conserved: $\partial_\mu I_\mu = 0$.

The order $(\partial J)^2$ terms which we have not written out explicitly are quartic in the fermion fields, and also involve two derivatives. Since $\mathcal{L}_\psi$ describes Dirac fermions in $2+1$ space-time dimensions, these quartic fermion terms are highly irrelevant, and rapidly vanish under the rescaling transformation in equation (7.43). Thus, the low energy description of the nodal liquid phase is exceedingly simple. It consists of four neutral Dirac fermion fields – two spin polarizations ($\alpha = 1, 2$) for each of the two pairs of nodes. Despite the free fermion description, the nodal liquid phase is highly *non-trivial* when re-expressed in terms of the underlying electron operators. Indeed, the ψ fermion operators are built from the quasiparticle operators Ψ in the d-wave superconductor, but are electrically neutral, due to the "gauge transformation" in equation (7.26).

In addition to the gapless nodons, one expects exotic *charged* excitations at finite energy in the nodal liquid. To see this, imagine applying an external dual "magnetic field" to the Ginzburg-Landau "superconductor", which corresponds to a non-zero chemical potential for the electrons. Being in the Meissner state, this "field" will be screened out, so that the internal field, b, which corresponds to deviations in the electron charge density from half-filling, will vanish. Clearly, this corresponds to a Mott insulator [45] with the Mott gap being proportional to the dual critical field. In a Type II superconductor, an internal magnetic field will be "quantized" into

flux-tubes carrying a quantum of flux [44]. For the dual Ginzburg-Landau theory, this corresponds to a quantization of electric charge, with a flux tube corresponding to charge e. Thus, in the nodal liquid one expects the presence of gapped finite energy excitations with charge e. These "holon" excitations are exotic since they carry *no* spin. The holon is the basic topological excitation that can be created in the hc/e vortex-condensate. The existence of a spin one-half neutral nodon excitation and a spinless charge e holon excitation in the nodal liquid, is a dramatic demonstration of spin-charge separation [41–43]. The excitations in the nodal liquid have the same quantum numbers as in the spin-charge separated gauge theories [46], but are weakly interacting, rather than strongly coupled by a gauge field.

9.1.1 Spin response

Although the nodons are electrically neutral they do carry *spin*, so the low-energy spin response in the nodal liquid can be computed from the Dirac Lagrangian $\mathcal{L}_\psi$. Moreover, since $\mathcal{L}_\psi$ was not altered under the duality transformation, the spin properties of the nodal liquid are essentially identical to those in the $d_{x^2-y^2}$ superconducting phase. As a simple example, consider the uniform magnetic spin susceptibility, χ. The uniform part of the electron spin operator is given as the conserved spin density in equation (7.41):

$$\mathbf{S}(\mathbf{x}) = \frac{1}{2}\psi^\dagger_{ja}(\mathbf{x})\boldsymbol{\sigma}\psi_{ja}(\mathbf{x}). \tag{9.4}$$

Being bi-linear in nodon operators spin correlation and response functions can be readily computed from the free nodon theory. For example, the uniform spin susceptibility is given by

$$\chi = \int_0^\infty \mathrm{d}E(-\partial f/\partial E)\rho_{\mathrm{n}}(E), \tag{9.5}$$

where the nodon density of states is $\rho_{\mathrm{n}}(E) = (\mathrm{const})E/v_{\mathrm{F}}v_\Delta$, and $f(E)$ is a Fermi function. One finds $\chi \sim T/v_{\mathrm{F}}v_\Delta$. There are also low energy spin excitations at wavevectors which span between two different nodes. The associated spin operators can be obtained by re-expressing the electron spin operator,

$$\mathbf{S}_q = \frac{1}{2}\sum_k c^\dagger_{k+q}\boldsymbol{\sigma}c_k, \tag{9.6}$$

in terms of the nodons. For example, the staggered magnetization operator, $\mathbf{S}_{\boldsymbol{\pi}}$, is found to be

$$\mathbf{S}_{\boldsymbol{\pi}} = \frac{1}{2}\left[\psi^\dagger(\tau^y\boldsymbol{\sigma}\sigma^y)\psi^\dagger + \mathrm{h.c.}\right]. \tag{9.7}$$

Notice that this operator is actually "anomalous" in terms of the conserved nodon charge.

In addition to carrying spin, the nodons carry energy, and so will contribute to the thermal transport. In the absence of scattering processes (such as Umklapp) the finite temperature nodon thermal conductivity is infinite. In practive, impurities will scatter the nodons and lead to a finite thermal conductivity. In fact, impurity scattering should also play an important role in modifying the spin response of the nodal liquid.

9.1.2 Charge response

The electrical charge properties of the nodal liquid are of course very different than in the superconductor. To see this, imagine changing the chemical potential away from $\mu = 0$ which corresponds to applying an external "magnetic" field to the dual Ginzburg-Landa theory: $\mathcal{L}_\mu = -\mu b$. Being in the "Meissner" phase, the electron density will stay "pinned" at half-filling for $\mu \leq \mu_c$, with μ_c the Ginzburg-Landau critical field. Despite the presence of this charge gap, there are low energy current fluctuations in the nodal liquid. Indeed, in this phase the electrical current operator is I_μ, which is bi-linear in the nodon fermions, ψ. To compute the electrical conductivity in the nodal liquid requires computing a two-point correlator of I_μ at zero wavevector (say in the $x-$direction) $I_x(q=0) = (\epsilon_0/\kappa_o v_s^2)\partial_t^2 J_x(q=0)$. But notice that $J_x(q=0)$ is proportional to a globally conserved nodon charge, since $J_x(\mathbf{x}) = (v_F/2)\psi_1^\dagger\psi_1$. Thus, when the nodon number is conserved one has $I_x(q=0) = 0$, and the nodons *do not* contribute to the electrical conductivity. When impurity (or Umklapp) scattering is present, however, the nodon number is no longer conserved, and the nodons will contribute to the real part of the electrical conductivity, but only at finite frequencies.

It is instructive to briefly consider the behavior of the electron Green's function, which can be accessed in photo-emission and tunneling experiments. The electron operator $c_\alpha(\mathbf{x})$ can be decomposed as a product of nodon and holon operators. For example, near the node at $\boldsymbol{K}_j$ one can write,

$$c_\alpha(\mathbf{x}) = e^{i\mathbf{K}_j\cdot\mathbf{x}} e^{i\phi(\mathbf{x})}\psi_{j1\alpha}(\mathbf{x}) + ... \tag{9.8}$$

where ψ is a nodon operator and $\exp(i\phi)$ can be interpreted as a holon destruction operator. In the nodal liquid phase, the electron Green's function, $G(\mathbf{x},t) = \langle c^\dagger(\mathbf{x},t)c(\mathbf{0},0)\rangle$ factorizes as,

$$G(\mathbf{x},t) = e^{i\mathbf{K}_j\cdot\mathbf{x}}\langle e^{-i\phi(\mathbf{x},t)}e^{i\phi(\mathbf{0},0)}\rangle G_n(\mathbf{x},t), \tag{9.9}$$

where the nodon Green's function is,

$$G_n(\mathbf{x},t) = \langle\psi_{j1\alpha}^\dagger(\mathbf{x},t)\psi_{j1\alpha}(\mathbf{0},0)\rangle\cdot \tag{9.10}$$

Although $G_{\rm n}(\mathbf{x},t)$ decays as a power law $|x|^{-2}$ and t^{-2}, since creating a holon costs a finite energy the holon Green's function is expected to be short-ranged, decaying exponentially in space and time. This indicates a gap in the electron spectral function at the Fermi energy.

9.2 Doping the nodal liquid

We briefly discuss the effects of doping charge into the nodal liquid phase. In a grand canonical ensemble this is achieved by changing the chemical potential, $\mu = A_0$. In the dual Ginzburg-Landau description of the vortices, a chemical potential acts as an applied dual field, as seen from equation (8.11), since

$$\mathcal{L}_\mu = -\mu b. \tag{9.11}$$

The dual magnetic field, $b = \epsilon_{ij}\partial_i a_j$, is the total electric charge in units of e. Provided the applied dual field, μ, is smaller than the critical field ($\mu_{\rm c}$) of the Ginzburg-Landau theory, the dual superconductor stays in the Meissner phase – which is the nodal liquid phase at half-filling. But for $\mu \geq \mu_{\rm c}$ dual flux will penetrate the Ginzburg-Landau superconductor, which corresponds to doping the nodal liquid. The form of the dual flux penetration will depend critically on whether the dual Ginzburg-Landau theory is Type I or Type II. Within a mean-field treatment this is determined by the ratio of the dual penetration length, λ_v, to the dual coherence length, ξ_v (where the subscript v denotes vortices). In particular, Type II behavior is expected if $\lambda_v/\xi_v \geq 1/\sqrt{2}$, and Type I behavior otherwise. In the Ginzburg-Landau description λ_v determines the size of a dual flux tube, which is essentially the size of a Cooper pair. We thus expect that λ_v will be roughly equal to the superconducting coherence length, ξ, which is perhaps $10-15$ Å in the cuprates. On the other hand, ξ_v is the size of the "vortex-core" in the dual vortex field, and presumably can be no smaller than the microscopic crystal lattice spacing, $\xi_v \geq 3-5$ Å. This reasoning suggest that λ_v/ξ_v is probably close to unity in the cuprates, so that either Type I or Type II behavior might be possible – and could be material dependent. We first consider such Type II doping, returning below to the case of a Type I Ginzburg-Landau theory.

9.2.1 Type II behavior

The phase diagram of a clean three-dimensional Type II superconductor is well understood [44]. Above the lower critical field, $H_{\rm c1}$, flux tubes penetrate, and form an Abrikosov flux lattice – usually triangular. As the applied field increases the flux tubes start overlapping, when their separation is closer than the penetration length. Upon approaching the upper

critical field H_{c2} their cores start overlapping, the Abrikosov flux lattice disappears, and the superconductivity is destroyed.

These results hold equally well for our dual Ginzburg-Landau superconductor, except that now the direction parallel to the applied field is actually imaginary time. Moreover, the Ginzburg-Landau order parameter describes quantum (hc/e) vortices, and the penetrating flux tubes are spinless charge e holons. Upon doping the nodal liquid with $\mu > \mu_{c1}$, charge is added to the 2d system, which corresponds to the penetration of dual magnetic flux. In this dual transcription, the resulting Abrikosov flux-lattice phase is a Wigner crystal of holons, with one holon per real space unit cell of the lattice. Upon further doping, at $\mu = \mu_{c2}$, the crystal of holons melts, and they condense – this is the d-wave superconductor.

In the holon Wigner crystal phase, translational symmetry is spontaneously broken. However, in a real material the Wigner crystal will have a preferred location, determined by impurities and perhaps crystal fields, which will tend to pin and immobilize the crystal. The resulting phase should be an electrical insulator.

A striking and unusual feature of the holon Wigner crystal is that it *co-exists* with the nodal liquid. We thereby arrive at a description of a rather remarkable new phase of matter. A Wigner crystal of doped holons co-exists with neutral gapless fermionic excitations – the nodons. In this co-existing phase, low energy spin and thermal properties will be dominated by the nodons. The behavior will be qualitatively similar to that in the undoped nodal liquid phase. It is possible that this phase underlies the physics of the pseudo-gap region of the high T_c cuprates.

9.2.2 Type I behavior

In a Type I superconductor, the applied field is expelled until the critical H_c is exceeded [44]. At this point there is a *first order* phase transition from the Meissner phase with all the flux expelled, to a normal metal phase in which (essentially) all the field penetrates. If our dual Ginzburg-Landau theory is I Type I, then analogous properties are expected. Specifically, as the chemical potential increases, the dual field – which is the holon density – remains at zero until a critical chemical potential μ_c is reached. At this point there is a first order phase transition, between the nodal liquid phase at half-filling, and a d-wave superconductor at finite doping, x_c. At fixed doping $x < x_c$, phase separation is impeded by long-ranged Coulomb interactions between the holons. The system will break apart into co-existing "micro-phases" of nodal liquid and d-wave superconductor. The configuration of the "micro-phases" will be determined by a complicated competition between the Coulomb energy and the (positive) energy of the domain walls. In practice, impurities will also probably play a very important role.

9.3 Closing remarks

The theoretical framework described above gives a skeletal description of the nodal liquid and, upon doping, the holon Wigner crystal. There are many important issues which will need to be addressed in detail to see if this novel Mott insulating phase gives a correct description of the low temperature pseudo-gap regime in the cuprates. At very low doping the cuprates are antiferromagnetic so it will clearly be necessary to incorporate magnetism into the theoretical framework. Perhaps even more important is assessing the role of impurities, which are expected to have rather dramatic effects both on the holon Wigner crystal and the gapless nodons. Impurities will tend to disorder the Wigner crystal and will scatter the nodons probably leading to a finite density of states and diffusive rather than ballistic motion. Since the nodons carry spin but no charge, a rather exotic "spin metal" phase is possible with a finite "spin conductivity" (but zero electrical conductivity) even at zero temperature. It is also possible that the impurities will localize the nodons, perhaps leading to a random singlet phase or a spin glass. An additional complication is that some materials might exhibit phase separation upon doping (Type I rather than Type II behavior) exhibiting micro-phase co-existence between the antiferromagnet and the d-wave superconductor, preempting the nodal liquid phase. It clearly remains as a future challenge to fully sort out the mysteries of the pseudo-gap regime.

A more general theme of these notes is that novel spin liquid phases can sometimes be more conveniently viewed as descendents of superconductors – rather than the more traditional route *via* magnetism. One can imagine quantum disordering other exotic superconducting phases besides the $d_{x^2-y^2}$ state, to obtain new spin liquid phases. Perhaps some of these phases will appear in other systems which exhibit finite angular momentum pairing, such as $3-$He and the heavy Fermion materials.

It gives me genuine pleasure to acknowledge my wonderful collaborators on the research described above. The renormalization group analysis of the two-leg ladder was carried out in collaboration with Hsiu-hau Lin and Leon Balents. The nodal liquid phase was introduced and analyzed in a collaboration with Chetan Nayak and Leon Balents. This research has been a true collective phenomena, to which I am deeply appreciative. I am also extremely grateful to Doug Scalapino for stimulating my interest in strongly correlated d-wave superconductors and for numerous discussions about Hubbard ladders. I would like to thank T. Senthil for sharing his insights about the effects of impurities in d-wave superconductors. This work has been supported by the National Science Foundation under grants Nos. PHY94-07194, DMR94-00142 and DMR95-28578.

Appendix

A Lattice duality

Duality plays a key role in understanding how to quantum disorder a superconductor, both in $1+1$ space-time dimensions (Sect. 5) and in $2+1$ (Sect. 8). The key idea involves exchanging the order parameter phase ϕ for vortex degrees of freedom. In $1+1$ dimensions these are point-like space-time vortices [31], whereas in $2+1$ there are point like vortices in space which propogate in time [39]. In Section 8 we chose to work directly in the continuum in implementing the $2+1$ duality transformation. However, the physics of duality is perhaps more accessible when carried out on the lattice. In this Appendix we show in some detail how lattice duality is implemented in both $1+1$ and $2+1$ dimensions [31, 39]. For simplicity we first Wick rotate to Euclidian space, and rescale imaginary time to set the charge velocity to one. The appropriate lattice model is then simply a 2d square lattice or 3d cubic lattice xy model. In the latter case, we also want to include a gauge-field, A, which is a sum of the physical electromagnetic field and the nodon current, as discussed in Section 8 – see equation (8.5).

The degrees of freedom which live on the sites of the square or cubic lattice (denoted by a vector of integers $\vec{x}$) are the phases $\phi_x \in [0, 2\pi]$. As usual, the gauge field lives on the links. Discrete lattice derivatives are denoted by

$$\triangle_\mu \phi_x = \phi_{x+\mu} - \phi_x, \tag{A.1}$$

where $\mu = x, y$ for the square lattice and $\mu = x, y, z$ for the cubic lattice and $x + \mu$ denotes the nearest neighbor site to $\vec{x}$ in the $\hat{\mu}$ direction. The gauge field is minimally coupled *via*,

$$\triangle_\mu \phi_x \to \triangle_\mu \phi_x + A^\mu_x. \tag{A.2}$$

Consider the partition function,

$$Z = \int_0^{2\pi} \prod_x \mathrm{d}\phi_x \exp\left[\sum_{x,\mu} V_\kappa(\triangle_\mu \phi_x)\right]. \tag{A.3}$$

Here the periodic "Villain" potential V_κ is given by,

$$\exp[V_\kappa(\triangle\phi)] = \sum_{J=-\infty}^{\infty} \mathrm{e}^{-\kappa J^2/2} \mathrm{e}^{iJ\triangle\phi}, \tag{A.4}$$

with integer J. When $\kappa \gg 1$ only the terms with $J = 0, \pm 1$ contribute appreciably in the sum and this reduces to the more familiar form:

$$V_\kappa(\triangle\phi) = K \cos(\triangle\phi), \tag{A.5}$$

with $K = 2\exp(-\kappa/2)$.

The partition function can thus be expressed as a sum over both ϕ and a vector of integers, $\vec{J}_x$, with components J_x^μ living on the *links* of the lattice:

$$Z = \int \prod_x \mathrm{d}\phi \sum_{[\vec{J}]} e^{-S} \equiv Tr_{\phi,\vec{J}}\ e^{-S}, \tag{A.6}$$

with action

$$S = S_0 + \sum_x i(\vec{\triangle} \cdot \vec{J}_x)\phi_x, \tag{A.7}$$

$$S_0 = \frac{\kappa}{2} \sum_x |\vec{J}_x|^2. \tag{A.8}$$

In this form the integration over ϕ can be explicitly performed giving

$$Z = Tr'_{\vec{J}}\ e^{-S_0}, \tag{A.9}$$

where the prime on the trace indicates a divergenceless *constraint* at each site of the lattice:

$$\vec{\triangle} \cdot \vec{J}_x = 0. \tag{A.10}$$

In the presence of a gauge field there is an additional term in the action of the form,

$$S_A = i \sum_x \vec{J}_x \cdot \vec{A}_x. \tag{A.11}$$

It is thus clear that the integer of vectors $\vec{J}$ can be interpreted as a conserved electrical current flowing on the links of the lattice. The divergenceless constraint on this electrical 3-current can be imposed automatically by re-expressing $\vec{J}$ as a curl of an appropriate *dual* field. Consider first the 2d case.

A.1 Two dimensions

To guarantee divergenceless we set the current equal to the (2d) curl of a scalar field, θ_x:

$$2\pi J_x^\mu = \epsilon_{\mu\nu} \triangle_\nu \theta_x, \tag{A.12}$$

so that the action becomes

$$S_0(\theta) = \frac{\kappa}{8\pi^2} \sum_{x,\mu} (\triangle_\mu \theta_x)^2. \tag{A.13}$$

To insure that $\vec{J}$ is an *integer* field, θ must be constrained to be 2π times an integer. This additional constraint can be imposed by introduction of yet another integer field, n_x, which will be interpreted as the (space-time) vortex density. The partition is thereby re-expressed as (dropping an unimportant multiplicative constant),

$$\tilde{Z} = \int_{-\infty}^{\infty} \prod_x \mathrm{d}\theta_x \sum_{[n_x]} e^{-S}, \tag{A.14}$$

with

$$S = S_0(\theta) + \sum_x \left[\frac{\tilde{\kappa}}{2} n_x^2 + i n_x \theta_x \right]. \tag{A.15}$$

For $\tilde{\kappa} = 0$ the summation over n_x gives a sum of delta functions restricting $\theta_x/2\pi$ to be integer. But we have softened this constraint, introducing a vortex "core" energy $\tilde{\kappa} \neq 0$.

At this stage one could perform the Gaussian integral over θ, to obtain a logarithmically interacting plasma of (space-time) vortices. Alternatively, for $\tilde{\kappa} \gg 1$ the summation over n_x can be performed giving,

$$S = S_0(\theta) - u \sum_x \cos(\theta_x), \tag{A.16}$$

with $u = 2\exp(-\tilde{\kappa}/2)$. Upon taking the continuum limit, $\theta_x \to \theta(x)$, one recovers the (Euclidian) sine-Gordon theory, $S = \int \mathrm{d}^2x \mathcal{L}$ with

$$\mathcal{L} = \frac{\kappa}{8\pi^2} (\vec{\nabla}\theta)^2 - u \cos(\theta). \tag{A.17}$$

After Wick rotating back to real time and restoring the velocity this takes the identical form to the dual Lagrangian considered for the 2-leg ladder in Section 5.

A.2 Three dimensions

In three dimensions the divergenceless integer 3-current $\vec{J}$ can be written as the curl of a *vector* field, $\vec{a}$:

$$2\pi \vec{J}_x = \vec{\triangle} \times \vec{a}_x. \tag{A.18}$$

As in 2d one imposes the integer constraint (softly) by introducing an integer vortex field, in this case a 3-vector $\vec{j}$, to express the partition function as,

$$\tilde{Z} = \int_{-\infty}^{\infty} \prod_x \mathrm{d}\vec{a}_x \sum_{[\vec{j}_x]} e^{-S}, \tag{A.19}$$

with

$$S = S_0(\vec{a}) + \sum_x \left[\frac{\tilde{\kappa}}{2} |\vec{j}_x|^2 - i\vec{j}_x \cdot \vec{a}_x \right], \tag{A.20}$$

$$S_0(\vec{a}) = \frac{\kappa}{8\pi^2} \sum_x |\vec{\triangle} \times \vec{a}_x|^2. \tag{A.21}$$

The integer vector field $\vec{j}$ is the vortex 3-current, "minimally" coupled to $\vec{a}$. To see that the vortex 3-current is conserved, it is convenient to decompose the vector field $\vec{a}$ into transverse and longitudinal pieces: $\vec{a} = \vec{a}_t - \vec{\triangle}\theta$, with θ_x a scalar field. The action becomes,

$$S = S_0(\vec{a}) + \sum_x \left[\frac{\tilde{\kappa}}{2} |\vec{j}_x|^2 + i\vec{j}_x \cdot (\vec{\triangle}\theta_x - \vec{a}_x) \right], \tag{A.22}$$

where we have dropped the subscript "t" on $\vec{a}$. The partition function follows from integrating over both $\vec{a}$ *and* θ and summing over integer $\vec{j}$. Integrating over θ leads to the expected condition: $\vec{\triangle} \cdot \vec{j} = 0$. Alternatively, for $\tilde{\kappa} \gg 1$ one can perform the summation over $\vec{j}$ to arrive at an action depending on θ and $\vec{a}$:

$$S = S_0(\vec{a}) - K \sum_{x,\mu} \cos(\triangle_\mu \theta_x - a_x^\mu), \tag{A.23}$$

with $K = 2\exp(-\tilde{\kappa}/2)$.

In the presence of a gauge field A^μ there is an additional term in the action of the form,

$$S_A = \frac{i}{2\pi} \sum_x (\vec{\triangle} \times \vec{a}_x) \cdot \vec{A}_x, \tag{A.24}$$

which follows directly from equations (A.11) and (A.18).

At this stage one can take the continuum limit, letting $\vec{a}_x \to \vec{a}(x)$ and $\theta_x \to \theta(x)$. Upon expanding the cosine for small argument one obtains $S = \int \mathrm{d}^3x \mathcal{L}$ with (Euclidian) Lagrangian

$$\mathcal{L} = \frac{\kappa}{8\pi^2} (\vec{\nabla} \times \vec{a})^2 + \frac{K}{2} (\vec{\nabla}\theta - \vec{a})^2. \tag{A.25}$$

In this dual representation, the vortex 3-current (which follows from $\partial\mathcal{L}/\partial\vec{a}$) is given by $\vec{j}^v = K(\vec{\nabla}\theta - \vec{a})$. Notice that the vortices are minimally coupled to the "vector potential" $\vec{a}$, whose curl equals the electrical 3-current. The field θ can be interpreted as the phase of a vortex operator. In fact it

is convenient to introduce such a complex vortex field before taking the continuum limit:

$$e^{i\theta_x} \rightarrow \Phi(\vec{x}). \tag{A.26}$$

The continuum limit can then be taking *retaining* the full periodicity of the cosine potential. The appropriate vortex Lagrangian replacing the second term in equation (A.25) is,

$$\mathcal{L}_v = \frac{K}{2}|(\vec{\nabla} - i\vec{a})\Phi|^2 + V_\Phi(|\Phi|). \tag{A.27}$$

The vortex current operator becomes,

$$\vec{j}^v = K\mathrm{Im}[\Phi^*(\vec{\nabla} - i\vec{a})\Phi]. \tag{A.28}$$

If the potential is expanded for small Φ as $V_\Phi(X) = r_\Phi X^2 + u_\Phi X^4$, the full dual theory is equivalent to a Ginzburg-Landau theory for a classical three-dimensional superconductor. Inclusion of the original gauge field A^μ leads to an additional term in the dual Lagrangian:

$$\mathcal{L}_A = \frac{i}{2\pi}(\vec{\nabla} \times \vec{a}) \cdot \vec{A}. \tag{A.29}$$

After Wick rotating back to real time and restoring the velocity, $\mathcal{L} + \mathcal{L}_A$ becomes identical to the dual vortex Lagrangian in Equations (8.9) and (8.11).

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