An Introduction to Theoretical Fluid Dynamics

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Notes on the course

The course meets on Wednesday, 9:20-11:30 am in Room 813 WWH.
Office Hours: TBA

BOOKS: The text is Landau and Lifshitz, Fluid Mechanics, available at the Bookstore. Acheson's Elementary Fluid Dynamics is a good elementary book. The reserve books given below will be available in the Courant library.

- Batchelor, G.K. Introduction to Fluid Dynamics, Cambridge Univsrsity Press 1967
- Landau and Lifshitz, Fluid Mechanis (2nd Ed.), Pergamon Press 1987.
- Milne-Thomson, L.M. Theoretical Hydrodynamics, McMillan (5th Ed.)
- Lighthill, M.J. An Informal Introduction to Theoretical Fluid Mechanics, Clarendon Press 1986.
- Prandtl, L. Essentials of Fluid Dynamics, Hafner 1952.
- Lamb, Hydrodynamics (6th Ed.), Cambridge University Press 1932
- Courant and Freidrichs, Supersonic Flow and Shock Waves, Interscience 1948.
- Meyer, An Introduction to Mathematical Fluids Dynamics, Dover 1971.
- D. J. Acheson, Elementary Fluid Dynamics, Clarendon 1990.

Chapter 1

The fluid continuum

This course will deal with a mathematical idealization of common fluids such as air or water. The main idealization is embodied in the notion of a *continuum* and our "fluids" will generally be identified with a certain connected set of points in R^N, where we will consider dimension N to be 1,2, or 3. Of course the fluids will move, so basically our subject is that of a moving continuum.

This description is an idealization which neglects the molecular structure of real fluids. *Liquids* are fluids characterized by random motions of molecules on the scale of $10^{-7} - 10^{-8}$ cm, and by a substantial resistance to compression. *Gases* consist of molecules moving over much larger distances, with mean free paths of the order of 10^{-3} cm, and are readily compressed. Both liquids and gases will fall within the scope of the theory of fluid motion which we will develop below. The theory will deal with observable properties such as velocity, density, and pressure. These properties must be understood as averages over volumes which contains many molecules but are small enough to be "infinitesimal" with respect to the length scale of variation of the property. We shall use the term *fluid parcel* to indicate such a small volume. The notion of a *particle* of fluid will also be used, but should not be confused with a molecule. For example, the time rate of change of position of a fluid particle will be the *fluid velocity*, which is an average velocity taken over a parcel and is distinct from molecular velocities. The continuum theory has wide applicability to the natural world, but there are certain situations where it is not satisfactory. Usually these will involve small domains where the molecular structure becomes important, such as shock waves or fluid interfaces.

1.1 Eulerian and Lagrangian descriptions

Let the independent variables (observables) describing a fluid be a function of position $\mathbf{x} = (x_1, \dots, x_N)$ in Euclidean space and time t. Suppose that at $t = 0$ the fluid is identified with an open set $\mathcal{S}_0$ of R^N. As the fluid moves, the particles of fluid will take up new positions, occupying the set $\mathcal{S}_t$ at time

t. We can introduce the map $\mathcal{M}_t, \mathcal{S}_0 \to \mathcal{S}_t$ to describe this change, and write $\mathcal{M}_t\mathcal{S}_0 = \mathcal{S}_t$. If $\mathbf{a} = (a_1, \dots, a_N)$ is a point of $\mathcal{S}_0$, we introduce the function $\mathbf{x} = \mathcal{X}(\mathbf{a}, t)$ as the position of a fluid particle at time t, which was located at $\mathbf{a}$ at time $t = 0$. The function $\mathcal{X}(\mathbf{a}, t)$ is called the *Lagrangian coordinate* of the fluid particle identified by the point $\mathbf{a}$. We remark that the"coordinate" $\mathbf{a}$ need not in fact be the initial position of a particle, although that is the most common choice and will be generally used here. But any unique labeling of the particles is acceptable.[1]

The *Lagrangian description* of a fluid emerges from this focus on the fluid properties associated with individual fluid particles. To "think Lagrangian" about a fluid, one must move with the fluid and sample the fluid properties in each moving parcel. The Lagrangian analysis of a fluid has certain conceptual and mathematical advantages, but it is often difficult to apply to useful examples. Also it is not directly related to experience, since measurements in a fluid tend to be performed at fixed points in space, as the fluid flows past the point.

If we therefore adopt the point of view that we will observe fluid properties at a fixed point $\mathbf{x}$ as a function of time, we must break the association with a given fluid particle and realize that as time flows different fluid particles will occupy the position $\mathbf{x}$. This will make sense as long as $\mathbf{x}$ remains within the set $\mathcal{S}_t$. Once properties are expressed as functions of $\mathbf{x}, t$ we have the *Eulerian description* of a fluid. For example, we might consider the fluid to fill all space and be at rest "at infinity". We then can consider the velocity $\mathbf{u}(\mathbf{x}, t)$ at each point of space, with $\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}, t) = 0$. Or, we might have a fixed rigid body with fluid flowing over it such that at infinity we have a fixed velocity $\mathbf{U}$. For points outside the body the fluid velocity will be defined and satisfy $\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}, t) = \mathbf{U}$.

It is of interest to compare these two descriptions of a fluid and understand their connections. The most obvious is the meaning of velocity: the definition is

$$\mathbf{x}_t = \frac{\partial \mathcal{X}}{\partial t}\Big|_{\mathbf{a}} = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t). \tag{1.1}$$

That is to say, following the particle we calculate the rate of change of position with respect to time. Given the Eulerian velocity field, the calculation of Lagrangian coordinates is therefore mathematically equivalent to solving the initial-value problem for the system (1.1) of ordinary differential equations for the function $\mathbf{x}(t)$, with the initial condition $\mathbf{x}(0) = \mathbf{a}$, the order of the system being the dimension of space. The special case of a steady flow leads to a system of *autonomous* ODEs.

Example 1.1: In two dimensions ($N = 2$), with fluid filling the plane, we take $\mathbf{u}(\mathbf{x}, t) = (u(x, y, t), v(x, y, t)) = (x, -y)$. This velocity field is independent of time, hence we call it a *steady flow*. To compute the Lagrangian coordinates of the fluid particle initially at $\mathbf{a} = (a, b)$ we solve:

$$\frac{\partial x}{\partial t} = x, x(0) = a, \quad \frac{\partial y}{\partial t} = -y, y(0) = b, \tag{1.2}$$

[1] We shall often use (x, y, z) in place of (x_1, x_2, x_3), and (a, b, c) in place of (a_1, a_2, a_3).

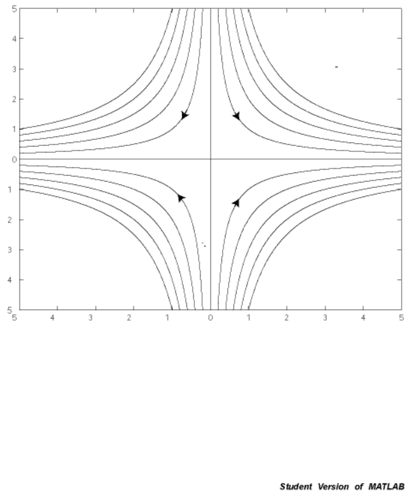

Figure 1.1: Stagnation-point flow

so that $\mathcal{X} = (ae^t, be^{-t})$. Note that, since $xy = ab$, the *particle paths* are hyperbolas; the curves independent of time, see figure 1.1. If we consider the fluid in $y > 0$ only and take $y = 0$ as a rigid wall, we have a flow which is impinging vertically on a wall. The point $x = y = 0$, where the velocity is zero, is called a *stagnation point*. This point is a hyperbolic point relative to particle paths. A flow of this kind occurs at the nose of a smooth body placed in a uniform current. Because this flow is steady, the hyperbolic particle paths are also called *streamlines*.

Example 1.2: Again in two dimensions, consider $(u, v) = (y, -x)$. Then $\frac{\partial x}{\partial t} = y$ and $\frac{\partial y}{\partial t} = -x$. Solving, the Lagrangian coordinates are $x = a\cos t + b\sin t, y = -a\sin t + b\cos t$, and the particle paths (and streamlines) are the circles $x^2 + y^2 = a^2 + b^2$. The motion on the streamlines is clockwise, and fluid particles located at some time on a ray x/y =constant remain on the same ray as it rotates clockwise once for every 2π units of time. This is *sold-body rotation*.

Example 1.3: If instead $(u, v) = (y/r^2, -x/r^2), r^2 = x^2 + y^2$, we again have particle paths which are circles, but the velocity becomes infinite at $r = 0$. This is an example of a flow representing a *point vortex*. We shall take up the study of vortices in chapter 3.

1.1.1 Particle paths, instantaneous streamlines, and streak lines

The present considerations are *kinematic*, meaning that we are assuming knowledge of fluid motion, through an Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ or else Lagrangian coordinates $\mathbf{x} = \mathcal{X}(\mathbf{a}, t)$, irrespective of the cause of the motion. One useful kinematic characterization of a fluid flow is the pattern of streamlines, as al-

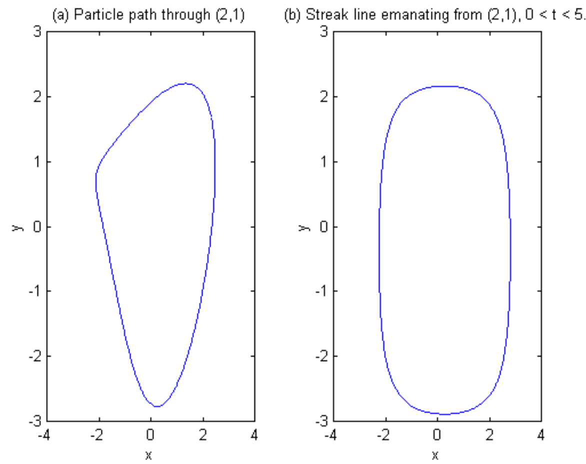

Figure 1.2: Particle path and streak line in example 1.4.

ready mentioned in the above examples. In steady flow the streamlines and particle paths coincide. In an unsteady flow this is not the case and the only useful recourse is to consider *instantaneous streamlines*, at a particular time. In three dimensions the instantaneous streamlines are the orbits of the $\mathbf{u}(\mathbf{x}, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$ at time t. These are the integral curves satisfying

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{1.3}$$

As time flows these streamlines will change in an unsteady flow, and the connection with particle paths is not obvious in flows of any complexity.

Visualization of flows in water is sometimes accomplished by introducing dye at a point in space. The dye can be thought of as labeling by color the fluid particle found at the point at a given time. As each point is labeled it moves along its particle path. The resulting *streak line* thus consists of all particles which at some time in the past were located at the point of injection of the dye. To describe a streak line mathematically we need to generalize the time of initiation of a particle path. Thus we introduce the *generalized Lagrangian coordinate* $\mathbf{x} = \mathcal{X}(\mathbf{a}, t, t_a)$, defined to be the position at time t of a particle that was located at $\mathbf{a}$ at time t_a. A streak line observed at time $t > 0$, which was started at time $t = 0$ say, is given by $\mathbf{x} = \mathcal{X}(\mathbf{a}, t, t_a), 0 < t_a < t$. Particle paths, instantaneous streamlines, and streak lines are all distinct objects in unsteady flows.

Example 1.4: Let $(u, v) = (y, -x + \epsilon \cos \omega t)$. For this flow the instantaneous streamlines satisfy $dx/y = dy/(-x + \epsilon \cos \omega t)$ and so are the circles $(x - \epsilon \cos \omega t)^2 + y^2 =$ constant. The generalized Lagrangian coordinates can be obtained from the general solution of a second-order ODE and takes the form

$$x = -\frac{\epsilon}{\omega^2 - 1}\cos\omega t + A\cos t + B\sin t,\ y = \frac{\epsilon\omega}{\omega^2-1}\sin\omega t + B\cos t - A\sin t, \quad (1.4)$$

where

$$A = -b\sin t_a + \frac{\epsilon\omega}{\omega^2-1}\sin\omega t_a \sin t_a + a\cos t_a + \frac{\epsilon}{\omega^2-1}\cos\omega t_a\cos t_a, \quad (1.5)$$

$$B = a\sin t_a + b\cos t_a - \frac{\epsilon}{\omega^2-1}\cos\omega t_a\sin t_a + \frac{\epsilon\omega}{\omega^2-1}\sin\omega t_a\cos t_a. \quad (1.6)$$

The particle path with $t_a = 0, \omega = 2, \epsilon = 1$ starting at the point $(2, 1)$ is given by

$$x = -\frac{1}{3}\cos 2t + \sin t + \frac{7}{3}\cos t, y = \cos t - \frac{7}{3}\sin t + \frac{2}{3}\sin 2t, \quad (1.7)$$

and is shown in figure 1.2(a). All particle paths are closed curves. The streak line emanating from (2,1) over the time interval $0 < t < 2\pi$ is shown in figure 1.2(b).

This last example is especially simple since the 2D system is linear and integrable explicitly. In general two-dimensional unsteady flows and three-dimensional steady flows can exhibit chaotic particle paths and streak lines.

Example 1.5: A nonlinear system exhibiting this complex behavior is the oscillating point vortex: $(u, v) = (y/r^2, -(x-\epsilon\cos\omega t)/r^2)$. We show an example of particle path and streak line in figure 1.3.

1.1.2 The Jacobian matrix

We will, with a few obvious exceptions, be taking all of our functions as infinitely differentiable wherever they are defined. In particular we assume that Lagrangian coordinates will be continuously differentiable with respect to the particle label **a**. Accordingly we may define the Jacobian of the Lagrangian map $\mathcal{M}_t$ by matrix

$$J_{ij} = \frac{\partial x_i}{\partial a_j}\Big|_t \quad (1.8)$$

Thus $dl_i = J_{ij}da_j$ is a differential vector which can be visualized as connecting two nearby fluid particles whose labels differ by da_j.[2] If $da_1\cdots da_N$ is the volume of a small fluid parcel, then $\mathrm{Det}(\mathbf{J})da_1\cdots a_N$ is the volume of that parcel under the map $\mathcal{M}_t$. Fluids which are *incompressible* must have the property that all fluid parcels preserve their volume, so that Det(**J**)= constant=1 when **a** denotes initial position, independently of $\mathbf{a}, t$. We then say that the Lagrangian map is volume preserving. For general compressible fluids Det(**J**) will vary in space and time.

Another important assumption that we shall make is that the map $\mathcal{M}_t$ is always invertible, Det(**J**)> 0. Thus when needed we can invert to express **a** as a function of $\mathbf{x}, t$.

[2]Here and elsewhere the summation convention is understood: unless otherwise started repeated indices are to be summed from 1 to N.

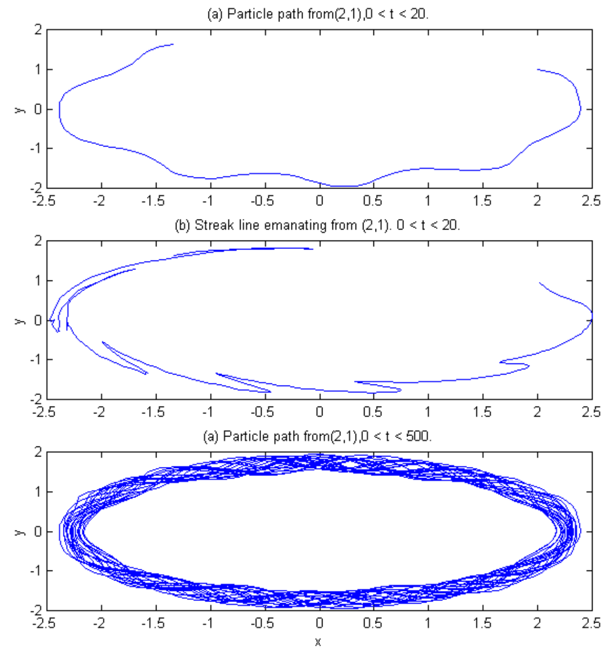

Figure 1.3: The oscillating vortex, $\epsilon = 1.5, \omega = 2$.

1.2 The material derivative

Suppose we have some scalar property $\mathcal{P}$ of the fluid that can be attached to a certain fluid parcel, e.g. temperature or density. Further, suppose that, as the parcel moves, this property is invariant in time. We can express this fact by the equation

$$\frac{\partial \mathcal{P}}{\partial t}\Big|_{\mathbf{a}} = 0, \tag{1.9}$$

since this means that the time derivative is taken with particle label fixed, i.e. taken as we move with the fluid particle in question. We will say that such an invariant scalar is *material*. A material invariant is one attached to a fluid particle. We now asked how this property should be expressed in Eulerian variables. That is, we select a point $\mathbf{x}$ in space and seek to express material invariance in terms of properties of the fluid *at this point*. Since the fluid is generally moving at the point, we need to bring in the velocity. The way to do this is to differentiate $\mathcal{P}(\mathbf{x}(\mathbf{a}, t), t)$, expressing the property as an Eulerian variable, using the chain rule:

$$\frac{\partial \mathcal{P}(\mathbf{x}(\mathbf{a}, t), t)}{\partial t}\Big|_{\mathbf{a}} = 0 = \frac{\partial \mathcal{P}}{\partial t}\Big|_{\mathbf{x}} + \frac{\partial x_i}{\partial t}\Big|_{\mathbf{a}} \frac{\partial \mathcal{P}}{\partial x_i}\Big|_{t} = \mathcal{P}_t + \mathbf{u} \cdot \nabla \mathcal{P}. \tag{1.10}$$

In fluid dynamics the Eulerian operator $\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla$ is called the *material derivative* or *substantive* derivative or *convective* derivative. Clearly it is a time derivative “following the fluid”, and translates the Lagrangian time derivative in terms of Eulerian properties of the fluid.

Example 1.6: The *acceleration* of a fluid parcel is defined as the material derivative of the velocity $\mathbf{u}$. In Lagrangian variables the acceleration is $\frac{\partial^2 \mathbf{x}}{\partial t^2}\Big|_{\mathbf{a}}$, and in Eulerian variables the acceleration is $\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}$.

Following a common convention we shall often write

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, \tag{1.11}$$

so the acceleration becomes $D\mathbf{u}/Dt$.

Example 1.7: We consider the material derivative of the determinant of the Jacobian $\mathbf{J}$. We may divide up the derivative of he determinant into a sum of N determinants, the first having the first row differentiated, the second having the next row differentiated, and so on. The first term is thus the determinant of the matrix

$$\begin{pmatrix} \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} & \cdots & \frac{\partial u_1}{\partial a_N} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \cdots & \frac{\partial x_2}{\partial a_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial a_1} & \frac{\partial x_N}{\partial a_2} & \cdots & \frac{\partial x_N}{\partial a_N} \end{pmatrix}. \tag{1.12}$$

If we expand the terms of the first row using he chain rule, e.g.

$$\frac{\partial u_1}{\partial a_1} = \frac{\partial u_1}{\partial x_1}\frac{\partial x_1}{\partial a_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial x_2}{\partial a_1} + \cdots + \frac{\partial u_1}{\partial x_N}\frac{\partial x_N}{\partial a_1}, \tag{1.13}$$

we see that we will get a contribution only from the terms involving $\frac{\partial u_1}{\partial x_1}$, since all other terms involve the determinant of a matrix with two identical rows. Thus the term involving the derivative of the top row gives the contribution $\frac{\partial u_1}{\partial x_1}\mathrm{Det}(\mathbf{J})$. Similarly, the derivatives of the second row gives the additive contribution $\frac{\partial u_2}{\partial x_2}\mathrm{Det}(\mathbf{J})$. Continuing, we obtain

$$\frac{D}{Dt}\mathrm{Det}\mathbf{J} = \mathrm{div}(\mathbf{u})\ \mathrm{Det}(\mathbf{J}). \tag{1.14}$$

Note that, since an incompressible fluid has $\mathrm{Det}(\mathbf{J}) = 1$, such a fluid must satisfy, by (1.14), $\mathrm{div}(\mathbf{u}) = 0$, which is the way an incompressible fluid is defined in Eulerian variables.

1.2.1 Solenoidal velocity fields

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The adjective *solenoidal* applied to a vector field is equivalent to “divergence-free”. We will used either $\mathrm{div}(\mathbf{u})$ or $\nabla\cdot\mathbf{u}$ to denote divergence. The incompressibility of a material with a solenoidal vector field means that the Lagrangian

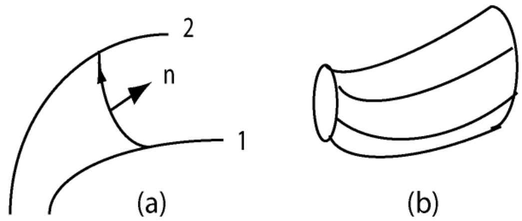

Figure 1.4: Solenoidal velocity fields. (a) Two streamlines in two dimensions. (b) A stream tube in three dimensions.

map $\mathcal{M}_t$ preserves volume and so whatever fluid moves into a region of space is matched by an equal amount of fluid moving out. In two dimensions the equation expressing the solenoidal condition is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.15}$$

If $\psi(x, y)$ posses continuous second derivatives we may satisfy (1.15) by setting

$$u = \frac{\partial \psi}{\partial y}. \quad v = -\frac{\partial \psi}{\partial x}. \tag{1.16}$$

The function ψ is called the *stream function* of the velocity field. The reason for the term is immediate: The instantaneous streamline passing through x, y has direction $(u(x, y), v(x, y))$ at this point. The normal to the streamline at this point is $\nabla\psi(x, y)$. But we see from (1.16) that $(u, v) \cdot \nabla\psi = 0$ there, so the lines of constant ψ are the instantaneous streamlines of (u, v).

Consider two streamlines $\psi = \psi_i, i = 1, 2$ and any oriented simple contour (no self-crossings) connecting one streamline to the other. The claim is then that the flux of fluid across this contour, from left to right seen by an observed facing in the direction of orientation of the contour is given by the difference of the values of the stream function, $\psi_2 - \psi_1$ if the contour is oriented to go from streamline 1 to streamline 2, see figure 1.4(a). Indeed, oriented as shown the line integral of flux is just $\int (u, v) \cdot (dy, -dx) = \int d\psi = \psi_2 - \psi_1$. In three dimensions, we similarly introduce a *stream tube*, consisting of a collection of streamlines, see figure (1.4)(b). The flux of fluid across any "face" cutting through the tube must be the same. This follows immediately by applying the divergence theorem to the integral of div $\mathbf{u}$ over the stream tube. Note that we are referring here to the flux of volume of fluid, not flux of mass.

In three dimensions there are various "stream functions" used when special symmetry allow them. An example of a class of solenoidal flows generated by two scalar functions is $\mathbf{u} = \nabla\alpha \times \nabla\beta$ where the intersections of the surfaces of constant $\alpha(x, y, z)$ and $\beta(x, y, z)$ are the streamlines. Since $\nabla\alpha \times \nabla\beta = \nabla \times (\alpha\nabla\beta)$ we see that these flows are indeed solenoidal.

1.2.2 The convection theorem

Suppose that $\mathcal{S}_t$ is a region of fluid particles and let $f(\mathbf{x},t)$ be a scalar function. Forming the volume integral over $\mathcal{S}_t$, $F = \int_{\mathcal{S}_t} f dV_{\mathbf{x}}$, we seek to compute $\frac{dF}{dt}$. Now $dV_{\mathbf{x}} = dx_1 \cdots dx_N = \mathrm{Det}(\mathbf{J}) da_1 \cdots a_N = \mathrm{Det}(\mathbf{J}) dV_{\mathbf{a}}$. Thus

$$\frac{dF}{dt} = \frac{d}{dt}\int_{\mathcal{S}_0} f(\mathbf{x}(\mathbf{a},t),t)\mathrm{Det}(\mathbf{J})dV_{\mathbf{a}} = \int_{\mathcal{S}_0} \mathrm{Det}(\mathbf{J})\frac{d}{dt} f(\mathbf{x}(\mathbf{a},t),t)dV_{\mathbf{a}}$$

$$+ \int_{\mathcal{S}_0} f(\mathbf{x}(\mathbf{a},t),t)\frac{d}{dt}\mathrm{Det}(\mathbf{J})dV_{\mathbf{a}} = \int_{\mathcal{S}_0} \Big[\frac{Df}{Dt} + f\mathrm{div}(\mathbf{u})\Big]\mathrm{Det}(\mathbf{J})dV_{\mathbf{a}},$$

and so

$$\frac{dF}{dt} = \int_{\mathcal{S}_t} \Big[\frac{Df}{Dt} + f\mathrm{div}(\mathbf{u})\Big] dV_{\mathbf{x}}. \tag{1.17}$$

The result (1.17) is called the *convection theorem*. We can contrast this calculation with one over a fixed finite region $\mathcal{R}$ of space with boundary $\partial\mathcal{R}$. In that case the rate of change of f contained in $\mathcal{R}$ is just

$$\frac{d}{dt}\int_{\mathcal{R}} f dV_{\mathbf{x}} = \int_{\mathcal{R}} \frac{\partial f}{\partial t} dV_{\mathbf{x}}. \tag{1.18}$$

The difference between the two calculations involves the *flux* of f through the boundary of the domain. Indeed we can write the convection theorem in the form

$$\frac{dF}{dt} = \int_{\mathcal{S}_t} \Big[\frac{\partial f}{\partial t} + \mathrm{div}(f\mathbf{u})\Big] dV_{\mathbf{x}}. \tag{1.19}$$

Using the divergence (or Gauss') theorem, and considering the instant when $\mathcal{S}_t = \mathcal{R}$, we have

$$\frac{dF}{dt} = \int_{\mathcal{R}} \frac{\partial f}{\partial t} dV_{\mathbf{x}} + \int_{\partial\mathcal{R}} f\mathbf{u}\cdot\mathbf{n} dS_{\mathbf{x}}, \tag{1.20}$$

where $\mathbf{n}$ is the outer normal to the region and $dS_{\mathbf{x}}$ is the area element of $\partial\mathcal{R}$. The second term on the right is flux of f out of the region $\mathcal{R}$. Thus the convection theorem incorporates into the change in f within a region, the flux of f into or out of the region, due to the motion of the boundary of the region. Once we identify f with a useful physical property of the fluid, the convection theorem will be useful for expressing the *conservation* of this property, see chapter 2.

1.2.3 Material vector fields: The Lie derivative

Certain vector fields in fluid mechanics, and notably the *vorticity field*, $\boldsymbol{\omega}(\mathbf{x},t) = \nabla \times \mathbf{u}$, see chapter 3, can in certain cases behave as a *material vector field*. To understand the concept of a material vector one must imagine the direction of the vector to be determined by nearby material points. It is wrong to think of a material vector as attached to a fluid particle and constant there. This would amount to a simple translation of the vector along the particle path.

Instead, we want the direction of the vector to be that of a differential segment connecting two nearby fluid particles, $dl_i = J_{ij} da_j$. Furthermore, the length of the material vector is to be proportional to this differential length as time evolves and the particles move. Consequently, once the particles are selected, the future orientation and length of a material vector will be completely determined by the Jacobian matrix of the flow.

Thus we define a material vector field as one of the form (in Lagrangian variables)

$$v_i(\mathbf{a}, t) = J_{ij}(\mathbf{a}, t) V_j(\mathbf{a}) \tag{1.21}$$

Of course, given the inverse $\mathbf{a}(\mathbf{x}, t)$ we can express v as a function of $\mathbf{x}, t$ to obtain its Eulerian structure.

We now determine the time rate of change of a material vector field following the fluid parcel. To obtain this we differentiate $v(\mathbf{a}, t)$ with respect to time for fixed $\mathbf{a}$, and develop the result using the chain rule:

$$\begin{aligned} \left.\frac{\partial v_i}{\partial t}\right|_{\mathbf{a}} &= \left.\frac{\partial J_{ij}}{\partial t}\right|_{\mathbf{a}} V_j(\mathbf{a}) = \frac{\partial u_i}{\partial a_j} V_j \\ &= \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial a_j} V_j = v_k \frac{\partial u_i}{\partial x_k}. \end{aligned} \tag{1.22}$$

Introducing the material derivative, we see that a material vector field satisfies the following equation in Eulerian variables:

$$\frac{D\mathbf{v}}{Dt} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{x}} + \mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{u} \equiv v_t + \mathcal{L}_{\mathbf{u}}\mathbf{v} = 0 \tag{1.23}$$

In differential geometry $\mathcal{L}_{\mathbf{u}}$ is called the Lie derivative of the vector field $\mathbf{v}$ with respect to the vector field $\mathbf{u}$.

The way this works can be understood by moving neighboring point along particle paths.

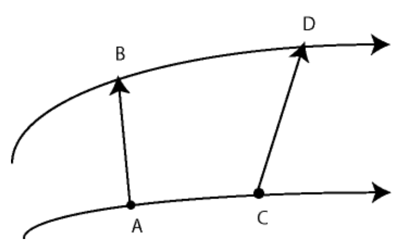

Figure 1.5: Computing the time derivative of a material vector.

Let $\mathbf{v} = \overline{AB} = \Delta\mathbf{x}$ be a small material vector at time t, see figure 1.5. At time Δtlater, the vector has become $\overline{CD}$. The curved lines are the particle paths through A, B of the vector field $\mathbf{u}(\mathbf{x}, t)$. Selecting A as $\mathbf{x}$, we see that after a small time interval Δt the point C is $A + \mathbf{u}(\mathbf{x}, t)\Delta t$ and D is the point $B + \mathbf{u}(\mathbf{x} + \Delta\mathbf{x}, t)\Delta t$. Consequently

$$\frac{\overline{CD} - \overline{AB}}{\Delta t} = \mathbf{u}(\mathbf{x} + \Delta\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t). \tag{1.24}$$

The left-hand side of (1.24) is approximately $D\mathbf{v}/Dt$, and right-hand side is approximately $\mathbf{v} \cdot \nabla \mathbf{u}$, so in the line $\Delta \mathbf{x}, \Delta t \to 0$ we get (1.23). A material vector field has the property that its magnitude can change by the stretching properties of the underlying flow, and its direction can change by the rotation of the fluid parcel.

Problem Set 1

1. Consider the flow in the (x, y) plane given by $u = -y, v = x + t$. (a) What is the instantaneous streamline through the origin at $t = 1$?(b) what is the path of the fluid particle initially at the origin, $0 < t < 6\pi$? (c) What is the streak line emanating form the origin, $0 < t < 6\pi$?

2. Consider the "point vortex " flow in two dimensions,

$$(u, v) = UL(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}),\ x^2 + y^2 \neq 0,$$

where U, L are reference values of speed and length. (a) Show that the Lagrangian coordinates for this flow may be written

$$x(a, b, t) = R_0 \cos(\omega t + \theta_0),\ y(a, b, t) = R_0 \sin(\omega t + \theta_0)$$

where $R_0^2 = a^2 + b^2, \theta_0 = \arctan(b/a)$, and $\omega = UL/R_0^2$. (b) Consider, at $t = 0$ a small rectangle of marked fluid particles determined by the points $A(L, 0), B(L + \Delta x, 0), C(L + \Delta x, \Delta y), D(L, \Delta y)$. If the points move with the fluid, once point A returns to its initial position what is the shape of the marked region? Since $(\Delta x, \Delta y)$ are small, you may assume the region remains a parallelogram. Do this, first, by computing the entry $\partial y/\partial a$ in the Jacobian, evaluated at $A(L, 0)$. Then verify your result by considering the "lag" of particle B as it moves on a slightly larger circle at a slightly slower speed, relative to particle A, for a time taken by A to complete one revolution.

3. As was noted in class, Lagrangian coordinates can use any unique labeling of fluid particles. To illustrate this, consider the Lagrangian coordinates in two dimensions

$$x(a, b, t) = a + \frac{1}{k} e^{kb} \sin k(a + ct),\ y = b - \frac{1}{k} e^{kb} \cos k(a + ct),$$

where k, c are constants. Note here a, b are *not* equal to (x, y) for any t_0 By examining the determinant of the Jacobian, verify that this gives a unique labeling of fluid particles provided that $b \neq 0$. What is the situation if $b = 0$?(These waves, which were discovered by Gerstner in 1802, represent gravity waves if $c^2 = g/k$ where g is the acceleration of gravity. They do not have any simple Eulerian representation. These waves are discussed in Lamb's book.)

4. In one dimension, the Eulerian velocity is given to be $u(x, t) = 2x/(1 + t)$.

(a) Find the Lagrangian coordinate $x(a,t)$. (b) Find the Lagrangian velocity as a function of a,t. (c) Find the Jacobean $\partial x/\partial a = J$ as a function of a,t.

5. For the stagnation-point flow $\mathbf{u} = (u,v) = U/L(x,-y)$, show that a fluid particle in the first quadrant which crosses the line $y = L$ at time $t = 0$, crosses the line $x = L$ at time $t = \frac{L}{U}\log(UL/\psi)$ on the streamline $Uxy/L = \psi$. Do this in two ways. First, consider the line integral of $\mathbf{u}\cdot\vec{ds}/(u^2+v^2)$ along a streamline. Second, use Lagrangian variables.

6. Let S be the surface of a deformable body in three dimension, and let $I = \int_S f\mathbf{n}dS$ for some scalar function f, $\mathbf{n}$ being the outward normal. Show that

$$\frac{d}{dt}\int f\mathbf{n}dS = \int_S \frac{\partial f}{\partial t}\mathbf{n}dS + \int_S (\mathbf{u}_b\cdot\mathbf{n})\nabla f dS. \tag{1.25}$$

(Hint: First convert to a volume integral between S and an outer surface S' which is *fixed*. Then differentiate and apply the convection theorem. Finally convert back to a surface integral.)

Chapter 2

Conservation of mass and momentum

2.1 Conservation of mass

Every fluid we consider is endowed with a non-negative *density*, usually denoted by ρ, which is in the Eulerian setting is a scalar function of $\mathbf{x}, t$. Its unit are mass per unit volume. Water has a density of about 1 gram per cubic centimeter. For air the density is about 10^{-3} grams per cubic centimeter, but of course pressure and temperature affect air density significantly. The air in a room of a thousand cubic meters= 10^9 cubic centimeters weighs about a thousand kilograms, or more than a ton!

2.1.1 Eulerian form

Let us suppose that mass is being added or subtracted from space as a function $q(\mathbf{x}, t)$, of dimensions mass per unit volume per unit time. The conservation of mass in a fixed region $\mathcal{R}$ can be expressed using (1.20) with $f = \rho$:

$$\frac{d}{dt}\int_{\mathcal{R}} \rho dV_{\mathbf{x}} = \int_{\mathcal{R}} \frac{\partial \rho}{\partial t} dV_{\mathbf{x}} + \int_{\partial \mathcal{R}} \rho \mathbf{u} \cdot \mathbf{n} dS_{\mathbf{x}}. \tag{2.1}$$

Now

$$\frac{d}{dt}\int_{\mathcal{R}} \rho dV_{\mathbf{x}} = \int_{\mathcal{R}} q dV_{\mathbf{x}} \tag{2.2}$$

and if we bring the surface integral in (2.1) back into the volume integral using the divergence theorem we arrive at

$$\int_{\mathcal{R}} \Big[\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{u}\rho) - q\Big] dV_{\mathbf{x}} = 0. \tag{2.3}$$

Since our functions are continuous and $\mathcal{R}$ is an arbitrary open set in R_N, the integrand in (2.3) must vanish, yielding the conservation of mass equation in the Eulerian form:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{u}\rho) = q. \tag{2.4}$$

Note that this last equation can also be written

$$\frac{D\rho}{Dt} + \rho \mathrm{div}\ \mathbf{u} = q. \tag{2.5}$$

The conservation of mass equation in either of these forms is sometimes called (for obscure reasons) the *equation of continuity.*

The form (2.5) shows that the material derivative of the density changes in two ways, either by sources and sinks of mass $q > 0$ or $q < 0$ respectively, or else by the non-vanishing of the divergence of the velocity field. A positive value of the divergence, as for $\mathbf{u} = (x, y, z)$, is associated with an expansive flow, thereby reducing local density. This can be examined more closely as follows. Let V be a small volume of fluid where the density is essentially constant. Then ρV is the mass within this fluid parcel, which is a material invariant $D(\rho V)/Dt = 0$. Thus $D\rho/Dt + \rho V^{-1} DV/Dt = 0$. Comparing this with (2.5) we have

$$\mathrm{div}\ \mathbf{u} = \frac{1}{V}\frac{DV}{Dt}. \tag{2.6}$$

Example 2.1: As we have seen in Chapter 1, an incompressible fluid satisfies div $\mathbf{u} = 0$. For such a fluid, free of sources or sinks of mass, we have

$$\frac{D\rho}{Dt} = 0, \tag{2.7}$$

that is, now density becomes a material property. This does not say that the density is constant everywhere in space, only that is constant at a given fluid parcel, as it moves about . (Note that we use parcel here to suggest that we have to average over a small volume to compute the density.) However a fluid of constant density without mass addition *must* be incompressible. This difference is important. Sea water is essentially incompressible but density changes due to salinity are an important part of the dynamics of the oceans.

2.1.2 Lagrangian form

If $q = 0$ the Lagrangian form of the conservation of mass is very simple because if we move with the fluid the density changes that we see are due to expansion and dilation of the fluid parcel, which is controlled by Det($\mathbf{J}$). Let a parcel have volume V_0 initially, with essentially constant initial density ρ_0. Then the mass of the parcel is $\rho_0 V_0$, and is a material invariant. At later times the density is ρ and the volume is V_0Det($\mathbf{J}$), so conservation of mass is expressed by

$$\mathrm{Det}\mathbf{J}(\mathbf{a}, t) = \frac{\rho_0}{\rho}. \tag{2.8}$$

If $q \neq 0$ the Lagrangian conservation of mass must be written

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{a}} \rho \mathrm{Det}(\mathbf{J}) = \mathrm{Det}(\mathbf{J}) q(\mathbf{x}(\mathbf{a}, t), t). \tag{2.9}$$

It is easy to get from Eulerian to Lagrangian form using (1.14). Assuming $q = 0$,

$$\frac{D\rho}{Dt} + \rho \text{ div } \mathbf{u} = 0 = \frac{D\rho}{Dt} + \rho \frac{D\text{Det}(\mathbf{J})/Dt}{\text{Det}(\mathbf{J})} = \frac{1}{\text{Det}(\mathbf{J})}\frac{D}{Dt}(\rho\text{Det}(\mathbf{J})) \tag{2.10}$$

and the connection is complete.

Example 2.2: Consider, in one dimension, the unsteady velocity field $u(x,t) = \frac{2xt}{1+t^2}$. We assume no sources of sinks of mass, and set $\rho(x,0) = x$. What is the density field at later times, in both Eulerian and Lagrangian forms? First note that this is a reasonable question, since we have a conservation of mass equation to evolve the density in time. First deriving the Lagrangian coordinates, we have

$$\frac{dx}{dt} = \frac{2xt}{1+t^2}, \quad x(0) = a. \tag{2.11}$$

The solution is $x = a(1+t^2)$. The Jacobian is then $J = 1+t^2$. The equation of conservation of mass in Lagrangian form, given that $\rho_0(a) = a$, is $\rho = a/(1+t^2)$. Since $a = x/(1+t^2)$, the Eulerian form of the density is $\rho = x(1+t^2)^2$. It is easy to check that this last expression satisfies the Eulerian conservation of mass equation in one dimension $\rho_t + (\rho u)_x = 0$.

Example 2.3 Consider the two-dimensional stagnation-point flow $(u,v) = (x,-y)$ with initial density $\rho_0(x,y) = x^2+y^2$ and $q = 0$. The flow is incompressible, so ρ is material. In Lagrangian form, $\rho(a,b,t) = a^2+b^2$. To find ρ as a function of x,y,t, we note that the Lagrangian coordinates of the flow are $(x,y) = (ae^t, be^{-t})$, and so

$$\rho(x,y,t) = (xe^{-t})^2 + (ye^t)^2 = x^2e^{-2t} + y^2e^{2t}. \tag{2.12}$$

The lines of constant density, which are initially circles centered at the origin, are flattened into ellipses by the flow.

2.1.3 Another convection identity

Frequently fluid properties are most conveniently thought of as densities per unit mass rather than per unit volume. If the conservation such a quantity, f say, is to be examined, we will need to consider ρf to get "f per unit volume" and so be able to compute total amount by integration over a volume. Consider then

$$\frac{d}{dt}\int_{\mathcal{S}_t} \rho f dV_{\mathbf{x}} = \int_{\mathcal{S}_t} \left[\frac{\partial \rho f}{\partial t} + \text{div}(\rho f \mathbf{u})\right] dV_{\mathbf{x}}. \tag{2.13}$$

We now assume conservation of mass with $q = 0$. From the product rule of differentiation we have $\text{div}(\rho f\mathbf{u}) = f\text{div}(\rho\mathbf{u}) + \rho\mathbf{u}\cdot\nabla f$, and so the integrand splits into a part which vanishes by conservation of mass, and a material derivative of f time the density:

$$\frac{d}{dt}\int_{\mathcal{S}_t} \rho f dV_{\mathbf{x}} = \int_{\mathcal{S}_t} \rho \frac{Df}{Dt} dV_{\mathbf{x}}. \tag{2.14}$$

Thus the effect of the multiplier ρ is to turn the derivative of the integral into an integral of a material derivative.

2.2 Conservation of momentum in an ideal fluid

The *momentum* of a fluid is defined to be $\rho\mathbf{u}$, per unit volume. Newton's second law of motion states that momentum is conserved by a mechanical system of masses if no forces act on the system. We are thus in a position to use (2.14), where the "sources and sinks" of momentum are *forces*.

If $\mathbf{F}(\mathbf{x}, t)$ is the force acting on the fluid, per unit volume, then we have immediately (assuming conservation of mass with $q = 0$),

$$\rho\frac{D\mathbf{u}}{Dt} = \mathbf{F}. \tag{2.15}$$

Since we have seen that $\frac{D\mathbf{u}}{Dt}$ is the fluid acceleration, (2.15) states Newton's Law that mass times acceleration equals force, in both magnitude and direction.

Of course the Lagrangian form of (2.15) is obtained by replacing the acceleration by its Lagrangian counterpart:

$$\rho\frac{\partial^2\mathbf{x}}{\partial t^2}\Big|_{\mathbf{a}} = \mathbf{F}. \tag{2.16}$$

The main issues involved with conservation of momentum are those connected with the forces which are on a parcel of fluid. There are many possible forces to consider: pressure, gravity, viscous, surface tension, electromotive, etc. Each has a physical origin and a mathematical model with a supporting set of observation and analysis. In the present chapter we consider only an *ideal fluid*. The only new fluid variable we will need to introduce is the *pressure*, a scalar function $p(\mathbf{x}, t)$.

In general the force $\mathbf{F}$ appearing in (2.15) is assumed to take the form

$$F_i = f_i + \frac{\partial\sigma_{ij}}{\partial x_j}. \tag{2.17}$$

Here $\mathbf{f}$ is a body force (exerted from the "outside"), and σ is a second-order tensor called the *stress tensor*. Integrated over a region $\mathcal{R}$, the force on the region is

$$\int_{\mathcal{R}} \mathbf{F} dV_{\mathbf{x}} = \int_{\mathcal{R}} \mathbf{f} dV_{\mathbf{x}} + \int_{\partial\mathcal{R}} \sigma \cdot \mathbf{n} dS_{\mathbf{x}}, \tag{2.18}$$

using the divergence theorem. We can thus see that the effect of the stress tensor is to produce a force on the boundary of any fluid parcel, the contribution from an area element to this force being $\sigma_{ij}n_j dS_{\mathbf{x}}$ for an outward normal $\mathbf{n}$. The remaining body force $\mathbf{f}$ will sometimes be taken to be a uniform gravitational field $\mathbf{f} = \rho\mathbf{g}$, where $\mathbf{g}$ = constant. On the surface of the earth gravity acts toward the Earth's center with a strength $g \approx 980\ cm/sec^2$. We also introduce a general force potential Φ, such that $\mathbf{f} = -\rho\nabla\Phi$.

2.2.1 The pressure

An ideal fluid is defined by a stress tensor of the form

$$\sigma_{ij} = -p\delta_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, \tag{2.19}$$

where $\delta_{ij} = 1, i = j, = 0$ otherwise. Thus when pressure is positive the force on the surface of a parcel is opposite to the outer normal, as intuition suggests. Note that now

$$\text{div } \sigma = -\nabla p. \tag{2.20}$$

For a compressible fluid the pressure accounts physically for the resistance to compression. But pressure persists as a fundamental source of surface forces for an incompressible fluid, and its physical meaning in the incompressible case is subtle.[1]

An ideal fluid with no mass addition and no body force thus satisfies

$$\rho\frac{D\mathbf{u}}{Dt} + \nabla p = 0, \tag{2.21}$$

together with

$$\frac{D\rho}{Dt} + \rho\text{div } \mathbf{u} = 0. \tag{2.22}$$

This system of equation for an ideal fluid are also often referred to as *Euler's equations*. The term *Euler flow* is also in wide use.

With Euler's system we have $N + 1$ equations for the $N + 2$ unknowns $u_1, \ldots, u_N, \rho, p$. Another equation will be needed to complete the system. One possibility is the incompressible assumption div $\mathbf{u} = 0$. A common option is to assume constant density. Then ρ is eliminated as an unknown and the conservation of mass equation is replaced by the incompressibility condition. For gases the missing relation is an equation of state, which brings in the thermodynamic properties of the fluid.

The pressure force as we have defined it above is *isotropic*, in the sense the pressure is the same independently of the orientation of the area element on which it acts. A simple two-dimensional diagram will illustrate why this is so, see figure 2.1. Suppose that the pressure is p_i on the face of length L_i. Equating forces, we have $p_1L_1\cos\theta = p_2L_2, p_1L_1\sin\theta = p_3L_3$. But $L_1\cos\theta = L_2, L_1\sin\theta = L_3$, so we see that $p_1 = p_2 = p_3$. So indeed the pressure sensed by a face does not depend upon the orientation of the face.

2.2.2 Lagrangian form of conservation of momentum

The Lagrangian form of the acceleration has been noted above. The momentum equation of an ideal fluid requires that we express ∇p as a Lagrangian variable.

[1] One aspect of the incompressible case should be noted here, namely that the pressure is arbitrary up to an additive constant. Consequently it is only pressure *differences* which matter. This is not the case for a compressible gas.

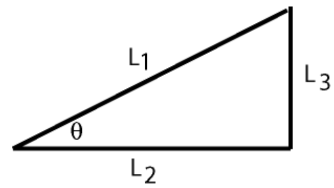

Figure 2.1: Isotropicity of pressure.

That is, if p is to be a function of $\mathbf{a}, t$ then since ∇ here is actually the $\mathbf{x}$ gradient $\nabla_{\mathbf{x}}$, we have $\nabla_{\mathbf{x}} p = \mathbf{J}^{-1}\nabla_{\mathbf{a}} p$. This appearance of the Jacobian is an awkward feature of Lagrangian fluid dynamics, and is one of the reasons that we shall emphasize Eulerian variables in discussing the dynamics of a fluid.

2.2.3 Hydrostatics: the Archimedean principle

Hydrostatics is concerned with fluids at rest ($\mathbf{u} = 0$), usually in the presence of gravity. We consider here only the case of a fluid stratified in one dimension. To fix the coordinates let the z-axis be vertical up, and $\mathbf{g} = -g\mathbf{i}_z$, where g is a positive constant. We suppose that the density is a function of z alone. This allows, for example, a body of water beneath a stratified atmosphere. Let a solid three-dimensional body (any deformation of a sphere for example) be submerged in the fluid. Archimedes principle says that the force exerted by the pressure on the surface of the body is equal to the total weight of the fluid displaced by the body. We want to establish this principle in the case considered.

Now the pressure satisfies $\nabla p = -g\rho(z)\mathbf{i}_z$. The pressure force is given by $\mathbf{F}_{pressure} = -\int p\mathbf{n}dS$ taken over the surface of the body. But this surface pressure is just the same as would be acting on a virtual surface within the fluid, no body present. Using the divergence theorem, we may convert this to an integral over the interior of this surface. Of course, there is no fluid within the body. We are just using the math to evaluate the surface integral. The result is $\mathbf{F}_{pressure} = g\mathbf{i}_z \int \rho dV$. This is a force upward equal to the weight of the displaced fluid, as stated.

2.3 Steady flow of a fluid of constant density

This special case gives us an opportunity to obtain some useful results rather easily in a class of problems of some importance. We shall allow a body force of the form $\mathbf{f} = -\rho\nabla\Phi$, so the momentum equation may be written, after division by the constant density,

$$\mathbf{u} \cdot \nabla\mathbf{u} + \rho^{-1}\nabla p + \nabla\Phi = 0. \tag{2.23}$$

We note now a vector identity which will be useful:

$$\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{B} \cdot \nabla\mathbf{A} = \nabla(\mathbf{A} \cdot \mathbf{B}). \tag{2.24}$$

Applying this to $\mathbf{A} = \mathbf{B} = \mathbf{u}$ we have

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.25}$$

Using (2.25) in (2.23) we have

$$\nabla(\rho^{-1}p + \Phi + \frac{1}{2}|\mathbf{u}|^2) = \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.26}$$

Taking the dot product with $\mathbf{u}$ on both sides we obtain

$$\mathbf{u} \cdot \nabla(\rho^{-1}p + \Phi + \frac{1}{2}|\mathbf{u}|^2) = 0. \tag{2.27}$$

The famous *Bernoulli theorem* for steady flows follows:*In the steady flow of an ideal fluid of constant density the quantity $H \equiv \rho^{-1}p + \Phi + \frac{1}{2}|\mathbf{u}|^2$, called the Bernoulli function, is constant on the streamlines of the flow.* The importance of this result is in the relation it gives us between velocity and pressure. Apart from the contribution of Φ, the constancy of H implies that an increase of velocity is accompanied by a decrease of the pressure. This is not an obvious dynamical consequence of the equations of motion, and it is interesting that we have derived it without referring to the solenoidal property of $\mathbf{u}$. Recall that the latter is implied by the constancy of density when there is no mass added or removed. If we make use of the solenoidal property then, using the identity $\nabla \cdot (\mathbf{A}\psi) = \psi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\phi$ for vector and scaler fields, we see that $\mathbf{u}H$ is also solenoidal, and so the flux of this quantity is conserved in stream tubes. This vector field arises when conservation of *mechanical energy*, relating changes in kinetic energy to the work done by forces, is studied, see problem 2.2.

It is helpful to apply the Bernoulli theorem to flow in a smooth rigid pipe of circular cross section and slowly varying diameter, with $\Phi = 0$. For an ideal (frictionless) fluid we may assume that the velocity is approximately constant over the section, this being reasonable if the slope of the wall of the pipe is small. The velocity may thus be taken as a scalar function $u(x)$. If the section area is $A(x)$, then the conservation of mass (and here, volume) implies that $uA \equiv Q =$ constant, so that $\rho^{-1}p + \frac{Q^2}{2}A^{-2} =$ constant. If we consider a contraction, as in figure 2.2., where the area and velocity go from A_1, u_1 to A_2, u_2, then the fluid speeds up to satisfy $A_1u_1 = A_2u_2 = Q$. To achieve this speedup in steady flow, a force must be acting on the fluid, here a pressure force. Conservation of momentum states the flux of momentum out minus the flux of momentum in must equal the pressure force on the fluid in the pipe between section 1 and section 2. Now $H = p/\rho + \frac{1}{2}(Q/A)^2$ is constant, so (if force is positive to the right) the two ends of the tube give a net pressure force $p_1A_1 - p_2A_2 = \rho Q^2/2(1/A_2 - 1/A_1)$ acting on the fluid. But there is also a pressure force along the curved part of the tube. This is seen to be $\int_{A_1}^{A_2} p dA = -\int_{A_1}^{A_2} \frac{\rho}{2}(Q/A)^2 dA = \rho Q^2/2(1/A_2 - 1/A_1)$. These two contributions are equal in our one-dimensional approximation, and their sum is $\rho Q^2(1/A_2 - 1/A_1)$ But the momentum out minus momentum in is $\rho(A_2u_2^2 - A_1u_1^2) = \rho Q^2(1/A_2 - 1/A_1)$ and is indeed

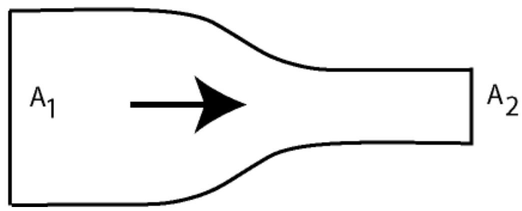

Figure 2.2: Steady flow through a contraction.

equal to the net pressure force. Intuitively then, to achieve the speedup of the fluid necessary to force the fluid through a contraction, and to maintain such a flow as steady in time, it is necessary to supply a larger pressure at station 1 than at station 2. Bernoulli's theorem captures this creation of momentum elegantly, but ultimately the physics comes down to pressure differences accelerating fluid parcels.

2.4 Intrinsic coordinates in steady flow

The one-dimensional analysis just given suggests looking briefly at the relations obtained in an arbitrary steady flow of an ideal fluid using the streamlines a part of the coordinate system. The resulting *intrinsic coordinates* are revealing of the dynamics of fluid parcels. Let $\mathbf{t}$ be the unit tangent vector to an oriented streamline. The we may write $\mathbf{u} = q\mathbf{t}, q = |\mathbf{u}|$. If s is arclength along the streamline, then

$$\frac{\partial \mathbf{u}}{\partial s} = \frac{\partial q}{\partial s}\mathbf{t} + q\frac{\partial \mathbf{t}}{\partial s} = \frac{\partial q}{\partial s}\mathbf{t} + q\kappa\mathbf{n}, \tag{2.28}$$

where $\mathbf{n}$ is the unit normal, κ the streamline curvature, and we have used the first Frenet-Serret formula. Now the operator $\mathbf{u} \cdot \nabla$ is just $q\frac{\partial}{\partial s}$, and so we have from (2.28)

$$\mathbf{u} \cdot \nabla \mathbf{u} = q\frac{\partial q}{\partial s}\mathbf{t} + q^2\kappa\mathbf{n}. \tag{2.29}$$

This shows that the acceleration in steady flow splits into a component along the streamline, determined by the variation of q, and a centripetal acceleration associated with streamline curvature. The equations of motion in intrinsic coordinates (zero body force) are therefore

$$\rho q\frac{\partial q}{\partial s} + \frac{\partial p}{\partial s} = 0, \quad \rho\kappa q^2 + \frac{\partial p}{\partial n} = 0. \tag{2.30}$$

What form does the solenoidal condition take in intrinsic coordinates? We consider this question in two dimensions. We have

$$\nabla \cdot \mathbf{u} = \nabla \cdot (q\mathbf{t}) = \mathbf{t} \cdot \nabla q + q\nabla \cdot \mathbf{t} = \frac{\partial q}{\partial s} + q\nabla \cdot \mathbf{t}. \tag{2.31}$$

Let us introduce an angle θ so that $\mathbf{t}(s) = (\cos\theta(s), \sin\theta(s))$. Then

$$\nabla \cdot \mathbf{t} = -\sin\theta\frac{\partial \theta}{\partial x} + \cos\theta\frac{\partial \theta}{\partial y} = \mathbf{n} \cdot \nabla\theta = \frac{\partial \theta}{\partial n}. \tag{2.32}$$

Since $\kappa = \frac{\partial \theta}{\partial s}$ is the streamline curvature, $\frac{\partial \theta}{\partial n}$, which we write as κ_n, is the curvature of the coordinate lines normal to the streamlines. Thus the solenoidal condition in two dimensions assumes the form

$$\frac{\partial q}{\partial s} + q\kappa_n = 0. \tag{2.33}$$

2.5 Potential flows with constant density

Another important and very large class of fluid flows are the so-called potential flows, defined as flows having a velocity field which is the gradient of a scalar *potential*, usually denoted by ϕ:

$$\mathbf{u} = \nabla\phi. \tag{2.34}$$

For simplicity we consider here only the case of constant density, but allow a body force $-\rho\nabla\Phi$ and permit the flow to be unsteady. Since we now also have that $\mathbf{u}$ is solenoidal, it follows that

$$\nabla \cdot \nabla\phi = \nabla^2\phi = 0. \tag{2.35}$$

Thus the velocity field is determined by solving Laplace's equation (2.35)

The momentum equation has not yet been needed, but it necessary in order to determine the pressure, given $\mathbf{u}$. The momentum equation is

$$\mathbf{u}_t + \nabla(\frac{1}{2}|\mathbf{u}|^2 + p/\rho + \Phi) = \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{2.36}$$

Since $\mathbf{u} = \nabla\phi$ we now have $\nabla \times \mathbf{u} = 0$ and therefore

$$\nabla(\phi_t + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \Phi) = 0, \tag{2.37}$$

or

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \Phi = h(t). \tag{2.38}$$

The arbitrary function $h(t)$ may in fact be set equal to zero; otherwise we can replace ϕ by $\phi - \int h dt$ without affecting $\mathbf{u}$. We see that (2.38) is another "Bernoulli constant", this time applicable to any connected region of space where the potential flow is defined. It allows us to compute the pressure in an unsteady potential flow, see problem 2.6.

2.6 Boundary conditions on an ideal fluid

As we have noted, a main physical property of real fluid which is not present for an ideal fluid is a viscosity. The ideal fluid is“slippery”, in the following sense. Suppose that adjacent to a solid wall the pressure varies along the wall. The only force a fluid parcel can experience is a pressure force associated with the pressure gradient. If the gradient at the wall is tangent to the wall, fluid will be accelerated and there will have to be a tangential component of velocity *at the wall.* This suggests that we cannot place any restriction on the tangential component of velocity at a rigid fixed boundary of the fluid.

On the other hand, by a rigid fixed wall we mean that fluid is unable to penetrate the wall, and so we will have to impose the condition $\mathbf{n}\cdot\mathbf{u} = u_n = 0$ on the wall. There is a subtlety here connected with our continuum approximation. It might be though that the fluid cannot penetrate *into* a rigid wall, but could it not be possible for the fluid to tear off the wall, forming a free interface next to an empty cavity? to see that this cannot be the case for smooth pressure fields, consider the reversed stagnation-point flow $(u, v) = (-x, y)$. On the upper y-axis we have a Bernoulli function $p/\rho + \frac{1}{2}y^2$. The gradient of pressure along this line is indeed accelerating the fluid away from the wall, but the fluid remains at rest at $x = y = 0$. We cannot really contemplate a pressure force on a particle, which might cause the particle to leave the wall, only on a parcel. In fact in this example fluid parcels near the y-axis are being compressed in the x-direction and stretched in the y-direction.

Thus, the appropriate boundary condition at a fixed rigid wall adjacent to an ideal fluid is

$$u_n = 0 \quad \text{on the wall.} \tag{2.39}$$

For a potential flow, this becomes

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on the wall.} \tag{2.40}$$

We shall find that these conditions at a rigid wall for an ideal fluid are sufficient to (usually uniquely) determine fluid flows in problems of practical importance.

Another way to express the appropriate boundary condition on a ideal fluid at a rigid wall is that *fluid particles on a wall stay on the wall.* This alternative is attractive because it is also true of a *moving* rigid wall, where the velocity component normal to the wall need not vanish at the wall. So what is the appropriate condition on a moving wall? To obtain this it is convenient to define the surface as a function of time by the equation $\Sigma(\mathbf{x}, t) = 0$. For a particle at position $\mathbf{x}_p(t)$ to be on the surface means that $\Sigma(\mathbf{x}_p(t), t) = 0$. Differentiating this expression with respect to time we obtain

$$\left.\frac{\partial\Sigma}{\partial t}\right|_{\mathbf{x}} + \mathbf{u}\cdot\nabla\Sigma = 0. \tag{2.41}$$

For example, let a rigid cylinder of radius a move in the x-direction with velocity U. Then $\Sigma = (x - Ut)^2 + y^2 - a^2$, and (2.41) becomes $-2U(x - Ut) + 2(x -$

$Ut)u + 2yv = 0$ Evaluating this on the surface of the cylinder, we get

$$u\cos\theta + v\sin\theta = U\cos\theta = u_n. \tag{2.42}$$

We remark that the same reasoning can be applied to the moving *interface* between two fluids. This interface may also be regarded as consisting of fluid particles that remain on the interface. We refer to this generalized boundary condition at a moving surface as a *surface condition*.

Finally, as part of this first look at the boundary condition of fluid dynamics, we should note that for unsteady fluid flows we will sometimes need to prescribe *initial conditions*, insuring that the fluid equations may be used to carry the solution forward in time.

Example 2.4: We consider an example of potential flow past a body in two dimensions, constant density, no body force. The body is the circular cylinder $r = a$, and the fluid " at infinity" has fixed velocity $(U, 0)$ In two dimensional polar coordinates, Laplace's equation has solutions of the form $\ln r, (r^n, r^{-n})(\cos\theta, \sin\theta),\ n = 1, 2, \ldots$. The potential $Ur\cos\theta = Ux$ has the correct behavior at infinity, and so we need a decaying solution which will insure the boundary condition $\frac{\partial\phi}{\partial r} = 0$ when $r = a$. The correct choice is clearly a multiple of $r^{-1}\cos\theta$ and we obtain

$$\phi = U\cos\theta(r + a^2/r) \tag{2.43}$$

Note that $U\cos a^2/r$ is the potential of a flow seen by an observer at rest relative to the fluid at infinity, when the cylinder moves relative to the fluid with a velocity $(-U, 0)$. We see that indeed this potential satisfies $\frac{\partial\phi}{\partial r}\big|_{r=a} = -U\cos\theta$ as required by (2.42). Streamlines both inside and outside the cylinder are shown in figure 2.3.

We have found a solution representing the desired flow, but is the solution unique? Perhaps surprisingly, the answer is no. The reason, associated with the fluid region being non-simply connected, will be discussed in chapter 4.

Example 2.5 An interesting case of unsteady potential flow occurs with deep water waves (constant density). The fluid at rest is a liquid in the domain $z < 0$ of R^3. Gravity acts downward so $\Phi = -gz$. The space above is taken as having no density and a uniform pressure p_0. If the water is disturbed, waves can form on the surface, which we will assume to be described by a function $z = Z(x, y, t)$ (no breaking of waves). Under appropriate initial conditions it turns out that we may assume the liquid velocity to be a potential flow. Thus our mathematical problem is to solve Laplace's equation in $z < Z(x, y, t)$ with a surface condition on ϕ and a pressure condition $p_{z=Z} = p_0$. For the latter we can use the Bernoulli theorem for unsteady potential flows, to obtain

$$p_0/\rho = \Big[-\phi_t - \frac{1}{2}|\nabla\phi|^2 + gz\Big]_{z=Z}. \tag{2.44}$$

The surface condition is $\frac{D}{Dt}(z - Z(x, y, t)) = 0$ or

$$\Big[z - Z_t - uZ_x - vZ_y\Big]_{z=Z} = 0. \tag{2.45}$$

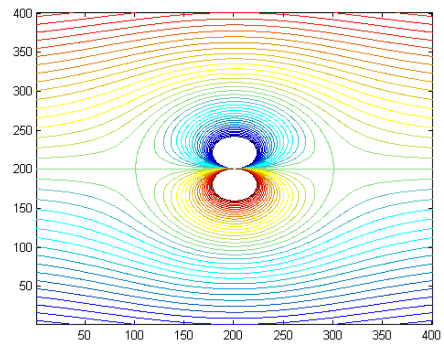

Figure 2.3: Potential flow past a circular cylinder.

The object is to find $\phi(x, y, z, t), Z(x, y, t)$, given e.g. that the water is initially at rest and that the fluid surface is at an initial elevation $z = Z_0(x, y)$. We will consider water waves in more detail in Chapter 9.

Problem set 2

1. For potential flow over a circular cylinder as discussed in class, with pressure equal to the constant p_∞ at infinity , find the static pressure on surface of the cylinder as a function of angle from the front stagnation point. (Use Bernoulli's theorem.) Evaluate the drag force (the force in the direction of the flow at infinite which acts on the cylinder), by integrating the pressure around the boundary. Verify that the drag force vanishes. This is an instance of *D'Alembert's paradox*, the vanishing of drag of bodies in steady potential flow.

2. For an ideal inviscid fluid of constant density, no gravity, the conservation of mechanical energy is studied by evaluating the time derivative of total kinetic energy in the form

$$\frac{d}{dt}\int_D \frac{1}{2}\rho|\mathbf{u}|^2 dV = \int_{\partial D} \mathbf{F}\cdot\mathbf{n}dS.$$

Here D is an arbitrary fixed domain with smooth boundary ∂D. What is the vector $\mathbf{F}$? Interpret the terms of $\mathbf{F}$ physically.

3. An open rectangular vessel of water is allowed to slide freely down a smooth frictionless plane inclined at an angle α to the horizontal, in a uniform vertical gravitational field of strength g. Find the inclination to the horizontal of the free surface of the water, given that it is a surface of constant pressure. We assume the fluid is at rest relative to an observer riding on the vessel. (Consider the acceleration of the fluid particles in the water and balance this against the gradient of pressure.)

4. Water (constant density) is to be pumped up a hill (gravity $= (0, 0, -g)$) through a pipe which tapers from an area A_1 at the low point to the smaller area A_2 at a point a vertical distance L higher. What is the pressure p_1 at the bottom, needed to pump at a volume rate Q if the pressure at the top is the atmospheric value p_0? (Express in terms of the given quantities. Assuming inviscid steady flow, use Bernoulli's theorem with gravity and conservation of mass. Assume that the flow velocity is uniform across the tube in computing fluid flux and pressure.)

5. For a *barotropic fluid*, pressure is a function of density alone, $p = p(\rho)$. In this case derive the appropriate form of Bernoulli's theorem for steady flow without gravity. If $p = k\rho^\gamma$ where γ, k are positive constants, show that $q^2 + \frac{2\gamma}{\gamma-1}\frac{p}{\rho}$ is constant on a streamline, where $q = |\mathbf{u}|$ is the speed.

6. Water fills a truncated cone as shown in the figure. Gravity acts down (the direction $-z$). The pressure at the top surface, of area A_2 is zero. The height of the container is H. At $t = 0$ the bottom, of area $A_1 < A_2$, is abruptly removed and the water begins to fall out. Note that at time $t = 0+$ the pressure at the bottom surface is also zero. The water has not moved but the acceleration is non-zero. We may assume the resulting motion is a potential flow. Thus the potential

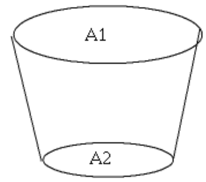

Figure 2.4: Truncated cone of fluid

$\phi(z, r, t)$ in cylindrical polars has the Taylor series $\phi(r, z, t) = t\Phi(r, z) + O(t^2)$, so $d\phi/dt = \Phi(r, z) + O(t)$. Using these facts, set up a mathematical problem for determining the pressure on the inside surface of cone at $t = 0+$. You should specify all boundary conditions. You do not have to solve the resulting problem, but can you guess what the surfaces Φ =constant would look like qualitatively? What is the force felt at $t = 0+$ by someone holding the cone, in the limits $A_1 \to 0$ and $A_1 \to A_2$?

Chapter 3

Vorticity

We have already encountered the vorticity field in the identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{3.1}$$

The vorticity field $\boldsymbol{\omega}(\mathbf{x}, t)$ is defined from the velocity field by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{3.2}$$

A potential flow is a flow with zero vorticity. The term *irrotational flow* is widely used. According to (3.1) the contribution to the acceleration coming from the gradient of velocity can be split into two components, one having a potential $\frac{1}{2}|\mathbf{u}|^2$, the other given as a cross product orthogonal to both the velocity and the vorticity. The latter component in older works in fluid dynamics has been called the *vortex force*.

We remark that, in analogy with stream lines, we shall refer to the flow lines of the vorticity field, i.e. the integral curves of the system

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}, \tag{3.3}$$

as (instantaneous) *vortex lines.* Similarly, in analogy with a stream tube in three dimensions, we will refer to a bundle of vortex lines a *vortex tube.*

This straightforward definition of the vorticity field gives little insight into its importance, either physically and theoretically. This chapter will be devoted to examining the vorticity field from a variety of viewpoints.

3.1 Local analysis of the velocity field

The first thing to be noted is that vorticity is fundamentally an Eulerian property since it involves spatial derivatives of the Eulerian velocity field. In a sense the analytical structure of the flow is being expanded to include the first derivatives of the velocity field. Suppose we expand the velocity field in a Taylor series about the fixed point $\mathbf{x}$:

$$u_i(\mathbf{x}+\mathbf{y},t) = u_i(\mathbf{x},t) + y_j \frac{\partial u_i}{\partial x_j}(\mathbf{x},t) + O(|\mathbf{y}|^2). \tag{3.4}$$

We can make the division

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big] + \frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big]. \tag{3.5}$$

The term first term on the right, $\frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big]$, is often denoted by e_{ij} and is the *rate-of-strain tensor* of the fluid. Here it will play a basic role when viscous stresses are considered (Chapter 5). The second term, $\frac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big]$, can be seen to be, in three dimensions, the matrix

$$\mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{3.6}$$

Example 3.1: In two dimensions, since u, v depend only on x, y, only one component of the vorticity is non-zero, $\omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. This is usually written simply as the scalar ω. Consider the two-dimensional flow $(u,v) = (y,0)$. In this case

$$\mathbf{e} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.7}$$

and $\omega = -1$. This is a simple "shear flow" with horizontal particle paths. Both $\mathbf{e}$ and $\mathbf{\Omega}$ are non-vanishing.

Example 3.2: Consider the flow $(u,v) = (-y,x)$. This is a simple solid-body rotation in the anti-clockwise sense. The vorticity is $\omega = 2$, and $\mathbf{e} = 0$.

These examples are a bit atypical because the vorticity is constant, but they emphasize that a close association of the vorticity with fluid rotation, a connection suggested by the skew-symmetric form of $\mathbf{\Omega}$, can be misleading.

Vorticity is a *point* property, but can only be defined by the limit operations implicit in the needed derivatives. So it is impossible to attach a physical meaning to "the vorticity of a particle". We *can* truncate (3.4) and consider the Lagrangian paths of fluid particles near $\mathbf{x}$. Since $\mathbf{e}$ is real symmetric, it may be diagonalized by a rotation to principal axes. Let the eigenvalues along the diagonal be $\lambda_i, i = 1,2,3$. We the may assume our coordinate system is such that $\mathbf{e}$ is the diagonal matrix $\mathbf{D}(\mathbf{x})$. Then the Lagrangian coordinates of the perturbed path $\mathbf{y}$ satisfies

$$\mathbf{y}_t = \mathbf{D}(\mathbf{x})\mathbf{y} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{y}. \tag{3.8}$$

These equations couple together the rotation associated with the vorticity at $\mathbf{x}$ with the straining field described by the first term. Note that the angular velocity associated with second term is $\frac{1}{2}\boldsymbol{\omega}$. The statement " vorticity at $\mathbf{x}$ equals twice the angular velocity of the fluid at $\mathbf{x}$" is often heard. But this statement in fact makes no sense, since an angular velocity cannot be attributed to a point. *Given the velocity field of a fluid, one can determine the effects of vorticity on the fluid only on a small open set, i.e. a fluid parcel.*

On the other hand it is true that when vorticity is sufficiently large there is sensible rotation observed in the fluid, and it *is* true that when one sees "rotation" in the fluid, then vorticity is present. In a sense this is the key to understanding its role, since it forces a definition of "rotation" in a fluid.

3.2 Circulation

Let C be a simple, smooth, oriented closed contour which is a deformation of a circle, hence the boundary of an oriented surface S. Now Stokes' theorem applied to the velocity field states that

$$\int_C \mathbf{u} \cdot d\mathbf{x} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{u}) dS, \tag{3.9}$$

where the direction of the normal $\mathbf{n}$ to S is chosen from the orientation of C by the "right-hand rule". We can interpret the right-hand side of (3.9) as the *flux of vorticity through S.* So it must be that the left-hand side is an expression of the effect of vorticity *on the velocity field*. We thus define *the fluid circulation of the velocity field* $\mathbf{u}$ *on the contour C* by

$$\Gamma_C = \oint_C \mathbf{u} \cdot d\mathbf{x}. \tag{3.10}$$

The circulation is going to be our measure of the rotation of the fluid.

The key "point" is that is that circulation is defined *globally*, *not* at a point. We need to consider an open set containing S in order to make this definition.

Example 3.3: Potential flows have the property that circulation vanishes on any closed contour, as long as $\mathbf{u}$ is well-behaved in an open set containing S. This is an obvious property of an irrotational flow.

Example 3.4 In two dimensions, the flow $(u, v) = \frac{1}{2\pi}(-y/r^2, x/r^2)$ is a point vortex. If C is a simple closed curve encircling the origin, then Γ_C is equal to the circulation on a circle centered at the origin, by independence of path since (u, v) is irrotational everywhere except at the origin. The circulation on a circle, taken counter-clockwise, is found to be unity. Indeed in polar form the velocity is given by $u_r = 0, u_\theta = \frac{1}{2\pi r}$. The circulation on the circle of radius r is thus $\frac{2\pi r}{2\pi r} = 1$. This flow is called the *point vortex of unit strength.*

3.3 Kelvin's theorem for a barotropic fluid

In chapters 12-14 we will be taking up the dynamics of general compressible fluids. The intervening discussion will deal with only a restricted class of compressible flows, the *barotropic fluids.* A barotropic fluid is defined by specifying pressure as a given function of the density, $p(\rho)$. This reduces the dependent variables of an ideal fluid to $\mathbf{u}, \rho$ and so the system of momentum and mass equations is closed.

Theorem 1 *(Kelvin's theorem) Let $C(t)$ be a simple close* material *curve in an ideal fluid with body force $-\rho\nabla\Phi$. Then, if either (i) $\rho =$ constant, or (ii) the fluid is barotropic, then the circulation $\Gamma_{C(t)}$ of $\mathbf{u}$ on C is invariant under the flow:*

$$\frac{d}{dt}\Gamma_{C(t)} = 0. \tag{3.11}$$

To prove this consider a parametrization $\mathbf{x}(\alpha, t)$ of $C(t)$, $0 \leq \alpha \leq A$. Then

$$\frac{d}{dt}\oint_C \mathbf{u}\cdot d\mathbf{x} = \frac{d}{dt}\int_0^A \mathbf{u}\cdot\frac{\partial\mathbf{x}}{\partial\alpha}d\alpha = \int_0^A \Big[\frac{D\mathbf{u}}{Dt}\cdot\frac{\partial\mathbf{x}}{\partial\alpha} + \mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial\alpha}\Big]d\alpha. \tag{3.12}$$

Making use of the momentum equation $\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$ we have

$$\frac{d\Gamma_C}{dt} = \int_0^A \Big[-(\frac{1}{\rho}\nabla p + \nabla\Phi)\cdot\frac{\partial\mathbf{x}}{\partial\alpha} + \mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial\alpha}\Big]d\alpha, \tag{3.13}$$

This becomes

$$\frac{d\Gamma_C}{dt} = \oint_C \Big[\frac{-dp}{\rho} + d(\frac{1}{2}|\mathbf{u}|^2 - \Phi)\Big]. \tag{3.14}$$

Now if ρ is a constant, or if the fluid is barotropic, the integrand may be written as perfect differential (in the barotropic case a differential of $-\int \rho^{-1}\frac{dp}{d\rho}d\rho + \frac{1}{2}|\mathbf{u}|^2 - \Phi$). Since all variables are assumed single-valued, the integral vanishes and the theorem is proved.

Kelvin's theorem is a cornerstone of ideal fluid theory since it expresses a global property of vorticity, namely the flux through a surface, as an invariant of the flow. We shall see that it is very useful in understanding the kinematics of vorticity.

3.4 The vorticity equation

In the present section we again assume that either $\rho =$ constant, or else the fluid is barotropic.

In either case it is of interest to consider an equation for vorticity, which can be obtained by taking the curl of

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0. \tag{3.15}$$

Under the conditions stated, this will give

$$\nabla\times\frac{D\mathbf{u}}{Dt} = 0. \tag{3.16}$$

Recalling $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega}$, we use the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A}. \tag{3.17}$$

For the case of constant density and no mass addition, both $\nabla \cdot \mathbf{u}$ and $\nabla \cdot \boldsymbol{\omega}$ vanish, with the result

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \tag{3.18}$$

For a barotropic fluid, we need to bring in conservation of mass to evaluate $\nabla \cdot \mathbf{u} = -\rho^{-1} D\rho/Dt$. We then get in place of (3.18)

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \frac{\boldsymbol{\omega}}{\rho} \frac{D\rho}{Dt}. \tag{3.19}$$

This can be rewritten as

$$\frac{D(\frac{\boldsymbol{\omega}}{\rho})}{Dt} = \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \mathbf{u}. \tag{3.20}$$

Now we want to compare (3.18) and (3.20) with (1.23), and observe that $\boldsymbol{\omega}$ in the first case and $\boldsymbol{\omega}/\rho$ is the second is a *material vector field* as we defined it in chapter 1. This is a deep and remarkable property of the vorticity field, which gives it its importance in fluid mechanics. It tells us, for example, that vorticity magnitude can be increased if two nearby fluid particles lying on the same vortex line move apart.

Example 3.5 In two dimensions $\boldsymbol{\omega} \cdot \nabla \mathbf{u} = 0$ and so the vorticity ω satisfies

$$\frac{D\omega}{Dt} = 0, \tag{3.21}$$

i.e. in two dimensions, for the cases studied here, vorticity is a scalar material invariant, whose value is always the same on a given fluid parcel.

In three dimensions the term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ is sometimes called the *vortex stretching term.* Its existence make two and three-dimensional vorticity behaviors entirely different.

There is a Lagrangian form of the vorticity equation, due to Gauss. We can obtain it here by recalling that $v_i(\mathbf{a}, t) = J_{ij}(\mathbf{a}, t) V_i(\mathbf{a})$ defines a material vector field. Let us assume that, given the initial velocity and therefore initial vorticity fields, vorticity may be solved for uniquely at some function time t using Euler's equations. Then, any material vector field assuming the assigned initial values for vorticity must be the unique vorticity field $\boldsymbol{\omega}$. However, if the initial vorticity is $\boldsymbol{\omega}_0(\mathbf{x})$, then a material vector field which takes on these initial values is $\mathbf{J}(\mathbf{a}, t) \cdot \boldsymbol{\omega}_0(\mathbf{a})$. By uniqueness, we must have

$$\omega_i(\mathbf{a}, t) = J_{ij}(\mathbf{a}, t) \omega_{0j}. \tag{3.22}$$

in the constant density case. For the barotropic case, given initial density $\rho_0(\mathbf{x})$, the corresponding equation is

$$\rho^{-1} \omega_i(\mathbf{a}, t) = \rho_0^{-1}(\mathbf{a}) J_{ij}(\mathbf{a}, t) \omega_{0j}. \tag{3.23}$$

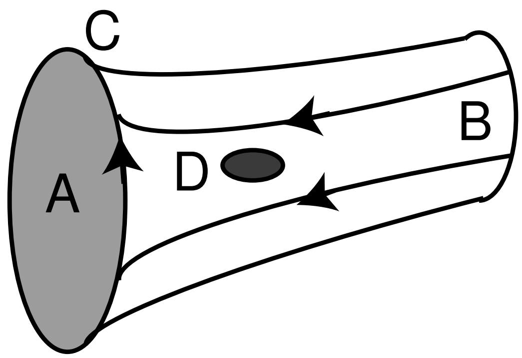

Figure 3.1: A segment of an oriented vortex tube.

This is Cauchy's "solution" of the vorticity equation . Of course nothing has been solved, only represented in terms of the unknown Jacobian. It is however a revealing relation which directly ties the changes in vorticity to the deformation experienced by a fluid parcel.

3.5 Helmholtz' Laws

In dicussing the behavior of vorticity in a fluid flow we will want to consider as our basic element a section of a vortex tube as shown in figure 3.1. Recall that a vortex tube is a bundle of vortex lines, each of the lines being the instantaneous flow lines of of the vorticity field.

In the mid-nineteenth century Helmholtz laid the foundations for the mechanics of vorticity. His conclusions can be summarized by the following three laws:

- Fluid parcels free of vorticity stay free of vorticity.
- Vortex lines are material lines.
- The strength of a vortex tube, to be defined below, is an invariant of the motion.

We have seen that the vorticity field, or the field divided by density in the barotropic case, is a material vector field. The vortex lines are the same in each case if $\boldsymbol{\omega}$ is the same. Hence particles on a particular vortex line at one time, remain on a line at a later time, and so the line is itself material. Thus the tube segment in figure 3.1 is bounded laterally by a surface of vortex lines. The small patch D in the surface thus carries no flux of vorticity. The bounding contour of this patch is a material curve, and by Kelvin's theorem the circulation on the contour is a material invariant. Since this circulation is initially zero by the absence of flux of vorticity through the patch, it will remain zero. Consequently the lateral boundary of a vortex tube remains a boundary of the tube.

In follows from the solenoidal property of vorticity and the divergence theorem that the flux of vorticity through the end surface A, must equal that through the end surface B. This flux is a property of a vortex tube, called the *vortex tube strength*. Note that this is independent of the compressibility or incompressibility of the fluid. The tube strength expresses simply a property of a solenoidal vector field.

To establish the third law of Helmholtz we must show that this strength is a material invariant. But this follows immediately from Kelvin's theorem, since the circulation on the contour C is a material invariant. This circulation, for the orientation of the contour shown in the figure, is equal to the vortex tube strength by Stokes' theorem, and we are done.

The first law is also established using Kelvin's theorem. Suppose that a flow is initially irrotational but at some time a fluid parcetionl is found where vorticity is non-zero. A small closed contour can then be found with non-vanishing circula is non-zero, by Kelvin's theorem. This contradicts the irrotationality of the initial flow.

Using these laws we may see how changes in the shape of a fluid parcel can change the magnitude of vorticity. In figure 3.2 we show a segment of small vortex tube which has changes under the flow from have length $L_1$and section area A_1, to new values A_2, L_2. If the density is constant, volume is conserved, $A_1L_1 = A_2L_2$. If the vorticity magnitudes are ω_1, ω_2, then invariance of the tube strength implies $\omega_1 A_1 = \omega_2 A_2$. Comparing these expressions, $\omega_2/\omega_1 = L_2/L_1$. Consequently, *for an ideal fluid of constant density the vorticity is proportional to vortex line length*. We understand here that by line length we are referring to the distance between to nearby fluid particles on the same vortex line. Thus the growth or decay of vorticity in ideal fluid flow is intimately connected to the stretching properties of the Lagrangian map.[1] Fluid tubulence is observed to contain small domains of very large vorticity, presumably created by this stretching.

For a compressible fluid the volume of te tube need not be invariant, but mass is conserved. Thus we have, introducing the initaial and final densities ρ_1, ρ_2,

$$\rho_1 A_1 L_1 = \rho_1 A_2 L_2, \quad \omega_1 A_1 = \omega_2 A_2. \tag{3.24}$$

It follows that

$$\frac{\omega_2/\rho_2}{\omega_1/\rho_1} = L_2/L_1. \tag{3.25}$$

Thus we see that it is the magnitude of the material field, whether $|\boldsymbol{\omega}|$ or $\rho^{-1}|\boldsymbol{\omega}|$, which is proportional to line length. Notice that in a compressible fluid vorticity may be increased by compressing a tube while holding the length fixed, so as to increase the density.

[1]This make chaotic flow, with positive Liapunov exponents, of great interest in amplifying vorticity.

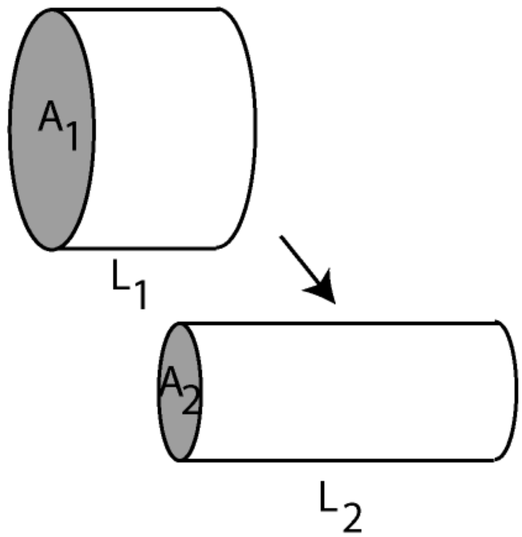

Figure 3.2: Deformation of a vortex tube under a flow.

3.6 The velocity field created by a given vorticity field

Suppose that in R^3 the vorticity field is non-zero in some region and vanishes at infinity. What is the velocity field or fields is created by this vorticity? It is clear that given a vorticity field $\boldsymbol{\omega}$, and a vector field $\mathbf{u}$ such that $\nabla \times \mathbf{u} = \boldsymbol{\omega}$, another vector field with the same property is given by $\mathbf{v} = \mathbf{u} + \nabla\phi$ for some scalar field ϕ, uniqueness is an issue. However, under appropriate conditions a unique construction is possible.

Theorem 2 *Let the given vorticity field be smooth and vanish strongly at infinity, e.g. for some* $R > 0$

$$|\boldsymbol{\omega}| \leq Cr^{-N}, \quad r > R, r = \sqrt{x^2 + y^2 + z^2} \tag{3.26}$$

Then there exists a unique solenoidal vector field $\mathbf{u}$ *such that* $\nabla \times \mathbf{u} = \boldsymbol{\omega}$ *and* $\lim_{r\to\infty} |\mathbf{u}| = 0$. *This vector field is given by*

$$\mathbf{u} = \frac{1}{4\pi} \int_{R^3} \frac{(\mathbf{y} - \mathbf{x}) \times \boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} dV_{\mathbf{y}}. \tag{3.27}$$

To prove this consider the vector field $\mathbf{v}$ defined by

$$\mathbf{v} = \frac{1}{4\pi} \int_{R^3} \frac{\boldsymbol{\omega}}{|\mathbf{x} - \mathbf{y}|} dV_{\mathbf{y}}. \tag{3.28}$$

This field exists and given (3.26) and can be differentiated if $\boldsymbol{\omega}$ is a smooth function. Let $\mathbf{u} = \nabla \times \mathbf{v}$. Now we have the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{3.29}$$

The right-hand side of (3.28) is the unique solution of the vector equation $\nabla^2 \mathbf{v} = \boldsymbol{\omega}$ which vanishes at infinity. Also

$$\begin{aligned} \operatorname{div} \int_{R^3} \boldsymbol{\omega} \cdot \nabla_{\mathbf{x}} \frac{1}{|\mathbf{x}-\mathbf{y}|} dV_{\mathbf{y}} &= -\int_{R^3} \boldsymbol{\omega} \cdot \nabla_{\mathbf{y}} \frac{1}{|\mathbf{x}-\mathbf{y}|} dV_{\mathbf{y}} \\ &= -\int_{R^3} \nabla_{\mathbf{y}} \cdot \Big[\frac{\boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\Big] dV_{\mathbf{y}} = 0 \end{aligned} \tag{3.30}$$

by the divergence theorem and the fact that the integral of $|\boldsymbol{\omega}|$ over $r = R$ be bounded in R in (3.26) holds. Thus $\mathbf{u}$ as defined by (3.27) satisfies $\nabla \times \mathbf{u} = \boldsymbol{\omega}$. as required. Also, this vector field is solenoidal since it is the curl of $\mathbf{v}$, and vanishes as $|\mathbf{x}| \to \infty$. And it is unique. Indeed if $\mathbf{u}'$ is another vector field with the same properties, then $\nabla \times (\mathbf{u} - \mathbf{u}') = 0$ and so $\mathbf{u} - \mathbf{u}' = \nabla\phi$ for some scale field whose gradient vanishes at infinity. But by the solenoidal property of $\mathbf{u}, \mathbf{u}'$ we see that $\nabla^2\phi = 0$, and this implies $\phi = constant$, giving the uniqueness of $\mathbf{u}$.

For compressible flows a general velocity field $\mathbf{w}$ with vorticity $\boldsymbol{\omega}$ will have the form $\mathbf{w} = \mathbf{u} + \nabla\phi$ where $\mathbf{u}$ is given by (3.27) and ϕ is an arbitrary scalar field.

The kernel

$$\frac{1}{4\pi} \frac{(\mathbf{y}-\mathbf{x}) \times (\cdot)}{|\mathbf{x}-\mathbf{y}|^3} \tag{3.31}$$

is interesting in the insight it gives into the creation of velocity as a cross product operation. The velocity induced by a small segment of vortex tube is orthogonal to both the direction of the tube and the vector joining the observation point to the vortex tube segment. A similar law relates magnetic field created by an electric cucrrent, where it is know as the *Biot-Savart law*.

3.7 Some examples of vortical flows

We end this chapter with a few examples of ideal fluid flows with non-zero vorticity.

3.7.1 Rankine's combined vortex

This old example is an interesting use of a vortical flow to model a "bath tub vortex", before the depression of the surface of the fluid develops a "hole". It will also give us an example of a flow with a free surface. The fluid is a liquid of constant density ρ with a free surface given by $z = Z(r)$ in cylindrical polar coordinates, see figure 3.3. The pressure above the free surface is the constant p_0. The body force is gravitational, $\mathbf{f} = -g\mathbf{i}_z$. The vorticity is a solid-body rotation in a vertical tube bounded by $r = a, z < Z$. The only nonzero velocity component it the θ-component u_θ.

In $r > a, z < Z$ Euler's equations will be solved by the field of a two-dimensional point vortex (actually a *line vortex*). This will be matched with a rigid rotation for $r < a$ so that velocity is continuous:

$$u_\theta = \begin{cases} \Omega a^2/r, & \text{when } r \geq a, \\ \Omega r, & \text{when } r < a. \end{cases} \tag{3.32}$$

Here Ω is the angular velocity of the core vortex. Now in the exterior region $r > a$ the flow is irrotational and so we have by the Bernoulli theorem for irrotational flows

$$\frac{p_{ext}}{\rho} = \frac{p_0}{\rho} - \frac{1}{2}\Omega^2 a^4 r^{-2} - gz, \tag{3.33}$$

for $z < Z$, where we have taken $Z = 0$ at $r = \infty$. The free surface is thus given for $r > a$ by

$$Z = -\frac{\Omega^2 a^4}{gr^2}. \tag{3.34}$$

Inside the vortex core, the equations reduce to

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} = \Omega^2 r, \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = g. \tag{3.35}$$

Thus

$$\frac{1}{2}\Omega^2 r^2 - gz + C \equiv \frac{p_{core}}{\rho}, \quad r < a, z < Z. \tag{3.36}$$

On the cylinder $r = a, z < Z$ we require that the $p_{core} = p_{ext}$, so

$$\frac{1}{2}\Omega^2 a^2 - gz + C = \frac{p_0}{\rho} - gz - \frac{1}{2}\Omega^2 a^2. \tag{3.37}$$

Therefore the constant C is given by

$$C = \frac{p_0}{\rho} - \Omega^2 a^2, \tag{3.38}$$

and

$$\frac{p_{core}}{\rho} = \frac{p_0}{\rho} - \Omega^2 a^2\Big(1 - \frac{r^2}{2a^2}\Big) - gz. \tag{3.39}$$

The free surface is then given by

$$Z = \begin{cases} -\frac{a^4\Omega^2}{2gr^2}, & \text{when } r \geq a, \\ \frac{\Omega^2 a^2}{g}\Big(\frac{r^2}{2a^2} - 1\Big), & \text{when } r < a./\text{cr} \end{cases} \tag{3.40}$$

We have used the adjective "combined" to emphasize that this vortex flow is an example of a solution of the equations of motions which is not smooth, since du_θ/dr is not continuous at $r = a, z < Z$. Since all other components

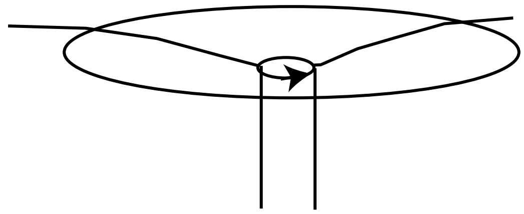

Figure 3.3: Rankine's combined vortex

of velocity are zero and the pressure is the only variable with a z-dependence, the equation are in fact satisfied everywhere. In a real, viscous fluid, if the ideal flow was taken as an initial condition, the irregularity at $r = a$ would be immediately smoothed out by viscous stresses. The ideal fluid solution would nonetheless be a good representation of the flow for some time, until the vortex core is substantially affected by the viscosity.

3.7.2 Steady propagation of a vortex dipole

We consider steady two-dimensional flow of an ideal fluid of constant density, no body force. Since then $\mathbf{u} \cdot \nabla\omega = 0$, introducing the stream function ψ, $(u, v) = (\psi_y, -\psi_x)$, we have

$$\psi_y(\nabla^2\psi)_x - \psi_x(\nabla^2\psi)_y = 0. \tag{3.41}$$

Consequently contours of constant ψ and of constant ω must agree, and so

$$\nabla^2\psi = f(\psi). \tag{3.42}$$

where the function f is arbitrary. We will look for solutions of the simplest kind, by choosing $f = -k^2\psi$, where k is a constant. Using polar coordinates, we look for solutions of the equation $\nabla^2\psi + k^2\psi$ in the disc $r < a$, which can match with the velocity in $r > a$ that is the same as irrotational flow past a circular body of radius a. That potential flow is easily re-expressed in terms of the stream function, since we see that in irrotational flow, where our function f vanishes, the stream function is harmonic. We then have

$$\psi = Uy\big(1 - \frac{a^2}{r^2}\big) = U\sin\theta\big(r - \frac{a^2}{r}\big). \tag{3.43}$$

Setting $\psi = h(r)\ \sin\theta$ in $\nabla^2\psi + k^2\psi = 0$ we we obtain the ODE for the Bessel functions of order 1. A solution regular in $r < a$ is therefore $h = CJ_1(kr)$. Ths

$$\psi = C\sin\theta J_1(kr). \tag{3.44}$$

Also

$$\omega = Ck^2\sin\theta J_1(kr). \tag{3.45}$$

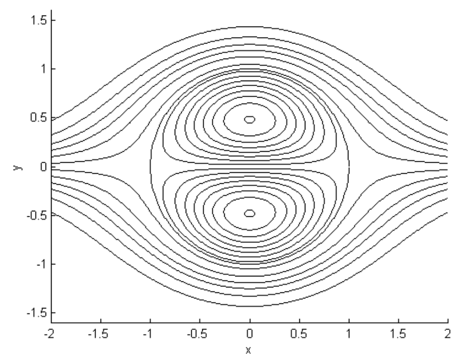

Figure 3.4: A propagating vortex dipole.

We have two constants to determine, and we will do this by requiring that both ω and u_θ be continuous on $r = a$. The condition on ω requires that $J_1(ka) = 0$. We thus choose ka to be the smallest zero if J_1, $ka \approx 3.83$.

The constant C is determined by the requirement that u_θ be continuous on $r = a$. Now $u_\theta = -\sin\theta\ \psi_y - \cos\theta\ \psi_x = -\psi_r$, and

$$\frac{d}{dr}J_1(kr) = -k^{-1}\frac{d^2}{dz^2}J_0(z)\Big|_{z=kr} = k^{-1}\Big(\frac{1}{z}\frac{dJ_0}{dz}+J_0\Big)_{z=kr} = k^{-1}\Big(-\frac{1}{z}J_1+J_0\Big)_{z=kr}. \tag{3.46}$$

Thus

$$\frac{d}{dr}J_1(kr)\Big|_{r=a} = k^{-1}J_0(ka). \tag{3.47}$$

The condition that ψ_r be continuous on $r = a$ thus becomes

$$C = 2k^{-1}\frac{U}{J_0(ak)}. \tag{3.48}$$

Thus

$$\omega = -\nabla^2\psi = \frac{2kU}{J_0(ak)}\sin\theta J_1(kr). \tag{3.49}$$

Since $J_0(3.83) \approx -.403$ we see that the constant multiplier in this last equation has a sign of opposite to that of U. Let us see if this makes sense. If U were negative, then the vorticity in the upper half of the disc would be positive. A positive vorticity implies an eddy rotating counterclockwise. This vorticity induces the vortex in the lower half of the disc to move to the right. Similarly the negative vorticity in the lower half of the disk causes the upper vortex to move to the right. Thus the vortex dipole propagates to the right, and in the frame moving with the dipole U is negative.

3.7.3 Axisymmetric flow

We turn now to a large class of vortical flows which are probably the simplest flows allowing vortex stretching, namely the axisymmetric Euler flows. These are solutions of Euler's equations in cylindrical polar coordinates (z, r, θ), under the assumption that all variables are independent of the polar angle θ. Euler's equations for the velocity $\mathbf{u} = (u_z, u_r, u_\theta)$ in cylindrical polar coordinates are

$$\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.50}$$

$$\frac{\partial u_r}{\partial t} + \mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.51}$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u} \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0, \tag{3.52}$$

where

$$\mathbf{u} \cdot \nabla(\cdot) = \Big[u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\Big](\cdot). \tag{3.53}$$

We take the density to be constant, so the solenoidal condition applies in the form

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0. \tag{3.54}$$

The vorticity vector is given by

$$(\omega_z, \omega_r, \omega_\theta) = \Big[\frac{1}{r}\frac{\partial r u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big]. \tag{3.55}$$

The vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \Big[\mathbf{u}\cdot\nabla\omega_z, \mathbf{u}\cdot\nabla\omega_r, \mathbf{u}\cdot\nabla\omega_\theta + \frac{u_\theta\omega_r}{r}\Big] - \Big[\boldsymbol{\omega}\cdot\nabla u_z, \boldsymbol{\omega}\cdot\nabla u_r, \boldsymbol{\omega}\cdot\nabla u_\theta + \frac{u_r\omega_\theta}{r}\Big] = 0. \tag{3.56}$$

In the axisymmetric case we thus have

$$\frac{\partial u_z}{\partial t} + \Big[u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r}\Big]u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.57}$$

$$\frac{\partial u_r}{\partial t} + \Big[u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r}\Big]u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.58}$$

$$\frac{\partial u_\theta}{\partial t} + \Big[u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r}\Big]u_\theta + \frac{u_r u_\theta}{r} = 0, \tag{3.59}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} = 0. \tag{3.60}$$

$$(\omega_z, \omega_r, \omega_\theta) = \Big[\frac{1}{r}\frac{\partial r u_\theta}{\partial r}, -\frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big] \tag{3.61}$$

If the *swirl* velocity component u_θ vanishes, the system simplifies further:

$$\frac{\partial u_z}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \tag{3.62}$$

$$\frac{\partial u_r}{\partial t} + \Big[u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}\Big] u_r + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3.63}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial r u_r}{r} = 0. \tag{3.64}$$

$$(\omega_z, \omega_r, \omega_\theta) = \Big[0, 0, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big]. \tag{3.65}$$

Note that the only nonzero component of vorticity is ω_θ. The vortex lines are therefore all rings with a common axis, the $z-$axis. The vorticity equation now has the form

$$\frac{\partial \omega_\theta}{\partial t} + u_z \frac{\partial \omega_\theta}{\partial z} + u_r \frac{\partial \omega_\theta}{\partial r} - \frac{u_r \omega_\theta}{r} = 0. \tag{3.66}$$

The last equation may be rewritten

$$\frac{D}{Dt}\frac{\omega_\theta}{r} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}. \tag{3.67}$$

Thus $\frac{\omega_\theta}{r}$ is a material invariant of the flow. We can easily interpret the meaning of this fact. A vortex ring of radius r has length $2\pi r$, and the vorticity associated with a given ring is a constant ω_θ. But the vorticity of a line is proportional to the line length (recall the increase of vorticity by line stretching). Thus the ratio $\frac{\omega_\theta}{2\pi r}$ must be constant on a given vortex ring. Since vortex rings move with the fluid, $\frac{\omega_\theta}{r}$ is a material invariant.

To compute axisymmetric flow without swirl we can introduce the stream function ψ for the solenoidal velocity in cylindrical polar coordinates:

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{3.68}$$

This ψ is often referred to as *the Stokes stream function*. Then

$$\omega_\theta = -\frac{1}{r}L(\psi), \quad L \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{3.69}$$

In the *steady* case, the vorticity equation gives

$$\Big[\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}\Big]\frac{1}{r^2}L(\psi) = 0. \tag{3.70}$$

Thus a family of steady solutions can be obtained by solving any equation of the form

$$L(\psi) = r^2 f(\psi), \tag{3.71}$$

where f is an arbitrary function, for the stream function ψ. The situation here is closely analogous to the steady two-dimensional case, see the previous subsection.

Now turning to axisymmetric flow *with* swirl, the instantaneous streamline and vortex lines can now be helices and a much larger class of Euler flows results. The same stream function applies. The swirl velocity satisfies, from (3.59)

$$\frac{Dru_\theta}{Dt} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_z\frac{\partial}{\partial z} + u_r\frac{\partial}{\partial r}. \tag{3.72}$$

We can understand the meaning of (3.72) using Kelvin's theorem. First note that a ring of fluid particles initially on a given circle C defined by initial values of z, r, will stay on the same circular ring as it evolves. The u_θ component takes the ring into itself, and the $(u_z, u_r, 0)$ sub-field determines the trajectory $C(t)$ of the ring, and thus the ring evolves as a material curve. Since u_θ is constant on the ring, the circulation on $C(t)$ is $2\pi r u_\theta$. By Kelvin's theorem, this circulation is a material invariant, and we obtain (3.72).

In the case of *steady* axisymmetic flow with swirl we see from (3.72) that we may take

$$ru_\theta = g(\psi), \tag{3.73}$$

where the function g is arbitrary. Bernoulli's theorem for steady flow with constant density gives

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} = H(\psi), \tag{3.74}$$

stating that the Bernoulli function H is constant on streamlines. From the momentum equation in the form $\nabla H - \mathbf{u}\times\boldsymbol{\omega} = 0$ we get, from the z-component e.g.:

$$u_r\omega_\theta - u_\theta\omega_r = \frac{\partial H}{\partial z}. \tag{3.75}$$

Using the expressions for the components of vorticity and expressing everything in terms of the stream function, we get from (3.73) and (3.75)

$$\frac{1}{r^2}\frac{\partial\psi}{\partial z} + \frac{1}{r^2}g\frac{dg}{d\psi}\frac{\partial\psi}{\partial z} = \frac{dH}{d\psi}\frac{\partial\psi}{\partial z}. \tag{3.76}$$

Eliminating the common factor $\frac{\partial\psi}{\partial z}$ and rearranging,

$$L(\psi) = r^2 f(\psi) - g\frac{dg}{d\psi}, \quad f(\psi) = \frac{dH}{d\psi}. \tag{3.77}$$

Thus two arbitrary functions, f, g are involved and any solution of (3.77) determines a steady solution in axisymmetric flow with swirl.

Problem set 3

1. Consider a fluid of constant density in two dimensions with gravity, and suppose that the vorticity $v_x - u_y$ is everywhere constant and equal to ω. Show

that the velocity field has the form $(u, v) = (\phi_x + \chi_y, \phi_y - \chi_x)$ where ϕ is harmonic and χ is any function of x, y (independent of t), satisfying $\nabla^2\chi = -\omega$. Show further that

$$\nabla(\phi_t + \frac{1}{2}q^2 + \omega\psi + p/\rho + gz) = 0$$

where ψ is the stream function for $\mathbf{u}$, i.e. $\mathbf{u} = (\psi_y, -\psi_x)$, and $q^2 = u^2 + v^2$.

2. Show that, for an incompressible fluid, but one where the density can vary independently of pressure (e.g. salty seawater), the vorticity equation is

$$\frac{D\omega}{Dt} = \omega \cdot \nabla\mathbf{u} + \rho^{-2}\nabla\rho \times \nabla p.$$

Interpret the last term on the right physically. (e.g. what happens if lines of constant p are $y =$ constant and lines of constant ρ are $x - y =$constant?). Try to understand how the term acts as a source of vorticity, i.e. causes vorticity to be created in the flow.

3. For steady two-dimensional flow of a fluid of constant density, we have

$$\rho\mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = 0, \nabla \cdot \mathbf{u} = 0.$$

Show that, if $\mathbf{u} = (\psi_y, -\psi_x)$, these equations imply

$$\nabla\psi \times \nabla(\nabla^2\psi) = 0.$$

Thus, show that a solution is obtained by giving a function $H(\psi)$ and then solving $\nabla^2\psi = H'(\psi)$. Show also that the pressure is given by $\frac{p}{\rho} = H(\psi) - \frac{1}{2}(\nabla\psi)^2$+constant.

4. Prove *Ertel's theorem* for a fluid of constant density: If f is a scalar material invariant, i.e. $Df/Dt = 0$, then $\omega \cdot \nabla f$ is also a material invariant, where $\omega = \nabla \times \mathbf{u}$ is the vorticity field.

5. A steady *Beltrami flow* is a velocity field $\mathbf{u}(\mathbf{x})$ for which the vorticity is always parallel to the velocity, i.e. $\nabla \times \mathbf{u} = f(\mathbf{x})\mathbf{u}$ for some scalar function f. Show that if a steady Beltrami field is also the steady velocity field of an inviscid fluid of constant density, the necessarily f is constant on streamlines. What is the corresponding pressure? Show that $\mathbf{u} = (Bsiny + C\cos z, C\sin z + A\cos x, A\sin x + B\cos y)$ is such a Beltrami field with $f = -1$. (This last flow an example of a velocity field yielding chaotic particle paths. This is typical of 3D Beltrami flows with constant f, according to a theorem of V. Arnold.)

6. Another formula exhibiting a vector field $\mathbf{u} = (u, v, w)$ whose curl is $\boldsymbol{\omega} = (\xi, \eta, \zeta)$, where $\nabla \cdot \boldsymbol{\omega} = 0$, is given by

$$u = z\int_0^1 t\eta(tx, ty, tz)dt - y\int_0^1 t\zeta(tx, ty, tz)dt,$$

$$v = x\int_0^1 t\zeta(tx,ty,tz)dt - z\int_0^1 t\xi(tx,ty,tz)dt,$$

$$w = y\int_0^1 t\xi(tx,ty,tz)dt - x\int_0^1 t\eta(tx,ty,tz)dt.$$

Verify this result. (Note that $\mathbf{u}$ will not in general be divergence-free, e.g. check $\xi = \zeta = 0, \eta = x$. A derivation of this formula, using differential forms, may be found in Flanders' book on the subject.)

7. In this problem the object is to find a 2D propagating vortex dipole structure analogous to that studied in subsection 3.6.2. In the present case, the structure will move clockwise on the circle of radius R with angular velocity Ω. Consider a rotating coordinate system and a circular structure of radius a, stationary and with center at $(0, R)$. Relative to the rotating system the velocity tends to $\Omega(-y,x) = \Omega(-y',x) + \Omega R(-1,0), y' = y - R$. It turns out that (assuming constant density), the momentum equation relative to the rotating frame can be reduced to that in the non-rotating frame in that the Coriolis force can be absorbed into the gradient of a modified pressure, see a later chapter. Thus we again take $\nabla^2\psi + k^2\psi = 0, r' < a$. Here $r' = \sqrt{(y')^2 + x^2}$. A new term proportional to $J_0(kr)$ must now be included. We require that u_θ aAand ω must be continuous on $r' = a$. Show that, relative to the rotating frame,

$$\psi = \begin{cases} -\frac{2R\Omega}{kJ_0(ak)}\sin\theta J_1(kr') + \frac{2\Omega}{k^2 J_0(ka)}J_0(kr'), & \text{if } r' < a, \\ -\frac{\Omega}{2}{r'}^2 - \Omega R(r' - a^2/r') + \Omega a^2 \ln r' + C, & \text{if } r' \geq a. \end{cases} \tag{3.78}$$

Chapter 4

Potential flow

Potential or irrotational flow theory is a cornerstone of fluid dynamics, for two reasons. Historically, its importance grew from the developments made possible by the theory of harmonic functions, and the many fluids problems thus made accessible within the theory. But a second, more important point is that potential flow is actually realized in nature, or at least approximated, in many situations of practical importance. Water waves provide an example. Here fluid initially at rest is set in motion by the passage of a wave. Kelvin's theorem insures that the resulting flow will be irrotational whenever the viscous stresses are negligible. We shall see in a later chapter that viscous stresses cannot in general be neglected near rigid boundaries. But often potential flow theory applies away from boundaries, as in effects on distant points of the rapid movements of a body through a fluid.

An example of potential flow in a barotropic fluid is provided by the theory of sound. There the potential is not harmonic, but the irrotational property is acquired by the smallness of the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in the momentum equation. The latter thus reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho} \nabla p \approx 0. \tag{4.1}$$

Since sound produces very small changes of density, here we may take ρ to be will approximated by the constant ambient density. Thus $\mathbf{u} = \nabla \phi$ with $\frac{\partial \phi}{\partial t} = -p/\rho$.

4.1 Harmonic flows

In a potential flow we have

$$\mathbf{u} = \nabla \phi. \tag{4.2}$$

We also have the Bernoulli relation (for body force $\mathbf{f} = -\rho \nabla \Phi$)

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + \int \frac{dp}{\rho} + \Phi = 0. \tag{4.3}$$

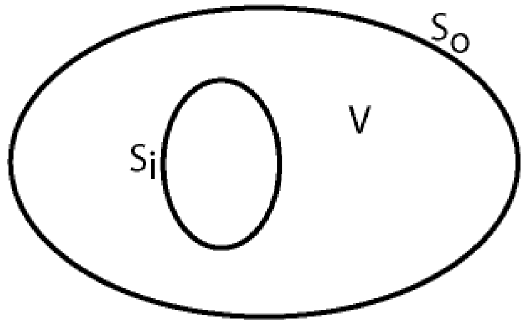

Figure 4.1: A domain V, bounded by surfaces $S_{i,o}$ where $\frac{\partial \phi}{\partial n}$ is prescribed.

Finally, we have conservation of mass

$$\rho_t + \nabla \cdot (\rho \nabla \phi) = 0. \tag{4.4}$$

The most extensive use of potential flow theory is to the case of constant density, where $\nabla \cdot \mathbf{u} = \nabla^2 \phi = 0$. These *harmonic flows* can thus make use of the highly developed mathematical theory of harmonic functions. in the problems we study here we shall usually consider explicit examples where existence is not an issue. On the other hand the question of uniqueness of harmonic flows is an important issue we discuss now. A typical problem is shown in figure 4.1.

A harmonic function ϕ has prescribed normal derivatives on inner and outer boundaries S_i, S_o of an annular region V. The difference $\mathbf{u}_d = \nabla \phi_d$ of two solutions of this problem will have zero normal derivatives on these boundaries. That the difference must in fact be zero throughout V can be established by noting that

$$\nabla \cdot (\phi_d \nabla \phi_d) = (\nabla \phi_d)^2 + \phi_d \nabla^2 \phi_d = (\nabla \phi_d)^2. \tag{4.5}$$

The left-hand side of (4.5) integrates to zero over V to zero by Gauss' theorem and the homogeneous boundary conditions of $\frac{\partial \phi_d}{\partial n}$. Thus $\int_V (\nabla \phi_d)^2 dV = 0$, implying $\mathbf{u}_d = 0$.

Implicit in this proof is the assumption that ϕ_d is a single-value function. A function ϕ is single-valued in V if and only if $\oint_C d\phi = 0$ on any closed contour C contained in V. In three dimensions this is insured by the fact that any such contour may be shrunk to a point in V. In two dimensions, the same conclusion applies to *simply-connected* domains. In non-simply connect domains uniqueness of harmonic flows in 2DS is not assured. Note for a harmonic flow

$$\oint_C d\phi = \oint_C \mathbf{u} \cdot d\mathbf{x} = \Gamma_C, \tag{4.6}$$

so that a potential which is not single valued is associated with a non-zero circulation on some contour. Since there is no vorticity within the domain of harmonicity, we must look outside of this domain to find the vorticity giving rise to the circulation.

Example 4.1: The point vortex of problem 1.2 is an example of a flow harmonic in a non-simply connected domain which excludes the origin. If $\mathbf{u} = \frac{1}{2\pi}(-y/r^2, x/r^2)$ then the potential is $\frac{\theta}{2\pi} + constant$ and the circulation on an simply closed contour oriented counter-clockwise is 1. This defines the *point vortex of unit circulation*. Here the vorticity is concentrated at the origin, outside the domain of harmonicity.

Example 4.2 Steady two-dimensional flow harmonic flow with velocity $(U, 0)$ at infinity, past a circular cylinder of radius a centered at the origin, is not unique. The flow of example 2.4 plus an arbitrary multiple of the point vortex flowof example 4.1 will again yield a flow with the same velocity at infinity, and still tangent to the boundary $r = a$:

$$\phi = Ux(1 + a^2/r^2) + \frac{\Gamma}{2\pi}\theta. \tag{4.7}$$

4.1.1 Two dimensions: complex variables

In two dimensions harmonic flows can be studied with the powerful apparatus of complex variable theory. We define the *complex potential* as an analytic function of the complex variable $z = x + iy$:

$$w(z) = \phi(x, y) + i\psi(x, y). \tag{4.8}$$

We will suppress t in our formulas in the case when the flow is unsteady. If we identify ϕ with the potential of a harmonic flow, and ψ with the stream function of the flow, then by our definitions of these quantities

$$(u, v) = (\phi_x, \phi_y) = (\psi_y, -\psi_x), \tag{4.9}$$

yielding the Cauchy-Riemann equations $\phi_x = \psi_y, \phi_y = -\psi_x$. The derivative of w gives the velocity components in the form

$$\frac{dw}{dz} = w'(z) = u(x, y) - iv(x, y). \tag{4.10}$$

Notice that the Cauchy-Riemann equations imply that $\nabla\phi \cdot \nabla\psi = 0$ at every point where the partials are defined, implying that the streamlines are there orthogonal to the lines of constant potential ϕ.

Example 4.3: The uniform flow at an angle α to the horizontal, with velocity $Q(\cos\alpha, \sin\alpha)$ is given by the complex potential $w = Qze^{-i\alpha}$.

Example 4.4: In complex notation the harmonic flow of example 4.2 may be written

$$w = U(z + a^2/z) - \frac{i\Gamma}{2\pi}\log z \tag{4.11}$$

where e.g. we take the principle branch of the logarithm function.

As a result of the identification of the complex potential with an analytic function of a complex variable, the conformal map becomes a valuable tool in

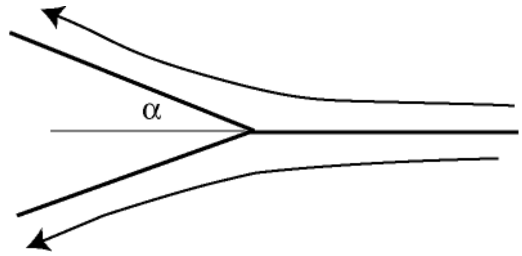

Figure 4.2: Flow onto a wedge of half-angle α.

the construction of potential flows. For this application we may start with the physical of z-plane, where the complex potential $w(z)$ is desired. A conformal map $z \to Z$ transforms boundaries and boundary conditions and leads to a problem which can be solved to obtain a complex potential $W(Z)$. Under the map values of ψ are preserved, so that streamlines map onto streamlines.

Example 4.5: The flow onto a wedge-shaped body (see figure 4.2). Consider in the Z plane the complex potential of a uniform flow,$-UZ, U > 0$. The region above upper surface of the wedge to the left, and the and the positive x-axis to the right, is mapped onto the upper half-plane $Y >$) by the function $Z = z^{\frac{\pi}{\pi-\alpha}}$. Thus $w(z) = -Uz^{\frac{\pi}{\pi-\alpha}}$.

Example 4.6: The map $z(Z) = Z + \frac{b^2}{Z}$ maps the circle of radius $a > b$ in the Z-plane onto the ellipse of semi-major axis $\frac{a^2+b^2}{a}$ and semi-minor (y)-axis $\frac{a^2-b^2}{a}$ in the z-plane. And the exterior is mapped onto the exterior. Uniform flow with velocity $(U, 0)$at infinity, past the circular cylinder $|Z| = a$, has complex potential $W(Z) = U(Z + a^2/Z)$. Inverting the map and requiring that $Z \approx z$ for large $|z|$ gives $Z = \frac{1}{2}(z + \sqrt{z^2 - 4b^2})$. Then $w(z) = W(Z(z))$ is the complex potential for uniform flow past the ellipse. Notice how the map satisfies $\frac{dz}{dZ} \to 1$ as $z \to \infty$ This insures that that infinity maps by the identity and so the uniform flow imposed on the circular cylinder is also imposed on the ellipse.

4.1.2 The circle theorem

We now state a result which gives the mathematical realization of the physical act of "placing a rigid body in an ideal fluid flow", at least in the two-dimensional case.

Theorem 3 *Let a harmonic flow have complex potential $f(z)$, analytic in the domain $|z| \leq a$. If a circular cylinder of radius a is place at the origin, then the new complex potential is $w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$.*

To show this we need to establish that the analytical properties of the new flow match those of the old, in particular that the analytic properties and the singularities in the flow are unchanged. Then we need to verify that the surface of the circle is a streamline. Taking the latter issue first, note that on the circle

$\frac{a^2}{\bar{z}} = z$, so that there we have $w = f(z) + \overline{f(z)}$, implying $\psi = 0$ and so the circle is a streamline. Next, we note that the added term is an analytic function of z if it is not singular at z. (If $f(z)$ is analytic at z, so is $\overline{f(\bar{z})}$. As for the location of singularities of w, since f is analytic in $|z| \leq a$ it follows that $f\Big(\frac{a^2}{z}\Big)$ is analytic in $|z| \geq a$, and the same is true of $\overline{f\Big(\frac{a^2}{\bar{z}}\Big)}$. Thus the only singularities of $w(z)$ in $|z| > a$ are those of $f(z)$.

Example 4.7: If a cylinder of radius a is placed in a uniform flow, then $f = Uz$ and $w = Uz + U\overline{(a^2/\bar{z})} = U(z + a^2/z)$ as we already know. If a cylinder is placed in the flow of a point source at $b > a$ on the x-axis, then $f(z) = \frac{Q}{2\pi}\ln(z - b)$ and

$$w(z) = \frac{Q}{2\pi}(\ln(z-b) + \overline{\ln\big(\frac{a^2}{\bar{z}} - b\big)} = \frac{Q}{2\pi}(\ln(z-b) + \ln(z - a^2/b) - \ln z) + C, \quad (4.12)$$

where C is a constant. From this form it may be verified that the imaginary part of w is constant when $z = ae^{i\theta}$. Note that the *image system* of the source, with singularities within the circle, consists of a source of strength Q at the image point a^2/b, and a source of strength $-Q$ at the origin.

Example 4.8: A point vortex at position z_k of circulation Γ_k has the complex potential $w_k(z) = -i\frac{\Gamma_k}{2\pi}\ln(z - z_k)$. A collection of N such vortices will have the potential $w(z) = \sum_{k=1}^{N} w_k(z)$. Since vorticity is a material scalar in two-dimensional ideal flow, and the delta-function concentration may be regarded as the limit of a small circular patch of constant vorticity, we expect that each vortex must move with he harmonic flow created at the vortex by the other $N - 1$ vortices. Thus the positions $z_k(t)$ of the vortices under this law of motion is governed by the system of N equations,

$$\overline{\frac{dz_j}{dt}} = \frac{-i}{2\pi} \sum_{k=1, k\neq j}^{N} \frac{\Gamma_k}{z - z_k}. \quad (4.13)$$

Note the conjugation on the left coming from the identity $w' = u - iv$.

4.1.3 The theorem of Blasius

An important calculation in fluid dynamics is the force exerted by the fluid on a rigid body. In two dimensions and in a steady harmonic flow this calculation can be carried out elegantly using the complex potential.

Theorem 4 *Let a steady uniform flow past a fixed two-dimensional body with bounding contour C be a harmonic flow with velocity potential $w(z)$. Then, if no external body forces are present, the force (X, Y) exerted by the fluid on the body is given by*

$$X - iY = \frac{i\rho}{2} \oint_C \Big(\frac{dw}{dz}\Big)^2 dz. \quad (4.14)$$

Here the integral is taken round the contour in the counter-clockwise sense. This formula, due to Blasius, reduces the force calculation to a complex contour integral. Since the flow is harmonic, the path of integration may be distorted to any simple closed contour encircling he body, enabling the method of residues to be applied. The exact technique will depend upon whether are not the are singularities in the flow exterior to the body.

To prove the result, first recall that $dX - idY = p(-dy - idx) = -ipd\bar{z}$. Also, Bernoulli's theorem for steady ideal flow applies, so that

$$p = -\frac{\rho}{2}\Big|\frac{dw}{dz}\Big|^2 + C, \tag{4.15}$$

where clearly the constant C will play no role. Thus

$$X - iY = \frac{i\rho}{2}\oint_C \frac{dw}{dz}\overline{\frac{dw}{dz}}d\bar{z}. \tag{4.16}$$

However, the contour C is a streamline, so that $d\psi = 0$ there, and so on C we have $\overline{\frac{dw}{dz}}d\bar{z} = d\bar{w} = dw = \frac{dw}{dz}dz$. using this in (4.16) we obtain (4.14).

Example 4.9: We have found in problem 2.1 that the force on a circular cylinder in a uniform flow is zero. To verify this using Blasius' theorem, we set $w = U\Big(z + \frac{a^2}{z}\Big)$ so that $U^2\Big(1 - \frac{a^2}{z^2}\Big)2$ is to be integrated around C. Since there is no term proprotional to z^{-1} in the Laurent expansion about the origin, the residue is zero and we get no contribution to the force integral.

Example 4.10: Consider a source of strength Q placed at $(b, 0)$ and introduce a circular cylinder of radius $a < b$ into the flow. From example 4.6 we have

$$\frac{dw}{dz} = \frac{1}{z-b} + \frac{1}{z - a^2/b} - \frac{1}{z}. \tag{4.17}$$

Squaring, we get

$$\frac{1}{(z-b)^2} + \frac{1}{(z-a^2/b)^2} + \frac{1}{z^2} + \frac{2}{(z-b)(z-a^2/b)} - \frac{2}{z(z-a^2/b)} - \frac{2}{z(z-b)}. \tag{4.18}$$

The first three terms to not contibute to the integral around the circle $|z| = a$. For the last three, the partial fraction decomposition is

$$\frac{A}{z-b} + \frac{B}{z-a^2/b} + \frac{C}{z}, \tag{4.19}$$

where we compute $A = \frac{2a^2}{(b^2-a^2)b}, B = \frac{2b^3}{a^2(a^2-b^2)}, C = \frac{2(a^2+b^2)}{a^2b}$. The contributions come from the poles within the circle and we have

$$X - iY = \frac{i\rho}{2}\frac{Q^2}{4\pi^2}2\pi i(B + C) = \frac{Q^2\rho}{2\pi}\frac{a^2}{b(b^2-a^2)}. \tag{4.20}$$

The cylinder is therefore feels a force of attraction toward the source.

This introduction to the use of complex variables in the analysis of two-dimenisonal harmonic flows will provide the groundwork for a discussion of lift and airfoil design, to be taken up in chapter 5.

4.2 Flows in three dimensions

We live in three dimensions, not two, and the "flow past body" problem in two dimensions introduces a domain which is not simply connected, with important consequences. The relation between two and three-dimensional flows is particularly significant in the generation of lift, as we shall see in chapter 5. In the present section we treat topics in three dimensions which are direct extensions of the two-dimensional results just given. They pertain to bodies, such as a sphere, which move in an irrotational, harmonic flow.

4.2.1 The simple source

The source of strength Q in three dimensions satisfies

$$\text{div}\,\mathbf{u} = Q\delta(\mathbf{x}), \quad \mathbf{u} = \nabla\phi. \tag{4.21}$$

Here $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional delta function. It has the following properties: (i) It vanishes if $\mathbf{x} \neq 0$. (ii) Any integral of $\delta(\mathbf{x})$ over an open region containing the origin yields unity. It is best to think of all relations involving delta functions and other distributions as limits of relations using smooth functions.

In our case, integrating $\nabla^2\phi = Q\delta(\mathbf{x})$ over a sphere of radius $R_0 > 0$ we get

$$\int_{R=R_0} \frac{\partial\phi}{\partial n}\, dS = Q. \tag{4.22}$$

Since $\nabla^2\phi = 0, \mathbf{x} \neq 0$, and since the delta function must be regarded as an isotropic distribution, having no exceptional direction, we make the guess (using now $\nabla^2\phi = R^{-1}d^2(R\phi)/dR^2$) that $\phi = C/R, R^2 = x^2 + y^2 + z^2$ for some constant C. Then (4.22) shows that $C = -\frac{Q}{4\pi}$. Thus the simple source in three dimensions, of strength Q, has the potential

$$\phi = -\frac{Q}{4\pi}\frac{1}{R}. \tag{4.23}$$

Note that Q is equal to the volume of fluid per unit time crossing any deformation of a spherical surface, assuming the deformed surface surounds the origin. [1]

[1] We indicate how to justify this calculation using a limit operation. Define the three-dimensional delta function by $\lim_{\epsilon\to 0}\delta_\epsilon(R)$ where $\delta_\epsilon = \frac{3}{2\pi\epsilon^3}\frac{1}{1+(R/\epsilon)^3}$. Solving $\nabla^2\phi_\epsilon = \delta_\epsilon = R^{-2}\frac{d}{dR}\Big(R^2\frac{d\phi_\epsilon}{dR}\Big)$, under the condition that ϕ_ϵ vanish at infinity, we obtain $\phi_\epsilon = -\frac{1}{4\pi R} + \int_R^\infty R^{-2}\Big[tan^{-1}(R^3\epsilon^{-3}) - \pi/2\Big]dR$. For any positive R the integral tends to zero as $\epsilon \to 0$.

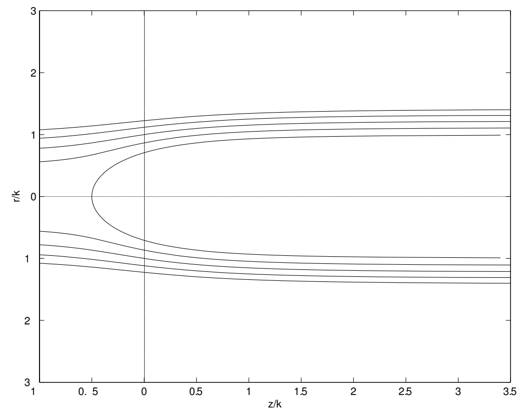

Figure 4.3: The Rankine fairing. All lengths are in units of k.

4.2.2 The Rankine fairing

We consider now a simple source of strength Q placed at the origin in a uniform flow $W\mathbf{i}_z$. The combined potential is then

$$\phi = Uz - \frac{Q}{4\pi}\frac{1}{R}. \tag{4.24}$$

The flow is clearly symmetric about the z-axis. In cylindrical polar coordinates $(z, r, \theta), r^2 = x^2 + y^2$ we introduce again the Stokes stream function ψ:

$$u_z = \phi_z = \frac{1}{r}\frac{\partial\psi}{\partial r},\ u_r = \phi_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}. \tag{4.25}$$

Thus for (4.24) we have

$$\frac{1}{r}\frac{\partial\psi}{\partial r} = U + \frac{Q}{4\pi}\frac{z}{R^3}. \tag{4.26}$$

Integrating,

$$\psi = Ur^2/2 - \frac{Q}{4\pi}\Big(\frac{z}{R} + 1\Big). \tag{4.27}$$

In (4.27) we have chosen the constant of integration to make $\psi = 0$ on the negative z-axis.

We show the stream surface $\psi = 0$, as well as several stream surfaces $\psi > 0$, in figure 4.3. This gives a good example of a uniform flow over a semi-infinite body. An interesting question is whether or not such a body would experience a force. We will find below that D'Alembert's paradox applies to *finite* bodies in three dimensions, that the drag force is zero, but it is not obvious that the result applies to bodies which are not finite.

We will use this question to illustrate the use of conservation of momentum to calculate force on a distant contour. In figure 4.4 the large sphere S of radius R_0 is centered at the origin and intersects the fairing on the at a circle bounding

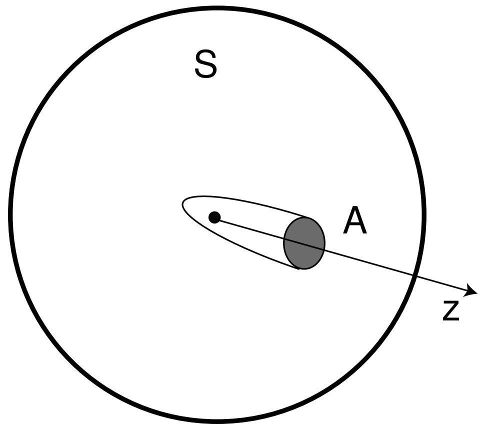

Figure 4.4: Geometry of the momentum integral for computation of the force on the Rankine fairing.

the disc A. Let S' be the spherical surface S minus hat part within the boundary of A. We are considering steady harmonic flow and so the momentum equation may be written

$$\frac{\partial}{\partial x_j}[\rho u_i\, u_j + p\,] = 0. \tag{4.28}$$

Let V' be the region bounded by S' and the piece of fairing enclosed. Integrating (4.28) over V' and using the divergence theorem., the contribution from the surface of the fairing is the integral $-\mathbf{n}p$ over this surface, where $\mathbf{n}$ is the outer normal *of the fairing*. Thus this contribution is the force $\mathbf{F}$ experienced by the enclosed piece of fairing, a force clearly directed along the z axis and therefore equal to the drag, $\mathbf{F} = D\mathbf{i}_z$. The remainder of the integral, taking only the z-component, takes the form of an integral over S minus the contribution from A. Thus conservation of momentum gives

$$D + \rho \int_S u_z \mathbf{u}\cdot\mathbf{R}/R + \frac{1}{2}\left[U^2 - |\mathbf{u}|^2\right]\frac{z}{R}dS - \rho I_A = 0. \tag{4.29}$$

We have here using the Bernoulli formula for the flow, $p + \frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}U^2$, the pressure at infinity being taken to be zero. Treating first the integral over S, we have

$$\mathbf{u} = U\mathbf{i}_z + \frac{Q}{4\pi}\frac{\mathbf{R}}{R^3}, \quad |\mathbf{u}|^2 = U^2 + \frac{UQ}{2\pi}\frac{z}{R^3} + \frac{Q^2}{16\pi^2}\frac{1}{R^4}. \tag{4.30}$$

Thus the integral in question becomes

$$\int_S \Big(U + \frac{Q}{4\pi}\frac{z}{R^3}\Big)\Big(\frac{Uz}{R} + \frac{Q}{4\pi}\frac{1}{R^2}\Big) - \frac{1}{4\pi}\Big(UQ\frac{z^2}{R^4} + \frac{1}{8\pi}\frac{Q^2 z}{R^5}\Big)dS. \tag{4.31}$$

We see that this last integral gives $UQ + \frac{1}{2}UQ - \frac{1}{2}UQ = UQ$. For the contribution I_A, we take the limit $R_0 \to \infty$ to obtain $I_A = U^2\pi r_\infty^2$, where r_∞ is the

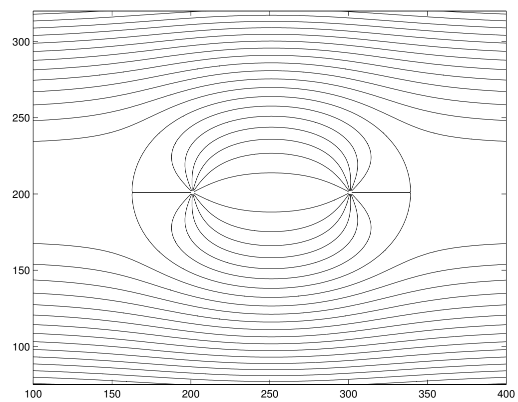

Figure 4.5: Flow around an airship.

asymptotic radius of the airing as $z \to \infty$. In this limit $D' \to D$, the total drag of the fairing. Thus the momentum integral method gives

$$D + \rho(UQ - U^2 \pi r_\infty^2) = 0. \tag{4.32}$$

But from (4.27) we see that the stream surface $\psi = 0$ is given by

$$z = \frac{r^2 - \frac{1}{2}k^2}{\sqrt{k^2 - r^2}}, \quad k^2 = \frac{Q}{\pi U}. \tag{4.33}$$

Thus $r_\infty = k$, and (4.32) becomes

$$D + \rho(UQ - UQ) = D = 0, \tag{4.34}$$

so the drag of the fairing is zero.

Example 4.11: The flow considered now typifies the early attempts to model the pressure distribution of an airship. The model consists of a source of strength Q at position $z = 0$ on the z-axis, and a equalizing sink (source of strength $-Q$) at the pint $z = 1$ on the z-axis. Since the source strengths cancel, a finite body is so defined when the singularities are place in the uniform flow $U\mathbf{i}_z$. It can be shown (see problem 4.7 below), that stream surfaces for the flow are given by constant values of

$$\Psi = \frac{U}{2} R^2 \sin^2\theta - \frac{Q}{4\pi}\Big(\cos\theta + \frac{1 - R\cos\theta}{\sqrt{R^2 - 2R\cos\theta + 1}}\Big), \tag{4.35}$$

where R, θ are spherical polars at the origin, with axial symmetry. We show the stream surfaces in figure 4.4.

4.2.3 The Butler sphere theorem.

The circle theorem for two-dimensional harmonic flows has a direct analog in three dimensions.

Theorem 5 *Consider an axisymmetric harmonic flow in spherical polars* (R, θ, φ), $u_\varphi = 0$, *with Stokes stream function* $\Psi(R, \theta)$ *vanishing at the origin:*

$$u_R = \frac{1}{R^2 \sin\theta}\frac{\partial\Psi}{\partial\theta}, u_\theta = \frac{-1}{R\sin\theta}\frac{\partial\Psi}{\partial R}. \tag{4.36}$$

If a rigid sphere of radius a is introduced into the flow at the origin, and if the singularities of Ψ *exceed a in distance from the origin, then the stream function of the resulting flow is*

$$\Psi_s = \Psi(R, \theta) - \frac{R}{a}\Psi(a^2/R, \theta). \tag{4.37}$$

It is clear that Ψ_s vanishes when $R = a$, so the surface of the sphere is a stream surface. Also the added term introduces no new singularities outside the sphere. Thus the theorrm is proved if we can verify that $\frac{R}{a}\Psi(a^2/R, \theta)$ represents a harmonic flow. In spherical polars with axial symmetry the only component of vorticity is

$$\omega_\varphi = \frac{1}{R}\Big[\frac{\partial(Ru_\theta)}{\partial R} - \frac{\partial u_R}{\partial\theta}\Big]. \tag{4.38}$$

Thus the condition on Ψ for an irrotational flow is

$$R^2\frac{\partial^2\Psi}{\partial R^2} + \sin\theta\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\Big) \equiv L_R\Psi = 0. \tag{4.39}$$

If $\frac{R}{a}\Psi(a^2/R, \theta)$ is inserted into (4.39) we can show that the equation is satisfied provided it is satisfied by $\Psi(R, \theta)$, see problem 4.8. Finally, since $\Psi(R, \theta)$ vanishes at the origin at least as R, $R\Psi(a^2/R, \theta$ is bounded at infinity and velocity component must decay as $O(R^{-2})$, so the uniform flow there is undisturbed.

Example 4.12: A sphere in a uniform flow $U\mathbf{i}_z$ has Stokes stream function

$$\Psi(R, \theta) = \frac{U}{2}R^2\sin^2\theta\Big[1 - \frac{a^3}{R^3}\Big]. \tag{4.40}$$

This translates into the following potential:

$$\phi = Uz\Big(1 + \frac{1}{2}\frac{a^3}{R^3}\Big). \tag{4.41}$$

Example 4.13: Consider a source of strength Q place on the z axis at $z = b$ and place a rigid sphere of radius $a < b$ at the origin. The streamfunction for this source which vanishes at the origin is

$$\Psi(R, \theta) = -\frac{Q}{4\pi}(\cos\theta_1 + 1), \tag{4.42}$$

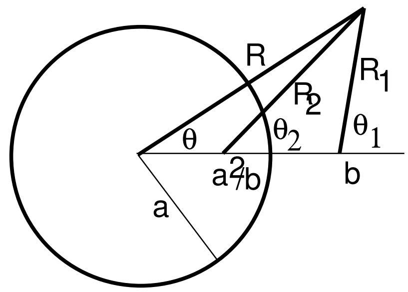

Figure 4.6: A a sphere of radius a in the presence of a source at $z = b > a$.

where θ_1 is defined in figure 4.6.

Now from the law of cosines and figure 4.6 we have

$$R\cos\theta - b = R_1\cos\theta_1, \quad R_1^2 = b^2 - 2bR\cos\theta + R^2. \tag{4.43}$$

Thus

$$\cos\theta_1 = \frac{R\cos\theta - b}{\sqrt{b^2 - 2bR\cos\theta + R^2}}. \tag{4.44}$$

Thus the stream function including the sphere, Ψ_s, is given by

$$\Psi_s = -\frac{Q}{4\pi}\Big[\frac{R\cos\theta - b}{\sqrt{b^2 - 2bR\cos\theta + R^2}} + 1\Big] + \frac{Q}{4\pi}\frac{R}{a}\Big[\frac{\frac{a^2}{R}\cos\theta - b}{\sqrt{b^2 - 2b\frac{a^2}{R}\cos\theta + \frac{a^4}{R^2}}} + 1\Big]. \tag{4.45}$$

Now, again using the law of cosines, $\sqrt{b^2 - 2b\frac{a^2}{R}\cos\theta + \frac{a^4}{R^2}} = bR_2/R$. Also we may use $R^2 = R_2^2 + 2\frac{a^2}{b}R\cos\theta - \frac{a^4}{b^2}$. Then Ψ_s may be brought into the form

$$\Psi_s = -\frac{Q}{4\pi}\Big[\frac{R\cos\theta - b}{R_1} + 1\Big] - \frac{a}{b}\frac{Q}{4\pi}\Big[\frac{R\cos\theta - \frac{a^2}{b}}{R_2}\Big] + \frac{Q}{4\pi}\Big[\frac{R - R_2}{a}\Big]. \tag{4.46}$$

The first term on the right is the source of strength Q at $z = b$. The second term is another source, of strength $\frac{a}{b}Q$, at the image point $z = a^2/b$. The last term can be understood as a line distribution of sinks of density $\frac{Q}{4\pi a}$, extending from the origin to the image point a^2/b. Indeed, if a point P on this line segment is associated with an angle θ_P, the the contribution from such a line of sinks would be

$$\frac{Q}{4\pi a}\int_0^{\frac{a^2}{b}} \cos\theta_P dz. \tag{4.47}$$

But $dR = -\cos\theta_P dz$, so the integral becomes

$$-\frac{Q}{4\pi a}\int_R^{R_2} dR = \frac{Q}{4\pi a}(R - R_2). \tag{4.48}$$

4.3 Apparent mass and the dynamics of a body in a fluid

Although harmonic flow is an idealization never realized exactly in actual fluids (except in some cases of super fluid dynamics), it is a good approximation in many fluid problems, particularly when rapid changes occur. A good example is the abrupt movement of a solid body through a fluid, for example a swimming stroke of the hand. We know from experience that a abrupt movement of the hand through water gives rise to a force opposing the movement. It is easy to see why this must be, within the theory of harmonic flows. An abrupt movement of the hand through still water causes the fluid to move relative to a observer fixed with the still fluid at infinity. This observer would therefore compute at the instant the hand is moving a finite kinetic energy of the fluid, whereas before the movement began the kinetic energy was zero. To produce this kinetic energy work must have been done, and so a force with a finite component opposite to the direction of motion must have occurred. We are here dealing only with the fluid, but if the body has mass the clearly a force is also needed to accelerate that mass. Thus both the body mass and the fluid movement contribute to the force experienced.

In a harmonic flow we shall show that, in the absence of external body forces, the force on a rigid body is proportional to its acceleration, and further the force contributed by the fluid can be expressed as an addition, *apparent* mass of the body. In other words the augmented force due to the presence of the surrounding fluid and the energy it acquires during motion of a body, can be explained as an inertial force associated with additional mass and the work done against that force. The term *virtual mass* is also used to denote this apparent mass. For a sphere, which has an isotropic geometry with no preferred direction, the apparent mass is just a scalar to be added to the physical mass. In general, however, the apparent mass associated with the momentum of a body in two or three dimensions will depend on the direction of the velocity vector. It thus must be a second order tensor, represented by the *apparent mass matrix*.

4.3.1 The kinetic energy of a moving body

Consider an ideal fluid at rest and introduce a moving rigid body, in two or three dimensions. An observer at rest relative to the fluid at infinity will se a disturbance of the flow which vanishes at infinity. It would be natural to compute compute the momentum of this motion by calculating the integral $\int \rho \mathbf{u} dV$ of the region exterior to the body. The problem is that such harmonic flows have an expansion at infinity of the form

$$\phi \sim a \ln r - \mathbf{A} \cdot \mathbf{r} r^{-2} + O(r^{-2}) \tag{4.49}$$

in two dimensions and

$$\phi \sim \frac{a}{R} - \mathbf{A} \cdot \mathbf{R} R^{-3} + O(R^{-3}) \tag{4.50}$$

in three dimensions. Thus

$$\rho \int \nabla\phi dV = \int_S \phi \mathbf{n} dS, \tag{4.51}$$

where S comprises both a surface in a neighborhood of infinity as well as the body surface, is not absolutely convergent as the distant surfaces recedes. We point out that $a = 0$ in two dimensions if the area of the body is fixed and there is no circulation about the body. In three dimensions a vanishes if the body has fixed volume, see problem 4.12.

But even if $a = 0$ and $\phi = O(R^{-1})$ the value of the integral is only conditionally convergent will depend on how one defines the distant surface. So the value attributed to the fluid momentum is ambiguous by this calculation.

An unambiguous result is however possible, if we instead focus on the kinetic energy and from it determine the incremental momentum created by a change in velocity. Let us fix the orientation of the body and consider its movement through space, without rotation. This *translation* is completely determined by a velocity vector $\mathbf{U}(t)$. The, from the discussion of section 2.6 we know that a harmonic flow will satisfy the instantaneous boundary condition

$$\frac{\partial\phi}{\partial n} = \mathbf{U}(t) \cdot \mathbf{n} \tag{4.52}$$

on the surface of the body. Now $\nabla^2\phi = 0$ is a linear equation, and so we see that there must exist a vthe Φ_i as encoding the effect of the shape of the body from all possible harmonic flows associated with translation of the body.

We may now compute the kinetic energy E of the fluid exterior to the body using

$$\mathbf{u} = U_i \nabla\Phi_i. \tag{4.53}$$

Thus

$$E(t) = \frac{1}{2} M_{ij} U_i U_j, \quad M_{ij} = \rho \int \nabla\Phi_i \cdot \nabla\Phi_j dV. \tag{4.54}$$

the integral being over the fluid domain. Clearly the matrix M_{ij} is symmetric, and thus

$$dE = M_{ij} U_j dU_i. \tag{4.55}$$

On the other hand the change of kinetic energy, dE, must equal, in the absence of external body forces, the work done by the force $\mathbf{F}$ which the body exerts on the fluid, $dE = \mathbf{F} \cdot \mathbf{U} dt$. But according to Newton's second law, the incremental momentum $d\mathbf{P}$ is given by $d\mathbf{P} = \mathbf{F} dt$. Consequently $dE = \mathbf{U} \cdot d\mathbf{P}$. From (4.54) we thus have

$$dE = M_{ij} dU_j U_i = dP_i U_i. \tag{4.56}$$

Since this holds for arbitrary $\mathbf{U}$ we must have $dP_i = M_{ij} dU_j$. Integrating and using the fact that M_{ij} is independent of time and $\mathbf{P} = 0$ when $\mathbf{U} = 0$ we obtain

$$P_i = M_{ij} U_j. \tag{4.57}$$

Thus we have reduced the problem of computing momentum, and then the inertial force, to calculating M_{ij}. Since M_{ij} arises here as an effective mass term associated with movement of the body, it is called the *apparent mass matrix.*

But the calculation of M_{ij} is not ambiguous since the integral for the kinetic energy converges absolutely, and we can deduce M_{ij} once the energy is written in the form (4.54). We write

$$E = \frac{\rho}{2}\int_V |\mathbf{u}|^2 dV = \frac{\rho}{2}\int_V (\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}+\mathbf{U})dV + \frac{\rho}{2}\int_V |\mathbf{U}|^2 dV. \tag{4.58}$$

The reason for this splitting is to exhibit $\mathbf{u}-\mathbf{U}$, whose normal component will vanish on the body by (4.52). Now $\mathbf{u}+\mathbf{U} = \nabla(\phi+\mathbf{U}\cdot\mathbf{x}$ and $\mathbf{u}-\mathbf{U}$ is solenoidal, so $\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}+\mathbf{U}\nabla\cdot\big[(\phi+\mathbf{U}\cdot\mathbf{x})(\mathbf{u}-\mathbf{U})\big]$. Thus, remembering that $|\mathbf{U}|^2$ is a constant, the application of the divergence theorem and use of (4.52) on the inner boundary aallows us to reduce (9.31) to

$$E = \frac{\rho}{2}\int_{S_o} (\phi+\mathbf{U}\cdot\mathbf{x})(\mathbf{u}-\mathbf{U})\cdot\mathbf{n}dS + |\mathbf{U}|^2(\mathcal{V}-\mathcal{V}_b), \tag{4.59}$$

where S_0 is the outer boundary, $\mathcal{V}$is the volume contained by S_o, and $\mathcal{V}_0$ is the volume of the body.

To compute the integral in (4.59) we need only the leading term of ϕ. Referring to (4.49),(4.50), we note that $a=0$ for a finite rigid body (or even for a flexible body of constant area/volume), see problem 4.11. Using

$$\phi = -\frac{\mathbf{A}\cdot\mathbf{x}}{|\mathbf{x}|^N}, \quad \mathbf{u} = \frac{-\mathbf{A}}{|\mathbf{x}|^N} + \frac{N\mathbf{A}\cdot\mathbf{x}\,\mathbf{x}}{|\mathbf{x}|^{N+2}} \tag{4.60}$$

in (4.59) we have

$$E \sim \frac{\rho}{2}\int_{S_o} \Big[\frac{-\mathbf{A}\cdot\mathbf{x}}{|\mathbf{x}|^N} + \mathbf{U}\cdot\mathbf{x}\Big]\Big[\frac{-\mathbf{A}}{|\mathbf{x}|^N} + \frac{N\mathbf{A}\cdot\mathbf{x}\,\mathbf{x}}{|\mathbf{x}|^{N+2}} - \mathbf{U}\Big]\cdot\mathbf{n}dS. \tag{4.61}$$

We are free to choose S_o to be a sphere of radius R_o. The term quadratic in $\mathbf{A}$ in (4.61) is $O(R_o^{1-2N})$and so the contribution is of order R_o^{-N} and will vanish in the limit. The term under the integral quadratic in $\mathbf{U}$ yields $-|\mathbf{U}|^2\mathcal{V}$, thus canceling part of the last term in (4.59). Finally two of the cross terms in $\mathbf{U},\mathbf{A}$ cancel out, the remaining term giving rthe contribution $2\pi\rho(N-1)\mathbf{A}\cdot\mathbf{U}$. Thus

$$E = \frac{\rho}{2}\big[2\pi\rho(N-1)\mathbf{A}\cdot\mathbf{U} - \mathcal{V}_b|\mathbf{U}|^2\big]. \tag{4.62}$$

Since $\phi = \Phi_i U_i$, we may write $A_i(t) = \rho^{-1}m_{ij}U_j$ where m_{ij} is dependent on body shape but not time. Then

$$E = \frac{1}{2}\big[2\pi(N-1)m_{ij} - \mathcal{V}_b\rho\delta_{ij}\big]U_iU_j. \tag{4.63}$$

Comparing (4.63) and (4.54) we obtain an expression for the apparent mass matrix:

$$M_{ij} = 2\pi(N-1)m_{ij} - \mathcal{V}_b\rho\delta_{ij}, \quad N=2,3. \tag{4.64}$$

We thus can obtain the apparent mass of a body by a knowledge of the expansion of ϕ in a neighborhood of infinity.

Given that we have computed a finite fluid momentum we are in a position to state

Theorem 6 *(D'Alembert's paradox) In a steady flow of a perfect fluid in three dimensions, and in steady flow in two dimensions for a body with zero circulation, the force experienced by the body is zero.*

Clearly if the flow is steady $d\mathbf{P}/dt = \mathbf{F} = 0$, and we are done. Of course the proof hinges on the existence of a finite fluid momentum associated with a single-value potential function.

Example 4.14: To find the apparent mass matrix of a elliptic cylinder in two dimensions, we may use example 4.6. In the Z-plane the complex potential for uniform flow $-Q(\cos\theta, \sin\theta)$ past the cylinder of radius $a > b$ is $W(Z) = -Qe^{-i\theta}Z - Qe^{i\theta}s^a/Z$. Since $Z = \frac{1}{2}(z + \sqrt{z^2 - 4b^2})$ we may expand at infinity to get

$$w(z) \sim -Qe^{-i\theta}z - Q\big[\frac{a^2e^{i\theta} - b^2e^{-i\theta}}{z}\big] + \dots, \tag{4.65}$$

so that

$$\mathbf{A} = \big[U(a^2-b^2), V(a^2+b^2)\big], \ \ (U, V) = Q(\cos\theta, \sin\theta). \tag{4.66}$$

Now the ellipse intersects the positive x-axis at its semi-major axis $\alpha = \frac{a^2+b^2}{a}$, and the positive y-axis at its semi-minor axis $\beta = \frac{a^2-b^2}{a}$. From (4.66) we obtain the apparent mass matrix

$$\mathbf{M} = 2\pi\rho\begin{pmatrix} a^2-b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} - \pi\frac{a^4-b^4}{a^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pi\rho\begin{pmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}. \tag{4.67}$$

In particular for a circular cylinder the apparent mass is just the mass of of the fluid displaced by the body.

An alternative expression for the apparent mass matrix in terms of an integral over the surface of the body rather than a distant surface is readily obtained in terms of the potential Φ_i. We have

$$E = \frac{\rho}{2}\int_V \nabla\Phi_i \cdot \nabla\Phi_j U_i U_j dV = \frac{\rho}{2}\int_V \nabla\cdot\Phi_j\nabla\Phi_i]dVU_iU_j. \tag{4.68}$$

Applying the divergence theorem to the integral, surfaces S_o, s_B, and observing that $\Phi_i\nabla\Phi_j = O(|\mathbf{x}|^{1-2N})$, we see that the receding surface integral will give zero contribution. Recalling that $\frac{\partial\phi}{\partial n} = \mathbf{U}\cdot\mathbf{n}$ on the body surface, we see that $\frac{\partial\Phi_i}{\partial n} = n_i$ *where the normal is directed out of the body surface.* In applying the divergence theorem the normal at the body is into the body, with the result that (4.54) applies with

$$M_{ij} = -\rho\int_{S_b}\Phi_j n_i dS, \ \ \mathbf{n} \text{ directed out of the body.} \tag{4.69}$$

It follows from (4.57) that the fluid momentum is given by

$$\mathbf{P} = -\rho \int_{S_b} \phi \mathbf{n} dS. \tag{4.70}$$

We can verify the fact that (4.70) gives the fluid momentum by taking its time derivative, using the result of problem 1.6:

$$\frac{d}{dt} \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} \frac{\partial \phi}{\partial t} \mathbf{n} dS + \int_{S_b} (\mathbf{u} \cdot \mathbf{n}) \nabla \phi dS. \tag{4.71}$$

Using the Bernoulli theorem for harmonic flow we have

$$\frac{d}{dt} \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} \Big[-\frac{p}{\rho} - \frac{1}{2} |\mathbf{u}|^2 \Big] \mathbf{n} - (\mathbf{u} \cdot \mathbf{n}) \mathbf{u} dS. \tag{4.72}$$

Converting the terms on the right involving $\mathbf{u}$ to a volume integral, we observe that the latter converges absolutely at infinity, as so we have, for the integration over the domain exterior to S_b,

$$\int_V \big[\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{2} \nabla |\mathbf{u}|^2 \big] dV = - \int_V \mathbf{u} \times (\nabla \times \mathbf{u}) dV = 0. \tag{4.73}$$

Therefore

$$-\frac{d}{dt} \rho \int_{S_b} \phi \mathbf{n} dS = \int_{S_b} p \mathbf{n} dS = \mathbf{F}, \tag{4.74}$$

where $\mathbf{F}$ is the force applied by the body to the fluid.

Finally we note again that the inertial force required to accelerate a body in a perfect fluid will contain a contribution from the actual mass of the body, M_b. This mass appears as an additional term $M_b \delta_{ij}$ in the expression (4.64) for the apparent mass matrix. The total momentum of the body including its apparent mass is thus $P_i = M_{ij} U_j + M_b U_i$ and Newton's second law becomes

$$\frac{dP_i}{dt} = F_i, \tag{4.75}$$

where $\mathbf{F}$ is the force applied to the body, to accelerate it and the surrounding fluid.

The case of time-dependent M_{ij}

.

Since we have used an energy method to define the apparent mass of a body, the foregoing has assumed that the orientation of the body in space was fixed as a function of time. Thus time has entered only though $\mathbf{U}(t)$. However a change of shape of the body allows a particularly simple way to alter its effective mass, which offers the possibility of inertial modes of locomotion, see the final section of this chapter. To treat the case of time-dependent M_{ij}, we simply adopt Newton's second law (4.75) as our basic assumption. The computation of kinetic energy of a deforming body must involve the contribution from the deformation as well as the contribution from translation. Proper accounting of this new component allows the energy mthod to be generalized, but we omit details. Since we now know how to compute M_{ij} for any shape based on its instantaneous translation, we may now apply (4.75) allowing for a time-dependent apparent mass.

4.3.2 Moment

We have so far restricted the motion of the body to translation, i.e. with no rotation relative to the fluid at infinity. In general a moment is experienced by a body in translational motion, so that in fact a free body will rotate and thereby give the apparent mass matrix a dependence upon time. The theory may be easily extended to include a time dependent apparent mass, due either to rotation and/or deformation of he body, see section 4.4. But even in steady translational motion of a body, a non-zero moment can result, see problem 4.14. (There is no D'Alembert paradox for moment.)

For example, in analogy with (4.69), the *apparent angular momentum* of the fluid exterior to a body is defined by

$$\mathbf{P}_A = -\rho \int_{S_b} \phi(\mathbf{x} \times \mathbf{n}) dS, \tag{4.76}$$

the normal being out of the body. It may be shown in a manner similar to that used for linear momentum that

$$\frac{d\mathbf{P}_A}{dt} = \mathbf{T}, \tag{4.77}$$

where $\mathbf{T}$ is the torque applied to the fluid by the body.

4.4 Deformable bodies and their locomotion

It might be thought that, in an ideal, or more suggestively, a "slippery" fluid, it would be impossible for a body to locomote, i.e. to "swim" by using some kind of mechanism involving changes of shape. The fact is, however, that inertial forces alone can allow a certain kind of locomotion. The key point is that the flow remains irrotational everywhere, and this will have the effect of disallowing the possibility of the body producing an average force on the fluid which can then accelerate the body. Rather, it is possible to locomote in the sense of getting from point A to point B, put without any finite average acceleration. If the body is assumed to deform periodically over some cycle of configurations, then the kind of locomotion we envision is of a finite, periodic translation (and possible rotation) of the body, repeated with each cycle of deformation.

We first note that the Newtonian relationships that we derived above for a rigid body carry over to an arbitrary deformable body, which for simplicity we take to have a fixed area/volume. This follows immediately from our verification of $\frac{d\mathbf{P}}{dt} = \mathbf{F}$ from (4.70), since we made no assumption about the velocity of the body surface.

Now the idea behind inertial swimming is to deform the body in a periodic cycle which causes a net translation. To simplify the problem we consider only a simple traslation of a suitable symmetric body along a line, e.g a body symmetric about the z-axis, translating with velocity $U(t)$along this axis. In general we cannot expect the velocity to remain of one sign, but over one cycle there will be a positive translation, say to the right. Let $U_m(t)$ be the velocity of the center of mass of the body, and let $U_v(t)$ be the velocity of the center of volume of the body. Also let P_D be the momentum of deformation of the body *relative to its center of volume*. If the total mass of the body is m, then $U_m(t)m$ is the momentum of the body mass. Consider now the momentum of the fluid. If the apparent mass of the body (now a scalar $M(t)$) is multiplied by $U_v(t)$, we get the fluid momentum associated with the instantaneous motion of the shape of the body at time t. Finally, we have the momentum associated with the motion of the boundary of the body relative to the center of volume. If the potential of this harmonic flow of deformation is ϕ_D, then the deformation momentum is $P_D(t) = -\rho \int_{S_b} \phi_D \mathbf{n} \cdot \mathbf{i} dS$. The total momentum if body and fluid is thus $mU_m(t) + M(t)U_v(t) + P_D$. If initially the fluid and body is at rest, then this momentum, which is conserved, must vanish, and it is for this reason that locomotion is possible.

Consider first a body of uniform density. so the center of mass and of volume coincide. The $U_m = U_v = U$ and

$$U(t) = \frac{P_D(t)}{m + M(t)}. \tag{4.78}$$

There is no reason for the right-hand side o (4.78) to have non-zero time average, and when it does not, we call this *locomotion by squirming*. To see squirming in action it is best to treat an simple case, see example 4.15 below.

Alternatively, we can imagine that the center of mass changes relative to the center of volume without and deformation. Then deformation occurs giving a new shape, then the center of mass again changes relative to the center of volume holding the boy fixed in the new shape. If the two shapes lead to different apparent masses, locomotion occurs by *recoil swimming*, see example 4.16.

Example 4.15: We show in figure 4.7(a) a squirming body of a simplified kind. The body consists of a thin vertical strip of length $L_1(t)$, and a horizontal part of length $L_2(t)$. The length will change as a function of time, think of L_2 as being extruded from the material of L_1. We neglect the width w of the strips except when computing mass and volume. The latter are constant, implying $L_1 + L_2 = L$ is constant. The density of the material is taken as ρ_b, so the total mass is $M_b = \rho_b w L$ and the total volume is wL.

A cycle begins with $L_1 = L$, when L_2 begins to grow to the right. If $X(t)$ denotes the position of the point P, then $(\rho_b w L_1 + \pi \rho L_1^2/4)\frac{dX}{dt}$ is the momentum of the fluid and vertical segment, where we have used the formula for apparent

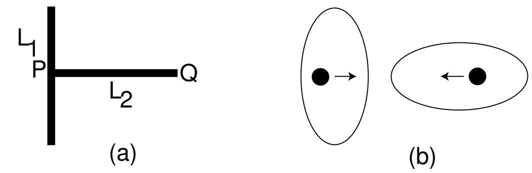

Figure 4.7: Swimming in an ideal fluid.

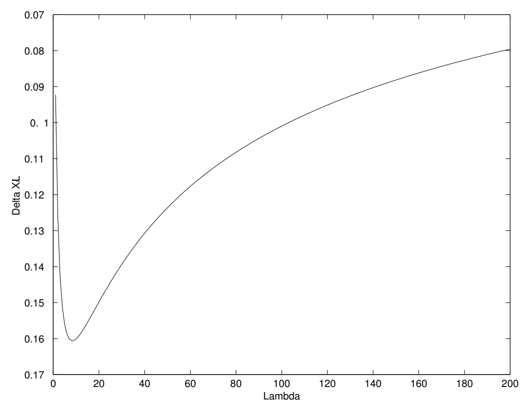

Figure 4.8: $\Delta X/L$ versus λ for the model squirmer of figure 4.7(a).

mass of a flat plate in 2D. The velocity of the extruded strip varies linearly from $\frac{dX}{dt}$ at P to $\frac{d(X+L_2)}{dt}$ at Q, so the momentum of the horizontal part is $\rho_b w L_2 d(X+\frac{1}{2}L_2)/dt$, where we neglect the apparent mass of the extruded strip. The first half of the cycle stops when $L_1 = 0$. Assuming the start is from fluid and body at rest, the sum of these momenta remains zero throughout the half-cycle:

$$(\rho_b w L + \pi\rho L_1^2/4)\frac{dX}{dt} + \frac{\rho_b w L_2}{2}\frac{dL_2}{dt} = 0. \tag{4.79}$$

If we et$L_2 = Lt/T, L_1 = L(1-t/T)$ where T is the half-period of the cycle, then we may obtain the change ΔX of X over the half-cycle by quadrature:

$$\Delta(X) = \frac{1}{\lambda}\ln(1+\lambda) - \frac{1}{\sqrt{\lambda}}\tan^{-1}(\sqrt{\lambda}), \quad \lambda = \frac{\pi\rho L}{4\rho_b w}. \tag{4.80}$$

We show this relation in figure 4.8 So we see that at the end of the half-cycle the point P has moved a distance $-\Delta X$ to the left. At this point, we imagine another half-cycle in which L_1 is created at the expense of L_2, but *at the point* Q. Observe that at the start of the second half-cycle Q is located a distance $L + \Delta X$ from the initial position of P. It can be seen from considerations of symmetry that the point Q will move to the left a distance $-\Delta X$ in time T over the second half-cycle. The the cycle is complete, $L_2 = L$, and the midpoint can be relabeled P. Thus the net advance to the right of the point P in time $2T$ has been $L + 2\Delta X$, which from figure 4.8 always exceeds about $.68L$.

Example 4.16: Recoil swimming can be illustrated by the 2D model of Figure (4.7)b. Let P denote the center of an elliptical surface of major,minor semi-axes α, β. Within this body is a mass M on a bar enabling it to be driven to the right or left. The weight of the shell and mechanism is m. Let the position of the center be $X(t)$ and the position of the mass be $x(t)$. At the beginning of the half-cycle the mass lies a distance $\beta/2$ to the right of P and the ellipse has its major axis vertical. The mass the moves to the left a distance β. Since momentunm is conserved, we have

$$(m + \rho\pi\alpha^2)\frac{dX}{dt} + M(\frac{d(X + x)}{dt} = 0. \tag{4.81}$$

Thus over a half-cycle $(m + \rho\pi\alpha^2)\Delta X + M(\Delta X + \Delta x) = 0$ or, since $\Delta x = \beta$,

$$\Delta X_1 = -\frac{M\beta}{m + M + \rho\pi\alpha^2}. \tag{4.82}$$

at this point the surface of the body deforms in a symmetric way, the points $(0, \pm\alpha/2$ moving down to $(0, \pm\beta/2$ and the points $(\pm\beta/2, 0$ moving our to $(\pm\alpha/2, 0)$, so that the major and minor axes get interchanged. There is no movement of P during this process. No the mass is moved back, a distance β to the left. We see that in this second half-cycle the displacement is

$$\Delta X_2 = \frac{M\beta}{m + M + \rho\pi\beta^2}. \tag{4.83}$$

The displacement over one cycle is then

$$\Delta X = \Delta X_1 + \Delta X_2 = \frac{M\beta}{m + M + \rho\pi\beta^2} - \frac{M\beta}{m + M + \rho\pi\alpha^2}, \tag{4.84}$$

which is positive since $\beta < \alpha$.

Problem set 4

1. (a) Show that the complex potential $w = Ue^{i\alpha}z$ determines a uniform flow making an angle α with respect to the x-axis and having speed U.

(b) Describe the flow field whose complex potential is given by

$$w = Uze^{i\alpha} + \frac{Ua^2e^{-i\alpha}}{z}.$$

2. Recall the system (4.13) governing the motion of point vortices in two dimensions. (a) Using these equations, show that two vortices of equal circulations Γ, a distance L apart, rotate on a circle with center at the midpoint of the line joining them, and find the speed of their motion.

(b) Show that two vortices of circulations Γ and $-\Gamma$, a distance L apart, move together on straight parallel lines perpendicular to the line joining them. Again find the speed of their motion.

3. Using the method of Blasius, show that the moment of a body in 2D potential flow, about the axis perpendicular to the plane (positive counterclockwise), is given by

$$M = -\frac{1}{2}\rho Re[\int_C z(dw/dz)^2 dz],$$

where Re denotes the real part and C is any simple closed curve about the body. Using this, verify by the residue method that the moment on a circular cylinder with a point vortex of circulation Γ at its center, in uniform flow, experiences zero moment.

4. Compute, using the Blasius formula, the force exerted by a point vortex at the point $c = be^{i\theta}, b > a$ upon a circular cylinder at the origin of radius a. The complex potential of a point vortex at c is $\frac{-\Gamma i}{2\pi}\ln(z-c)$. (Use the circle theorem and residues). Verify that the cylinder is pushed away from the vortex.

5. Prove Kelvin's minimum energy theorem: In a simply-connected domain V let $\mathbf{u} = \nabla\phi, \nabla^2\phi = 0$, with $\partial\phi/\partial n = f$ on the boundary S of V. (This $\mathbf{u}$ is unique in a simply-connected domain). If $\mathbf{v}$ is any differentiable vector field satisfying $\nabla \cdot \mathbf{v} = 0$ in V and $\mathbf{v}\cdot\mathbf{n} = f$ on S, then

$$\int_V |\mathbf{v}|^2 dV \geq \int_V |\mathbf{u}|^2 dV.$$

(Hint: Let $\mathbf{v} = \mathbf{u} + \mathbf{w}$, and apply the divergence theorem to the cross term.)

6. Establish (4.33) and work though he details of the proof of zero drag of the Rankine fairing using the momentum integral method, as outlined in section 4.2.2.

7. In spherical polar coordinates (r, θ, φ) a Stokes stream function Ψ may be defined by $u_R = \frac{1}{R^2 \sin\theta}\frac{\partial\Psi}{\partial\theta}, u_\theta = \frac{-1}{R\sin\theta}\frac{\partial\Psi}{\partial R}$.) Show that in spherical polar coordinates, the stream function Ψ for a source of strength Q, placed at the origin, normalized so that $\Psi = 0$ on $\theta = 0$, is given by $\Psi = \frac{Q}{4\pi}(1-\cos\theta)$. Verify that the stream function in spherical polars for the airship model consisting of equal source and sink of strength Q, the source at the origin and the sink at $R = 1, \theta = 0$, in a uniform stream with stream function $\frac{1}{2}UR^2(\sin\theta)^2$, is given by (4.35). (Suggestion: The sink will involve the angle with respect to $R = 1, \theta = 0$. Use the law of cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$ for a triangle with θ opposite side c) to express Ψ in terms of R, θ.)

8. In the Butler sphere theorem, we needed the following result: Show that $\Psi_1(R,\theta) \equiv \frac{R}{a}\Psi(\frac{a^2}{R},\theta)$ is the stream function of an irrotational, axisymmetric flow in spherical polar cordinates, provided that $\Psi(R,\theta)$ is such a flow. (Hint: Show that $L_R\Psi_1(R,\theta) = \frac{R}{a}L_{R_1}\Psi(R_1,\theta)$, where $R_1 = a^2/R$. Here L_R is defined by (4.39).)

9. (Reading, Milne-Thomson sec. 13.52 on "stationary vortex filaments in the presence of a circular cylinder" in 3rd edition.) Consider the following model of flow past a circular cylinder of radius a with two eddies downstream of the body. Consider two point vortices, of opposite strengths, the upper vortex having clockwise circulation $-\Gamma$(i.e. $\Gamma > 0$) located at the point $c = be^{i\theta}$, thus adding a term $(i\Gamma/2\pi)\ln(z-c)$ to the complex potential w, the other being having circulation Γ at the point $\bar{c} = be^{-i\theta}$. Here $b > a > 0$.

Using the circle theorem, write down the complex potential for the entire flow field, and determine by differentiation the complex velocity. Sketch the positions of the vortices and all vortex singularities within the cylinder, indicating their strengths.

10. Continuing problem 9, verify that $x = \pm a, y = 0$ remain stagnation points of the flow. Show that the vortices will remain stationary behind the cylinder (i.e. not move with the flow) provided that

$$U(1-\frac{a^2}{c^2}) = \frac{i\Gamma}{2\pi}\frac{(c^2-a^2)(b^2-a^2)+(c-\bar{c})^2a^2}{(c-\bar{c})(c^2-a^2)(b^2-a^2)}.$$

Show (by dividing both sides of the last equation by their conjugates and simplifying the result) that this relation implies $b - a^2/b = 2b\sin\theta$, that is, the distance between the exterior vortices is equal to the distance between a vortex and its image vortex.

11. Show that the apparent mass matrix for a sphere is $M_0/2\delta_{ij}$ where M_0 is the mass of fluid displaced by the sphere.

12. Show that for a body which may have a time-dependent shape but is of fixed area/volume, the quantity a in (4.49),(4.50) must vanish.

13. Using the alternative definition (4.69), show that M_{ij} is a symmetric matrix.

14. Let the elliptic cylinder of examples 4.14 and 5.13 be place in a steady uniform flow (U,V). Show, using the result of problem 4.3, that the moment experienced by the cylinder is $-\pi\rho UV(\alpha^2-\beta^2)$, α,β being the major and minor semi-axes of the ellipse.

Chapter 5

Lift and drag in ideal fluids

We take up now the study of the effects of vorticity on ideal fluid flow. One of the most interesting and subtle properties of ideal fluid flow theory is its relation to the physical properties of real, viscous fluids as the viscosity tends to zero. We will consider viscous fluids in chapters 6-8. In the present chapter we shall need to comment on some aspects of the role of viscous stresses in determining the relevance of the ideal fluid and the applicability of Euler's equations when vorticity and circulation do not vanish.

Our main point is to draw a distinction between the *limit process flow* obtained from a real fluid flow in the limit of vanishing viscosity, and the *ideal fluid flow* theory which results from setting viscosity formally equal to zero. Because of the nature of the mathematical form of viscous stresses, involving the *spatial rate of change of the velocity*, viscous stresses can be non-negligible at arbitrarily small viscosity when the velocity changes sufficiently rapidly. In ideal fluid theory the fluid velocity is assumed to be tangent to any fixed solid boundary abutting the flow. If this surface undergoes rapid changes in slope, as at a corner, large viscous stresses can develop. To relieve these stresses the flow can change, and we shall give examples of this below. The effect can persist even as viscosity vanishes. In fact surfaces need not be present. The persistent effect of viscosity also occurs in fluid away from boundaries, when the fluid is in turbulent motion. In that case the small spatial scales come from the stretching of vortex tubes by the flow. When a rigid body moves rapidly through a fluid it will often create vorticity, which is then embedded in the otherwise irrotational flow, and we shall explore examples of this. These considerations lead to ideal flow models which incorporate the effects of the viscosity of the real fluid, despite the fact that viscosity has been expelled from the equations of motion. Since the theoretical basis for dealing precisely with limit process flows is not well developed, the models we will study are fairly crude approximations. Nevertheless they adequately capture the essential physics and have important applications.

The title of this chapter emphasizes the most important applications of these ideas, to the concepts of lift and drag in aerodynamics. A closely related problem is the generation of thrust in flapping flight and by swimming fish. We begin with the calculation of lift of two-dimensional airfoils, then consider a model of a lifting three-dimensional wing. We also show in that case that drag is realized in an ideal (but not irrotational) fluid. Both lift and drag will be associated with vorticity. This vorticity may occur within a body, in which case it is called *bound vorticity*, or it may be in the flow exterior to the body, where it behaves as a material vector field. It is then called *free vorticity.* For a body moving with constant velocity through a quiescent ideal fluid can create a *vortical wake* stretching out behind it, a familiar example being the trailing vortices behind a high-flying jet. Vortical wakes are also creating by birds in forward flight, and by swimming fish.

It is easy to see how drag might be associated with a vortical wake. First we need to clarify these terms. *Drag* is conventionally equal to the component of force experienced by a body, parallel to the direction of motion. *Lift* is the component normal to the direction of motion, positive if opposite to gravity. If a body is pulled through the fluid with speed U and creates vorticity at a steady rate, this vorticity is carried off to infinity, and in the absence of viscosity there will be no decay. Consequently the associated flux of energy F_E is a loss to the system, which must be replenished by the work done against drag D, $UD = F_E$. A weightless self-propelled body in motion with constant average velocity is, according to Newton, not exerting any average force on the fluid. So in steady flight the drag must be balanced by thrust developed by a propeller, a jet engine, or a flapping fin. If the weight of a body minus the buoyancy (Archimedean) force is nonzero, then an unaccelerated body will exert a net force downward (against gravity) to compensate this net weight. The equilibrium can thus be expressed as thrust= drag and lift= weight in steady translation. In this case the energy in the vortical wake must equal to the work being done on the fluid by the body as it moves, to enable the flying or swimming, whether this is by a propeller, flapping wings, or a tail fin.

5.1 Lift in two dimensions and the Kutta-Joukowsky condition

In an ideal fluid any force on a two-dimensional body must be a result of the pressure exerted on the body. Accoring to the Bernoulli theorem for steady flow, the distribution of the pressure force over the surface of the body is directly related to the distribution of velocity there. This viewpoint can then lead to an attempt to understand the creation of lift as being a result of higher velocity over the top of the airfoil than over the bottom. Such an explanation, although mathematically correct, offers no hint of *why* the observed velocity distribution occurs.

To understand the basis of lift in two dimensions it is helpful to consider the simplest of "airfoils", namely a simple flat plate. Of course this choice is special since each end point of the plate is an extreme corner where the tangent

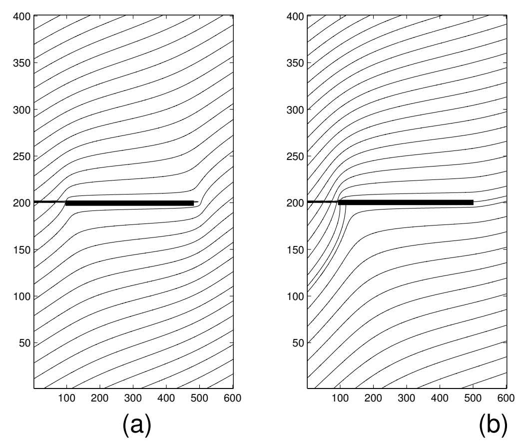

Figure 5.1: Streamlines for uniform flow past a flat plat, $\alpha = .2$. (a) Zero circulation. (b) Circulation determined by the Kutta-Joukowsky condition.

to the surface changes in direction by 180^o. We will later consider more realistic airfoils.

Now the circle of radius a into the Z-plane is mapped into the doubly-covered segment $|x| \leq 2a$ in the z-plane by $z = Z + a^2/Z$. We know also that uniform potential flow with velocity $(U, V) = Q(\cos\alpha, \sin\alpha)$ past the circular cylinder is not unique; the general solution is

$$W(Z) = Q[e^{-i\alpha}Z + a^2 e^{i\alpha} Z^{-1}] - \frac{i\Gamma}{2\pi} \ln Z, \tag{5.1}$$

where Γ is the circulation about the cylinder. With $w(z) = W(Z(z))$ we can then consider the streamline pattern about the plate for various Γ. The angle α, in the language of aeronautics, is called the *angle of attack* of the airfoil, here a flat plate.

We show in figure 5.1(a) the case of zero circulation $\Gamma = 0$. There are points of zero velocity, or *stagnation points* on the surface of the plate. The flow is forced around the endpoints so as to maintain the tangency of velocity at the body, and it is easy to see that there are singularities of velocity and pressure at $z = (\pm 2a, 0)$. In figure 5.1(b) we show the same flow with a negative Γ, chosen to move the stagnation point on the upper surface to the point $(2a, 0)$ and to eliminate the singularity there. The singularity at the point $(-2a, 0)$ remains.

We must now ask, what does one observe in a wind tunnel? Under conditions where ideal flow theory should prevail, except very close to the body surface, it is observed that the flow does come away smoothly from the downstream or "trailing" edge of the plate, as in figure 5.1(b). Experiments also show that the flow art the upstream or "leading" edge of the plate is actually as shown in figure 5.2, with *separation* of the boundary streamline at the leading edge, and reattachment further back, a small circulating eddy being enclosed by the separating streamline.

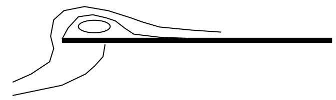

Figure 5.2: Leading-edge separation from a flat plate.

What we thus see is a definite upstream-downstream asymmetry of the flow in its response to the singular points of the boundary. [1] The flow seeks to make a smooth flow off the trailing edge, but accommodates itself on the leading edge by forming a separation bubble which effectively gives a smooth shape to the upstream end of the surface.

It can be assumed that this observed flow actually persists in the limit of arbitrarily small viscosity. In fact, hydrodynamic instabilities will generally prevent one from every observing the limit process as a steady flow, but the assumed limit would presumably apply to the unstable steady branch of solutions of the equations of the real fluid.

The condition which selects, among all possible values of Γ, the unique value which eliminates the singularity at the trailing edge of the plate, is called the *Kutta-Joukowsky condition*. To apply it in the present case, we note that

$$\frac{dw}{dz} = \Big[Q[e^{-i\alpha} - a^2 e^{i\alpha} Z^{-2}] - \frac{i\Gamma}{2\pi} Z^{-1}\Big]\frac{dZ}{dz}$$

$$= \Big[Q[e^{-i\alpha} - a^2 e^{i\alpha} Z^{-2}] - \frac{i\Gamma}{2\pi} Z^{-1}\Big]\Big[\frac{\sqrt{z^2-4a^2}+z}{2\sqrt{z^2-4a^2}}\Big]. \tag{5.2}$$

The terms within the first set of large brackets must sum to zero at $Z = a$ if the singularity at $z = 2a$ is to be removed. Thus we find

$$\Gamma = -4\pi Q a \sin\alpha \tag{5.3}$$

from the Kutta-Joukowsky condition. Once this condition is applied, the ideal flow theory matches the observations at the trailing edge, but fails to account for the separation bubble at the leading edge. This turns out not to be a serious discrepancy since the bubble acts to smooth the pressure distribution and mimic the smoothing of the airfoil leading edge.

To see that the resulting flow gives rise to a lift force, we compute the force on the body from Blasius' formula (4.14). From the residue of $(dw/dz)^2(z)$ we find for pour flat plate problem (see problem 5.1(a))

$$X - iY = \rho\Gamma Q(\sin\alpha + i\cos\alpha). \tag{5.4}$$

This is a force of magnitude $L = 4\pi\rho a Q^2 \sin\alpha$, which is orthogonal to the free stream velocity for the geometry of figure 5.1(b), and is upward for positive α by (5.3) $(\Gamma < 0)$, so it is indeed a lift. It is *not* orthogonal to the plate itself,

[1]We shall see in chapter 8 that this asymmetry can be traced to the parabolic nature of the partial-differential equation for the viscous boundary layer on the surface of the body.

which raises the paradoxical situation where a pressure force, presumably always orthogonal to the surface, seems to be in violation of that fact. The resolution of this paradox involves a careful analysis of the singularity near the leading edge. The airfoils considered in the next section have a smooth leading edge, and the flate plate may be regarded as the limit of a family of such smoothed foils. Now for each member of the family, it is found that the pressure distribution around the smooth nose in fact produces a component of force parallel to the plate, which is precisely the magnitude needed to make the lift vector orthogonal to the free stream velocity. This "leading edge suction force" is preserved in the limit, even though the "edge" the disappears, and gives the result (5.1) for the flat plate.

The result (5.1) is known as the *Kutta-Joukowsky theorem.* It is central to airfoil theory because it is a general result independent of the actual shape of the airfoil. The reason for this generality can be understood once on sees how the residue computation goes for the flat plate. For any foil at angle of attack in a uniform stream the expansion of $w(z)$ near infinity will have the form

$$w = Qze^{-\alpha} - \frac{i\Gamma}{2\pi}\log z + \frac{A}{z} + \frac{B}{z^2} + \dots, \tag{5.5}$$

which follows from the definition of circulation. The subsequent terms involving $A, B, \dots$ are determined by the particular shape of the airfoil. When we compute the residue of $(dw/dz)^2$ at infinity we obtain $-\frac{iQ}{\pi}e^{-i\alpha}$ and this leads to . Thus the lift computation for any airfoil really amounts only to a determination of of the circulation required by the Kutta-Joukowsky condition.

Although the K-J condition gives a unique circulation and lift for an airfoil, it remains an approximation to reality. Rapid movements of an airfoil can produce momentary flows which differ from that obtained under the K-J condition, and may be close to the flow with zero circulation in figure 5.1(a). In fact the "true" K-J theory, which would allow the "correct" ideal flow representing the slightly viscous flow under arbitrary movements of a body, remains an important, outstanding unsolved problem of fluid dynamics, and lies at the heart of a rigorous theory of vortex shedding from surfaces.

5.2 Smoothing the leading edge: Joukowski airfoils

We have noted that the leading edge of a flat plate is not well suited to the smooth flow that we wanted to establish around an airfoil by the application of the Kutta-Joukowsky condition. Airfoil designers therefore prefer a shape which maintains the sharp trailing edge, so as to "force" the Kutta-Joukowsky condition there, but which also provides a smooth leading edge around which the flow may pass without detachment.

Remarkably, such foils can be obtained by a simple modification of the conformal map associated with the flat plate flow. Instead of considering flow

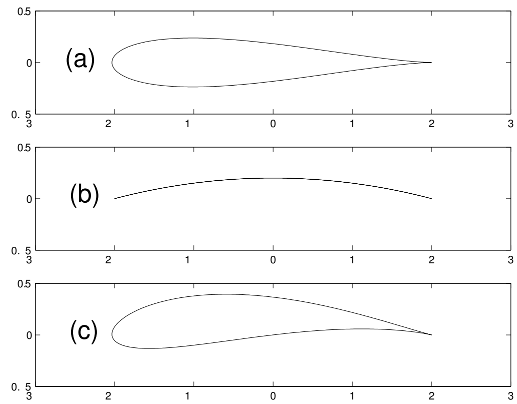

Figure 5.3: Joukowsky airfoils, a=1. (a) $\epsilon = -.1, \nabla = 0$ (b) $\epsilon = 0, \nabla = .1$ (c) $\epsilon = -.1, \nabla = .1$.

around the circular cylinder of radius a and center at the origin, we consider the flow past a circular cylinder with center at $Z_0 = \epsilon + i\delta$ and radius $c = \sqrt{(a-\epsilon)^2 + \delta^2}$, with $\epsilon < 0, \delta > 0$. We show in figure 5.3 the foil shapes that result from various choices of ϵ, δ. Note that ϵ determines the foil thickness, and δ its *camber*, or the arc the foil makes relative to the x-axis. The geometry of the Z-plane is shown in figure 5.4. The trailing edge is a cusp.

It is not difficult to modify the force calculation to accommodate the Joukowsky family of profiles, and there results a lift force orthogonal to the free stream velocity, but with magnitude (see problem 5.1(b))

$$= 4\pi\rho c Q^2 \sin(\alpha + \beta), \tan\beta = \frac{\delta}{a - \epsilon}. \tag{5.6}$$

Note that the effect of camber is to change the angle of attack at which the lift vanishes.

The moment on a Joukowsky airfoil can be computed by residue theory using the formula given in problem 4.3. To work this out we have

$$\left(\frac{dw}{dz}\right)^2 = \Big[Qe^{-i\alpha} - \frac{c^2}{z^2}Qe^{i\alpha} - \frac{i\Gamma}{2\pi z} - \frac{i\Gamma Z_0}{2\pi z^2}\Big]^2\Big[1 - \frac{a^2}{z^2}\Big]^2 +)(z^{-3}), \tag{5.7}$$

and so the residue at infinity of $z(dw/dz)^2$ is $-2Q^2a^2e^{-21\alpha} - 2Q^2c^2 - \frac{i}{\pi}Q\Gamma Z_0 e^{-i\alpha} - \frac{1}{4\pi^2}\Gamma^2$. Thus

$$M = -\frac{1}{2}\rho\Re\Big[(2\pi i)\big[-2Q^2a^2e^{-2i\alpha} - 2Q^2c^2 - \frac{i}{\pi}Q\Gamma Z_0 e^{-i\alpha} - \frac{1}{4\pi^2}\Gamma^2\big]. \tag{5.8}$$

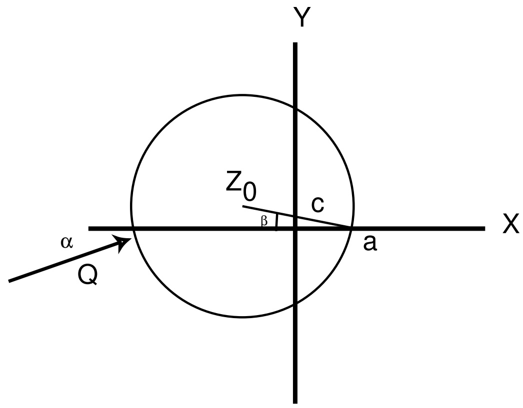

Figure 5.4: Geometry of the Z-plane for the Joukowsky airfoils. $Z_0 = \epsilon + i\delta$.

After substituing $Z_0 = Z_0 - a + a = -ce^{-i\beta} + a$ we obtain

$$M = -2\pi\rho Q^2 \left[a^2 \sin 2\alpha + c^2 \sin 2(\alpha + \beta) - 2ac \cos\alpha \sin(\alpha + \beta)\right]. \qquad (5.9)$$

Recall moment is positive in the counter-clockwise direction and (5.9) refers to a Joukowski airfoil with the trailing edge to the right. If $\beta = 0$ and α, δ are small, then $a \approx C$ and $M \approx -4\pi\rho Q^2 a^2 \alpha \approx -aL$. This places center of lift at approximately $z = -a$. The length of the foil, known as the *chord*, $\approx 4a$, so the center of lift is approximately at the quarter-chord point. For many aircraft the position of the center of gravity is is located near this point to provide stability to forward flight.

If one looks at wind tunnel data for Joukowsky airfoils, or for the many other foil designs of a similar kind, it is found that the predictions for lift at small angles of attack is reasonably good, especially in slope. Usually one plots a *lift coefficient* $C_L = \frac{L}{\frac{1}{2}\rho Q^2 4a}$ versus α. For the Joukowsky foils $C_L = 2\pi\frac{\sin(\alpha+\beta)}{\cos\beta}$. Realized lift is usually somewhat smaller than predicted. More dramatic is the failure of the theory to account for *airfoil stall*, a fall-off lift with increasing α, which usually begins for α in the range $10 - 15^o$. Stall is a result of separation of the flow from the foil, again a manifestation of the effects of viscosity. Usually the flow becomes unsteady as well, so an aircraft experiences buffeting and an abrupt loss of lift. Aircraft designers introduce modifications of three-dimensional wings, such as twist, reducing the angle of attack of outboard wing sections relative to inboard, to minimize the control problems and make the stall a more gradual phenomenon as angle of attack increases.

5.3 Unsteady and quasi-steady motion of an airfoil

Unsteady motion of an airfoil occurs during th take-off and maneuvering of an airplane, and in the flapping of the wings of birds and insects. A interesting thought experiment is to imagine an airfoil at positive angle of attack and leading edge to the left, to be suddenly accelerated from rest to the velocity $(-Q, 0)$. After the flow has settled down, and observer moving with the foil would see a steady flow $(Q, 0)$ past the foil and would measure a lift, hence a circulation $\Gamma > 0$. Now repeat the experiment with a large material contour initially encircling the foil, see figure 5.5. The initial circulation on this contour is is zero since the fluid is at rest. After the acceleration to a fixed velocity, there exists a negative circulation about the foil. However, according to Kelvin's theorem, the circulation about the image of the large initial contour, now distorted by the motion of the foil, must remain zero. (The contour is a material curve.) Since we know the foil has negative circulation, there must be other vorticity within the contour contributing positive circulation. Observation of the acceleration of foils shows that this missing vorticity occurs at the initial acceleration of the foil. Positive vorticity is rapidly shed at the trailing edge, to form a coherent *starting vortex* whose circulation exactly cancels the circulation bound to the foil in steady flight, as we show in figure 5.5. This is an example of the *unsteady aerodynamics* of an airfoil. Such unsteady motions will generally involve shedding of vorticity from the trailing edge, and the shed vorticity will then influence the flow external to the foil. The shed vorticity moves with the fluid, and must be accounted for in calculating the forces on the foil.

A measure of unsteadiness is a parameter of the form $\frac{L}{TU}$ where L is some typical length, T a time over which a cycle of motion is performed, and U a speed of flight. If this dimensionless number is of order unity or larger, the the resulting flow is sqid to be fully unsteady. If the number is small, the flow is said to be *quasi steady*. A dragonfly may beat its wings once in $T = 1/40$ second and have a wing chord L=1 cm. If it moves at U=40 cm/sec then $\frac{L}{TU} = 1$ and the flow is unsteady. A pigeon with a wingbeat each 1/5 sec, a wing chord of 10 cm, flying at 3 m/sec has $\frac{L}{TU} = 1/6$, so its flight might be considered quasi-steady.

Let us devise a quasi-steady theory of forward fight of a flapping wing. While it is true that birds are flapping their wings to fly, the fact is that the main reason for flapping is to produce *thrust*, so as to overcome drag. A flapping Joukowsky airfoil at angle of attack $\alpha = -\beta$ produces only thrust. To acquire lift it is only necessary to increase the angle of attack while maintaining the flapping at that angle of attack. But of course to develop lift the wing must be moving! So the initiation of flappong flight is a kind of "bootstrap" operation where special wing movements may be needed.

To understand thrust production without any net lift consider a simple flat plate in uniform flow, which maintains itself horizontal while moving up and down in a periodic motion, see figure 5.6. Remember that as the plate moves we sshall assume quasi-steady aerodynamics, meaning that the *instantaneous*

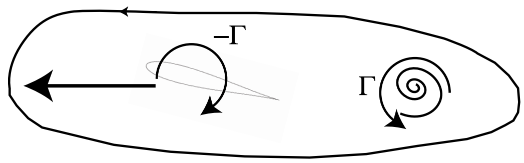

Figure 5.5: The starting vortex shed by a lifting foil abruptly accelerated to a constant velocity.

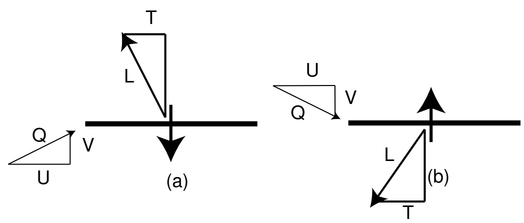

Figure 5.6: Thrust production by quasi-steady flapping of a flat plate.

flow about the plate will be the steady flow corresponding to the instantaneous velocity the wing sees approaching it, and we assume the Kutta-Joukowsky condition applies. Thus in figure 5.6(a) the wing is moving down with speed V, and so sees an effective angle of attack $\alpha, \tan\alpha = V/U$. The "lift" vector $\mathbf{L}$ is orthogonal to this the instantaneous approach velocity vector (U, V), which produces a thrust component $T = L\sin\alpha = 4\pi a Q^2 \sin^2\alpha$. In fig 5.6(b) the wind moves up, but the same expression for thrust results. Thus the average in time of the thrust is positive, $\overline{T} = 4\pi a Q^2 \overline{\sin^2\alpha}$.

We remark that in quasi-steady flapping flight there is a steady stream of vorticity shed from the the trailing edge of the foil, but it is swept downstream so fast that its effect on the flow is small.

5.4 Drag in two-dimensional ideal flow

In the present section we give two examples of the modeling of drag in an ideal fluid. Recall that for irrotational flow the drag force will vanish in two or three dimensions. In fact, a body in a real fluid will experience drag. We will see how drag in two-dimensional flow can result from vorticity in the fluid.

5.4.1 The Von Kármán vortex street

Experiments with flow past a circular cylinder in a wind tunnel, and numerical calculation in two dimensions, show that as the velocity of the stream increases, a point is reached where the flow becomes unsteady and vortices are shed into the flow, alternating between the top and bottom of the cylinder, see figure

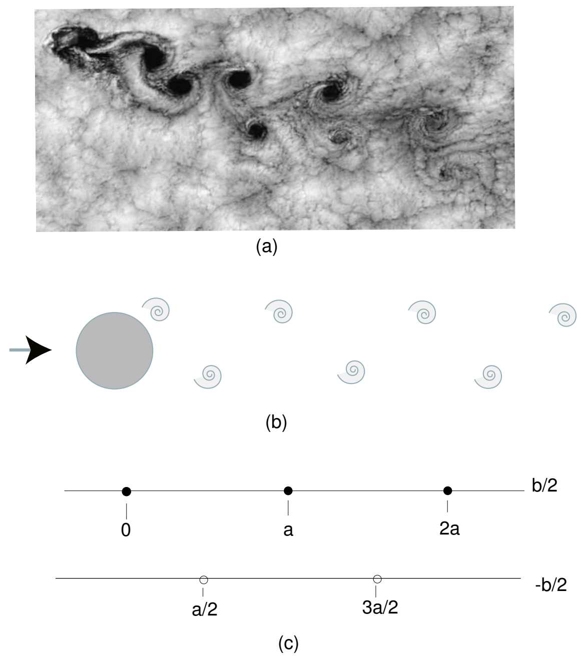

Figure 5.7: (a) Kármán vortex street in the atmosphere due to motion past an island off the Chilean coast. These atmospheric motions are very nearly two-dimensional. (b) Schematic of vortex shedding from a circular cylinder. (c) The doubly infinite street. The upper vortices carry circulation $-\gamma$. The lower vortices carry circulation $\gamma > 0$.

5.7(a). These vortices (really patches of vorticity) are the carried by the flow downstream, forming a vortical wake. This wake carries energy downstream, and cylinder experiences a drag. The time dependence can give rise to an oscillating lateral force, and one manifestation is the "singing" of wires in a wind.

Von Kármán developed a simple model for such a wake, called now the *Kármán vortex street.* It consists of a periodic array of point vortices of strengths $\pm\gamma$, extending from the cylinder to downstream infinity. It can be most conveniently analyzed by extending the street to upstream infinity as well. So the model is of the wake well downstream of the cylinder, see figure 5.7(b). The We show in figure 5.7(c) Von Kármán's doubly infinite vortex street. To study this flow, consider first a single finite line of vortices of circulation γ, spaced a distance a apart on the x-axis. The velocity potential is

$$w_N = -\frac{i\gamma}{2\pi}\sum_{n=-N}^{+N}\log(z-na) = -\frac{i\gamma}{2\pi}\log\left[\frac{\pi z}{a}\Pi_{n=1}^{N}\big(1-\frac{z^2}{n^2a^2}\big)\right] + \text{ constant.} \tag{5.10}$$

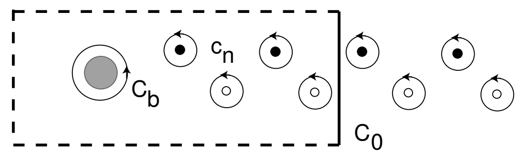

Figure 5.8: Contours for calculating drag for the vortex street.

Using the identity

$$\sin z = z\big(1 - \frac{z^2}{\pi^2}\big)\big(1 - \frac{z^2}{2^2\pi^2}\big)\big(1 - \frac{z^2}{3^2\pi^2}\big)\cdots, \tag{5.11}$$

we get, in the limit $N \to \infty$ for a suitable additive constant,

$$w_n \to w_\infty = -\frac{i\gamma}{2\pi}\log\sin\frac{\pi z}{a}. \tag{5.12}$$

For the vortex street shown in figure 5.7(c), we thus have

$$w_\infty = \frac{i\gamma}{2\pi}\log\Big(\frac{\sin\frac{\pi}{a}(z - ib/2)}{\sin\frac{\pi}{a}(z - a/2 + ib/2)}\Big). \tag{5.13}$$

Since the vortices are being shed by a body in a flow $(U, 0)$, relative to the body the complete velocity potential is

$$w = Uz + w_\infty. \tag{5.14}$$

Two see how the vortices are moving relative to an observer fixed with the body, we can, by symmetry, consider the velocity at $(0, b/2)$ for the system minus the vortex at that point. Thus the vorticies move with velocity

$$\lim_{z\to ib/2}\Big[\frac{dw}{dz} - \frac{i\gamma}{2\pi}\frac{1}{z - ib/2}\Big] = U - V. \tag{5.15}$$

Evaluating this limit, we find

$$V = \frac{\gamma}{2a}\tanh\frac{\pi b}{a}. \tag{5.16}$$

In experiments V is considerably less than U, so the vortices move downstream with a speed slightly less than the free stream speed. For a circular cylinder of diameter D the vortices of like sign are shed with a frequency f where $fd/U \approx .2$. Thus $(U - V)/f = a \approx 5D$.

The drag force can be computed using the Blasius formula for force, but with an added effect due to the fact that vortices are being steadily added as they are shed from the body. We describe the ideas involved with this calculation without all the details. Relative to an observer *fixed with the vortex street* the velocity at infinity is V, the body is moving to the left with speed $U - V$. Imagine a rectangular boundary $C_0 = ABCD$ surrounding the the entire region, as shown in figure 5.8. The dotted sides will eventually move off move off to infinity and the solid line will be placed at a position far downstream where the street will be effectively doubly infinite. The small positively oriented contours c_n surround the vortices, and the contour C_b is the body contour. The solid right side of the outer rectangular contour does not intersect any vortex. At a particular time the the Blasius theorem may be applied to yield

$$X - iY = -\oint_{\sum c_n} \Big(\frac{dw}{dz}\Big)^2 dz + \oint_{C_0} \Big(\frac{dw}{dz}\Big)^2 dz, \tag{5.17}$$

where the sum is over the c_n within C_0. In this frame the potential seen on AB,

$$w_V = Vz + w_\infty, \tag{5.18}$$

will be essentially independent of time of the street is taken as doubly infinite. The contributions from the first integral in (5.17) are seen to contribute only to Y, since the residues are just $2V\frac{\pm i\gamma}{2\pi}\gamma$. (These contributions would allow us to deduce an oscillating vertical force on the body.) The second term in (5.17) gets a contribution in the limit only from the right vertical side of the outer contour, and we obtain the following contribution to the drag from (5.17):

$$D_1 = \Re\frac{i\rho}{2}\int_{-i\infty}^{+i\infty} \Big(\frac{dw_V}{dz}\Big)^2 dz = \frac{\gamma^2\rho}{2\pi a}\Big(1 - \frac{\pi b}{a}\tanh\frac{\pi b}{a}\Big). \tag{5.19}$$

However there is also momentum being created as a function of time by the shedding of vortices within C_0. At this point we must do an approximate calculation, for the shed vortices break the symmetry of a doubly infinite street. We can approximate the calculation by determining the $x-$momentum per unit length of the street from w_∞, m say, then determining the momentum shed per cycle period T as ma/T. Since $(U - V)T = a$, the contribution will be $D_2 = -m(U - V)$ since positive drag contributes negative x-momentum.

To compute ma, we need only consider two adjacent vortices of opposite sign. Thus

$$\begin{aligned} ma &= \rho\int \frac{dw_\infty}{dz} dS \\ &= \rho\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \Big(\frac{i\gamma}{2\pi(x+iy-ib)} - \frac{i\gamma}{2\pi(x+iy-a/2+ib)}\Big) dx. \\ &= \frac{i\rho\gamma}{2\pi}\int_{-\infty}^{+\infty} \big[\log(x+iy-ib) - \log(x+iy-a/2+ib)\big]_{x=-\infty}^{x=+\infty} dy. \end{aligned} \tag{5.20}$$

Now the integrand in (5.20) gets contributions from the change of the argument of the log terms as x goes from $-\infty$ to $+\infty$. This is seen to give $+2\pi i$ when $|y| < b$ and zero when $|y| > b$. Thus we have

$$ma = -\rho\gamma b, \tag{5.21}$$

and

$$D_2 = \rho b\gamma(U - V)/a. \tag{5.22}$$

Thus the drag of the body is

$$D = D_1 + D_2 = \frac{\gamma^2\rho}{2\pi a}\Big(1 - \frac{\pi b}{a}\tanh\frac{\pi b}{a}\Big) + \rho b\gamma(U - V)/a$$

$$= \frac{\rho\gamma b}{a}(U - 2V) + \frac{\rho\gamma^2}{2\pi a}. \tag{5.23}$$

5.4.2 Free streamline theory of flow normal to a flat plate

There is another body of theory in two-dimensional ideal fluid flow involving streamlines on which velocity is discontinuous, and where these lines of discontinuity are embedded in the flow exterior to any boundaries. These *free streamline* theories effectively embed free vorticity in an otherwise irrotational flow field. Suppose that one one side of a streamline, as the streamline is approached, the velocity is non-zero, but on the other side the velocity is identically zero and pressure is constant. If the flow is steady and Bernoulli's theorem applies then on the flow side $p + \frac{\rho}{2}|\mathbf{u}|^2$ is constant on the streamline. We now assert that at such a streamline pressure must be continuous. Otherwise a difference of pressure would act across a sheet, with no inertia to support such a force by a finite acceleration. Thus it must be that $|\mathbf{u}| = q$ is constant on the free streamline.

We will now examine a model, due to Kirchoff, which seeks to represent the detached flow that is observed behind bluff bodies in a uniform stream. The theory will deal with a steady flow, even though the observed flows are always time-dependent. The structure is shown in figure 5.9 in the case of flow broadside onto a finite flat plate. Two free separation streamlines leave the tips of the plate and extend to infinity aft of the body. The region behind the plate, between the free streamlines, is a cavity or "dead water" region, where velocity is zero and pressure a constant p_0. Well upstream the velocity is $(U, 0)$ and the pressure is p_0. Thus $p + \frac{\rho}{2}q^2 = p_0 + \frac{\rho}{2}U^2$ and in particular $q = U$ on the free streamlines.

Ths solution to this flow problem involves a interesting technique in conformal mapping, which exploit a correspondence between identical maps in distinct variables, allowing a direct connection between these variables and an equation determining the complex potential. The procedure is sometimes referred to as a *hodograph method* because the velocity components appear in the definition of an intermediate complex variable. We now describe the series of maps involved and the connections between them.

We first note that $w = \phi(x, y) + i\psi(x, y)$, whatever form it may take, maps the z or physical plane shown in figure 5.9 onto the w-plane as shown in figure 5.10(a). The body is here a streamline $\psi = 0$. We next map the w plane onto the Z plane as shown in figure 5.10(b). The map is defined by

$$w = \frac{C}{2}Z^2 \tag{5.24}$$

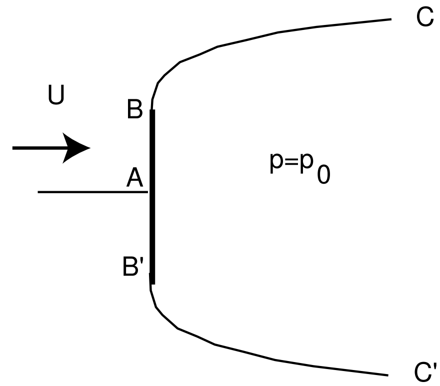

Figure 5.9: Free streamline flow onto a flat plate, a=1.

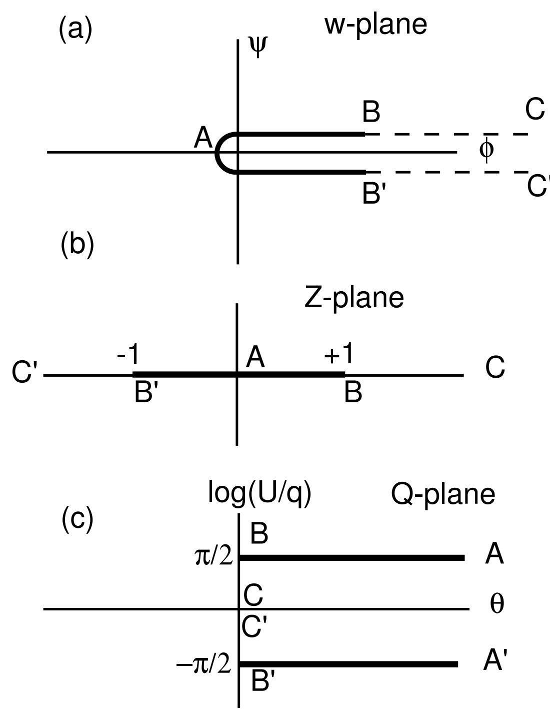

Figure 5.10: The conformal mappings for the Kirchoff solution.

where the constant C will need to be chosen to make the points B, B' map onto $(1,0)$ and $(-1,0)$.

Next, consider the variable

$$Q = \ln \frac{U}{q} + i\theta, \tag{5.25}$$

where $q = \sqrt{u^2+v^2}, \theta = \tan^{-1}\frac{v}{u}$ with $u - iv = dw/dz$.. The Q plane will be the hodograph plane. The mapping of $Y = 0$ to the hodograph plane is especially simple since either the angle or the speed is constant. Thus we are bound to get a polygon. Since we know how to map a polygon onto the upper half-plane, we can connect Z to Q.

We show the Q plane in figure 5.10(c). The map from z to Q is a Schwarz-Christoffel map, given by

$$\frac{dQ}{dZ} = \frac{1}{Z\sqrt{Z^2-1}} \times \text{ constant}. \tag{5.26}$$

The integral may ve calculated using a substitution $Z = 1/\cosh X$. We obtain

$$Q = C_1 \cosh^{-1}\frac{1}{Z} + C_2 = C_1 \log\left[\frac{1}{Z} + \sqrt{\frac{1}{Z^2} - 1}\right] + C_2 = Q(Z), \tag{5.27}$$

where $C_{1,2}$ are constants.

But $Q(e^{i\pi}) = -C_1 i\pi + C_2 = -i\pi/2, Q(1) = C_2 = i\pi/2$, giving

$$Q = \log\left(\frac{1}{Z} + \sqrt{\frac{1}{Z^2} - 1}\right) + \frac{i\pi}{2}. \tag{5.28}$$

Since $Q = \log \frac{U}{\frac{dw}{dz}}$ we have using (5.28)

$$U\frac{dz}{dw} = iZ^{-1}[1 + \sqrt{1-Z^2}]. \tag{5.29}$$

Also $\frac{dw}{dZ} = CZ$, so

$$U\frac{dz}{dZ} = iC(1 + \sqrt{1-Z^2}). \tag{5.30}$$

If the width of the plate is L, then

$$\int_{-1}^{+1} U\frac{dz}{dZ}dZ = iUL = iC\int_{-1}^{+1}(1+\sqrt{1-Z^2}dZ = iC(2+\pi/2). \tag{5.31}$$

This determines C and gives

$$w = \frac{UL}{4+\pi}Z^2. \tag{5.32}$$

Since we also have

$$U^{-1}\frac{dw}{dz} = \frac{iZ}{1+\sqrt{1-Z^2}}, \tag{5.33}$$

we have defined implicitly $w(z)$.

We will now show that, because of the cavity, the plate experiences a drag. The drag is given by

$$D = \int_{plate} p dy = -i \int p dz = \frac{-i\rho}{2} \int (U^2 - q^2) dz. \tag{5.34}$$

Now on the front face of the plate $q^2 = v^2 = \left(\frac{\partial \phi}{\partial y}\right)^2 = -(dw/dz)^2$, and so , using (5.32) and (5.30) we have

$$D = \frac{\rho U^2 L}{2} - \frac{i\rho}{2} \int_{-1}^{+1} (dw/dZ)^2 (dZ/dz) dZ$$

$$= \frac{\rho U^2 L}{2} - \frac{\rho U L}{4+\pi} \int_{-1}^{+1} (1 - \sqrt{1 - Z^2}) dZ = \frac{\rho U^2 L}{2} \frac{4-\pi}{4+\pi} = \frac{\rho U^2 L \pi}{\pi + 4}. \tag{5.35}$$

This drag is close to what is observed when a flat plate is placed in a stream and a wake cavity forms. As we have already noted observed bluff body flows are time dependent and of course the cavity is finite in extent. Nevertheless the Kirchoff solution is a classic example of fluid modeling, exhibiting many features of observed flows and providing a good example of the role of free streamlines in the production of drag.

5.5 The 3D wing: Prandtl's lifting line theory

Airplanes and birds fly in three dimensions. We will now explore how lift and drag arise in the real world. Since D'Alembert's paradox now implies neither lift nor drag is possible in irrotational flow, it is clear that lift and/or drag imply the existence of vorticity in the fluid.

We start by reviewing the vorticity stucture of a 2D Airfoil, in particular a flat plate at angle of attack with the Kutta-Joukowsky condition applied. We know the complex potential in the flat plate problem from section 5.1:

$$w(z) = W(Z(z)), W(Z) = Q[e^{-i\alpha} Z + a^2 e^{i\alpha} Z^{-1}] - \frac{i\Gamma}{2\pi} \ln Z, \tag{5.36}$$

with

$$Z(z) = \frac{1}{2}[z + \sqrt{z^2 - 4a^2}], \Gamma = -4\pi a \sin \alpha. \tag{5.37}$$

Since the airfoil has zero thickness, vorticity must be concentrated on the line segment $|x| < 2a, y = 0$. Now $v = 0$ on the segment, so the vorticity $\omega = v_x - u_y$ is given by $-u_y$, and we shall see that u is discontinuous on the segment. Thus the vorticity of the flat plate is proportional to $\delta(y)$, and the total vorticity at a given value of x must be computed as $\gamma(x) = -u(x, 0+) + u(x, 0-)$, with

$$\Gamma = \int_{-2a}^{2a} \gamma(x) dx. \tag{5.38}$$

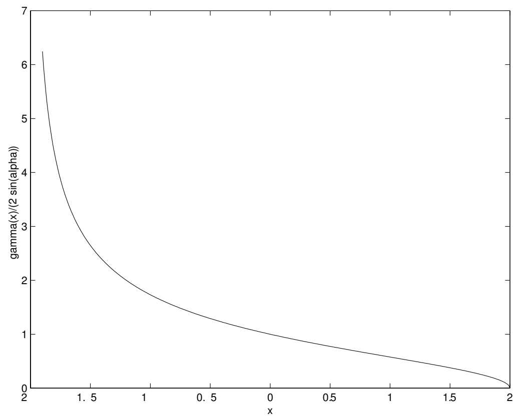

Figure 5.11: The distribution of vorticity on a flat plate, in units of $-2\sin\alpha$.

The plate is therefore said to contain a *vortex sheet of strength* $\gamma(x)$.

Using (5.36) and (5.37), and the fact that $\sqrt{z^2 - 4a^2} = \pm i\sqrt{4a^2 - x^2}$ when $z = (x, 0\pm)$ we obtain (see problem 5.5)

$$Q^{-1}u(x, 0\pm) = \pm\sqrt{\frac{2a - x}{2a + x}}\sin\alpha + \cos\alpha, \gamma(x) = -2Q\sqrt{\frac{2a - x}{2a + x}}\sin\alpha. \quad (5.39)$$

We show $\gamma(x)$ in figure 5.11.

The vorticity of this foil is said to be *bound* to the foil, meaning that it exists "in the plate" and is not present in the fluid. Suppose now that we consider a three-dimensional wing, as shown in figure 5.12. If the wing is sliced by a plane $y =$ constant, we obtain a 2D airfoil section. For example, it might be a Joukowsky section, with its chord c, thickness, camber, and local angle of attack, all functions of y. The direction y is called the *spanwise* direction. The *wingspan* is here $2b$. Since all of the airfoil parameters can vary down the span, we expect the lifting properties of the wing to be a function of y. We also expect that near the center of the wing, the section AB of the figure 5.13(a), the flow should behave as if the section were approximately a two-dimensional airfoil, with vorticity bound to the foil and carrying an associated circulation and lift. However as we move to the tips of tdhe wing, eventually the section lift must go to zero, if for no other reason than that the section chord goes to zero. Since vorticity is a solenoidal vector, the question has to be, what happens to the vortex lines which were bound to the center section? The answer, suggested in figure 5.13(a), is that the the decrease in the lift distribution as one moves from center to tip, vortex lines turn and are shed into the wake of the wing, thereby reducing the section circulation. Thus there is a sheet of vorticity emerging from the trailing edge of the wing. For many wings the lift decreases to zero rapidly near the tips, so that substantial free vorticity is released near the tips, and this is the source of the "tip vortices" seen in the wake of high-flying jets.

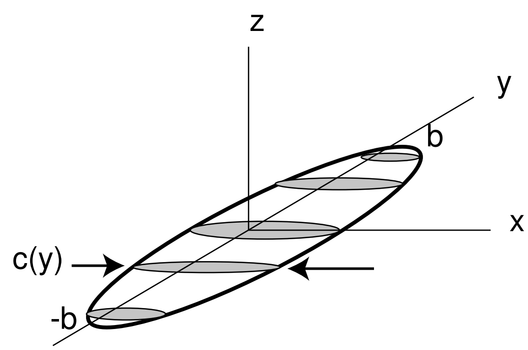

Figure 5.12: The 3D wing.

To understand this shedding process, consider figure 5.13(b). We consider a strip of wing sections of width dy. The wing is assumed to be changing so slowly in the spanwise direction that each such strip acts as if it were a 2D airfoil. On this section the lift will then be $l(y)dy$ where $l(y)$ is the 2D lift of the local section. Let the pressure on the upper and lower surfaces be $p_{\pm}(x)$. Then

$$l(y) = -\int_{chord}(p_+(x,y) - p_-(x,y))dx = -\int [p]dy = -\rho U\Gamma(y), \tag{5.40}$$

where $\Gamma(y)$ is the circulation of the local section and the brackets denote the jump from bottom to top surface. Now consider

$$\frac{dl}{dy} = -\int_{chord}\frac{\partial[p]}{\partial y}dx = \int_{chord}\rho\frac{D[v]}{Dt}dx, \tag{5.41}$$

where v is the spanwise velocity component and we are assuming an ideal fluid of constant density. If we assume that the spanwise acceleration is so small that a fluid particles near the wing surface, passing over or under the wing, acquire a spanwise velocity that is small compared to U, then we may substitute $dx = Udt$ and evaluate the last integral as a time integral to obtain

$$\rho U[\Delta v], \tag{5.42}$$

giving the jump in spanwise velocity developed by a fluid particles flowing over the top and bottom surfaces, $\Delta v_{\pm}$ being the spanwise velocities developed at the trailing edge of the section. Since we expect the lift to decrease as we move toward each tip, the directions of spanwise flow are indicated in figure 5.13(b) for a piece of the left wing looking upstream from the rear of the wing. Since the pressures are *increasing* on the upper surface toward the tip the flow is driven away from the tip. On the bottom surface the spanwise flow is in the opposite direction (the dotted arrow in figure 5.13(b), since on this surface loss of lift is associated with a *decrease* in pressure.

We thus have

$$\frac{dl}{dy} = \rho U \frac{d\Gamma}{dy} = \rho U[\Delta v] \tag{5.43}$$

or

$$\frac{d\Gamma}{dy} = [\Delta v]. \tag{5.44}$$

Here Γ, the circulation on a section, is positive if the spanwise vorticity is in the direction of positive y. Since Γ is the local circulation of a section, (5.44) relates the spanwise change of local lift to the exitence of a discontinuity in spanwise velocity at the strailing edge of the wing.

This discontinuity, $[\Delta v]$, is associated with the production of a vortex sheet at the trailing edge in the x-velocity component $\omega_x = w^y - v_z$. Integrating from $z = 0-$ to $z = 0+$ at the trailing edge, we have $\int \omega_x dz = -[\Delta v]$. This means that $-[\Delta v]dy$ is incremental vorticity shed into the wake at the trailing edge at section y due to the change of l with y. For the left half-wing, as seen from an observer behind the wing looking upstream, $[\Delta v]dy$ is positive if lift increases with y there. Thus $-[\Delta v]dy$ is negative, and so the shed vorticity represents a turning downstream of some of the vortex lines bound to the wing, as shown for $y < 0$ in figure 5.13(a). Similarly, for the right half-wing the decrease of lift with increasing y causes $-[\Delta v]dy$ to be positive.

We now examine the model of the 3D wing created by Prandtl, who sought as a simple means of deducing the lift and drag of a wing, given the section properties of the wing. This model is sometimes called the *lifting line* model. The idea is basically to regard the wing as long and thin. An *aspect ratio* can be defined for a wing planform (projection onto the x, y plane) by $=$ wingspan^2/wingarea $= 4b^2/A$ where b is the half-span. Mathematically, the Prandtl model is an asymptotic approximation to the fluid dynamics of a 3D wing in the limit $\to \infty$. The situation is as shown in figure 5.13(c). Because in this limit the chord is small compared to the wingspan, the bound vorticity can be though of as confined to a line, but the circulation about this line becomes a function of y, namely $\Gamma(y)$. If, to fix ideas, ee take each section of the wsing to be a Joukowski foil, then we know from (5.6) that $l(y) = 4\pi\rho c(y)Q^2 \sin(\alpha + \beta)$. We will, along with Prandtl, make the assumption that the angles α, β are small, so that $\sin(\alpha + \beta) \approx \alpha + \beta$ and $Q \approx U$. Then, with the orientation of the coordinate system of 5.13(c) we have, approximately,

$$\Gamma(y) = 4\pi c(y)U(\alpha(y) + \beta(y)). \tag{5.45}$$

We now need to make a crucial reinterpretation of α. Owing to the shed vorticity of the wake, the *effective* angle of attack, that is, the angle made byt the oncoming stream at the particular section, will be dependent upon the z-component of velocity induced at that section by the shed vorticity. If this veclocity is $w(y)$, then the (small) effective angle of attack is give by

$$\alpha_{eff} = \alpha + \frac{w}{U}, \tag{5.46}$$

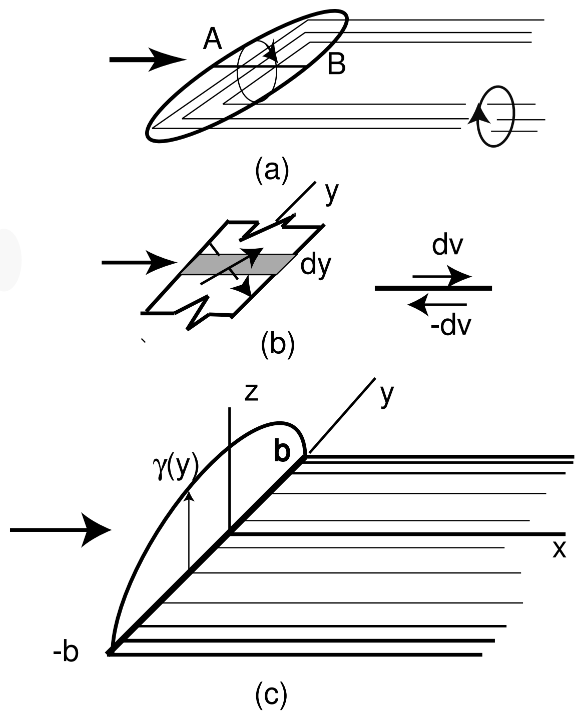

Figure 5.13: (a) The vorticity shed from a 3D lifting wing. (b) The origin of the shed vorticity. (c) Prandtl's lifting line model.

where α is the angle made by the section relative to the velocity at true infinity. In other words, the induced w near the wing, the so called *downwash*, changes the apparent "velocity at infinity" from its true value to α_{eff}, and each section will "see" a different approach angle. [2] Now w is an as yet unknown function of y, while β, c are given functions of y determined by the section properties.

With

$$\Gamma(y) = 4\pi c(y)U(\alpha + \frac{w(y)}{U} + \beta(y)) \tag{5.47}$$

we are now in a position to use the Bior-Savart expression for velocity in terms of vorticity, to determine $w(y)$ from $\Gamma(y)$. Recall the the shed x-component of vorticity at each section is $-d\Gamma(y) = \frac{-d\Gamma}{dy}dy$. Now a doubly-infinite vortex induces a velocity given by the 2D point vortex flow. Such a line, carrying unit circulation, parallel to the x-axis at position $y = \eta$ in the $z = 0$ plane, will induce a velocity

$$\frac{1}{2\pi}\frac{1}{y-\eta} \tag{5.48}$$

at any point y on the bound vortex. Since the shed vortex is only semi-infinite, this induced vorticity is reduced by a factor $\frac{1}{2}$. Since the circulation shed at section η is $-\frac{d\Gamma}{dy}dy$ evaluated at $y = \eta$, we have

$$w = -\frac{1}{4\pi}\int_{-b}^{+b}\frac{\frac{d\Gamma}{dy}(\eta)}{y-\eta}d\eta. \tag{5.49}$$

Thus, from (12.30) we obtain

$$\Gamma(y) = 4\pi c(y)U\Big[\alpha + \beta(y) - \frac{1}{4\pi U}\int_{-b}^{+b}\frac{\frac{d\Gamma}{dy}(\eta)}{y-\eta}d\eta, \tag{5.50}$$

which is an integral equation form $\Gamma(y)$.

The beauty of this model is the direct insight it gives into an important fact about three-dimensional aerodydnamics, namely the creation of drag in a perfect fluid model. Observed that whenever the $w(y)$ is negative, which is generally the case for normal wings, the effective angle of attack is less than α. Since our 2D airfoil theory tells us that the local lift is perpendicular to the "flow at infinity", here the apparent or effective flow at infinity, we see that the local lift vector is rotated slightly so as to produce a component in the direction of positive x. This is a drag component, and the summation over all sections will give rise to the wing drag. This drag, since is is caused by the downwash induced at the wing sectionn by the vortical wake, is called the *induced drag*.

We now indicate how to solve the integral equation (5.50) and calculate the lift and induced drag of our 3D wing. We set $y = -b\cos\theta, 0 \le \theta \le \pi$, and

[2] If $\frac{d\Gamma}{dy} \ge 0$ on the left half-wing and $\frac{d\Gamma}{dy} \le 0$ on the right half-wing, then, then the shed vorticity is such as to make $w(y) \le 0$ everywhere at the lifting line, hence the term "downwash".

suppose that Γ is an even function of y, so it can be is represented by a Fourier series

$$\Gamma = Ub \sum_{n=0}^{\infty} B_{2n+1} \sin(2n+1)\theta. \tag{5.51}$$

Then

$$\frac{d\Gamma}{dy} dy = \frac{d\Gamma}{d\theta} d\theta = Ub \sum_{n=0}^{\infty} B_{2n+1} \cos(2n+1)\theta d\theta. \tag{5.52}$$

Using this in (5.50) we obtain the definite integral

$$\int_0^{\pi} \frac{\cos m\theta'}{\cos\theta - \cos\theta'} d\theta' = -\pi \frac{\sin m\theta}{\sin\theta}, \tag{5.53}$$

The verification of which we leave as problem 5.6. Thus, if $c(y) = C(\theta)$ and $c(y)[\alpha + \beta(y)] = D(\theta)$, (5.50) becomes

$$b \sum_{n=0}^{\infty} B_{2n+1} \sin\theta \sin(2n+1)\theta$$

$$-\pi C(\theta) \sum_{n=0}^{\infty} (2n+1) B_{2n+1} \sin(2n+1)\theta = 4\pi \sin\theta D(\theta). \tag{5.54}$$

Given $C(\theta)$ and $D(\theta)$, we are in a position to express all terms as Fourier series in $\sin(2n+1)\theta$ and solve the resulting linear system for the B_{2n+1}.

Given a solution the lift is

$$L = \rho U \int_{-b}^{+b} \Gamma(y) dy = \rho U b \int_0^{\pi} \Gamma(\theta) \sin\theta d\theta = \frac{\pi}{2} \rho U^2 b^2 B_1. \tag{5.55}$$

From small w/U, then *induced drag* is given by

$$D_{ind} = -\rho \int_{-b}^{+b} w\Gamma dy = \frac{\pi}{8} \rho U^2 b^2 \sum_{n=0}^{\infty} (2n+1) B_{2n+1}^2. \tag{5.56}$$

Problem set 5

1. (a) Verify (5.1) for the flat plate with the Kutta-Joukowski condition applied. (b) Verify that for the Joukowsky family of airfoils the lift is given by (5.6), and that the change comes from the new value of the circulation as determined by the K-J condition.

2. Consider the Joukowski airfoil with $Z_0 = bi$ $a > b > 0$. (a) Show that the airfoil is an arc of the circle with center at $(0, -(a^2 - b^2)i/b$ and radius $(a^2 + b^2)/b$. (b) With Kutta condition applied to the trailing edge, at what angle of attack (as a function of b) is the lift zero?

3. Let the airfoil parameters other than chord (i.e. k, β) be independent of y, the coordinate along the span of the wing. Also, assume the planform is symmetric about the line $x = 0$ in the $x - y$ plane. Using Prandtl's lifting-line theory, show that for a given lift the minimal induced drag occurs for a wing having an elliptical planform. Show in this case that the coefficient of induced drag $C_{D_i} = 2 \times drag/(\rho U^2 S)$ and lift coefficient $C_L = 2 \times lift/(\rho U^2 S)$ are related by

$$C_{D_i} = C_L^2/(\pi \quad).$$

Here S is the wing area and is the aspect ratio $4b^2/S$. (Some of the WW II fighters, notably the Spitfire, adopted an approximately elliptical wing.)

4. This problem will study flow past a slender axisymmetric body whose surface is given (in cylindrical polar coordinates), by $r = R(z), 0 \le z \le L$. Here $R(z)$ is continuous, and positive except at $0, L$ where it vanishes. By "slender" we mean that $\max_{0 \le x \le L} R << L$. The body is placed in the uniform flow $(u_z, u_r, u_\theta) = (U, 0, 0)$. We are interested in the steady, axisymmetric potential flow past the body. It can be shown that such a body perturbs the free stream by only a small amount, so that in particular, $u_z \approx U$ everywhere. On the other hand the flow must be tangent at the body, which implies $\phi_r(z, R(z)) \approx U dR/dz, 0 < z < L$.

We look for a representation of ϕ as a distribution of sources with strength $f(z)$. Thus

$$\phi(z, r) = -\frac{1}{4\pi}\int_0^L \frac{f(\zeta)}{\sqrt{(z-\zeta)^2 + r^2}} d\zeta.$$

(a) Compute $\frac{\partial \phi}{\partial r}$, and investigate the resulting integral as $r \to 0, 0 < z < L$. Argue that the dominant contribution comes near $\zeta = z$, and hence show that $\frac{\partial \phi}{\partial r} \approx \frac{1}{4\pi}\frac{f(z)}{r}\int_{-\infty}^{+\infty}(1+s^2)^{-3/2} ds$ for $r << L$.

(b) From the above tangency condition, deduce that $f(z) \approx dA/dz$ where $A(z) = \pi R^2$ is the cross-sectional area of the body.

(c) By expanding the above expression for ϕ for large z, r, show that in the neighborhood of infinity

$$\phi \approx -\frac{1}{4\pi}\frac{z}{(z^2+r^2)^{3/2}}\int_0^L A(\zeta) d\zeta, z^2 + r^2 \to \infty.$$

5. Verify (5.39).

6. Verify (5.53). (Suggestion: Let $z = e^{i\theta'}, \zeta = e^{i\theta}$, and convert the integral to one on a contour around the unit circle in the z-plane. You will want to indent the contour at poles on the boundary. Evaluate using theory.)

7. The *Trefftz plane* is a virtual plane orthogonal to the x-axis in figure 5.13 and placed at large x downstream of the wing. The vortical wake in Prandtl's model may be regarded as intersecting the Trefftz plane on the segment $I : |y| \le b, z = 0$. The induced drag may be calculated in the Trefftz plane as follows.

Adopt the energy balance that UD_{ind}, the rate of working done by the induced drag, is equal to UE, the flux of wake energy through the Trefftz plane (TP). Here

$$E = \frac{\rho}{2}\int_{TP} (\nabla\phi)^2 dydz, \tag{5.57}$$

where $\nabla\phi = (0, v, w) = (0, \phi_y, \phi_z)$. That is, in the Trefftz plane the velocity perturbations of the free stream are dominated by the induced velocities of the, now doubly infinite line vortices. Use the fact that $\phi_z = w$ is continuous on Y but ϕ is discontinuous there to show that

$$D_{ind} = -\frac{\rho}{2}\int_{-b}^{+b} w_{TP}[\phi]_{TP}dy, \tag{5.58}$$

where w_{TP} is twice the downwash computed at the lifting line in Prandtl's model, and $[\phi]_{TP} = \phi(y.0+) - \phi(y, 0-)$ on I. From this result show that (5.56) follows from the definition of circulation.

Chapter 6

Viscosity and the Navier-Stokes equations

6.1 The Newtonian stress tensor

Generally real fluids are not inviscid or ideal. [1] Modifications of Euler's equations, needed to account for real fluid effects at the continuum level, introduce additional forces in the momentum balance equations. There exists a great variety of real fluids which can be treated at the continuum level, differing in what we shall call their *rheology*. Basically the problem is to identify the forces experienced by a fluid parcel as it is moved about and deformed according to the mathematical description we have developed. Because of the molecular structure of various fluid materials, the nature of these forces can vary considerably and there are many rheological models which attempt to capture the observed properties of fluids under deformation.

The simplest of these rheologies is the *Newtonian viscous fluid*. To understand the assumptions let us restrict attention to the determination of a viscous stress tensor at $\mathbf{x}, t$, which depends only upon the fluid properties within a fluid parcel at that point and time. One could of course imagine fluids where some local average over space determines stress at a point. Also it is easy enough to find fluids with a memory, where the stress at a particular time depends upon the stress history at the point in question.

It is reasonable to assume that the forces dues to the rheology of the fluid are developed by the deformation of fluid parcels, and hence could be determined by the velocity field. If we allow only point properties, deformation of parcels must involve more than just the velocity itself; first and higher-order partial derivatives with respect to the spatial coordinates could be important. (The time derivative of velocity has already been taken into consideration in the acceleration terms.) A moment's thought shows viscous forces cannot depend

[1] In quantum mechanics the superfluid is in many respects an ideal fluid, but the laws governing vorticity, for example, need to be modified.

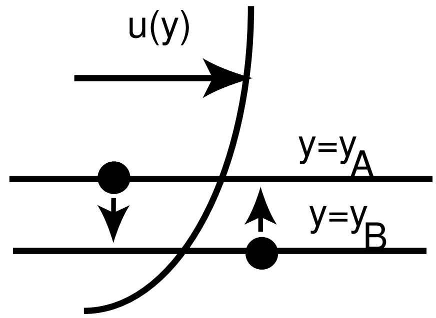

Figure 6.1: Momentum exchange by molecules between lamina in a shear flow.

on velocity. The bulk translation of the fluid with constant velocity produces no force. Thus the deformation of a small fluid parcel must be responsible for the viscous force, and the dominant measure of this deformation should come from the first derivatives of the velocity field, i.e. from the components of the velocity derivative matrix $\frac{\partial u_i}{\partial x_j}$. The Newtonian viscous fluid is one where the stress tensor is *linear* in the components of the velocity derivative matrix, with a stress tensor whose specific form will depend on other physical conditions.

To see why a linear relation of this might capture the dominant rheology of many fluids consider a flow $(u, v) = (u(y), 0)$.Each different plane or *lamina* of fluid, $y=$ constant, moves with a particular velocity. Now consider the two lamina $y = y_A, y_B$ as shown in figure 6.1, moving at velocities $u_B < u_A$. If a molecule moves from B to A, then it is moving from an environment with velocity u_B to an environment with a larger velocity u_A. Consequently it must be accelerated to match the new velcity. According to Newton, a force is therefore applied to the lamina $y = y_A$ in the direction of *negative* x. Similarly a molecule moving from y_A to y_B must slow down, exerting a force on lamina $y = y_B$ in the direction of *positive* x. Thus these exchanges of molecules would tend to reduce the velocity difference between the two lamina. [2]

This tendency to reduce the difference in velocities can be thought of as a force applied to each lamina. Thus if we insert a virtual surface at some position y, a force should be exerted on the surface, in the positive x direction if $du/dy(y) > 0$. Generally we expect the gradients of the velocity components to vary on a length scale L comparable to some macroscopic scale- the size of a container, the size of a body around which the fluid flows, etc. On the other hand the scale of the molecular events envisaged above is very small compared to the macroscopic scale. Thus it is reasonable to assume that the force on the

[2]Perhaps a more direct analogy would be two boats gliding along on the water on parallel paths, one moving faster than the other. If, at the instant they are side by side, an occupant of the fast boat jumps into the slow boat, the slower boat with speed up, and similarly an occupant of the slow boat can slow up the fast boat by jumping into it.

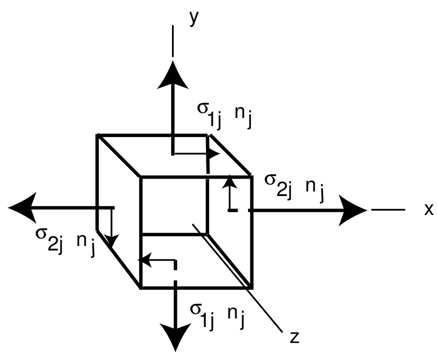

Figure 6.2: Showing why $\sigma_{12} = \sigma_{21}$. The forces are per unit area. The area of each face is Δ^2.

lamina is dominated by the first derivative,

$$F(y) = \mu \frac{du}{dy}. \tag{6.1}$$

The constant of proportionality, μ, is called the *viscosity*, and a fluid obeying this law is called a *Newtonian viscous fluid*.

We have considered so far only a simple planar flow $(u(y), 0)$. In general all of the components of the velocity derivative matrix need to be considered in the construction of the viscous stress tensor. Let us write

$$\sigma_{ij} = -p\delta_{ij} + d_{ij}. \tag{6.2}$$

That is, we have simply split off the pressure contribution and exhibited the *deviatoric stress tensor* d_{ij}, which contains the viscous stress. We first show that d_{ij}, and hence σ_{ij}, must be a *symmetric* tensor. We can do that by considering figure 6.2. We show a square parcel of fluid of side Δ. We show those forces on each face which exert a torque about the z-axis. We see that the torque is $\Delta^3(\sigma_{21} - \sigma_{12})$, since each face has area Δ^2 and each of the four forces considered has a moment of $\Delta/2$ about the z-axis. Now this torque must be balanced by the angular acceleration of the parcel about the z-axis. Now the moment of inertial of the parcel is a multiple of Δ^4. As $\Delta \to 0$ we see that the angular acceleration must tend to infinity as Δ^{-1}. It follows that the only way to have stability of a parcel is for $\sigma_{21} = \sigma_{12}$. The same argument applies to moments about the other axes.

A final requirement we shall place on d_{ij}, so a further condition on the fluids we shall study, is that there should be no preferred direction, the condition of *isotropy*. The conditions of isotropy of symmetric matrices of second order then imply that d_{ij} can satisfy these while being linear in the components of the velocity derivative matrix only if it has either of two forms:

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad \frac{\partial u_k}{\partial x_k}\delta_{ij}. \tag{6.3}$$

For a Newtonian fluid the linearity implies that the most general allowable deviatoric stress has the form

$$d_{ij} = \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\Big) + \mu' \frac{\partial u_k}{\partial x_k}\delta_{ij}. \tag{6.4}$$

Notice that we have divided the two terms so that the first term, proportional to μ, has zero trace. Thus if $\mu' = 0$, the deviatoric stress contributes nothing to the normal force on an area element; this is given solely by the pressure force. The possibility of a normal force distinct from the pressure force is allowed by the second term of (6.4). We have attached the term viscosity to μ, so μ' is usually called the *second viscosity*. Often it is taken as zero, an approximation that is generally valid for liquids. The condition $\mu' = 0$ is equivalent to what is sometimes called the *Stokes relation*. In gases in particular μ' may be positive, in which case the thermodynamic pressure and the normal stresses are distinct.

It should be noted that if we had simply taken d_{ij} to be proportional to the velocity derivative matrix, then the splitting (3.5) would show that only e_{ij} could possibly appear, since otherwise uniform rotation of the fluid would produce a force orthogonal to the rotation axis, , which is never observed. The second term in (6.4) then follows as the only isotropic symmetric tensor linear in the velocity derivative which could be included as a contribution to "pressure".

In this course we shall be dealing with two special cases of (6.4). The first is an incompressible fluid, in which case

$$\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad \text{(incompressible fluid)}. \tag{6.5}$$

Note that with the incompressibility

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i. \tag{6.6}$$

The second case is compressible flow in one space dimension. Then $\mathbf{u} = (u(x,t),0,0)$ and the only non-zero component of the stress tensor is

$$\sigma_{11} = -p + \mu''\frac{\partial u}{\partial x}, \quad \mu'' = \frac{4}{3}\mu + \mu', \quad \text{one} - \text{dimensional gas flow}. \tag{6.7}$$

The momentum balance equation in the form

$$\rho\frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{6.8}$$

together with the stress tensor given by (6.4), defines the momentum equation for the *Navier-Stokes* equations. These are the most commonly used equations for the modeling of the rheology of fluids. The have been found to apply to a wide variety of practical problems, but it is important to realize their limitations. First, for highly rarified gases the mean free path of molecules of the gas can become so large that the concept of a fluid parcel, small with respect to the macroscopic scale but large with respect to mean free path, becomes untenable. Also, many common fluids, honey being an example, are non-Newtonian and can exhibit effects not captured by the Navier-Stokes equations. Finally, whenever a flow involves very small domains of transition, the Navier-Stokes model may break down. Example of this occurs in shock waves in gases, where changes occur over a distance of only several mean free paths, and in the interface between fluids, which can involve transitions over distances comparable to intermolecular scales. In these problems a multi-scale analysis is usually needed, which can bridge the macroscopic-molecular divide.

Finally, we point out that the viscosities in this model will generally depend upon temperature, but for simplicity we shall neglect this variation, and in particular for the incompressible case we always take μ to be constant. Also we shall often exhibit the *kinematic viscosity* $\nu = \frac{\mu}{\rho}$ in place of μ. We remark that ν has dimensions length^2/ time, as can be verified from the momentum equations after division by ρ.

6.2 Some examples of incompressible viscous flow

We now take the density and viscosity to be constant and consider several exact solutions of the incompressible Navier-Stokes equations. We shall be dealing with fixed or moving rigid boundaries and we need the following assumption regarding the boundary condition on the velocity in the Navier-Stokes model:

Assumption (The non-slip condition): At a rigid boundary the relative motion of fluid and boundary will vanish.

Thus at a non-moving rigid wall the velocity of the fluid will be zero, while at any point on a moving boundary the fluid velocity must equal the velocity of that point of the boundary. This condition is valid for gases and fluids in situations where the stress tensor is well approximated by (6.4). It can fail in small domains and in rarified gases, where some slip may occur.

6.2.1 Couette flow

Imagine two rigid planes $y = 0, H$ where the no-slip condition will be applied. The plane $y = H$ moves in the x-direction with constant velocity U, while the plane $y = 0$ is stationary. The flow is steady, so the velocity field must be a function of y alone. Assuming constant density, $\mathbf{u} = (u(y), 0)$ and $p_x = 0$ we obtain a momentum balance if

$$-\mu u_{yy} = 0. \tag{6.9}$$

Thus given that $u(0) = 0, u(H) = U$, we have $u = Uy/H$. We see that the viscous stress is here constant and equal to $\mu U/H$. This is the force per unit area felt by the plane $y = 0$. No pressure gradient is needed to sustain this stress field. Couette flow is the simplest exact solution of the Navier-Stokes equations with non-zero viscous stress.

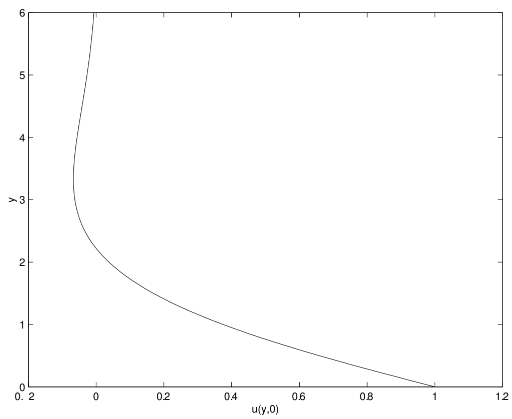

Figure 6.3: The velocity in the Rayleigh problem at t=0 mod 2π, y in units of $\sqrt{\mu/\omega}$.

6.2.2 The Rayleigh problem

A related unsteady problem results from the time dependent motion in the x-direction with velocity $U(t)$of the plane $y = 0$. A no-slip condition is applied on this plane. A fluid of constant density occupies the semi-infinite domain $y > 0$. In this case an exact solution of the Navier-Stokes equations is provided by $\mathbf{u} = (u(y,t), 0), p = 0$, with

$$u_t - \mu u_{yy} = 0,\ u(0) = U(t). \tag{6.10}$$

In the case $U(t) = U_0 \cos \omega t$ we see that $u(y,t) = \Re(e^{i\omega t} f(y))$ where $f(y)$ is the complex-valued function of y satisfying

$$i\omega f - \mu f_{yy} = 0,\ f(0) = U_0. \tag{6.11}$$

We shall also require that $u(\infty) = 0$. Thus

$$u = \Re U_0 e^{i\omega t - (1+i)y\sqrt{\frac{\omega}{2\nu}}} = U_0 \cos\big(\omega t - \sqrt{\frac{\omega}{2\nu}} y\big) e^{-y\sqrt{\frac{\omega}{2\nu}}}. \tag{6.12}$$

We show the velocity field in figure 6.3. Note that the oscillation dies away extremely rapidly, with barely one reversal before decay is almost complete.

6.2.3 Poiseuille flow

We consider now a flow in a cylindrical geometry. A Newtonian viscous fluid of constant density is in steady motion down a cylindrical tube of radius R and of infinite extent in both directions. Because of viscous stresses at the walls of the tube, we expect there to be a pressure gradient down the tube. Let the axis of the tube be the z-axis, r the radial variable, and $\mathbf{u} = (u_z, u_r, u_\theta) = (u_z(r), 0, 0)$ the velocity field in cylindrical polar coordinates. We note here, for future reference, the form of the Navier-Stokes equations in these coordinates:

$$\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu \nabla^2 u_z, \tag{6.13}$$

$$\frac{\partial u_r}{\partial t} + \mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = \nu\Big(L u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\Big), \tag{6.14}$$

$$\frac{\partial u_\theta}{\partial t} + \mathbf{u} \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} + \frac{1}{r\rho}\frac{\partial p}{\partial \theta} = \nu\Big(L u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\Big), \tag{6.15}$$

$$\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0. \tag{6.16}$$

Here

$$u \cdot \nabla = u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}, \tag{6.17}$$

$$\nabla^2 = \frac{\partial^2(\cdot)}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial(\cdot)}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2(\cdot)}{\partial \theta^2}, \quad L = \nabla^2 - \frac{1}{r^2}. \tag{6.18}$$

For the problem at hand, we set $p = -Gz+$ constant to obtain the following equation for $u_z(r)$:

$$\mu \nabla^2 u_z = -G = \mu\Big(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\Big). \tag{6.19}$$

The no-slip condition applies at $r = R$, so the relevant solution of (6.19) is

$$u_z = \frac{G}{4\mu}(R^2 - r^2). \tag{6.20}$$

Thus the velocity profile is parabolic. The total flux down the tube is

$$Q \equiv 2\pi \int_0^R r u_z dr = \frac{\pi G R^4}{8\mu}. \tag{6.21}$$

If a tube of length L is subjected to a pressure difference Δp at the two ends, then we can expect to drive a total volume flow or flux $Q = \frac{\pi \Delta p R^4}{8\mu L}$ down the tube. The rate W at which work is done to force the fluid down a tube of length L is the pressure difference between the ends of the tube times the volume flow rate Q, i.e.

$$W = \frac{\pi G^2 L R^4}{8\mu} \tag{6.22}$$

Poiseuille flow can be easily observed in the laboratory, particularly in tubes of small radius, and measurements of flow rates through small tubes provides one way of determining a fluid's viscosity. Of course all tubes are finite, the velocity profile (6.20) is not established at once when fluid is introduced into a tube. This *entry effect* can persist for substantial distances down the tube, depending on the viscosity and the tube radius, and also on the velocity profile at the entrance. Another interesting question concerns the *stability* of Poiseuille flow in a doubly infinite pipe; this was studied by the engineer Osborne Reynolds in the 1870's. He observed instability and transition to turbulence in long tubes. An application of Poiseuille flow of some importance is to blood flow; and in the arterial system there are many branches which are too short to escape significant entry effects.

A generalization of Poiseuille flow to an arbitrary cylinder, bounded by generators parallel to the z-axis and having a cross section S is easily obtained. The equation for u_z is now

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = -G/\mu, \quad u_z = 0 \text{ on } \partial S. \tag{6.23}$$

The solution is necessarily ≥ 0 for $G > 0$ and can be found by standard methods for the inhomogeneous Laplace equation.

6.2.4 Flow down an incline

We consider now the flow of a viscous fluid down an incline, see figure 6.4. The velocity has the form $(u, v, w) = (u(z), 0, 0)$ and the pressure is a function of z alone. The fluid is forced down the incline by the gravitational body force. The equations to be satisfied are

$$\rho g \sin\alpha + \mu \frac{d^2 u}{dz^2} = 0, \quad \frac{dp}{dz} + \rho g \cos\alpha = 0. \tag{6.24}$$

On the free surface $z = H$ the stress must equal the normal stress due to the constant pressure, p_0 say, above the fluid. Thus $\sigma_{xz} = \nu \frac{du}{dz} = 0$ and $\sigma_{zz} = -p = -p_0$ when $z = H$. Since the no-slip condition applies, we have $u(0) = 0$. Therefore

$$u = \frac{\rho g \sin\alpha}{2\mu} z(2H - z), \quad p = p_0 + \rho g(H - z)\cos\alpha. \tag{6.25}$$

The volume flow per unit length in the y-direction is

$$\int_0^U u dz = \frac{gH^3 \sin\alpha}{3\nu}. \tag{6.26}$$

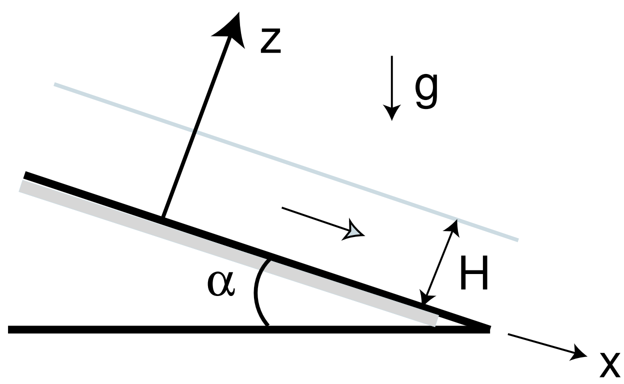

Figure 6.4: Flow of a viscous fluid down an incline.

6.2.5 Flow with circular streamlines

We consider a velocity field in cylindrical polar coordinates of the form $(u_z, u_r, u_\theta) = (0, 0, u_\theta(r, t))$, with $p = p(r, t)$. From (6.13)-(6.18) the equation for u_θ is

$$\frac{\partial u_\theta}{\partial t} = \nu\Big(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\Big), \tag{6.27}$$

with the equation

$$\frac{\partial r}{\partial r} = \frac{\rho}{r} u_\theta^2 \tag{6.28}$$

determining the pressure. The vorticity is

$$\omega = \frac{1}{r}\frac{\partial r u_\theta}{\partial r}. \tag{6.29}$$

From (6.27) we then find an equation for the vorticity

$$\frac{\partial \omega}{\partial t} = \nu\Big(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\Big) = \nu \nabla^2 \omega. \tag{6.30}$$

This equation, which is the symmetric form of the heat equation in two space dimensions, may be used to study the decay of a point vortex in two dimensions, see problem 6.2.

6.2.6 The Burgers vortex

Th implication of (6.30) is that vorticity confined to circular streamlines in two dimensions will diffuse like heat, never reaching a non-trivial steady state in R^2. We now consider a solution of the Navier-Stokes equations which involves a two-dimensional vorticity field $\omega = (\omega_z, \omega_r, \omega_\theta)) = (\omega(r), 0, 0)$. The idea is to prevent the vorticity from diffusing by placing it in a steady irrotational flow field of the form $(u_z, u_r, u_\theta) = (\alpha z, -\alpha r/2, 0)$. Thus the full velocity field has the form

$$(u_z, u_r, u_\theta) = (\alpha z, -\alpha r/2, u_\theta(r, t)). \tag{6.31}$$

Now the z-component of the vortiity equation is, with (6.31),

$$\frac{\partial \omega}{\partial t} - \frac{\alpha}{2}\frac{r}{}\frac{\partial \omega}{\partial r} - \alpha\omega = \nu \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \omega}{\partial r}\Big), \quad \omega = \frac{1}{r}\frac{\partial r u_\theta}{\partial r}. \tag{6.32}$$

First note that if $\nu = 0$, so that there is no diffusion of ω, we my solve the equation to obtain

$$\omega = e^{\alpha t} F(r^2 e^{\alpha t}), \tag{6.33}$$

where $F(r^2)$ is the initial value of ω. This solution exhibits the exponential growth of vorticity coming from the stretching of vortex tubes in the straining flow $(\alpha z, -\alpha r/2, 0)$.

If now we restore the viscosity, we look for a *steady* solution of (6.32), representing a vortex in for which diffusion is balanced by the advection of vorticity toward the z-axis. We have

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{\alpha}{2}r^2\omega + \nu r\frac{\partial \omega}{\partial r}\Big) = 0. \tag{6.34}$$

Integrating and enforcing the condition2 that $r^2\omega$ and $r\frac{\partial \omega}{\partial r}$ vanish when $r = \infty$, we have

$$\frac{\alpha}{2} r\omega + \nu\frac{d\omega}{dr} = 0. \tag{6.35}$$

Thus

$$\omega(r) = Ce^{\frac{-\alpha r^2}{4\nu}}, \tag{6.36}$$

so that

$$u_\theta = \frac{\Gamma}{2\pi}\frac{1 - e^{\frac{-\alpha r^2}{4\nu}}}{r}, \tag{6.37}$$

where we have redefined the constant to exhibit the total circulation of the vortex. Note that as ν decreases the size of the vortex tubes shrinks. With Γ fixed this would mean that the vorticity of the tube is increased.

6.2.7 Stagnation-point flow

In this example we attempt to modify the two-dimensional stagnation point flow with streamfunction $UL^{-1}xy$ to a solution in $y > 0$ of the Navier-Stokes equations with constant density, satisfying the no-slip condition on $y = 0$. The vorticity will satisfy

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} - \nu\Big(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\Big) = 0. \tag{6.38}$$

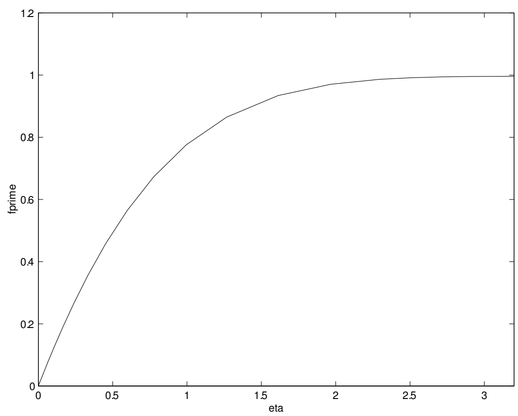

Figure 6.5: f' versus η for the viscous stagnatioin point flow.

If we set $\psi = UL^{-1}xF(y)$, then $\omega = -UL^{-1}yF''$. Insertion in (6.38) gives

$$F'F'' - FF''' - Re^{-1}F'''', \tag{6.39}$$

where $Re = UL/\nu$. The boundary conditions are that $F(0) = F'(0) = 0$ to make ψ, u, v vanish on the wall $y = 0$, and $F \sim y$ as $y \to \infty$, so that we obtain the irrotational stagnation point flow at $y = \infty$.

One integration of (6.39) can be carried out to obtain

$$F'^2 - FF'' - Re^{-1}F''' = 1. \tag{6.40}$$

With $F = Re^{-1/2}f(\eta), \eta = Re^{1/2}y$, (6.40) becomes

$$f'^2 - ff'' - f''' = 1, \tag{6.41}$$

with conditions $f'(\infty) = 1, f(0) = f'(0) = 0$. We show in figure 6.5 the solution $f'(\eta)$ of this ODE problem. This represents a gradual transition through a layer of thickness of order $\sqrt{UL/nu}$ between the null velocity on the boundary and the velocity $U(x/L)$ which u has at the wall in the irrotational stagnation point flow. We shall be returning to a discussion of such transition layers in chapter 7, where we take up the study of boundary layers.

6.3 Dynamical similarity

In the stagnation point example just considered, the dimensional combination $Re = UL/\nu$ has occurred as a parameter. This parameter, called the *Reynolds* *number* in honor of Osborne Reynolds, arose because we chose to exhibit the problem in a dimensionless notation. Consider now the Navier-Stokes equations with constant density it their dimensional form:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{6.42}$$

We may define dimensionless (starred) variables as follows:

$$\mathbf{u}^* = \mathbf{u}/U, \mathbf{x}^* = \mathbf{x}/L, p^* = p/\rho U^2. \tag{6.43}$$

Here U, L are assumed to be a velocity and length characteristic of the problem being studied. In the case of flow past a body, L might be a body diameter and U the flow speed at infinity. In these starred variables it is easily checked that the equations become

$$\frac{\partial \mathbf{u}^*}{\partial t} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + \nabla^* p^* - \frac{1}{Re} \nabla^{*2} \mathbf{u}^* = 0, \quad \nabla^* \cdot \mathbf{u}^* = 0. \tag{6.44}$$

Thus Re survives as the only dimensionless parameter in the equations. For a given value of Re a given problem will have a solution or solutions which are fully determined by the value of Re.[3] Nevertheless the set of solutions is fully determined by Re and Re alone. Thus we are able to make a correspondence between various problems having different U and L but the same value of Re. We call this correspondence *dynamical self-similarity.* Two flows which are self-similar in this respect become identical which expressed in the starred, dimensionless variables (6.43). In a sense the statement "the viscosity ν is small" conveys no dynamical information, although the intended implication might be that $Re \gg 1$. If L is also "small", then it could well be that $Re = 1$ or $e \ll 1$. The only meaningful way to state that a fluid is "almost inviscid" is through the Reynolds number, $Re \gg 1$. If we want to consider fluids whose viscosity is dominant compared to inertial forces, we should require $Re \ll 1$. These remarks underline the oft-repeated definition of Re as "the ratio of inertial to viscous forces". This is because

$$\frac{\rho \mathbf{u} \cdot \nabla \mathbf{u}}{\mu \nabla^2 \mathbf{u}} = Re \frac{\mathbf{u}^* \cdot \nabla^* \mathbf{u}^*}{\nabla^{*2} \mathbf{u}^*} \sim Re \tag{6.45}$$

since we regard all starred variables as of order unity.

example 6.1: The drag D per unit length of a circular cylinder of radius L in a two-dimensional uniform flow of speed U must satisfy $D = \rho U^2 L F(Re)$ for some function F. Note that we are assuming here that cylinders are fully determined by their radius. In experiments other factors, such as surface material or roughness, slight ellipticity, etc. must be considered.

Problem set 6

[3]It is not always the case that well-formulated boundary-value problems for the Navier-Stokes equations have unique solutions. See the example of viscous flow in a diverging channel, page 79 of Landau an Lifshitz.

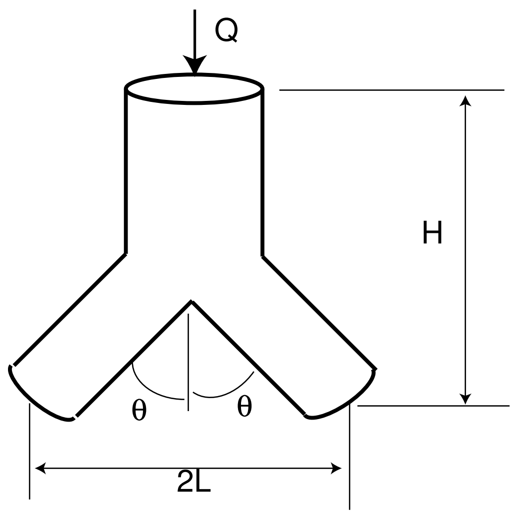

Figure 6.6: Bifurcating Poiseuill flow. Assume a parabolic profile in each section.

1. Consider the following optimzation problem: A Newtonian viscous fluid of constant density flows through a cylindrical tube of radius R_1, which then bifurcates into two straight tubes of radius R_2, see the figure. A volume flow Q is introduced into the upper tube, which divides into flows of equal flux $Q/2$ at the bifurcation. Because of the material composition of the tubes, it is desirable that the wall stress $\mu du/dr$, evaluated at the wall, be the same in both tubes. If L and H are given and fixed, what is the angle θ which minimizes the rate of working required to sustain the flow Q?. Be sure to verify that you have a true minimum.

2. Look for a solution of (6.30) of the form $\omega = t^{-1}F(r/\sqrt{t})$, satisfying $\omega(\infty, t) = 0$, $2\pi \int_0^\infty r\omega(r,t)dr = 1, t >)$. Show, by computing u_θ with $u_\theta(\infty, t) = 0$, that this represents the decay of a point vortex of unit strength in a vbiscous fluid, i.e.

$$\lim_{t\to 0+} u_\theta(r,t) = \frac{1}{2\pi r}, r > 0. \tag{6.46}$$

3. A Navier-Stokes fluid has constant ρ, μ, no body forces. Consider a motion in a fixed bounded domain V with no-slip condition on its rigid boundary. Show that

$$dE/dt = -\Phi, E = \int_V \rho|\mathbf{u}|^2/2dV, \Phi = \mu \int_V (\nabla \times \mathbf{u})^2 dV.$$

This shows that for such a fluid kinetic energy is converted into heat at a rate $\Phi(t)$. This last function of time gives the net *viscous dissipation* for the fluid contained in V. (Hint: $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}$. Also $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \nabla \times \mathbf{A} \cdot \mathbf{B} - \nabla \times \mathbf{B} \cdot \mathbf{A}$.)

4. In two dimensions, with streamfunction ψ, where $(u, v) = (\psi_y, -\psi_x)$, show that the incompressible Navier-Stokes equations without body forces for a fluid of constant ρ, μ reduce to

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{(\partial(\psi, \nabla^2\psi)}{\partial(x, y)} - \nu\nabla^4\psi = 0.$$

In terms of ψ, what are the boundary conditions on a rigid boundary if the no-slip condition is satisfied there?

5. Find the time-periodic 2D flow in a channel $-H < y < H$, filled with viscous incompressible fluid, given that the pressure gradient is $dp/dx = A + B\cos(\omega t)$, where A, B, ω are constants. This is an oscillating 2D Poiseuille flow. You may assume that $u(y, t)$ is even in y and periodic in t with period $2\pi/\omega$.

6. verify (6.33).

7. The plane $z = 0$ is rotating about the z-axis with an angular velocity Ω. A Newtonian viscous fluid of constant density and viscosity occupies $z > 0$ and the fluid satisfies the no-slip condition on the plane, i.e. at $z = 0$ the fluid rotates with the plane. By centrifugal effect we expect the fluid near the plane to be thrown out radially and a compensating flow of fluid downward toward the plane.

Using cylindrical polar coordinates, look for a steady solution of the Navier-Stokes equations of the form

$$(u_z, u_r, u_\theta) = (f(z), rg(z), rh(z)). \tag{6.47}$$

We assume that the velocity component u_θ vanishes as $z \to \infty$. Show that then

$$\frac{p}{\rho} = \nu\frac{df}{dz} - \frac{1}{2}f^2 + F, \tag{6.48}$$

where F is a function of r alone. Now argue that, if $h(\infty) = 0$, i.e. no rotation at infinity, then F must in fact be a constant. From the r and θ component of the momentum equation together with $\nabla \cdot \mathbf{u} = 0$, find equations for f, g, h and justify the following conditions:

$$f = \frac{df}{dz} = 0, h = \Omega, \quad z = 0; \quad f', h \to 0, \quad z \to \infty. \tag{6.49}$$

(The solution of these equations is discussed on pp. 75-76 of L&L and 290-92 of Batchelor.)

Chapter 7

Stokes flow

We have seen in section 6.3 that the dimensionless form of the Navier-Stokes equations for a Newtonian viscous fluid of constant density and constant viscosity is, now dropping the stars,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re} \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{7.1}$$

The Reynolds number Re is the only dimensionless parameter in the equations of motion. In the present chapter we shall investigate the fluid dynamics resulting from the *a priori* assumption that the Reynolds number is very small compared to unity, $Re \ll 1$. Since $Re = UL/\nu$, the smallness of Re can be achieved by considering extremely small length scales, or by dealing with a very viscous liquid, or by treating flows of very small velocity, so-called *creeping flows*.

The choice $Re \ll 1$ is an very interesting and important assumption, for it is relevant to many practical problems, especially in a world where many products of technology, including those manipulating fluids, are shrinking in size. A particularly interesting application is to the swimming of micro-organisms. In all of these areas we shall, with this assumption, unveil a special dynamical regime which is usually referred to as *Stokes flow*, in honor of George Stokes, who initiated investigations into this class of fluid problems. We shall also refer to this general area of fluid dynamics as the *Stokesian realm*, in contrast to the theories of inviscid flow, which might be termed the *Eulerian realm*.

What are the principle characteristics of the Stokesian realm? Since Re is indicative of the ratio of inertial to viscous forces, the assumption of small Re will mean that viscous forces dominate the dynamics. That suggests that we may be able to drop entirely the term $D\mathbf{u}/Dt$ from the Navier-Stokes equations, rendering the system *linear*. This will indeed be the case, with some caveats discussed below. The linearity of the problem will be a major simplification.

Looking at (7.1)in the form

$$Re\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p\Big) = \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{7.2}$$

It is tempting to say that the smallness of Re means that we can neglect the left-hand side of the first equation, leading to the reduced (linear) system

$$\nabla^2\mathbf{u} = 0, \quad \nabla\cdot\mathbf{u} = 0. \tag{7.3}$$

Indeed solutions of (7.3) belong to the Stokesian realm and are legitimate.

Example 7.1: Consider the velocity field $\mathbf{u} = \frac{\mathbf{A}\times\mathbf{R}}{R^3}$ in three dimensions with $\mathbf{A}$ a constant vector and $\mathbf{R} = (x, y, z)$. Note that $\mathbf{u} = \nabla\times\frac{\mathbf{A}}{R}$, and so $\nabla\cdot\mathbf{u} = 0$ and also $\nabla^2\mathbf{u} = 0, R > 0$ since $\frac{1}{R}$ is a harmonic function there. This in fact an interesting example of a Stokes flow. Consider a sphere of radius a rotating in a viscous fluid with angular velocity Ω. The on the surface of the sphere the velocity is $\Omega\times\mathbf{R}$ if the no-slip condition holds. Comparing this with our example we see that if $\mathbf{A} = \Omega a^3$ we satisfy this condition with a Stokes flow. Thus we have solved the Stokes flow problem of a sphere spinning in an infinite expanse of viscous fluid.

It is not difficult to see, however, that (7.3) does not encompass all of the Stokes flows of interest. The reason is that the pressure has been expelled from the system, whereas there is no physical reason for this. If, in the process of writing the dimensionless equations, we had defined the dimensionless pressure as $pL/(\mu U)$ instead of $p/(\rho U^2$, (7.2) would be changed to

$$Re\Big(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\Big) + \nabla p = \nabla^2\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0, \tag{7.4}$$

leading in the limit $\Re \to 0$ to

$$\nabla p - \nabla^2\mathbf{u} = 0, \quad \nabla\cdot\mathbf{u} = 0. \tag{7.5}$$

We see that any solution of (7.5) will have the form $\mathbf{u} = \nabla\phi + \mathbf{v}$ where $\nabla^2\phi = p$ and $\nabla^2\mathbf{v} = 0, \nabla\cdot\mathbf{v} = -p$. This larger class of flows, valid for Re small, are called *Stokes flows*. The special family of flows with zero pressure form a small subset of all Stokes flows.

7.0.1 Some caveats

We noted above that the dropping of the inertial terms in Stokes flow might have to be questioned in some cases, and we consider these exceptions now. First, it can happen that there is more than one possible Reynolds number which can be formed, involving one or more distinct lengths, and/or a frequency of oscillation, etc. It can then happen that the time derivative of $\mathbf{u}$ needs to be kept even though the $\mathbf{u}\cdot\nabla\mathbf{u}$ nonlinear term may be dropped. An example is a wall adjacent too a viscous fluid, executing a standing wave with amplitude A, frequency ω and wavelength L. If $\omega L^2/\nu$ is of order unity, and we take $U = \omega L$, then the Reynolds number UL/ν is of order unity and no terms may be dropped. However the actual velocity is of order ωA, and if $A \ll L$ then the nonlinear terms are negligible.

Another unusual situation is associated with the non-uniformity of the Stokes equations in three dimensions near infinity, in steady flow past a finite body.

Even through the Reynolds number is small, the fall off of the velocity as R^{-1} (associated with the fundamental solution of the Stokes equations) means that near infinity the perturbation of the free stream speed U is or order R^{-1}. Thus the $\mathbf{u}\cdot\nabla\mathbf{u}$ term is $O(U^2/R^2))$ while the viscous term is $O(\nu U/R^3)$. The ratio is UR/ν, which means that when $R \sim \nu/U$ the stokes equations cannot govern the perturbational velocity. The momentum equation needed to replace the Stokes equation contains the term $U\frac{\partial \mathbf{u}}{\partial x}$. We shall remark later on the need for this new set of equations, the *Oseen equations*, in connection with two-dimensional Stokes flow.

7.1 Solution of the Stokes equations

Returning to dimensional equations, the Stokes equations are

$$\nabla p - \mu\nabla^2\mathbf{u} = 0, \quad \nabla\cdot\mathbf{u} = 0. \tag{7.6}$$

From the divergence of $\nabla p - \nabla^2\mathbf{u} = 0$, using the solenoidal property of $\mathbf{u}$, we see that $\nabla^2 p = 0$, and hence that $\nabla^4\mathbf{u} = \nabla^2\nabla^2\mathbf{u} = 0$. The curl of this equation gives also $\nabla^2\nabla\times\mathbf{u} = 0$. The components of $\mathbf{u}$ thus solve the *biharmonic equation* $\nabla^4\phi = 0$ as well as the solenoidal condition, and the vorticity is a harmonic vector field. We shall combine these constraints now and set up a procedure for constructing solutions from a scalar biharmonic equation.

We first set

$$u_i = \Big(\frac{\partial^2\chi}{\partial x_i\partial x_j} - \delta_{ij}\chi\Big)a_j, p = \mu\frac{\partial\nabla^2\chi}{\partial x_j}a_j, \tag{7.7}$$

where $\mathbf{a}$ is a constant vector. Inserting these expressions into (7.6) we see that the equations are satisfied identically provided that

$$\nabla^4\chi = 0. \tag{7.8}$$

A second class of solution, having zero pressure, has the form

$$\varepsilon_{ijk}\frac{\partial\phi}{\partial j}a_k, \quad \nabla^2\phi = 0, \tag{7.9}$$

for a constant vector $\mathbf{a}$, where $\varepsilon_{ijk} = 1$ for subscripts which are an even permutation of 123, and is -1 otherwise. The solutions (7.9) include example 7.1, with $\mathbf{A} = \mathbf{a}$ and $\phi = R^{-1}$.

Example 7.2: The fundamental solution of the Stokes equations in three dimensions corresponds to a point force $\mathbf{F}\delta(\mathbf{x})$ on the right of the momentum equation, $\mathbf{F}$ a constant vector:

$$\nabla p - \mu\nabla^2\mathbf{u} = \mathbf{F}\delta(\mathbf{x}), \quad \nabla\cdot\mathbf{u} = 0. \tag{7.10}$$

Setting $\mathbf{a} = \mathbf{F}$ in (7.7) we must have

$$\mu\nabla^4\chi = \delta(\mathbf{x}). \tag{7.11}$$

We know the fundamental solution of $\nabla^2\phi = 0$, satisfying $\nabla^2\phi = \delta(\mathbf{x})$ and vanishing at infinity is $-\frac{1}{4\pi R}$ in three dimensions. Thus

$$\nabla^2\chi = -\frac{1}{4\pi}\frac{1}{R} = \frac{\mu}{R}\frac{d^2R\chi}{dR^2}, \tag{7.12}$$

and so

$$\chi = -\frac{1}{8\pi\mu}R + A + BR^{-1}. \tag{7.13}$$

The singular component is incompatible with (7.11) and the constant A may be set equal to zero without changing $\mathbf{u}$, and so $\chi = -\frac{1}{8\pi\mu}R$. Then we find

$$u_i = \frac{1}{8\pi\mu}\Big(\frac{x_i x_j}{R^3} + \frac{\delta_{ij}}{R}\Big)F_j, p = \frac{1}{4\pi}\frac{x_j F_j}{R^3}. \tag{7.14}$$

The particular Stokes flow (7.14) is often referred to as a *Stokeslet.*

7.2 Uniqueness of Stokes flows

Consider Stokes flow within a volume V having boundary S. Let the boundary have velocity $\mathbf{u}_S$. By the no-slip condition (which certainly applies when viscous forces are dominant), the fluid velocity $\mathbf{u}$ must equal $\mathbf{u}_S$ on the boundary. Suppose now that there are two solutions $u_{1,2}$ to the problem of solving (7.6) with this boundary condition on S. Then $\mathbf{v} = u_1 - u_2$ will vanish on S while solving (7.6). But then

$$\int_V \mathbf{v}\cdot(\nabla p - \mu\nabla^2\mathbf{v})dV = 0 = \int_V \frac{\partial}{\partial x_j}\Big(v_j p - \mu v_i \frac{\partial v_i}{\partial x_j}\Big)dV + \mu\int_V \Big(\frac{\partial v_i}{\partial x_j}\Big)^2 dV, \tag{7.15}$$

where the solenoidal property of $\mathbf{v}$ has been used. The first integral on the right vanishes under the divergence theorem because of the vanishing of $\mathbf{v}$ on S. The second is non-negative (with understood summation over i, j), and vanishes only if $\mathbf{v} = 0$. We remark that the non-negative term is equal to the rate of dissipation of kinetic energy into heat as a result of viscous stresses, for the velocity field $\mathbf{v}$. This dissipation can vanish only if the velocity is identically zero.

The solution of the Stokes equations is not easy in most geometries, and frequently the coordinate system appropriate to the problem will suggest the best formulation. We illustrate this process in the next section.

7.3 Stokes' solution for uniform flow past a sphere

We now consider the classic solution of the Stokes equations representing the uniform motion of a sphere of radius a in an infinite expanse of fluid. We shall first consider this problem using the natural coordinates for the available symmetry, namely spherical polar coordinates. Then we shall re-derive the solution using (7.7). The velocity field in spherical coordinates has the form $(u_R, u_\theta, u_\phi) = (u_R, u_\theta, 0)$ and the solenoidal condition is

$$\frac{1}{R}\frac{\partial R^2 u_R}{\partial R} + \frac{1}{\sin\theta R}\frac{\partial \sin\theta u_\theta}{\partial\theta} = 0. \tag{7.16}$$

We thus introduce the Stokes steam function Ψ,

$$u_R = \frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \quad u_\theta = -\frac{1}{R\sin\theta}\frac{\partial\Psi}{\partial R}. \tag{7.17}$$

Now Stokes' equations in spherical coordinates are

$$\frac{\partial p}{\partial R} = \mu\Big(\nabla^2 u_R - \frac{2u_R}{R^2} - \frac{2}{R^2\sin\theta}\frac{\partial\sin\theta u_\theta}{\partial\theta}\Big), \tag{7.18}$$

$$\frac{1}{R}\frac{\partial p}{\partial\theta} = \mu\Big(\nabla^2 u_\theta + \frac{2}{R^2}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R^2\sin^2\theta}\Big), \tag{7.19}$$

together with (7.16). The vorticity is $(0, 0, \omega_\phi)$, where

$$\omega_\phi = -\frac{1}{R\sin\theta}\mathcal{L}\Psi, \tag{7.20}$$

where

$$\mathcal{L} = \frac{\partial^2}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big). \tag{7.21}$$

Now from the form $\nabla p + \mu\nabla\times\nabla\times\mathbf{u} = 0$ of the momentum equation, we have the alternative form

$$\frac{\partial p}{\partial R} = -\frac{\mu}{R\sin\theta}\frac{\partial}{\partial\theta}\omega_\phi\sin\theta, \tag{7.22}$$

$$\frac{1}{R}\frac{\partial p}{\partial\theta} = \frac{\mu}{R}\frac{\partial}{\partial R}R\omega_\phi. \tag{7.23}$$

Eliminating the pressure and using (7.20) we obtain

$$\frac{1}{R^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\mathcal{L}\Psi + \frac{\partial}{\partial R}\frac{1}{\sin\theta}\frac{\partial}{\partial R}\mathcal{L}\Psi = 0. \tag{7.24}$$

We seek to solve (7.24) with the conditions

$$u_R = u_\theta = 0,\ R = a, \quad \Psi \sim \frac{1}{2}R^2\sin^2\theta U,\ R\to\infty. \tag{7.25}$$

We now separate variables in the form

$$\Psi = \sin^2\theta f(R), \tag{7.26}$$

to obtain from (7.24)

$$\Big(\frac{\partial^2}{\partial R^2} - \frac{2}{R^2}\Big)^2 f = 0. \tag{7.27}$$

Trying $f = R^\lambda$ we get $(\lambda^2-1)(\lambda-2)(\lambda-4) = 0$ and therefore the general solution of (7.27) is

$$f = \frac{A}{R} + BR + CR^2 + DR^4. \tag{7.28}$$

From the behavior needed for large R, $D = 0, C = U/2$. The two conditions at $R = a$,namely $f(a) = f'(a) = 0$, then require that

$$A = \frac{1}{4}Ua^3, \quad B = -\frac{3}{4}Ua. \tag{7.29}$$

Thus

$$\Psi = \frac{1}{4}U\Big(\frac{a^3}{R} - 3aR + 2R^2\Big)\sin^2\theta. \tag{7.30}$$

7.3.1 Drag

To find the drag on the sphere, we need the following stress component evaluated on $R = a$:

$$\sigma_{RR} = -p + 2\mu\frac{\partial u_R}{\partial R}, \quad \sigma_{R\theta} = \mu R\frac{\partial}{\partial R}\Big(\frac{u_\theta}{R}\Big) + \frac{\mu}{R}\frac{\partial u_R}{\partial\theta}. \tag{7.31}$$

Given these functions the drag D is determined by

$$D = a^2\int_0^{2\pi}\int_0^\pi [\sigma_{RR}\cos\theta - \sigma_{R\theta}\sin\theta]\sin\theta d\theta d\phi. \tag{7.32}$$

Now from (7.23) the pressure is determined by

$$\frac{1}{R}\frac{\partial p}{\partial\theta} = -\frac{\mu}{R\sin\theta}\frac{\partial}{\partial R}\sin^2\theta\Big(f_{RR} - \frac{2}{R^2}f\Big), \tag{7.33}$$

or, using (7.30),

$$p = -\frac{3}{2}\mu Ua\frac{\cos\theta}{R^2} + p_\infty. \tag{7.34}$$

Also

$$u_R = \frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta} = \frac{U\cos\theta}{2R^2}(a^3/R - 3aR + 2R^2), \tag{7.35}$$

$$u_\theta = -\frac{1}{R\sin\theta}\frac{\partial\Psi}{\partial R} = -\frac{U\sin\theta}{4R}(-a^3/R - 3a + 4R). \tag{7.36}$$

Thus

$$D = 2\pi a^2\int_0^\pi\Bigg[\Big[\frac{3}{2}\mu Ua\frac{\cos\theta}{R^2} - p_\infty + 2\mu\cos\theta\big(\frac{-3a^3}{R^4} - \frac{3a}{R}\big)_{R=a}\Big]\cos\theta$$

$$+\frac{\mu U\sin^2\theta}{4}\big(\frac{3a^2}{R^4} + \frac{3a}{R^2}\big)_{R=a}\Bigg]\sin\theta d\theta. \tag{7.37}$$

Thus

$$D = \underbrace{3\pi\mu aU \int_0^\pi \cos^2\theta \sin\theta d\theta}_{\text{pressure}} + \underbrace{3\pi\mu aU \int_0^\pi \sin^3\theta d\theta}_{\text{viscous}},$$

$$= 2\pi\mu aU + 4\pi\mu aU = 6\pi\mu aU. \tag{7.38}$$

That is, one-third of the drag is due to pressure forces, two-thirds to viscous forces.

7.3.2 An alternative derivation

We can re-derive Stokes' solution for a sphere by realizing that at large distances from the sphere the flow field must consist of a uniform flow plus the fundamental solution for a force $-6\pi\mu Ua\mathbf{i}$. This must be added a term or terms which will account for the finite sphere size. Given the symmetry we try a dipole term proportional to $\nabla(x/R^3)$. We thus postulate

$$\mathbf{u} = U\mathbf{i} - \frac{6\pi\mu aU}{8\pi\mu}\Big(\frac{x\mathbf{R}}{R^3} + \frac{\mathbf{i}}{R}\Big) + C\Big(\frac{\mathbf{i}}{R^3} - 3\frac{x\mathbf{R}}{R^5}\Big), \tag{7.39}$$

where C remains to be determined. By inspection we see that $C = -\frac{1}{4}a^2U$ makes $\mathbf{u} = 0$ on $R = a$, so we are done! The pressure is as given previously $p = -\frac{3}{2}\mu Uax/R^3 + p_\infty$, and is entirely associated with the fundamental part of the solution.

7.4 Two-dimensions: Stokes' paradox

The fundamental solution of the Stokes equations in two dimensions sets up as given in example 7.2, except that the biharmonic equation is to be solved in two dimensions. If Radial symmetry is again assumed, we may try to solve the problem equivalent to flow past a sphere, i.e. Stokes flow past a circular cylinder of radius a. If the pressure is eliminated from the Stokes equations in two dimensions, we get

$$\mu\nabla^2\omega = \mu\nabla^4\psi = 0. \tag{7.40}$$

in terms of the two-dimensional stream function ψ. We the set $\psi = \sin\theta f(r)$ to separate variables in polar coordinates, leading to

$$\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\Big]^2 f = 0. \tag{7.41}$$

We are now in a position to study flow over a circular cylinder of radius a. The no-slip condition at the surface of the cylinder requires that $\psi(a) = \frac{\partial\psi}{\partial r}(a) = 0$, while the attaining of a free stream $\mathbf{u} = (U, 0)$ at infinity requires that $f \sim Ur, r \to \infty$. Now by quadrature we can find the most general solution of (7.41) as

$$f(r) = Ar^3 + Br\ln r + Cr + Dr^{-1}. \tag{7.42}$$

The condition at infinity requires that $A = B = 0$. The no-slip conditions then yield

$$Ca + Da^{-1} = 0, \quad C - Da^{-2} = 0, \tag{7.43}$$

which imply $C = D = 0$. *There is no satisfactory steady solution of the two-dimensional Stokes equations representing flow of an unbounded fluid past a circular cylinder.* This result, known as *Stokes paradox*, underlines the profound effect that dimension can play in fluid dynamics.

What is the reason for this non-existence? We can get some idea of what is going on by introducing a finite circle $r = R$ on which we make $\mathbf{u} = (U, 0)$. Then there *does* exist a function $f(r)$ satisfying $f(a) = f'(a) = 0, f(R) = R, f'(R) = 1$.[1] We shall obtain as asymptotic approximation for large R/a to this solution by setting $A = 0$ in (7.42) and satisfying the conditions at $r = a$ with the remaining terms. Then we obtain

$$f \sim B[r \ln r - (\ln a + 1/2)r + \frac{1}{2}a^2/r]. \tag{7.44}$$

We then make $f(R) \sim UR, R/a \to \infty$ by setting $B = U/\ln(R/a)$. Then also $f'(R) \sim 1 + o(1), R/a \to \infty$, so all conditions are satisfied exactly or asymptotically for large R/a. Thus

$$f \sim \frac{U}{\ln(R/a)}\big[r(ln(r/a) - 1/2) + \frac{1}{2}a^2/r\big]. \tag{7.45}$$

A a fixed value of $r/a > 1$ we see that $f \to 0$ as $R/a \to \infty$. It is only when $\frac{\ln(r/a)}{\ln(R/a)}$ become $O(1)$ that order UR values of f, and hence order U values of velocity, are realized. Thus a cylindrical body in creeping through a viscous fluid will tend to carry with it a large stagnant body of fluid, and there is no solution of the boundary-value problem for an infinite domain in Stokes flow.

This paradox results from a failure to properly account for the balance of forces in a viscous fluid at large distances from a translating body, however small the Reynolds number of translation may be. If the velocity of translation is U and the body size L The remedy for this paradox is involves a problem of singular perturbation wherein the regions distant from the cylinder see a disturbance from a point force. Let the velocity at some point a distance $R \gg a$ from the body be q. The the inertial forces at this point sill be approximately $\sim \rho Uq/R$, (since we should linearize $\mathbf{u} \cdot \nabla \mathbf{u}$ about the free stream velocity). Also the viscous forces there are of order $\mu q/R^2$. These two estimates are comparable when $R/L \sim \nu/(UL) = 1/Re$. Thus when $Re < 1$ and we try to apply the Stokes equations, there is always distant points where the neglect of the inertial terms fails to be valid.

In the case of three dimensions, we have Stokes' solution for a sphere and we know that at distances $O(1/Re)$ the perturbation velocity caused by the sphere is small, or order Re. Thus, the Stokes approximation fails in a region

[1] If $\psi_y = U, \psi_x = 0$ on a circle $r = R$, then $f/r + (f/r)'(y^2/r) = U, (f/r)'(xy/r) = 0$ when $r = R$, by differentiation of $\sin\theta f$. Thus $f(R) = R$ and $f'(R) = 1$.

where the free stream velocity is essentially unperturbed, and there is no Stokes paradox. In two dimensions, the perturbation caused by the cylinder persists out to distances of order $1/Re$. Thus the Stokes equations fail to be uniformly valid in a domain large enough to allow necessary conditions at infinity to be satisfied.

The remedy for this paradox in two dimensions involves a proper accounting for the singular nature of the limit $Re \to 0$ in the neighborhood of infinity. At distances $r \sim Re^{-1}$ the appropriate equations are found to be

$$\rho U \frac{\partial \mathbf{u}}{\partial x} + \nabla p - \mu \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{7.46}$$

This system is known as *Oseen's equations*. Oseen proposed them as a way of approximately accounting for fluid inertia in problems where there is an ambient free stream $U\mathbf{i}$. Their advantage is of course that they comprise a *linear* system of equations. The fact remains that they arise rigorously to appropriately treat viscous flow in the limit of small Reynolds numbers, in a way that expels any paradox associated with large distances.

To summarize, in creeping flow the Stokes model works well in three dimensions; near the body the equations are exact, and far from the body the non-uniformity, leading to the replacement of the Stokes equations by the Oseen equations, is of no consequence and Stokes' solution for a sphere is valid. In two dimensions the distant effect of a cylinder must be determined from Oseen's model. It is only by looking at that solution, expanded near the position of the cylinder, that we can determine the appropriate solution of Stokes' equations in two dimensions; this solution remains otherwise undetermined by virtue of the Stokes paradox.

7.5 Time-reversibility in Stokes flow

Consider a viscous fluid contained in some finite region V bounded by surface or surfaces ∂V. If Stokes flow prevails, and if the boundary moves, each point of ∂V being assigned a boundary velocity $\mathbf{u}_b$, then we have a boundary-value problem for the Stokes equations, whose solution will provide the instantaneous velocity of every fluid particle in V. We assume the existence of this solution, and we have seen that this solution must be unique.and verify now that it will be unique.

Thus in Stokes flow the instantaneous velocity of a fluid particle at P is determined uniquely by the instantaneous velocities of all points on the boundary of the fluid domain. Let us now assume a motion of the boundary through a sequence of configurations $\mathcal{C}(t)$. Each $\mathcal{C}$ represents a point in *configuration space*, and the motion can be thought of as a path in configuration space with time as a parameter. Indeed "time" has no dynamical significance. A path from A to B in configuration space can be taken quickly or slowly. In general, let the configuration at time t be given by $\mathcal{C}(\tau(t))$, where $\tau(0) = 0, \tau(1) = 1$ but is otherwise an arbitrary differentiable function of time. If the point P has velocity $\mathbf{u}_P(t), 0 \leq t \leq 1$ when $\tau(t) = t$, then in general $\mathbf{u}_P(t) = \dot{\tau}(t)\mathbf{u}_P(\tau(t))$. The vector displacement of the point P under this sequence of configurations is

$$\Delta_P = \int_{t=0}^{t=1} \dot{\tau}(t)\mathbf{u}_P(\tau(t))dt = \int_0^1 \mathbf{u}(\tau)d\tau \tag{7.47}$$

and so is independent of the choice of τ. Another way to say this is that the displacement depends upon the ordering of the sequence of configurations but not on the timing of the sequence.

The displacement *does* however depend in general on the *path* taken in configuration space in going from configuration $\mathcal{C}_0$ to $\mathcal{C}_1$. We now give an example of this dependence.

Example 7.3: We must find two paths in configuration space having the same starting and finishing configurations (i.e. the boundary points coincide in each case), but for which the displacement of some fluid particle is not the same. Consider then a two-dimensional geometry with fluid contained in the circular annulus $a < r < b$. Let the inner cylinder of radius a rotate with time so that the angle made by some fixed point on the cylinder is $\theta(t)$ relative to a reference axis. The outer circle $r = b$ sis fixed. The instantaneous velocity of each point of the fluid. Given that $\frac{\partial p}{\partial \theta} = 0$ and that the velocity is $0.u_\theta(r)$, the function $u_\theta(r)$ satisfies (from the Stokes form of (6.15)) $Lu_\theta = 0$. Integrating and applying boundary conditions, the fluid velocity in the annulus is

$$u_\theta = \frac{a\dot{\theta}}{a^2 - b^2}r - \frac{ab^2\dot{\theta}}{a^2 - b^2}r^{-1}. \tag{7.48}$$

Consider not two paths which leave the position of the point of the inner circle unchanged. In the first, θ rotates from 0 to $\pi/4$ in one direction, then from $\pi/4$ back to zero in the other direction. Clearly every fluid particle will return to its original position after these two moves. For the second path, rotate the cylinder through 2π. Again every point of the boundary returns to its starting point, but now every point of fluid in $a < r < b$ moves through an angle θ which is positive and less than 2π. Thus only the points on the two circles $r = a, b$ are in their starting positions at the end of the rotation.

Note that in this example the first path, returning all fluid particles to their starting positions, is special in that *the sequence of configurations in the second movement is simply a reversal of the sequence of configurations in the first movement* (a rotation through angle $\pi/4$. A moments reflection shows that zero particle displacement is a necessary consequence of this kind path- a sequence followed by the reverse sequence. And note that the timing of each of these sequences may be different.

The second path, a full rotation of the inner circle, involves no such reversal. In fact if the direction of rotation is reversed, the fluids point move in the opposite direction. If we now let these two paths be repeated periodically, say every one unit of time t, then in the first case fluid particles move back and forth periodically with no net displacement, while in the second case particles move on circles with a fixed displacement for each unit of time. Notice now an importance difference in the time symmetry of these two cases. If time is run backwards in the first case, we again see fluid particles moving back and forth with no net displacement. In the second case, reversal of time leads to steady rotation of particles in the opposite direction. We may say that the flow in the first case exhibits *time reversal symmetry*, while in the second case it does not exhibit this symmetry. In general, a periodic boundary motion exhibiting time reversal symmetry cannot lead to net motion of any fluid particle over one period, as determined by the resulting time-periodic Stokes flow. On the other hand, if net motion is observed, the boundary motion cannot be symmetric under time reversal.

However a motion that is not symmetric under time reversal may in fact not produce any displacement of fluid particles.

Example 7.4 In the previous example, let both circles rotate through 2π with $\dot{\theta}_b = \frac{b}{a}\dot{\theta}_a$. The boundary motion does not then exhibit time-reversal symmetry, and in fact the fluid can be seen to be in a solid body rotation. Thus every fluid particle returns to its starting position.

Theorem 7 *Time reversal symmetry of periodic boundary motion is sufficient to insure that all fluid particles return periodically to their starting positions. If particles do not return periodically to their starting position, the boundary motion cannot be time-symmetric.*

7.6 Stokesian locomotion and the scallop theorem

One of the most important and interesting applications of Stokes flow hydrodynamics is to the swimming of micro-organisms. Most micro-organisms move by a periodic or near periodic motion of organelles such as cilia and flagella. The aim of this waving of organelles is usually to move the organisms from point A to point B, a process complementary to a variable boundary which moves the fluid about but does not itself locomote. Indeed time-reversal symmetry plays an key role in the selection of swimming strategies.

Theorem 8 *(The scallop theorem) Suppose that a small swimming body in an infinite expanse of fluid is observed to execute a periodic cycle of configurations, relative to a coordinate system moving with constant velocity* $\mathbf{U}$ *relative to the fluid at infinity. Suppose that the fluid dynamics is that of Stokes flow. If the sequence of configurations is indistinguishable from the time reversed sequence, then* $\mathbf{U} = 0$ *and the body does not locomote.*

The reasoning here is that actual time reversal of the swimming motions would lead to locomotion with velocity $-\mathbf{U}$. But if the two motions are indistinguishable then $\mathbf{U} = -\mathbf{U}$ and so $\mathbf{U} = 0$. The name of the theorem derives from the non-locomotion of a scallop in Stokes flow that simple opens and closes its shell periodically. In Stokes flow this would lead to a back and forth motion along a line (assuming suitable symmetry of shape of the shell), with no net locomotion.

In nature the breaking of time-reversal symmetry takes many forms. Flagella tend to propagate waves from head to tail. The wave direction gives the arrow of time, and it reverses, along with the swimming velocity, under time reversal. Cilia also execute complicated forward and return strokes which are not time symmetric.

Problem set 7

1. Consider the uniform slow motion with speed U of a viscous fluid past a spherical bubble of radius a, filled with air. Do this by modifying the Stokes flow analysis for a rigid sphere as follows. The no slip condition is to be replaced on $r = a$ by the condition that both u_r and the tangential stress $\sigma_{r\theta}$ vanish. (This latter condition applies since there is no fluid within the bubble to support this stress.) Show in particular that

$$\Psi = \frac{U}{2}(r^2 - ar)\sin^2\theta$$

and that the drag on the bubble is $D = 4\pi\mu U a$. Note: On page 235 of Batchelor see the analysis for a bubble filled with a second liquid of viscosity $\bar{\mu}$. The present problem is for $\bar{\mu} = 0$.

2. Prove that Stokes flow past a given, rigid body is unique, as follows. Show if $p_1, \mathbf{u}_1$ and $p_2, \mathbf{u}_2$ are two solutions of

$$\nabla p - \mu\nabla^2\mathbf{u} = 0, \nabla\cdot\mathbf{u} = 0,$$

satisfying $u_i = -U_i$ on the body and

$$\mathbf{u} \sim= O(1/r), \frac{\partial u_i}{\partial x_j}, p \sim O(1/r^2)$$

as $r \to \infty$, then the two solutions must agree. (Hint: Consider the integral of $\partial/\partial x_i(w_j \partial w_j/\partial x_i)$ over the region exterior to the body, where $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$.)

3. Two small spheres of radius a and density ρ_s are falling in a viscous fluid with centers at P and Q. The line PQ has length $L \gg a$ and is perpendicular to gravity. Using the Stokeslet approximation to the Stokes solution past a sphere, and assuming that each sphere sees the unperturbed Stokes flow of the other sphere, show that the spheres fall with the same speed

$$U \approx U_s(1 + ka/L + O(a^2/L^2)),$$

and determine the number k. Here $U_s = 2a^2g/9\nu(\rho_s/\rho - 1)$ is the settling speed of a single sphere in Stokes flow.

Chapter 8

The boundary layer

The concept of the boundary layer is a classic example of an applied science greatly influencing the development of mathematical methods of wide applicability. The key idea was introduced in a 10 minute address in 1904 by Ludwig Prandtl, then a 29 year old professor in Hanover, Germany. Prandtl had done experiments in the flow of water over bodies, and sought to understand the effect of the small viscosity on the flow. Realizing that the no-slip condition had to apply at the surface of the body, his observations led him to the conclusion that the flow was brought to rest in a thin layer adjacent to the rigid surface. His reasoning suggested that the Navier-Stokes equations should have a somewhat simpler form owing to the thinness of this layer. This led to the equations of the viscous boundary layer. Boundary-layer methods now occupy a fundamental place in many asymptotic problems for partial differential equations.

8.1 The limit of large Re

Let us consider the steady viscous two-dimensional flow over a flat plate aligned with a uniform stream $(U, 0)$. In dimensionless variables the steady Navier-

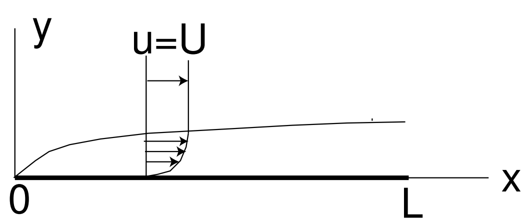

Figure 8.1: Boundary layer on a finite flat plate.

Stokes equations in two dimensions may be written

$$\mathbf{u} \cdot \nabla u + p_x - \frac{1}{Re}\nabla^2 u = 0, \tag{8.1}$$

$$\mathbf{u} \cdot \nabla v + p_y - \frac{1}{Re}\nabla^2 v = 0, \tag{8.2}$$

$$u_x + v_y = 0. \tag{8.3}$$

We are dealing with the geometry of figure 8.1. The boundary layer is seen to grow in thickness as x moves from 0 to L. This suggests that the term $\mathbf{u} \cdot \nabla u$ in (8.1) has been properly estimated as of order U^2/L in the dimensionless formulation, and so should be taken as $O(1)$ at large Re in (8.1). If this term is to balance the viscous stress term, then the natural choice, since the boundary layer on the plate is observed to be so thin, is to assume that the y-derivatives of u are so large that the balance is with $\frac{1}{Re}u_{yy}$. Thus it makes sense to define an *stretched* variable $\bar{y} = \sqrt{Re}y$. If we now apply the stretched variable to (8.3), still taking u_x as of order unity, then in order to keep this essential equation intact we must compensate the stretched variable $\bar{y}$ by a streched form of the y-velocity component:

$$\bar{v} = \sqrt{Re}v. \tag{8.4}$$

Prandtl would have been comfortable with this last definition. The boundary layer on the plate was so thin that there could have been only a small velocity component normal to its surface. Thus the continuity equation will survive our limit $Re \to \infty$:

$$u_x + \bar{v}_{\bar{y}} = 0. \tag{8.5}$$

Returning now to consideration of (8.1), retain the pressure term p_x as $O(1)$ as well so that the simplified equation, obtained in the limit $Re \to \infty$ in the stretched variables, amounts to dropping the term $\frac{1}{Re}u_{xx}$:

$$uu_x + \bar{v}u_{\bar{y}} + p_x - u_{\bar{y}\bar{y}} = 0. \tag{8.6}$$

Finally, using these stretched variables in (8.2) we have

$$p_{\bar{y}} = -\frac{1}{Re}(u\bar{v}_x + \bar{v}\bar{v}_{\bar{y}} - \bar{v}_{\bar{y}\bar{y}}) + \frac{1}{Re^2}\bar{v}_{xx}. \tag{8.7}$$

Thus in the limit $Re \to \infty$ the vertical momentum equation reduces to

$$p_{\bar{y}} = 0. \tag{8.8}$$

We thus see from (8.8) that the pressure does not change as we move vertically through the thin boundary layer. That is, the pressure throughout the boundary layer at a station x must be the pressure outside the layer. At this point a crucial contact is made with inviscid fluid theory. The "pressure outside the boundary layer" should be determined by the inviscid theory, since the boundary layer is thin and will presumably not disturb the inviscid flow very

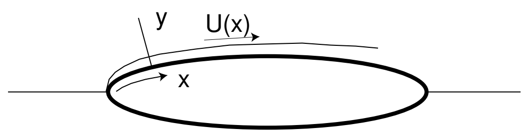

Figure 8.2: Boundary layer over a general body with varying $U(x)$.

much. In particular for a flat plate the Euler flow is the uniform stream- the plate has no effect- and so the pressure has its constant free-stream value.

Prandtl's striking insight is clearer when we consider flow past a general smooth body, as in figure 8.2. Since the boundary layer is again taken as thin in the neighborhood of the body, curvilinear coordinates may be introduced, with x the arc length along curves paralleling the body surface and y the coordinate normal to these curves. In the stretched variables, and in the limit for large Re, it turns out that we again get (8.6)-(8.8), only now (8.8) must be interpreted to mean that the pressure is what would be computed from the inviscid flow past the body. If p_0, U_0 are the free stream values of p, u, then Bernoulli's theorem for steady flow yields along the body surface

$$p_{euler} = p_0 + \frac{1}{2} - \frac{1}{2}U^2(x) = p(x), \tag{8.9}$$

and it is this $p(x)$ which now applies in the boundary layer, by (8.8). Thus the inviscid flow past the body determines the pressure variation which is then impose on the boundary layer through the now known function p_x in (8.6).

We note that the system (8.6)=(8.8) are usually called the *Prandtl boundary-layer equations*.

We are giving here the essence of Prandtl's idea without any indication of possible problems in implementing it for an arbitrary body. The main problem which will arise is that of *boundary layer separation*. It turns out that the function $p(x)$, which is determined by the inviscid flow, may lead to a boundary layer which cannot be continued indefinitely along the surface of the body. What can and does occur is a breaking away of the boundary layer from the surface, the ejection of vorticity into the free stream, and the creation of free separation streamline similar to the free streamline of the Kirchoff flow we considered in chapter 6. Separation is part of the stalling of an airfoil at high angles of attack, for example.

8.2 Blasius' solution for a semi-infinite flat plate

We now give the famous Blasius solution of the boundary layer past a semi-infinite flat plate; geometrically the problem is that of figure 8.1 with $L = \infty$. The fact that the plate is infnite will mean that the boundary layer extends to infinity. We will comment on this later. For the moment simply note that we have now expelled the length L from the problem, even though we used it previously to define a Reynolds number, which number we then let tend to infinity. Without a length in the problem, however, it becomes much simpler to solve, because the no-slip conditions applies on the entive line $x > 0, \bar{y} = 0$.

We recall that for the aligned flat plate the pressure in Prandtl's boundary layer equations is zero, so we seek to solve

$$uu_x + \bar{v}u_{\bar{y}} - u_{\bar{y}\bar{y}} = 0, \quad u_x + \bar{v}_{\bar{y}} = 0, \tag{8.10}$$

subject to the conditions $u = \bar{v} = 0, \bar{y} = 0, x > 0$, and $u \to 1$ as $\bar{y} \to \infty, x > 0$. We can satisfy the solenoidal condition in the usual way with a boundary-layer stream function $\bar{\psi} = \sqrt{Re}\psi$ such that $u = \bar{\psi}_{\bar{y}}, \bar{v} = -\bar{\psi}_x$. We then observe that our problem has a *self-similar* structure in the following sense. The equations and conditions are invariant under the group of "stretching" transformations

$$x \to Ax, \bar{y} \to B\bar{y}, \bar{\psi} \to C\bar{\psi}, \tag{8.11}$$

provided that $A = B^2$ and $B = C$. Indeed, the condition $u = 1$ transforms to $uC/B = 1$ so we must have $B = C$. Also the term uu_x scales like $\frac{C^2}{AB^2}$ while $u_{\bar{y}\bar{y}}$ scales like C/B^3, and the equality of these two factors requires $A = BC$. The remaining terms follow suit and so (8.10) and the conditions are invariant under the stated conditions $A = B^2 = C^2$. Now the combination $\eta = y/\sqrt{x}$ is then invariant under (8.11), and therefore so is the equation $\bar{\psi} = \sqrt{x}F(\eta)$ for any function F. If we assume a $\bar{\psi}$ of this form and substitute it into

$$\bar{\psi}_{\bar{y}}\bar{\psi}_{x\bar{y}} - \bar{\psi}_x\bar{\psi}_{\bar{y}\bar{y}} - \bar{\psi}_{\bar{y}\bar{y}\bar{y}} = 0, \tag{8.12}$$

it is straightforward to show that we get

$$-\frac{1}{x}\left[\frac{1}{2}FF'' + F'''\right] = 0. \tag{8.13}$$

The conditions to be satisfied are then

$$F(0) = F'(0) = 0, \quad F' \to 1, \ \eta \to \infty. \tag{8.14}$$

The simplest way to solve this problem is to replace it by the following initial-value problem:

$$\frac{1}{2}GG'' + G''' = 0, G(0) = G'(0) = 0, G''(0) = 1. \tag{8.15}$$

When this problem is solved (a simple matter using ode45 in MATLAB on an interval $0 < \eta < 5$ say, we obtain values of $G'(\eta)$ similar to figure 8.3 (the solution of the actual problem) but asymptoting to $c = 2.0854$ instead of 1. However if $G(\eta)$ is a solution of our equation, so is $AG(A\eta)$ for any constant A. Since $G \sim c\eta + o(\eta), \eta \to \infty$, we set

$$F(\eta) = c^{-1/2}G(c^{-1/2}\eta). \tag{8.16}$$

This give the curve for $F'(\eta)$ shown in figure 8.3. One finds

$$F(\eta) \sim \eta - 1.7208 + o(1), \quad \eta \to \infty, \tag{8.17}$$

and also $F''(0) = c^{-3/2} = .332$

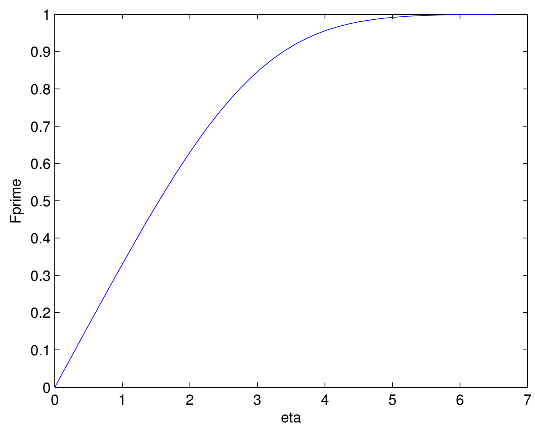

Figure 8.3: The Blasius boundary layer velocity profile.

8.2.1 Discussion of the Blasius solution

Recalling that the dimensional form of the stream function is $UL\psi$, the dimensional form of ψ is $UL\psi = ULR^{-1/2}\bar{\psi}$. In terms of x, y which have dimensions, $\bar{\psi} = (x/L)^{1/2} F\Big(\frac{\sqrt{R}y/L}{\sqrt{(x/L)}}\Big)$. Thus with dimensions fully restored the stream function may be written

$$\sqrt{U\nu x} F\Big(y\sqrt{\frac{U}{\nu x}}\Big). \tag{8.18}$$

confirming the fact that the problem we have solved is free of a length L. From the asymptotic behavior of F for large η we then have the dimensional stream function for large y in the form

$$Uy - 1.7208\sqrt{U\nu x} + o(1), y \to \infty. \tag{8.19}$$

This combination of terms vanishes when $y = 1.7208\sqrt{\nu x/U}$. This shows that well away from the plate the streamlines look like those over a thin parabolic cylinder. This process of "lifting" the distant streamlines makes the plate look like it has some thickness, which grows downstream as $\sqrt{x}$. This thickness, which has been given the term *displacement thickness*, can be understood from the nature of the volume flux in the boundary layer. As the boundary layer grows with increasing x more and more fluid parcels originally moving with the free-stream velocity U, are found to be moving more slowly. This depleted volume flux near the wall, which increases with x, must be compensated by an outward full of volume away from the wall. It is this outward flux which lifts the streamlines to their parabolic form.

The displacement thickness can be given a precise definition as follows. Let

$$\delta(x) = \int_0^\infty (1 - u/U)dy. \tag{8.20}$$

Then $Ud\delta(x) = V(x)dx$ where $V(x)$ is the compensating upward velocity is equal to the integral of $U - u$ through the layer, which is the reduced volume flux through the boundary layer. But according to (8.18) $u = UF'\Big(y\sqrt{\frac{U}{\nu x}}\Big)$ and so, since $F(\eta) \sim \eta - 1.7208 + o(1)$ we have

$$\delta(x) = \sqrt{\frac{\nu x}{U}} \lim_{\eta\to\infty} (\eta - F(\eta)) = 1.7208\sqrt{\frac{\nu x}{U}}. \tag{8.21}$$

Thus

$$V(x) = 1.7208U\delta'(x) = .8604\sqrt{\frac{U\nu}{x}} = dy/dx, \tag{8.22}$$

where $y(x) = 1.7208\sqrt{\nu x/U}$ is the zero streamline of the "effective body" whose thickness we may now identify with the displacement thickness as defined.

It is interesting to ask what error is being made if we substitute the Blasius solution into the full Navier-Stokes equations and look at the remainder. We consider here only the dimensionless form of the x-momentum equation. There the terms we expelled to get the boundary layer equation were $\frac{1}{R}u_{xx}$ and p_x. Substituting $u = F'(\eta)$ we obtain the exact equation

$$p_x - \frac{1}{x}\Big[\frac{1}{2}FF'' + F'''\Big] - \frac{\eta}{4x^2R}\Big[3F'' + \eta F'''\Big] = 0. \tag{8.23}$$

We see that the second bracketed term fails to be smaller that the first when $xR = O(1)$. Thus near the front edge of the plate the boundary layer equations are not uniformly valid. In a small circular domain of order $1/R$ in radius about the origin, the full Navier-Stokes equations can be shown to govern the fluid flow. This small non-uniformity does not affect the validity elsewhere, however. We can assert this because of the existence of the Blasius solution, and the fact that experimental measurements confirm its validity at large R.

As a final remark concerning the Blasius solution, we note that the *finite* flat plate, of length L, can be approached with exactly the same apparatus. Although the Prandtl boundary-layer equations fail to hold at $x = L$ as well as $x = 0$, the development of the layer on the plate is unaltered to first order. In paticular the drag on the plate, accounting fro both sides is given by

$$D = 2\mu \int_0^L u_y(x,0)dx = 2\mu U \int_0^L \sqrt{\frac{U}{\nu x}} F''(0)dx \tag{8.24}$$

This yields

$$D = 2\mu U \cdot 2\sqrt{\frac{UL}{\nu}} \cdot .332 = 1.328\rho U^2/\sqrt{R}. \tag{8.25}$$

Thus friction drag on a plate is $O(R^{-1/2})$ at large R, at least in a laminar flow.

8.2.2 The Falkner-Skan family of boundary layers

An immediate generalization of the Blasius solution is to boundary layers whose pressure gradient is some power of x. From the Bernoulli equation for steady flow, a gradient $p_x = -mA^2x^{2m-1}$ results from an external stream with velocity $U(x) = Ax^m$. We remark that for positive m there is an associated physical Euler flow problem. Such a velocity variation occurs on the surface of an infinite wedge aligned with a constant free stream, provided that the half-angle of the wedge is $\frac{m}{m+1}\pi$. Then x is measured along the surface of the wedge, and y is measured perpendicular to the surface. So again there is no length in the problem. The equations to be solved are then, in dimensional form,

$$uu_x + vu_y - mA^2x^{2m-1} - \nu u_{yy} = 0, \quad u_x + v_y = 0. \tag{8.26}$$

Since there is no length, we are led to look for a similarity solution . If we try $\psi = x^\alpha F(y/x^\beta)$, then the factors of x coming from insertion into (8.26) will cancel, leaving an ordinary differential equation, provided that

$$\alpha == \frac{1+m}{2}, \quad \beta = \frac{1-m}{2}. \tag{8.27}$$

Setting

$$\psi = AKx^{\frac{1+m}{2}}F(\eta), \eta = \frac{y}{Kx^{\frac{1-m}{2}}}, K = \sqrt{\frac{\nu}{(m+1)A}}, \tag{8.28}$$

the equation which results (see problem 8.1) is

$$F''' + \frac{1}{2}FF'' + \frac{m}{1+m}(1 - F'^2) = 0. \tag{8.29}$$

The boundary conditions are again $F(0) = F'(0) = 0, F'(\infty) = 1$.

We show in figure 8.4 several profiles for various m. For m positive existence and uniqueness of the solution has be established, and the profiles become somewhat steeper. The cases $m > 0$ are said to correspond to a *favorable pressure gradient*, $U'(x) > 0$ and $p'(x) < 0$. The boundary layer can be said to respond favorably to a pressure which decreases in the streamwise direction. When m becomes negative, the story is significantly different. Uniqueness of the profile can be lost, although profiles such that $u \geq 0$ for all η can be shown to be unique. In figure 8.4 we show the limiting case of such non-negative profiles, occurring when $m = -.0904$. Note that $F''(0) = 0$ for this profile. This implies du/dy vanishes at the wall, and so the viscous friction force is zero there. Positive pressure gradients are said to be *unfavorable*, and can lead to the phenomenon of *boundary layer separation*. We will return to the separation problem below. Here the suggestion is that $m < .0904$ would lead to a boundary layer which has a *negative* value of u_y at the wall, and so would involve a region of reversed flow; the streamline $\psi = 0$ must actually bifurcate from the wall, so the term "separation" is appropriate.

We may summarize the general picture of high Reynolds number flow, as provided by the boundary-layer concept, as follows. For a general finite body

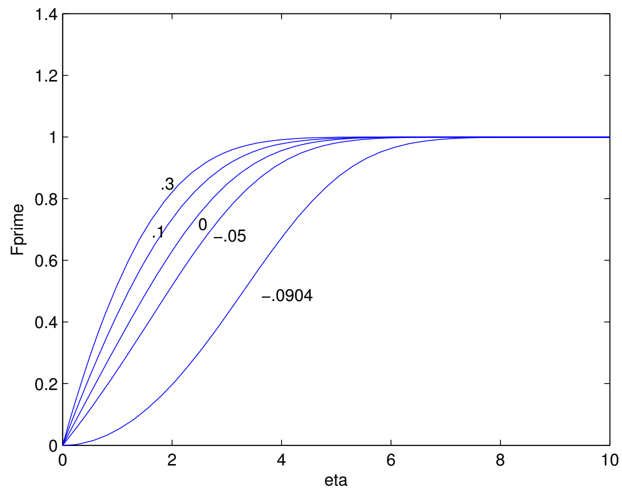

Figure 8.4: The Falkner-Skan profiles for various m.

in a flow, there should be a portion of the surface of the body, upstream of any point or point of separation of the boundary layer, where the flow is that of an inviscid fluid except within a small layer adjacent to the body, called the boundary layer. Within the boundary layer, the pressure gradient is imposed by the inviscid exterior flow. At the same time the boundary layer modifies the inviscid flow slightly due to its displacement thickness. The picture is clouded by separation, and the tendency of high Reynolds number flows to be unstable and hence time dependent.

Finally, with the example which follows we indicate how boundary-layer techniques can arise in a somewhat different context.

Example 8.1

We give here as example of the application of boundary-layer ideas to a different physical problem. The idea is to model a laminar two-dimensional steady jet issuing from a small slit in a wall, see figure 8.5. We are going to treat the jet as "thin" when $Re \gg 1$, and so apply Prandtl's reasoning to obtain again his boundary layer equations. Since $p_{\bar{y}} = 0$ the is invariant through the jet, and assuming that at $\bar{y} = \infty$ we have uniform conditions, we may assert that p is independent of x, as in Blasius' semi-infinite plate problem. There is no length in the problem (ignoring the small width of the slit), so again we are led to try a solution of the form $\psi = x^\alpha F(y/x^\beta)$. The condition that $\mathbf{u} \cdot \nabla u$ and u_{yy} have common factors of x requires that $\alpha + \beta = 1$. We do not have a nonzero value assumed by F' at infinity, as in the Blasius problem. However there is a new physical constraint. Since the pressure is constant throughout, there are no forces available to cause the net flux of x-momentum to vary as a

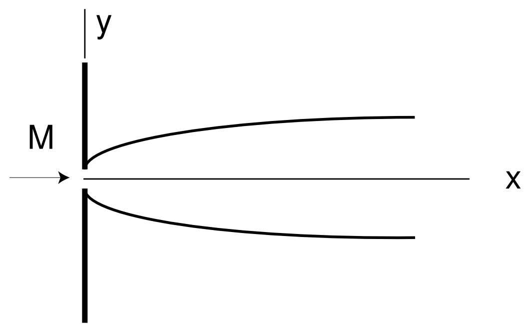

Figure 8.5: A two-dimensional laminar jet emerges from a slit in a wall.

function of x. Consequently the integral (omitting a constant factor of ρ)

$$M = \int_{-\infty}^{+\infty} u^2 dy \tag{8.30}$$

must be independent of x. This requires that $\beta = 2\alpha$, so that $\alpha = 1/3, \beta = 2/3$.[1]

Substituting $\psi = x^{1/3}F(y/x^{2/3})$ into the dimensional equation for ψ,

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = 0, \tag{8.31}$$

we get the equation

$$\nu F''' + \frac{1}{3}(FF')' = 0. \tag{8.32}$$

We require that $F'', F' \to 0$ as $\eta \to \infty$ and

$$\int_{-\infty}^{+\infty} F'^2 d\eta = M. \tag{8.33}$$

Integrating twice,

$$\frac{1}{6}F^2 + \nu F' = \frac{1}{6}F_\infty^2, \quad F_\infty = F(\infty). \tag{8.34}$$

The integral yields

$$F = F_\infty \tanh\left(\frac{F_\infty \eta}{6\nu}\right). \tag{8.35}$$

Applying the condition (8.33) we obtain

$$\frac{\nu}{4}F_\infty^3 = M, \tag{8.36}$$

[1]To get this by a stretching group, $x \to Ax, y \to By, \psi \to C\psi$, the momentum equation requires $A = BC$ as in the Blasius solution, but the momentum flux constraint is invariant when $C^2 = B$, so then $C^3 = A$. Thus $y/x^{2/3}$ is invariant, and ψ must be proportional to $x^{1/3}$.

which determine F_∞ in terms of M. The velocity component u, which dominates in the jet, is given by

$$u = \frac{F_\infty^2}{6\nu x^{1/3}} \frac{1}{\cosh^2\left(\frac{F_\infty \eta}{6\nu}\right)}. \tag{8.37}$$

Note that the jet spreads as $x^{2/3}$ and decays as $x^{-1/3}$. In practice it is difficult to establish a laminar jet because of instabilities, and the jets obtained in the laboratory are usually turbulent.

8.3 Boundary-layer analysis as a matching problem

We now digress somewhat to indicate some of the mathematical ideas that have grown our of Prandtl's approach to high Reynolds number flow. We have suggested that there is a kind of interaction at work between an "outer", inviscid flow, and an "inner" boundary-layer flow. That is, the pressure gradient is fundamentally an outer condition imposed on the boundary layer. On the other hand the boundary-layer modifies somewhat the strreamlines well away from the body, in the inviscid flow. We now explore a model problem in one space dimension, involving a *singular perturbation* of an ordinary differential equation. The small parameter ϵ will replace $1/Re$, and the problem is not one of fluid dynamics; nevertheless there will be an inner solution and an outer solution that will be analogous to our viscous boundary layer and out outer inviscid flow. We suggest that the model indicates how a more formal approach to boundary layer theory might proceed, although we shall not pursue this further here.

The model problem is the following: let $f(x) = f(x, \epsilon)$ satisfy

$$\epsilon f'' + f' = a, \quad 0 < a < 1, \;\; 0 \leq x \leq 1, \quad y(0) = 0, y(1) = 1. \tag{8.38}$$

The "singular" adjective is usually applied to problems where the limiting operation, in this case $\epsilon \to 0$ reduces the order of the differential equation, in our case from order two to order one.

We first define our "outer problem", analogous to the inviscid Euler flow. We bound x away from zero, $0 < A \leq x \leq 1$, and apply the limit $\epsilon \to 0$ to the model equation. This gives the reduced system

$$f' = a. \tag{8.39}$$

We apply the condition at $x - 1$ to the solution of this reduced equation, yielding

$$f_{\text{outer}} = ax + 1 - a. \tag{8.40}$$

We see that f_{outer} does *not* satisfy the condition on f at $x = 0$. This adjustment will happen in a boundary layer near $x = 0$. So we consider with Prandtl how to deal with the combination ϵf_{xx}. If derivatives become large this combination need not be small. On the other hand f_x can also be large, so that it is tempting to suppose that at least minimally ϵf_{xx} and f_x must be the same size. This suggests function of x/ϵ, so we define the stretched variable $\bar{x} = x/\epsilon$. [2]

Using the stretched variable our equation takes the form

$$f_{\bar{x}\bar{x}} + f_{\bar{x}} = \epsilon a. \tag{8.41}$$

We no consider the limit $\epsilon \to 0$ with $0 \leq \bar{x} < B < \infty$, obtaining the limiting equation

$$f_{\bar{x}\bar{x}} + f_{\bar{x}} = 0. \tag{8.42}$$

This is our model of the Prandtl boundary layer. We require that its solution vanish at $\bar{x} = 0$, so that

$$f_{\text{inner}} = C(1 - e^{-\bar{x}}). \tag{8.43}$$

Here C is an undetermined constant. Note that $f_{\text{inner}} \to C$ as $\bar{x} \to \infty$, so that we have the model equivalent of obtaining the "velocity at infinity" for the viscous boundary layer. Since f is suppose to be represent by f_{outer} away from $x = 0$, it is natural to identify C with the limit of f_{outer} for small x. This yields

$$C = 1 - a. \tag{8.44}$$

This is usually stated as a *matching condition*:

$$\lim_{\bar{x}\to\infty} f_{\text{inner}} = \lim_{x\to 0} f_{\text{outer}}. \tag{8.45}$$

An approximation to $f(x, \epsilon)$ which applies to the entire interval can be obtained by adding with inner and outer solutions, provided we account for any terms that are common to both. The common part in our problem is just $1 - a$. We define the approximate *composite* solution by

$$f_{comp} = f_{\text{inner}} + f_{\text{outer}} - 1 + a = ax + (1 - a)(1 - e^{-\frac{x}{\epsilon}}). \tag{8.46}$$

It is interesting to compare our approximation with the exact solution of the model problem, namely

$$f(x, \epsilon) = ax + \frac{(1 - a)}{1 - e^{-\frac{1}{\epsilon}}}(1 - e^{-\frac{x}{\epsilon}}). \tag{8.47}$$

The difference is of order $e^{-\frac{1}{\epsilon}}$ uniformly over the domain.

Anyone wishing to explore further the use of singular perturbations in fluid dynamics should consult the book *Perturbation Methods in Fluid Mechanics*, by Milton D. Van Dyke. For boundary-layer theory these methods culminated in an analytical attack on the problem of separation, which we explore briefly in the final section of this chapter.

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[2] The fact that we do not have a square root in defining a stretched variable, as we did for the Reynolds number in the Prandtl boundary layer, reflects the vast difference in the fluid equations and the model equation. This however is a relatively unimportant difference.

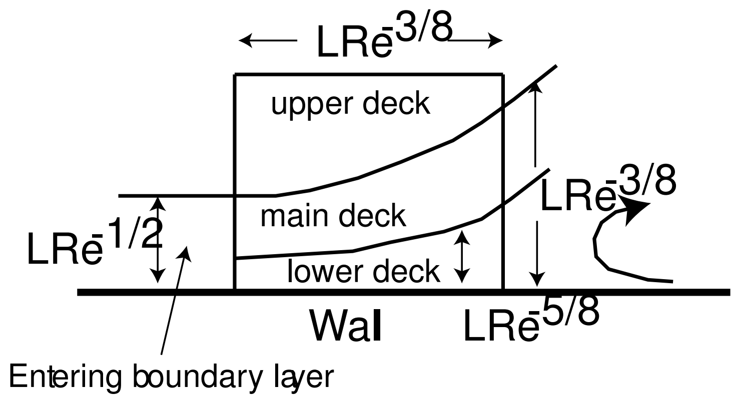

Figure 8.6: The triple deck

8.4 Separation

One of the great accomplishments of 20th century fluid dynamics was an understanding of the fundamental mechanisms of separation of a boundary layer in the limit of large R. This work, due to Stewartson, Williams, Messiter, Neiland, Smith, Sychev, Kaplun, and others, led to a full description of the mechanism of separation in a class of problems of wide applicability. A review of much of the work on the separation problem may be found in Stewartwon, K., *D'Alembert's paradox*, SIAM Review vol. 23, No. 3, p.308 (1981).

The main result of this effort has been the so-called *triple deck theory.* The name applies to the layering of domains of different orders of magnitude, in the neighborhood of the point of separation. We show the structure of the triple deck in figure 8.6. The main point to be made in discussing triple deck is that the layered structure results from a *nonlinear interaction between the boundary layer and the pressure gradient.* In other words, separation represents a breaking of the inner-outer separation of the pressure gradient from the boundary layer responding to the pressure gradient. Within the triple-deck region the boundary layer is modifying the pressure gradient, which in turn is affecting the boundary layer. Entering from the left of the main deck is the profile of the boundary layer as it has evolved through a length we call L in the figure. Thus the main deck has thickness $LR^{-1/2}$. The thinner lower deck is a region where the full boundary layer equations apply, with viscous stress important and reversal of the flow occurring following separation. Over a Δx of order $LR^{-3/8}$ the boundary layer is essentially raised by the same order, forming the upper layer. During this lifting the main deck profile is unchanged by viscosity, since it is traversing such a small domain. This lifting of the boundary layer modifies the pressure gradient locally, and this penetrates down to the lower layer, providing the feeedback that completes the cycle.

Unfortunately this brief description of separation does not do justice to the analysis involved, nor to the insight that was needed to determine the construction of the triple deck, nor to the many related questions that have been tackled with this machinery.

Problem set 8

2. Verify (8.27) and (8.29).

2. *Oseen's equations* are sometimes also proposed as a *model* of the Navier-Stokes, equations, in the study of steady viscous flow past a body. Oseen's equations, for a flow with velocity $(U, 0, 0)$ at infinity, are

$$U\frac{\partial \mathbf{u}}{\partial x} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{u} = 0, \nabla \cdot \mathbf{u} = 0.$$

(a) Show that in this model, if viscous stresses are neglected, the vorticity is a function of y, z alone.

(b) For the Oseen model, and for a flat plate aligned with the flow, carry out Prandtl's simplifications for deriving the boundary-layer equations in two dimensions, given that the boundary condition of no slip is retained at the body. That is, find the form of the boundary layer on a flat plate of length L aligned with the flow at infinity, according to Oseen's model, and show that in the boundary layer the the x-component of velocity, u, satisfies

$$U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial y^2} = 0.$$

What are the boundary conditions on u for the flat-plate problem? Find the solution, by assuming that u is a function of $y\sqrt{\frac{U}{\nu x}}$, for $0 < x < L$.

(c) Compute the drag coefficient of the plate (drag divided by $\rho U^2 L$, and remember there are two sides), in the Oseen model.

3. What are the boundary-layer equations for the boundary-layer on the front portion of a circular cylinder of radius a, when the free stream velocity is $(-U, 0, 0)$? (Use cylindrical polar coordinates). What is the role of the pressure in the problem? Be sure to include the effect of the pressure as an explicit function in your momentum equation, the latter being determined by the potential flow past a circular cylinder studied previously. Show that, by defining $x = a\theta, \bar{y} = (r - a)\sqrt{R}$ in the derivation of the boundary-layer equations, the equations are equivalent to a boundary layer on a flat plate aligned with the free stream, in rectangular coordinates, but with pressure a given function of x.

4. For a *cylindrical* jet emerging from a hole in a plane wall, we have a problem analogous to the 2D jet considered in class. Consider only the boundary-layer limit. (a) Show that

$$\frac{\partial}{\partial z}(u_z^2) + \frac{1}{r}\frac{\partial}{\partial r}(ru_r u_z) - \frac{\nu}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) = 0,$$

and hence that the momentum M is a constant, where

$$M = 2\pi\rho \int_0^\infty r u_z^2 dr.$$

(b) Letting $(u_z, u_r) = (1/r)(\psi_r, -\psi_z)$ where $\psi(0, z) = 0$ show that we must have $\psi = zf(\eta),\ \eta = r^2/z^2$. Determine the equation for f and thus show that the boundary-layer limit has the form

$$f = 4\nu\frac{\eta}{\eta + \eta_0},$$

where η_0 is a constant. Express η_0 in terms of M, the momentum flux of the jet defined above.

5.consider the Prandtl boundary-layer equations with $U(x) = 1/x$, so $p(x)/\rho = p_\infty - 1/(2x^2)$. Verify that the similarity solution has the form $\psi = f(\eta), \eta = y/x$. Find the equation for f. Show that there is no continuously differentiable solution of the equation which satisfies $f(0) = f'(0) = 0$ and $f' \to 1, f'' \to 0$ as $\eta \to \infty$. (Hint: Obtain an equation for $g = f'$.)

Chapter 9

Energy

9.1 Mechanical energy

We recall the two conservation laws:

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{9.1}$$

Conservation of momentum:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \rho \frac{Du_i}{Dt} = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}, \ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla. \tag{9.2}$$

Here σ_{ij} are the components of the viscous stress tensor: [1]

$$\sigma_{ij} = -p\delta_{ij} + \mu\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla \cdot \mathbf{u}\Big]. \tag{9.3}$$

We now want to use these results to compute the rate of change of total kinetic energy within a fixed fluid volume V. We see that

$$\frac{d}{dt}\int_V \frac{1}{2}\rho u^2 dV = \int_V \big[\rho u_i \frac{\partial u_i}{\partial t} + \frac{1}{2}\frac{\partial \rho}{\partial t}u^2\big] dV$$

$$= \int_V \Big(\big[-\rho u_j \frac{\partial u_i}{x_j} + \frac{\partial \sigma_{ij}}{\partial x_j} + F_i\big] u_i - \frac{1}{2}\frac{\partial \rho u_j}{\partial x_j} u^2\Big) dV$$

$$= \int_V \big[u_i F_i + p\nabla \cdot \mathbf{u}\big] dV - \Phi + \int_S \big[-\frac{1}{2}\rho u_j u^2 + u_i \sigma_{ij}\big] n_j dS, \tag{9.4}$$

[1] You will find that Landau and Lifshitz allow a deviation from the Stokes relation. Recall that this relation makes the trace of the non-pressure part of the stress tensor (the deviatoric stress) zero. This deviation is accomplished by adding a term $\mu' \delta_{ij}\nabla \cdot \mathbf{u}$ to the stress tensor, where μ' is called the *second viscosity.* For simplicity, and sionce the exact form is not material to the problems we shall study, we neglect this second viscosity in the present notes. However for applications to real gases it should be retained as the Stokes relation does not hold for many gases.

where S is the boundary of V and Φ is the total viscous dissipation in V:

$$\Phi = \int_\phi dV,\ \phi = \mu\Big[\frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)^2 - \frac{2}{3}(\nabla \cdot \mathbf{u})^2\Big]. \tag{9.5}$$

We note from (9.4) that the contributions in order come from the work done by body forces, the work done by pressure in compression, viscous heating, flux of kinetic energy through S, and work done by stresses on S. We refer to (9.4) as the *mechanical* energy equation, since we have use only conservation of mass and momentum.

To put this expression into a different form we now complete the fluid equations by assuming a barotropic fluid, $p = p(\rho)$. Then

$$\int_V p\nabla \cdot \mathbf{u} dV = \int_S p\mathbf{u} \cdot \mathbf{n} dS - \int_V \mathbf{u} \cdot \nabla p dV. \tag{9.6}$$

But

$$\int_V \mathbf{u} \cdot \nabla p dV = \int \rho \mathbf{u} \cdot \nabla \int \frac{1}{\rho} p'(\rho) d\rho dV$$

$$= \int_S \rho \int \frac{1}{\rho} p'(\rho) d\rho \mathbf{u} \cdot \mathbf{n} dS + \int_V \frac{\partial \rho}{\partial t} \int \frac{1}{\rho} p'(\rho) d\rho dV. \tag{9.7}$$

We now define $g(\rho)$ by

$$\int \frac{1}{\rho} p'(\rho) d\rho = g'(\rho), g(0) = 0. \tag{9.8}$$

and define e by

$$g = \rho e. \tag{9.9}$$

Noting that

$$\frac{d}{d\rho}\big(p - \rho \int \frac{1}{\rho} p'(\rho) d\rho\big) = -\int \frac{1}{\rho} p'(\rho) d\rho = -g' = -(\rho e)', \tag{9.10}$$

we have that

$$p - \rho \int \frac{1}{\rho} p'(\rho) d\rho = -\rho e. \tag{9.11}$$

Thus

$$\int_V p\nabla \cdot \mathbf{u} dV = -\int_S \rho e \mathbf{u} \cdot \mathbf{n} dS - \int_V \frac{\partial \rho}{\partial t} (\rho e)' dV. \tag{9.12}$$

Using this in (9.4) we obtain

$$\frac{d}{dt} \int_V E dV + \int_S E \mathbf{u} \cdot \mathbf{n} dS = -\Phi + \int_V u_i F_i dV + \int_S u_i \sigma_{ij} n_j dS, \tag{9.13}$$

where

$$E = \rho\big(e + \frac{1}{2} u^2\big). \tag{9.14}$$

Note that if $\mu = 0$ and $F_i = 0$ then (9.13) reduces to a conservation law of the form

$$\frac{d}{dt}\int_V E dV + \int_S (E+p)\mathbf{u}\cdot\mathbf{n} dS = 0. \tag{9.15}$$

We note that E should have the meaning of energy, and we shall refer to e as the *internal energy* of the fluid (per unit mass). Then (9.13) can be viewed as as expression of the first law of thermodynamics $\Delta E = \Delta Q - W$, where ΔE is the change of energy of an isolated system (not flux of energy through the boundary), ΔQ is the heat *added* to the system, and W is the work done *by* the system.

The form of (9.13) can however be used as a model for formulating a more general energy equation, and we shall do this after first reviewing some of the basic concepts of reversible thermodynamics.

9.2 Elements of classical thermodynamics

Thermodynamics deals with transformations of energy within an isolated system. These transformations are determined by *thermodynamic variables.* These come in two types: *Extensive* variables are proportional to the amount of material involved. Examples are internal energy, entropy, heat. *Intensive* variables are not proportional to quantity. Examples are pressure, density, temperature.

We have just introduced two new scalar fields, the *absolute temperature* T, and the *specific entropy* s. We shall also make use of *specific volume* v, defined by $v = 1/\rho$.

We now discuss the thermodynamics of gases. In general we shall assume the existence of an *equation of state* of the gas, connecting p, ρ, T. An important example is the equation of state of an *ideal* or *perfect* gas, defined by

$$pv = RT. \tag{9.16}$$

Here R is a constant associated with the particular gas. In general all thermodynamic variables are determined by ρ, p and T. With an equation of state, in principle we can regard any variable as a function of two independent variables.

We can now view our thermodynamic system as a small volume of gas which can do work by changing volume, can absorb and give off heat, and can change its internal energy. The first law then takes the differential form

$$dQ = de + pdv. \tag{9.17}$$

It is important to understand that we are considering here small changes which take place in such a way that irreversible dissipative processes are not present. For example, when a volume changes the gas has some velocity, and there could be resulting viscous dissipation. W are assuming that the operations are performed so that such effects are negligible. We then say that the system is *reversible*. If the changes are such that $dQ = 0$, we say that the system is *adiabatic*.

We define the following *specific heats* of the gas: The specific heat at constant pressure is defined by

$$c_p = \frac{\partial Q}{\partial T}\Big|_{dp=0} = \Big(\frac{\partial e}{\partial T}\Big)_p + p\Big(\frac{\partial v}{\partial T}\Big)_p. \tag{9.18}$$

Note that for an ideal gas $p\Big(\frac{\partial v}{\partial T}\Big)_p = R$.

The specific heat an constant volume is defined by

$$c_v = \frac{\partial Q}{\partial T}\Big|_{dv=0} = \Big(\frac{\partial e}{\partial T}\Big)_v. \tag{9.19}$$

We will make use of these presently.

The second law of thermodynamics for reversible systems establishes the existence of the thermodynamics variable s, the specific entropy, such that

$$dQ = Tds. \tag{9.20}$$

Thus we have the basic thermodynamic relation

$$Tds = de + pdv. \tag{9.21}$$

We know make use of (9.21) to establish an important property of an ideal gas, namely that its internal energy is a function of T alone. To see this, note from (9.21) that

$$\Big(\frac{\partial e}{\partial s}\Big)_v = T, \Big(\frac{\partial e}{\partial v}\Big)_s = -p. \tag{9.22}$$

Thus

$$R\Big(\frac{\partial e}{\partial s}\Big)_v + v\Big(\frac{\partial e}{\partial v}\Big)_s = 0. \tag{9.23}$$

Thus e is a function of $s - R\ln v$ alone. Then, by the first of (9.22), $T = e'(s - R\ln v)$, implying $s - R\ln v$ is a function of T alone, and therefore e is also a function of T alone. Thus the derivative of e with respect to T at constant volume is the same as the derivative at constant pressure. By the definition of the specific heats, we have

$$c_p - c_v = R. \tag{9.24}$$

For an ideal gas it then follows that c_p and c_v differ by a constant. If both specific heats are constants, so that $e = c_v T$, it is customary to define that ratio

$$\gamma = c_p/c_v. \tag{9.25}$$

For air γ is about 1.4.

The case of constant specific heats gives rise to a useful model gas. Indeed we then have

$$ds = c_v\frac{dT}{T} + R\frac{dv}{v}. \tag{9.26}$$

Note that here the right-hand side explicitly verifies the existence of the differential ds. Using the equation of state of an ideal gas, the last equation may be integrated to obtain

$$p = k(s)\rho^\gamma,\ k(s) = Ke^{s/c_v}, \tag{9.27}$$

where K is a constant. The relation $p = k\rho^\gamma$ defines a *polytropic gas*.

9.3 The energy equation

The fundamental variables of compressible fluid mechanics of ideal gases are $\mathbf{u}, \rho, p, T$. We have three of momentum equations, one conservation of mass equations, and an equation of state. We need one more scalar equation to complete the system, and this will be an equation of conservation of energy. Guided by the mechanical energy equation, we are led to introduce the total energy per unit mass as $e + \frac{1}{2}u^2 = E/\rho$, and express energy conservation by the following relation:

$$\frac{d}{dt}\int_V EdV + \int_S E\mathbf{u}\cdot\mathbf{n}dS = \int_S u_i\sigma_{ij}n_j dS + \int_V F_i u_i dV + \int_S \lambda\nabla T\cdot\mathbf{n}dS. \quad (9.28)$$

We have on the right the working of body and surface forces and the heat flux to the system. The latter is based upon the assumption of Fick's law of heat condition, stating that heat flux is proportional to the gradient of temperature. We have introduced λ as the factor of proportionality. Given that heat flows from higher to lower temperature, λ as defined is a positive function, most often of ρ, T.

We now use (9.4) to eliminate some of the terms involving kinetic energy. Note the main idea here. Once we recognize that the energy of the fluid involves both kinetic and internal parts, we are prepared to write the first law as above. Then we make use of (9.4) to move to a more "thermodynamic" formulation. Proceeding we see easily that (9.28) becomes

$$\frac{d}{dt}\int_V \rho e dV + \int_S \rho e\mathbf{u}\cdot\mathbf{n}dS = \int_S \lambda\nabla T\cdot\mathbf{n}dS + \Phi - \int p\nabla\cdot\mathbf{u}dV. \qquad (9.29)$$

This implies the local equation

$$\rho\frac{De}{Dt} - \nabla\cdot\lambda\nabla T - \Phi + p\nabla\cdot\mathbf{u} = 0. \qquad (9.30)$$

Using $Tds = de + pdv$ and the equation of conservation of mass, the last equation may be written

$$\rho T\frac{Ds}{Dt} = \nabla\cdot\lambda\nabla T + \Phi. \qquad (9.31)$$

This is immediately recognizable as having on the right precisely the heat inputs associated with changes of entropy.

There are other forms taken by the energy equation in addition to (9.30) and (9.31). These are easiest to derive using Maxwell's relations. To get each such relation we exhibit a principle function and from it obtain a differentiation identity, by using $Tds = de + pdv$ in a form exhibiting the principle function. For example if e is the principle function, then

$$\left(\frac{\partial e}{\partial s}\right)_v = T,\ \left(\frac{\partial e}{\partial v}\right)_s = -p. \qquad (9.32)$$

Then the Maxwell relation is obtained by cross differentiation:

$$\left(\frac{\partial T}{\partial v}\right)_s = -\left(\frac{\partial p}{\partial s}\right)_v. \tag{9.33}$$

We define the next principle function by $h = e + pv$, the *specific enthalpy*. Then $Tds = dh - vdp$. Thus

$$\left(\frac{\partial h}{\partial s}\right)_p = T,\ \left(\frac{\partial h}{\partial p}\right)_s = v, \tag{9.34}$$

giving the relation

$$\left(\frac{\partial T}{\partial p}\right)_s = \left(\frac{\partial v}{\partial s}\right)_p. \tag{9.35}$$

The principle function and the corresponding Maxwell relation in the two remaining cases are:

The free energy $F = e - Ts$, yielding

$$\left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial s}{\partial v}\right)_T. \tag{9.36}$$

The free enthalpy $G = h - Ts$, yielding the relation

$$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p. \tag{9.37}$$

We illustrate the use of these relations by noting that

$$\begin{aligned} ds &= \left(\frac{\partial s}{\partial T}\right)_p dT + \left(\frac{\partial s}{\partial p}\right)_T dp \\ &= c_p \frac{dT}{T} - \left(\frac{\partial v}{\partial T}\right)_p dp, \end{aligned} \tag{9.38}$$

where we have used (9.37). Now for a perfect gas $\left(\frac{\partial v}{\partial T}\right)_p = R/p$, so that (9.31) may be written

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = \nabla \cdot \lambda \nabla T + \phi. \tag{9.39}$$

In particular if c_p, λ, μ are known functions of temperature say, then we have with the addition of (9.39)a closed system of six equations for $\mathbf{u}, p, \rho, T$.

Chapter 10

Gas dynamics I

10.1 Some basic relations for the non dissipative case $\mu = \lambda = 0$

In these case local conservation of energy may be written

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}E) = \mathbf{u} \cdot \mathbf{F} - \nabla \cdot (\mathbf{u}p). \tag{10.1}$$

Using conservation of mass we have

$$\begin{aligned}\frac{D(e + \frac{1}{2}u^2)}{Dt} &= \frac{1}{\rho}\Big(\mathbf{u} \cdot \mathbf{F} - \mathbf{u} \cdot \nabla p + \frac{p}{\rho}\frac{D\rho}{Dt}\Big) \\ &= \frac{1}{\rho}\Big(\mathbf{u} \cdot \mathbf{F} + \frac{\partial p}{\partial t}\Big) - \frac{D}{Dt}\frac{p}{\rho}.\end{aligned}$$

Thus

$$\frac{D}{Dt}\Big(e + \frac{1}{2}u^2 + \frac{p}{\rho}\Big) = \frac{1}{\rho}\Big(\mathbf{u} \cdot \mathbf{F} + \frac{\partial p}{\partial t}\Big). \tag{10.2}$$

If now the flow is *steady*, and $\mathbf{F} = -\rho\nabla\Psi$, then we obtain a Bernoulli equation in the form

$$H \equiv e + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi = \text{ constant} \tag{10.3}$$

on streamlines of the flow.

To see how H changes from streamline to streamline in steady flow, note that

$$dH = d\big(\frac{1}{2}u^2 + h + \Psi\big) = d\big(\frac{1}{2}u^2 + \Psi\big) + Tds + vdp, \tag{10.4}$$

so that we may write

$$\nabla H = \nabla(\frac{1}{2}u^2 + \Psi\big) + T\nabla s + \frac{1}{\rho}\nabla p. \tag{10.5}$$

But in steady flow with $\mu = 0$ we have $\rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = -\rho\nabla\Psi$, or

$$\nabla(\frac{1}{2}u^2 + \Psi) + \frac{1}{\rho}\nabla p = \mathbf{u}\times\omega, \tag{10.6}$$

where $\omega = \nabla\times\mathbf{u}$ is the vorticity vector. Using the last equation in (10.5) we obtain *Crocco's relation*:

$$\nabla H - T\nabla s = \mathbf{u}\times\omega. \tag{10.7}$$

A flow in which $Ds/Dt = 0$ is called *isentropic*. From (9.31) we see that $\mu = \lambda = 0$ implies isentropic flow. If in addition s is constant throughout space, the flow is said to be *homentropic*. Wee see from (10.7) that in homentropic flow we have

$$\nabla H = \mathbf{u}\times\omega. \tag{10.8}$$

Note also that in homentropic flow the Bernoulli relation (10.3) becomes (since $dh = vdp$)

$$\int\frac{c^2}{\rho}d\rho + \frac{1}{2}u^2 + \Psi = \text{ constant} \tag{10.9}$$

on streamlines. here

$$c^2 = \left(\frac{\partial p}{\partial\rho}\right)_s \tag{10.10}$$

is the speed of sound in the gas.

10.1.1 Kelvin's theorem in a compressible medium

Following the calculation of the rate of change of circulation which we carried out in the incompressible case, consider the circulation integral over a material contour C:

$$\frac{d}{dt}\oint_{C(t)}\mathbf{u}\cdot d\mathbf{x} = \frac{d}{dt}\oint_{C(t)}\mathbf{u}\cdot\frac{\partial\mathbf{x}}{\partial\alpha}d\alpha, \tag{10.11}$$

where α is a Lagrangian parameter for the curve. then

$$\frac{d}{dt}\oint_{C(t)}\mathbf{u}\cdot d\mathbf{x} = \oint_C\frac{D\mathbf{u}}{Dt}\cdot\mathbf{dx} + \oint_C\mathbf{u}\cdot d\mathbf{u}. \tag{10.12}$$

Using $D\mathbf{u}/Dt = -\nabla p/\rho - \nabla\Psi$, we get after disposing of perfect differentials,

$$\frac{d}{dt}\oint_{C(t)}\mathbf{u}\cdot d\mathbf{x} = \int_S\frac{1}{\rho^2}(\nabla\rho\times\nabla p)\cdot\mathbf{n}dS. \tag{10.13}$$

Here S is any oriented surface spanning C. In a perfect gas, $Tds = c_v dT + pdv$, so that

$$T\nabla T\times\nabla s = \nabla\frac{p}{\rho R}\times p\nabla\frac{1}{\rho} = -\frac{T}{\rho^2}\nabla p\times\nabla\rho. \tag{10.14}$$

Thus

$$\frac{d}{dt}\oint_{C(t)}\mathbf{u}\cdot d\mathbf{x} = \int_S(\nabla T\times\nabla s)\cdot\mathbf{n}dS. \tag{10.15}$$

10.1.2 Examples

We now give a brief summary of two examples of systems of compressible flow equations of practical importance. We first consider the equations of *acoustics*. This is the theory of sound propagation. The disturbances of the air are so small that viscous and heat conduction effects may be neglected to first approximation, and the flow taken as homentropic. Since disturbances are small, we write $\rho = \rho_0 + \rho', p = p_0 + p', \mathbf{u} = \mathbf{u}'$where subscript "0" denotes constant ambient conditions. If the ambient speed of sound is

$$\left(\frac{\partial p}{\partial \rho}\right)_s\Big|_0 = c_0^2, \tag{10.16}$$

we assume $\rho'/\rho_0, p'/p_0 \ \|\mathbf{u}'|/c_0$ are all small. Also we see that $p' \approx c_0^2 \rho'$. With no body force, the mass and momentum equations give us

$$\frac{\partial \mathbf{u}'}{\partial t} + \frac{c_0^2}{\rho_0}\nabla\rho' = 0, \ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0. \tag{10.17}$$

here we have neglected terms quadratic in primed quantities. Thus we obtain acoustics as a linearization of the compressible flow equations about a homogeneous ambient gas at rest.

Combining (10.17) we obtain

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2\right)(\rho', \mathbf{u}') = 0. \tag{10.18}$$

Thus we obtain the wave equation for the flow perturbations. If sound waves arise from still air, Kelvin's theorem guarantees that $\mathbf{u}' = \nabla\phi$, where ϕ will also satisfy the wave equation, with

$$p' = c_0^2\rho' = -\frac{1}{\rho_0}\frac{\partial \phi}{\partial t}. \tag{10.19}$$

The second example of compressible flow is 2D steady isentropic flow of a polytropic gas with $\mu = \lambda = \Psi = 0$. Then

$$\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = 0, \nabla\cdot(\rho\mathbf{u}) = 0. \tag{10.20}$$

Let $\mathbf{u} = (u, v), q^2 = u^2 + v^2$. In this case the Bernoulli relation hold in the form

$$\frac{1}{2}q^2 + \int\frac{dp}{\rho} = \ \text{constant} \tag{10.21}$$

on streamlines. For a polytropic gas we have

$$\int\frac{dp}{\rho} = k\frac{\gamma}{\gamma - 1}\rho^{\gamma-1} = \frac{1}{\gamma - 1}c^2. \tag{10.22}$$

Now we have, in component form

$$uu_x + vu_y + \frac{c^2}{\rho}\rho_x = 0, uv_x + vv_y + \frac{c^2}{\rho}\rho_y = 0, \tag{10.23}$$

and

$$u_x + v_y + u(\rho_x/\rho) + v(\rho_y/\rho) = 0 \tag{10.24}$$

Substituting for the ρ terms using (10.23), we obtain

$$(c^2 - u^2)u_x + (c^2 - v^2)v_y - uv(v_x + u_y) = 0. \tag{10.25}$$

If now we assume irrotational flow, $v_x = u_y$, so that $(u, v) = (\phi_x, \phi_y)$, then we have the system

$$(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_y^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0, \tag{10.26}$$

$$\phi_x^2 + \phi_y^2 + \frac{2}{\gamma - 1}c^2 = \text{ constant.} \tag{10.27}$$

10.2 The theory of sound

The study of acoustics is of interest as the fundamental problem of linearized gas dynamics. we have seen that the wave equation results. In the present section we drop the subscript "0" and write

$$\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi = 0, \tag{10.28}$$

where c is a constant phase speed of sound waves.

We first consider the one-dimensional case, and the initial-value problem on $-\infty < x < +\infty$. The natural initial conditions are for the gas velocity and the pressure or density, implying that both ϕ and ϕ_t should be supplied initially. Thus the problem is formulated as follows:

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = 0,\ \phi(x, 0) = f(x),\ \phi_t(x, 0) = g(x). \tag{10.29}$$

The general solution is easily seen to have the form

$$\phi = F(x - ct) + G(x + ct), \tag{10.30}$$

using the initial conditions to solve for F, G we obtain easily *D'Alembert's solution*:

$$\phi(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{10.31}$$

In the (x, t) plane, a given point (x_0, t_0) in $t > 0$ in influenced only by the initial data on that interval of the x-axis lying between the points of intersection with the axis of the two lines $x - x_0 = \pm c(t - t_0)$. This interval is called the *domain of dependence* of (x_0, t_0). Conversely a given point (x_0, t_0) in $t \geq 0$ can influence on the point with the wedge bounded by the two lines $x - x_0 = \pm c(t - t_0)$ with $t - t_0 \geq 0$. This wedge is called the *range of influence* of (x_0, t_0). These lines are also known as the *characteristics* through the point (x_0, t_0).

10.2.1 The fundamental solution in 3D

We first note that the three dimensions, under the condition of spherical symmetry, the wave equation has the form

$$\frac{\partial^2\phi}{\partial t^2} - c^2\Big(\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}\Big) = 0. \tag{10.32}$$

here $r^2 = x^2 + y^2 + z^2$. Note that we can rewrite this as

$$(r\phi)_{tt} - (r\phi)_{rr} = 0, \tag{10.33}$$

Thus we can reduce the 3D problem to the 1D problem if the symmetry holds.

Now we are interested in solving the 3D wave equation with a distribution as a forcing function, an with null initial conditions. In particular we seek the solution of

$$\phi_{tt} - c^2\nabla^2\phi = \delta(\mathbf{x})\delta(t), \tag{10.34}$$

with $\phi(\mathbf{x}, 0-) = \phi_t(\mathbf{x}, 0-) = 0$. Since the 3D delta function imposes no deviation from spherical symmetry, we assume this symmetry and solve the problem as a 1D problem. When $t > 0$ we see from the 1D problem that

$$\phi = \frac{1}{r}[F(t - r/c) + G(t + r/c)]. \tag{10.35}$$

(The change $r - ct$ to $t - r/c$ is immaterial but will be convenient here.) Now the term in G represents "incoming" signals propagating toward the origin from ∞. Such wave are unphysical in the present case. Think of the delta function a disturbance localized in space and time, like a firecracker set off at the origin and at $t = 0$. It should produce only outgoing signals. So we set $G = 0$. Also, near the origin $F(t - r/c) \approx F(t)$, so the $\delta(\mathbf{x})\delta(t)$ distribution would result, using $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$, provided

$$F(t - r/c) = \frac{1}{4\pi c^2}\delta(t - r/c). \tag{10.36}$$

Another way to be this is to integrate the left-hand side of (10.34) over the ball $r \le \epsilon$, use the divergence theorem, and let $\epsilon \to 0$.

So we define the fundamental solution of the 3D wave equation by

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi c^2 r}\delta(t - r/c). \tag{10.37}$$

10.2.2 The bursting balloon problem

To illustrate solution in three dimensions consider the following initial conditions under radial symmetry. We assume that the pressure perturbation p satisfies

$$p(\mathbf{x}, 0) = \begin{cases} p_b, & \text{if } 0 < r < r_b, \\ 0, & \text{if } r > r_b. \end{cases} \equiv N(r) \tag{10.38}$$

Here p_b is a positive constant representing the initial pressure in the balloon. Now $p_t = c^2\rho_t = -c^2\rho_0\nabla^2\phi$. If the velocity of the gas is to be zero initially, as we must assume in the case of a fixed balloon, then

$$p_t(\mathbf{x}, 0) = 0. \tag{10.39}$$

Since rp satisfies the 1D wave equation, and P is presumably bounded at $r = 0$, we extend the solution to negative r by making rp an odd function. Then the initial value problem for rp is will defined in the D'Alembert sense and the solution is

$$rp = \frac{1}{2}\big[N(r - ct) + N(r + ct)\big]. \tag{10.40}$$

Note that we have both incoming and outgoing waves since the initial condition is over a finite domain. For large time, however, the incoming wave does not contribute and the pressure is a decaying "N" wave of width $2r_b$ centered at $r = ct$.

10.2.3 Kirchoff's solution

We now take up the solution of the general initial value problem for the wave equation in 3D:

$$\phi_{tt} - c^2\nabla^2\phi = 0,\ \phi(\mathbf{x}, 0) = f(\mathbf{x}),\ \phi_t(\mathbf{x}, 0) = g(\mathbf{x}). \tag{10.41}$$

This can be accomplished from two ingenious steps. We first note that if ϕ solves the wave equation with the initial conditions $f = 0, g = h$, the ϕ_t solves the wave equation with $f = h, g = 0$. Indeed$\phi_{tt} = c^2\nabla^2\phi$ tends to zero as $t \to 0$ since this is a property of ϕ.

The second step is to note that the solution ϕ with $f = 0, g = h$ is give by

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi tc^2}\int_{S(\mathbf{x},t)} h(\mathbf{x}')dS'. \tag{10.42}$$

The meaning of S here is indicated in figure 10.1. To verify that this is the solution, note first that we are integrating over a spherical surface of radius $4\pi c^2t^2$, Given that h is bounded, division by t still leaves a factor t, so we obtain 0 in the limit as $t \to 0$. Also

$$\phi = \frac{t}{4\pi}\int_{|\mathbf{y}|=1} h(\mathbf{x} + \mathbf{y}tc)dS_y \tag{10.43}$$

by a simple change of variable.Thus

$$\phi_t = \frac{1}{4\pi}\int_{|\mathbf{y}|=1} h(\mathbf{x} + \mathbf{y}tc)dS_y + \frac{t}{4\pi}\int_{|\mathbf{y}|=1} c\mathbf{y}\cdot\nabla h(\mathbf{x} + \mathbf{y}tc)dS_y. \tag{10.44}$$

The first term on the right clearly tends to $h(\mathbf{x})$ as $t \to 0$, while the second term tends to zero providing that h is a sufficiently well-behaved function.

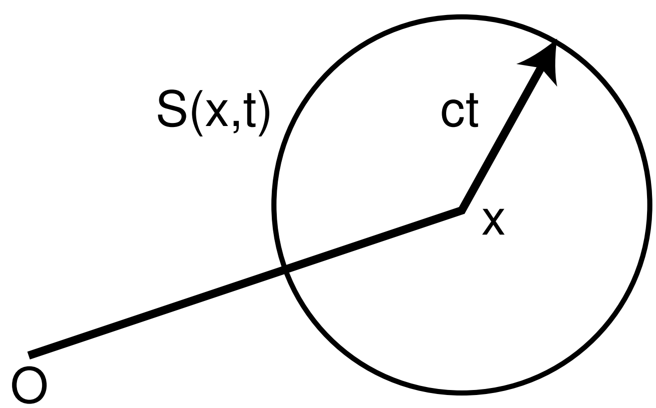

Figure 10.1: Definition of $S(\mathbf{x}, t)$ in the Kirchoff solution.

We now show that (10.43) solves the wave equation. Using the divergence theorem we can write (10.44) in the form

$$\phi_t = \frac{1}{4\pi}\int_{|\mathbf{y}|=1} h(\mathbf{x}+\mathbf{y}tc)dS_y + \frac{1}{4\pi ct}\int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}')dV', \tag{10.45}$$

where $V(\mathbf{x},t)$ denotes the sphere of radius ct centered at $\mathbf{x}$. Then

$$\phi_t = \frac{1}{4\pi}\int_{|\mathbf{y}|=1} h(\mathbf{x}+\mathbf{y}tc)dS_y + \frac{1}{4\pi ct}\int_0^{ct}\int_{S_{\rho(\mathbf{x})}} \nabla^2 h(\mathbf{y})dS_y d\rho, \tag{10.46}$$

where $S_{\rho(\mathbf{x})}$ is the spherical surface of radius ρ centered at $\mathbf{x}$.

Now we can compute

$$\phi_{tt} = \frac{c}{4\pi}\int_{|\mathbf{y}|=1} \mathbf{y}\cdot\nabla h(\mathbf{x}+\mathbf{y}tc)dS_y - \frac{1}{4\pi ct^2}\int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}')dV' + \frac{1}{4\pi t}\int_{S(\mathbf{x},t)} \nabla^2 h(\mathbf{y})dS_y$$

$$= \frac{1}{4\pi ct^2}\int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}')dV' - \frac{1}{4\pi ct^2}\int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}')dV' + \frac{1}{4\pi t}\int_{S(\mathbf{x},t)} \nabla^2 h(\mathbf{y})dS_y$$

$$= \frac{1}{4\pi t}\int_{S(\mathbf{x},t)} \nabla^2 h(\mathbf{y})dS_y.$$

$$= \frac{c^2 t}{4\pi}\int_{|\mathbf{y}|=1} h(\mathbf{x}+\mathbf{y}tc)dS_y = c^2\nabla^2\phi. \tag{10.47}$$

Thus we have shown that ϕ satisfies the wave equation.

Given these facts we may write down Kirchoff's solution to the initial value problem with initial data f, g:

$$\phi(\mathbf{x},t) = \frac{1}{4\pi tc^2}\int_{S(\mathbf{x},t)} g(\mathbf{x}')dS' + \frac{\partial}{\partial t}\frac{1}{4\pi tc^2}\int_{S(\mathbf{x},t)} f(\mathbf{x}')dS'. \tag{10.48}$$

Although we have seen that the domain of dependence of a point in space at a future time is in fact a finite segment of the line in one dimension, the corresponding statement in 3D, that the domain of dependence is a finite region of 3-space, is false. The actual domain of dependence is the surface of a sphere of radius ct, centered at $\mathbf{x}$. This fact is know as *Huygen's principle*.

We note that the bursting balloon problem can be solved directly using the Kirchoff formula. A nice exercise is to compare this method with the 1D solution we gave above.

Moving sound sources give rise to different sound field depending upon whether or not the source is moving slower of faster that the speed of sound. In the latter case, a source moving to the left along the x-axis with a speed $U > c$ will produce sound waves having a conical envelope, see figure 10.2.

Here $\sin\alpha = c/U = 1/M$ where $M = U/c$ is the *Mach number*. A moving of a slender body through a compressible fluid at supersonic speeds can be thought of as a sound source. The effect of the body is then confined to within the conical surface of figure 10.2. This surface is called the *Mach cone*.

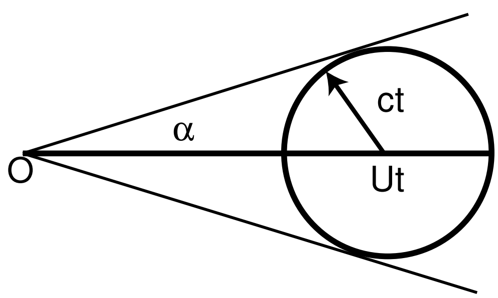

Figure 10.2: Supersonic motion of a sound source.

10.2.4 Weakly nonlinear acoustics in 1D

We have seen that sound propagation in 1D involves the characteristics $x \pm ct =$ constant, representing to directions of propagation. If a sound pulse traveling in one of these directions is followed, over time weak nonlinear effects can become important, and a nonlinear equation is needed to describe the compressive waves. In this section we shall derive the equation that replaces the simple linear wave equation $\phi_t \pm c\phi_x = 0$ associated with the two families of characteristics.

We shall suppose that the disturbance is moving to the right, i.e. is in linear theory a function of $x - ct$ alone. The characteristic coordinates

$$\xi = x - ct, \ \eta = x + ct \tag{10.49}$$

can be used in place of x, t provided $c > 0$. Then

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}, \; \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}. \tag{10.50}$$

Thus with the linear theory our right-moving disturbance is annihilated by the operator

$$\frac{\partial}{\partial \eta} = \frac{1}{2}\frac{\partial}{c\partial t} + \frac{\partial}{\partial x}. \tag{10.51}$$

We shall be therefore looking a compressive wave which, owing to nonlinearity, has a nonzero but small variation with respect to η. The variation with respect to ξ will involves small effects, both from nonlinearity and from the viscous stresses.

If the variables are again $\rho_0 + \rho', p_0 + p'$, and u', the exact conservation of mass equation is

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = -\frac{\partial(\rho' u')}{\partial x}. \tag{10.52}$$

To get the proper form of the momentum equation we expand the pressure as a function of ρ, assuming that we have a polytropic gas. With $p = h\rho^\gamma$ we have the Taylor series

$$p = p_0 + c^2 \rho' \frac{(\gamma - 1)}{\rho_0}\frac{(\rho')^2}{2} + \dots. \tag{10.53}$$

Here we have use $c^2 = \gamma k \rho_0^{\gamma-1}$. Thus the momentum equation takes the form, through terms quadratic in primed quantities,

$$\rho_0 \frac{\partial u'}{\partial t} + c^2 \frac{\partial \rho'}{\partial x} = -\rho' \frac{\partial u'}{\partial t} - \rho_0 u' \frac{\partial u'}{\partial x} - \frac{\gamma - 1}{\rho_0} c^2 \rho' \frac{\partial \rho'}{\partial x} + \frac{4\mu}{3}\frac{\partial^2 u'}{\partial x^2}. \tag{10.54}$$

Note that the viscous stress term comes from the difference $2\mu - \frac{2}{3}\mu$ in the coefficient of $\frac{\partial u'}{\partial x}$ in the 1D stress tensor. We assume here that μ is a constant.

To derive a nonlinear equation for the propagating disturbance we proceed in two steps. First, eliminate the ξ differentiations from the linear parts of the two equations. This will yield a equation with a first derivative term in η, along with the viscosity term and a collection of quadratic nonlinearities in u', ρ'. Then we use the approximate *linear* relation between u' and ρ' to eliminate ρ' in favor of u' in these terms. The result will be a nonlinear equation for u' in will all terms are small but comparable.

The linear relation used in the nonlinear terms comes from

$$\rho_0 \frac{\partial u'}{\partial t} = -c^2 \frac{\partial \rho'}{\partial x} \tag{10.55}$$

Since dependence upon η is weak, the last relation expressed in ξ, η variables becomes

$$-c\rho_0 \frac{\partial u'}{\partial \xi} = -c^2 \frac{\partial \rho'}{\partial \xi}, \tag{10.56}$$

so that

$$\rho' \approx \frac{\rho_0}{c} u' \tag{10.57}$$

in the nonlinear terms as well as in any derivative with respect to η.

Now in characteristic coordinates the linear parts of the equations take the forms

$$\frac{\partial}{\partial \xi}(-c\rho' + \rho_0 u') + \frac{\partial}{\partial \eta}(c\rho' + \rho_0 u') = -\frac{\partial(\rho' u')}{\partial x}. \tag{10.58}$$

$$\frac{\partial}{\partial \xi}(-c\rho_0 u' + c^2\rho') + \frac{\partial}{\partial \eta}(c\rho_0 u' + c^2\rho') = \dots, \tag{10.59}$$

where the RHS consists of nonlinear and viscous terms. Dividing the second of these by c and adding the two equations we get

$$2\frac{\partial}{\partial \eta}(c\rho' + \rho_0 u') = -\frac{\partial(\rho' u')}{\partial x} - \frac{1}{c}\rho'\frac{\partial u'}{\partial t} - \frac{\rho_0}{c}u'\frac{\partial u'}{\partial x} - \frac{\gamma - 1}{\rho_0}c\rho'\frac{\partial \rho'}{\partial x} + \frac{4\mu}{3c}\frac{\partial^2 u'}{\partial x^2}. \tag{10.60}$$

Since the LHS here involves now only the $\eta derivative$, we may use (10.57) to eliminate ρ', and similarly with all terms on the RHS. Also we express x and t derivatives on the RHS in terms of ξ. Thus we have

$$4\rho_0\frac{\partial u'}{\partial \eta} = -2\frac{\rho_0}{c}u'\frac{\partial u'}{\partial \xi} + \frac{\rho_0}{c}u'\frac{\partial u'}{\partial \xi} - +\frac{\rho_0}{c}u'\frac{\partial u'}{\partial \xi} - \frac{(\gamma-1)\rho_0}{c}u'\frac{\partial u'}{\partial \xi} + \frac{4\mu}{3c}\frac{\partial^2 u'}{\partial \xi^2}. \tag{10.61}$$

Thus

$$2c\frac{\partial u'}{\partial \eta} + \frac{\gamma+1}{2}u'\frac{\partial u'}{\partial \xi} - \frac{2\mu}{3\rho_0}\frac{\partial^2 u'}{\partial \xi^2} = 0. \tag{10.62}$$

Now

$$\frac{\partial}{\partial \eta} = \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right), \tag{10.63}$$

so

$$\frac{\partial u'}{\partial t}|_x + c\frac{\partial u'}{\partial x}|_t + \frac{\gamma+1}{2}u'\frac{\partial u'}{\partial \xi} - \frac{2\mu}{3\rho_0}\frac{\partial^2 u'}{\partial \xi^2} = 0. \tag{10.64}$$

Now this involves a linear operator describing the time derivative relative to an observer moving with the speed c. The linear operator is now the time derivative holding ξ fixed. Thus

$$\frac{\partial u'}{\partial t}|_\xi + \frac{\gamma+1}{2}u'\frac{\partial u'}{\partial \xi} - \frac{2\mu}{3\rho_0}\frac{\partial^2 u'}{\partial \xi^2} = 0. \tag{10.65}$$

The velocity perturbation u', we emphasize, is that relative to the fluid at rest at infinity. Moving with the wave the gas is seen to move with velocity $u = u' - c$ or u' really denotes $u + c$ where u is the velocity seen by the moving observer.

What we have in (10.65) is the viscous form of *Burgers' equation*. It is a nonlinear wave equation incorporating viscous dissipation but not dispersion. By suitable scaling it may be brought into the form

$$u_t + uu_x - \nu u_{xx} = 0. \tag{10.66}$$

If the viscous term is dropped we have the inviscid Burgers wave equation,

$$u_t + uu_x = 0. \tag{10.67}$$

This equation is much studied as a prototypical nonlinear wave equation. We review the method of characteristics for such equations in the next section.

Chapter 11

Gas dynamics II

11.1 Nonlinear waves in one dimension

The simplest scalar wave equation can be written in the conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0, \tag{11.1}$$

equivalent to

$$\frac{\partial u}{\partial t} + v(u)\frac{\partial u}{\partial x} = 0, \quad v(u) = F'(u). \tag{11.2}$$

The last equation can be regarded as stating that an observer moving with the velocity $v(u)$ observes that u does not change. The particle path of the observer is called a *characteristic curve*. Since u is constant on the characteristic and the velocity v is a function of u alone we see that the characteristic is a straight line in the x, t- plane. If $u(x, 0) = u_0(x)$, the characteristics are given by the family

$$x = v(u_0(x_0))t + x_0. \tag{11.3}$$

Here x_0 acts like a Lagrangian coordinate, marking the intersection of the characteristic with the initial line $t = 0$.

As an example of the solution of the initial-value problem using characteristics, consider the equation

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0, \quad u_0(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x > 1 \end{cases}. \tag{11.4}$$

First observe that the characteristics are vertical line in $x < 0$, so that $u = 0$ in $x < 0, t > 0$. Similarly the characteristics are the line $x = t + x_0$ when $x_0 > 1$, so that $u = 1$ when $x > 1 + t$. Solving $x = x_0^2 t + x_0$ for $x_0(x, t)$, we arrive at the following solution in the middle region $0 < x < 1 + t$:

$$u(x,t) = \frac{-1 + \sqrt{1 + 4xt}}{2t}. \tag{11.5}$$

Now we modify the initial condition to

$$u_0(x) = \begin{cases} 0, & \text{if } x < 0, \\ x/\epsilon, & \text{if } 0 < x < \epsilon, \\ 1, & \text{if } x > \epsilon \end{cases}. \tag{11.6}$$

The solution is then, since the characteristics in the middle region are $x = (x_0/\epsilon)^2 t + x_0$

$$u(x,t) = \frac{\epsilon}{2t}\left[-1 + \sqrt{1 + 4xt/\epsilon^2}\right]. \tag{11.7}$$

letting $\epsilon \to 0$ in (11.7) we obtain

$$u \to \sqrt{\frac{x}{t}}. \tag{11.8}$$

This solution, existing in the wedge $0 < x/t < 1$ of the x,t-plane, is called an *expansion fan.* Given the discontinuous initial condition

$$u = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0, \end{cases} \tag{11.9}$$

we can solve for the expansion fan directly by noting that u must be a function of $\eta = x/t$ Substituting $u = f(\eta)$ in our equation, we obtain

$$-\eta f' + f^2 f' = 0, \tag{11.10}$$

implying $f = \pm\sqrt{\eta}$. The positive sign is needed to make the solution continuous at the edges of the fan.

11.1.1 Dynamics of a polytropic gas

We have the following equation for a polytropic gas in one dimension, in the absence of dissipative processes and assuming constant entropy:

$$u_t + uu_x + \frac{c^2}{\rho}\rho_x, \quad \rho_t + u\rho_x + \rho u_x = 0. \tag{11.11}$$

Here $c^2 = k\gamma\rho^{\gamma-1}$. If we define the column vector $[u \;\; \rho]^T = w$, the system may be written $w_t + A \cdot w_x$ where

$$A = \begin{pmatrix} u & c^2/\rho \\ \rho & u \end{pmatrix}. \tag{11.12}$$

We now try to find analogs of the characteristic lines $x \pm ct =$ constant which arose in acoustics in one space dimension. We want to find curves on which some physical quantity is invariant. Suppose that v is a right eigenvector of A^T (transpose of A), $A^T \cdot v = \lambda v$. We want to show that the eigenvalue λ plays a role analogous to the acoustic sound velocity.

Indeed, we see that

$$v^T \cdot [w_t + A \cdot w_x] = v^T \cdot w_t + A_T \cdot v \cdot w_x = v^T \cdot w_t + \lambda v^T \cdot w_x = 0. \quad (11.13)$$

Now suppose that we can find an integrating factor ϕ such that $\phi v^T \cdot dw = dF$. The we would have

$$F_t + \lambda F_x = 0. \quad (11.14)$$

Thus $dx/dt = \lambda$ would define a characteristic curve in the x, t- plane on which $F =$ constant. The quantity F is called a *Riemann invariant.*

Thus we solve the eigenvalue equation

$$det(A^T - \lambda I) = \left| \begin{pmatrix} u-\lambda & \rho \\ c^2/\rho & u-\lambda \end{pmatrix} \right| = 0. \quad (11.15)$$

Then $(u - \lambda)^2 = c^2$, or

$$\lambda = u \pm c \equiv \lambda_\pm. \quad (11.16)$$

We see that the characteristic speeds are indeed related to sound velocity, but now altered by the doppler shift introduced by the fluid velocity. (Unlike light through space, the speed of sound does depend upon the motion of the observer. Sound moves relative to the compressible fluid in which it exists.)

Thus the following eigenvectors are obtained:

$$\lambda_+ : \quad \begin{pmatrix} -c & \rho \\ c^2/\rho & -c \end{pmatrix} v_+ = 0, \quad v_+^T = [\rho \;\; c], \quad (11.17)$$

$$\lambda_- : \quad \begin{pmatrix} c & \rho \\ c^2/\rho & c \end{pmatrix} v_- = 0, \quad v_-^T = [\rho \;\; -c], \quad (11.18)$$

We now choose ϕ:

$$\phi[\rho \;\; \pm c] \begin{bmatrix} du \\ d\rho \end{bmatrix} = dF_\pm. \quad (11.19)$$

Since c is a function of ρ we see that we may take $\phi = 1/\rho$, to obtain

$$F_\pm = u \pm \int \frac{c}{\rho} d\rho, \quad (11.20)$$

which may be brought into the form

$$F_\pm = u \pm \frac{2}{\gamma - 1} c. \quad (11.21)$$

Thus we find that the Riemann invariants $u \pm \frac{2}{\gamma-1} c$ are constant on the curves $\frac{dx}{dt} = u \pm c$:

$$\left[\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right]\left[u \pm \frac{2}{\gamma - 1} c\right] = 0. \quad (11.22)$$

11.1.2 Simple waves

Any region of the x, t-plane which is adjacent to a region where are fluid variables are constant (i.e. a region at a constant state), but which is not itself a region of constant state, will be called a *simple wave* region, or SWR. The characteristic families of curves associated with $\lambda_\pm$ will be denoted by $C_\pm$. Curves of both families will generally propagate through a region. In a simple wave region one family of characteristics penetrates into the region of constant state, so that one of the two invariants $F_\pm$ will be known to be constant over a SWR. Suppose that F_- is constant over the SWR. Then any C_+ characteristic in the SWR not only carries a constant value of F_+ but also a constant value of F_-, and this implies a constant value of $u + c$ (see the definitions (11.21) of $F_\pm$). Thus *in a SWR where F_- is constant the C_+ characteristics are straight lines*, and similarly for the C_- characteristics over a SWR where F_+ is constant.

Let us suppose that a simple wave region involving constant F_- involves $u > 0$, so fluid particles move upward in the x, t-plane. All of the C_+ characteristics have positive slope. They may either converge on diverge. In the latter case we have the situation shown in figure 11.1(a). Since $u + c > u$, fluid particles must cross the C_+ characteristics from right to left. Moving along this path, a fluid particle experiences steadily decreasing values of $u + c$. We assume now that $\gamma > 1$. Since $u = \frac{2c}{\gamma-1} +$ constant by the constancy of F_-, we see that in this motion of the fluid particle c, and hence ρ, is decreasing. Thus the fluid is becoming less dense, or expanding. We have in figure 11.1(a) what we shall call a *forward-facing expansion wave.* Similarly in figure 11.1(b) F_+ is constant in the SWR, and $u - c$ is constant on each C_- characteristic. These are again an expansion waves, and we term them backward- facing. Forward and backward-facing compression waves are similarly obtained when C_+ characteristics converge and C_- characteristics diverge.

11.1.3 Example of a SWR: pull-back of a piston

We consider the movement of a piston in a tube with gas to the right, see figure 11.2. The motion of the piston is described by $x = X(t)$, the movement being to the left, $dX/dt < 0$. If we take $X(t) = -at^2/2$, then $u = -at$ on the piston. We assume that initially $u = 0, \rho = \rho_0$ in the tube.

On the C_- characteristics, we have $u - \frac{2c}{\gamma-1} = F_- = \frac{-2c_0}{\gamma-1}$, or

$$c = c_0 + \frac{\gamma - 1}{2}u. \tag{11.23}$$

Also on C_+ characteristics we have $u+\frac{2c}{\gamma-1}$ constant. By this fact and (11.23) we have $2u + \frac{2c_0}{\gamma-1}$ is constant on the C_+ characteristics. But since $u = u_p = -at$ at the piston surface, this determines the constant value of u . Let the C_+ characteristic in question intersect the piston path at $t = t_0$. Then the equation of this characteristic is

$$\frac{dx}{dt} = u + c = \frac{\gamma + 1}{2}u + c_0 = -\frac{\gamma + 1}{2}at_0 + c_0. \tag{11.24}$$

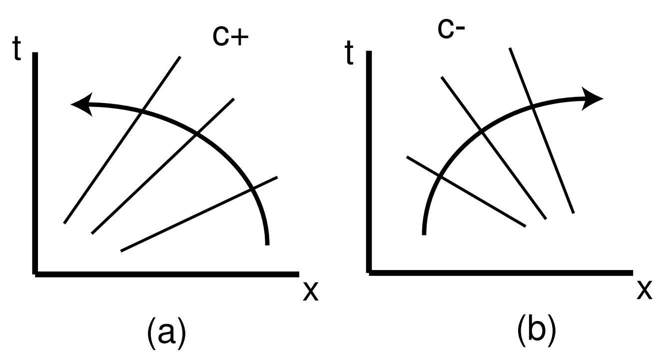

Figure 11.1: Simple expansion waves, the curves indicating the direction of particle paths. (a) Forward-facing; (b) Backward facing.

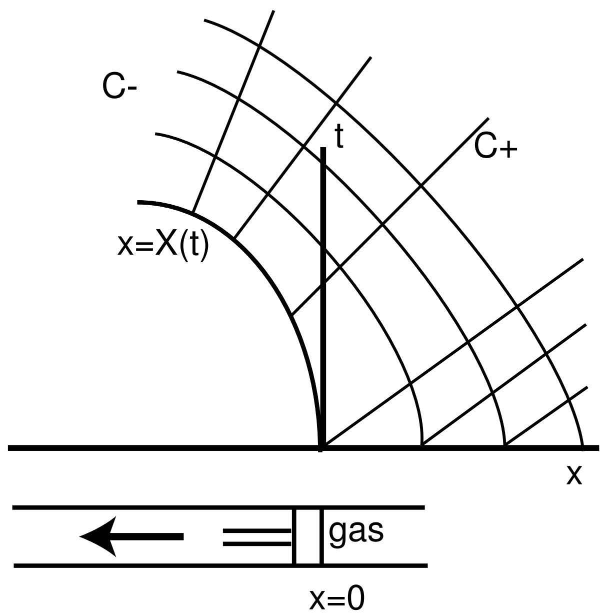

Figure 11.2: Pull-back of a piston, illustrating a simple wave region.

Thus

$$x = -a\frac{\gamma+1}{2}t_0 t + \frac{a\gamma}{2}t_0^2 + c_0(t - t_0). \qquad (11.25)$$

If we solve the last equation for $t_0(X,T)$ we obtain

$$t_0 = \frac{1}{a\gamma}\Big[c_0 + \frac{at(\gamma+1)}{2} - \sqrt{\big(c_0 + \frac{at(\gamma+1)}{2}\big)^2 + 2a\gamma(x - c_0 t)}\Big]. \qquad (11.26)$$

Then $u = -at_0(x,t)$ in the simple wave region, c being given by (11.23).

Note that according to (11.23), $c = 0$ when $t = t^*$, where $at^* = \frac{2}{\gamma-1}c_0$. This piston speed is the limiting speed the gas can obtain. For $t > t^*$ the piston pulls away from a vacuum region bounded by an interface moving with speed $-at^*$.

If we consider the case of instantaneous motion of the velocity with speed u_p, the C_+ characteristics emerge from the origin as an expansion fan. Their equation is

$$\frac{x}{t} = u + c = c_0 + \frac{\gamma+1}{2}u, \qquad (11.27)$$

so that

$$u = \frac{2}{\gamma+1}\Big[\frac{x}{t} - 2c_0\Big]. \qquad (11.28)$$

To compute the paths $\xi(t)$ of fluid particles in this example, we must solve

$$\frac{d\xi}{dt} = \frac{2}{\gamma+1}\Big[\frac{\xi}{t} - 2c_0\Big]. \qquad (11.29)$$

A particle begins to move with the rightmost wave of the expansion fan, namely the line $x = c_0 t$, meets the initial particle position. Thus (11.29) must be solved with the initial condition $\xi(t_0) = c_0 t_0$. The solution is

$$\xi(t) = \frac{-2c_0}{\gamma-1}t + c_0 t_0\frac{\gamma+1}{\gamma-1}(t/t_0)^{\frac{2}{\gamma+1}}. \qquad (11.30)$$

For the location of the C_- characteristics we must solve

$$\frac{dx}{dt} = u - c = \frac{3-\gamma}{2}u - c_0 = \frac{3-\gamma}{\gamma+1}(x/t - c_0) - c_0, \qquad (11.31)$$

with the initial condition $x(t_0) = c_0 t_0$. There results

$$x(t) = \frac{-2c_0}{\gamma-1}t + c_0 t_0\frac{\gamma+1}{\gamma-1}(t/t_0)^{\frac{3-\gamma}{1-\gamma}}. \qquad (11.32)$$

11.2 Linearized supersonic flow

We have seen above that 2D irrotational inviscid homentropic flow of a polytropic gas satisfies the system

$$(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_y^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0, \qquad (11.33)$$

$$\phi_x^2 + \phi_y^2 + \frac{2}{\gamma - 1}c^2 = \quad \text{constant}. \tag{11.34}$$

We are interested in the motion of thin bodies which do not disturb the ambient fluid very much, The assumption of small perturbations, and the corresponding linearized theory of compressible flow, allows us to consider some steady flow problems of practical interest which are analogs of sound propagation problems.

We assume that the air moves with a speed U past the body, from left to right in the direction of the x-axis. Then the potential is taken to have the form

$$\phi = U_0 x + \phi', \tag{11.35}$$

where $|\phi'_x| \ll U_0$. It is easy to derive the linearized form of (11.33), since the second-derivative terms must be primed quantities. The other factors are then evaluated at the ambient conditions, $c \approx c_0$, $\phi_x \approx U_)$. Thus we obtain

$$(M^2 - 1)\phi'_{xx} - \phi'_{yy}. \tag{11.36}$$

Here

$$M = U_0/c_0 \tag{11.37}$$

is the *Mach number* of the ambient flow. We note that in the linear theory the pressure is obtained from

$$U_0 \frac{\partial u'}{\partial x} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{11.38}$$

or

$$p' \approx -U_0 \rho_0 \phi'_x. \tag{11.39}$$

We now drop the prime from ϕ'. The density perturbation is then $\rho' = -c_0^{-2} U_0 \rho \phi_x$.

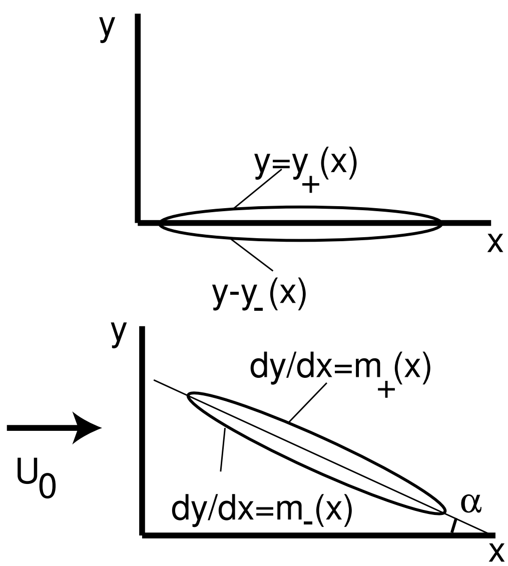

Figure 11.3: Thin airfoil geometry.

11.2.1 Thin airfoil theory

We consider first the 2D supersonic flow over a thin airfoil.

Linearized supersonic flow results when $M > 1$, linearized subsonic flow when $M < 1$. The *transonic regime* $M \approx 1$ is special and needs to be examined as a special case.

We show the geometry of a thin airfoil in figure 11.3. We assume that the slopes $dy_\pm/dx$ and the angle of attack α are small. In this case

$$m_\pm(x) \approx -\alpha + dy_\pm/dx. \tag{11.40}$$

let the chord of the airfoil be l, so we consider $0 < x < l$. We note that

$$\frac{1}{l}\int_0^l (m_+ + m_-)dx = -2\alpha, \quad \frac{1}{l}\int_0^l (m_+^2 + m_-^2)dx = -2\alpha^2 + \frac{1}{l}\int_0^l (y_+^2 + y_-^2)dx. \tag{11.41}$$

The analysis now makes use the following fact analogous to the linear wave equation in 1D: the linear operator factors as

$$\left[\sqrt{M^2-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]\left[\sqrt{M^2-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]. \tag{11.42}$$

Thus $\phi = f(x - y\sqrt{M^2-1}) + g(x + y\sqrt{M^2-1})$, where f, g are arbitrary functions. We now need to use physical reasoning choose the right form of solution. In linearized supersonic flow past an airfoil the disturbances made by the foil propagate out relative to the fluid at the speed of sound, but are simultaneously carried downstream with speed U_0. In supersonic flow the foil cannot therefore cause disturbances of the fluid upstream of the body. Consequently the characteristic lines $x \pm y\sqrt{M^2-1} =$ constant, which carry disturbances away from the foil, must always point downstream. Thus in the half space above the foil the correct choice is $\phi = f(x - y\sqrt{M^2-1})$, while in the space below it the correct choice is $\phi = g(x + y\sqrt{M^2-1})$. To determine these functions, we must make the flow tangent to the foil surface. Since we are dealing with thin airfoils and small angles, the condition of tangency can be applied, approximately, at $y = 0$. Thus we have the tangency conditions

$$\frac{\phi_y}{U_0}|_{y=0+} = m_+(x) = -\sqrt{M^2-1}U_0^{-1}f'(x), \tag{11.43}$$

$$\frac{\phi_y}{U_0}|_{y=0-} = m_-(x) = \sqrt{M^2-1}U_0^{-1}g'(x), \tag{11.44}$$

Of interest to engineers is the lift and drag of a foil. To compute these we first need the pressures

$$p'_+(x) = -U_0\rho_0 u'(x, 0+) = -U_0\rho_0 f'(x), \quad p'_-(x) = -U_0\rho_0 g'(x). \tag{11.45}$$

This yields

$$p'_\pm = \pm\frac{U_0^2\rho_0}{\sqrt{M^2-1}}(-\alpha + dy_\pm/dx). \tag{11.46}$$

Then

$$\text{Lift} = \int_0^l (p'_- - p'_+)dx = \frac{2\alpha\rho_0 U_0^2 l}{\sqrt{M^2-1}}, \tag{11.47}$$

$$\text{Drag} = \int_0^l (p'_+ m_+ - p'_- m_-)dx = \frac{\rho_0 U_0^2 l}{\sqrt{M^2-1}}[2\alpha^2 + \frac{1}{l}\int_0^l [(dy_+/dx)^2 + (dy_-/dx)^2]dx. \tag{11.48}$$

Note that now inviscid theory gives a positive drag. We recall that for incompressible potential flow we obtained zero drag (D'Alembert's paradox).

In supersonic flow, the characteristics carry finite signals to infinity. In fact the disturbances are being created so that the rate of increase of kinetic energy per unit time is just equal to the drag times U_0. This drag is often called *wave drag* because it is associated with characteristics, usually called in this context *Mach waves*, which propagate to infinity.

What happens if we solve for compressible flow past a body in the subsonic case $M < 1$? In the case of thin airfoil theory, it is easy to see that we must get zero drag. The reason is that the equation we are now solving may be written $\phi_{xx} + \phi_{\bar{y}\bar{y}} = 0$ where $\bar{y} = \sqrt{1-M^2}y$. The boundary conditions are at $y = \bar{y} = 0$ so in the new variables we have a problem equivalent to that of an incompressible potential flow.

In fact compressible potential flow past any finite body will give zero drag so long as the flow field velocity never exceeds the local speed of sound, i.e. the fluid stays locally subsonic everywhere. In that case no shock waves can form, there is no dissipation, and D'Alembert's paradox remains.

11.2.2 Slender body theory

Another case of interest is the steady supersonic flow past a slender body of revolution. If the ambient flow is along the z-axis in cylindrical polar coordinates x, r, the body we consider is a slender body of revolution about the z-axis. It is easy to show that the appropriate wave equation (coming from the linearized equations $\rho_0 U_0 \frac{\partial u'}{\partial z} + c_0^2 \frac{\partial \rho'}{\partial z} = 0, U_0 \frac{\partial \rho'}{\partial z} + \rho_0 \nabla \cdot \mathbf{u'} = 0$), is

$$\beta^2 \phi_{zz} - \phi_{rr} - \frac{1}{r}\phi_r = 0, \quad \beta = \sqrt{M^2 - 1}. \tag{11.49}$$

To find a fundamental solution of this equation, not that

$$a^2 \phi_{xx} + b^2 \phi_{yy} + c^2 \phi_{xx} = 0 \tag{11.50}$$

clearly has a "sink-like" solution $[(x/a)^2 + (y/b)^2 + (z/c)^2]^{-1}$, equivalent to the simple sink solution (-4π time the fundamental solution) $1/\sqrt{(x^2 + y^2 + z^2)}$ of Laplace's equation in 3D. This holds for arbitrary complex numbers a, b, c. It follows that a solution of (11.49) is given by

$$S(z, r) = \frac{1}{\sqrt{z^2 - \beta^2 r^2}}. \tag{11.51}$$

Note that this is a real quantity only if $\beta r < z$, where S is singular. We therefore want to complete the definition of S by setting

$$S(z, r) = 0, \quad \beta r > z. \tag{11.52}$$

Suppose now that we superimpose these solutions by distributing them on the interval $(0, l)$ of the z-axis,

$$\phi = \int_0 \frac{f(\zeta)}{\sqrt{(z-\zeta)^2 - \beta^2 r^2}} d\zeta. \tag{11.53}$$

However notice that if we are interested in the solution on the surface $z-\beta r = C$, then there can be no contributions from values of ζ exceeding C. We therefore propose a potential

$$\phi(z,r) = \begin{cases} \int_0^{z-\beta r} \frac{f(\zeta)}{\sqrt{(z-\zeta)^2-\beta^2 r^2}} d\zeta, & \text{if } 0 < z-\beta r < 1, \\ \int_0^1 \frac{f(\zeta)}{\sqrt{(z-\zeta)^2-\beta^2 r^2}} d\zeta, & \text{if } z-\beta r > 1. \end{cases} \tag{11.54}$$

where we now require $f(0) = 0$.

We can in fact verify that (11.54) gives us a solution of (11.49) for any admissible f, but will leave this verification as a problem.

Consider now the behavior of ϕ near the body. When r is small the main contribution comes from the vicinity of $\zeta = z$, so we may extract $f(z)$ and use the change of variables $\zeta = z - \beta r \cosh\lambda$ to obtain

$$\phi \approx f(z) \int_0^{\cosh^{-1}(z/\beta r)} d\lambda = f(z)\cosh^{-1}(z/\beta r) \sim f(z)\log(2z/\beta r). \tag{11.55}$$

Now let the body be described by $r = R(z), 0 < z < l$. The tangency condition is then

$$r\frac{\phi_r}{U_0}\Big|_{r=R} = R\frac{dR}{dz} \approx -f(z)/U_0. \tag{11.56}$$

If $A(a)$ denotes cross-sectional area, then we have

$$f(z) = -\frac{1}{2\pi} U_0 \frac{dA}{dz}. \tag{11.57}$$

The calculation of drag is a bit more complicated here, and we give the result for the case where $R(0) = R(l) = 0$. Then

$$\text{Drag} = \frac{\rho_0 U_0^2}{4\pi} \int_0^l \int_0^l A''(z)A''(\zeta)\frac{1}{\log|z-\zeta|} dz d\zeta. \tag{11.58}$$

The form of this emphasizes the importance of have a smooth distribution of area. in order to minimize drag.

An alternative formulation and proof of the drag formula

To prove (11.58) it is convenient to reformulate the problem in terms of a stream-function. We go back to the basic equations for steady homentropic potential flow in cylindrical polar coordinates.

$$\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0, \quad \frac{\partial r\rho u_z}{\partial z} + \frac{\partial r\rho u_r}{\partial r} = 0, \tag{11.59}$$

$$u_z^2 + u_r^2 + \frac{2}{\gamma - 1}c^2 = \text{constant}. \tag{11.60}$$

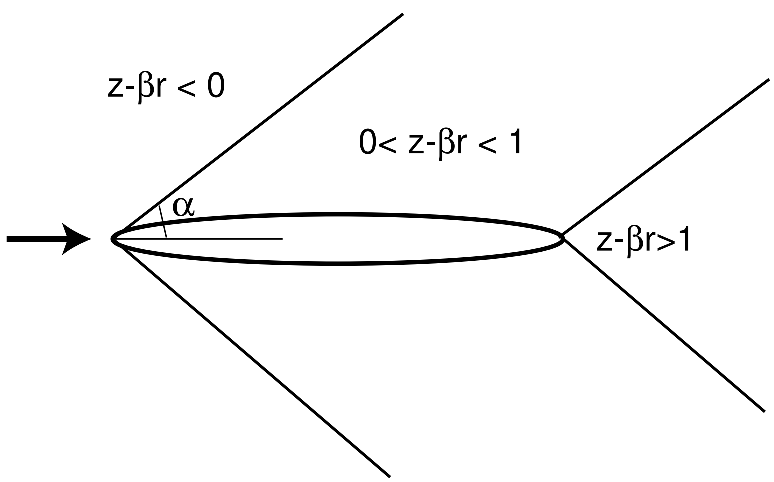

Figure 11.4: Steady supersonic flow past a slender body of revolution. Here $\tan\alpha = 1/\sqrt{M^2-1}$.

From the second of (11.59) we introduce the streamfunction ψ,

$$r\rho u_z = \frac{\partial\psi}{\partial r}, r\rho u_r = -\frac{\partial\psi}{\partial z}. \tag{11.61}$$

We then expand the equations as follows:

$$\psi = \frac{U_0}{2}r^2 + \psi', \rho = \rho_0 + \rho', \tag{11.62}$$

and linearize. The result is the equation for ψ',

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi'}{\partial r} - \beta^2\frac{\partial^2\psi'}{\partial z^2} = 0. \tag{11.63}$$

Now the boundary condition in terms of the stream function is that ψ equal zero on the slender body $r = R(z)$. Approximately, this gives

$$\rho_0\frac{U_0}{2}r^2 + \psi'(z,0) \approx 0. \tag{11.64}$$

We also want no disturbance upstream, so ψ' and $\prime_z$ should vanish on $z = 0, r > 0$. A solution of this problem is given by

$$\psi' = -\frac{\rho_0 U_0}{2\pi}\int_0^{z-\beta r}\sqrt{(z-\zeta)^2 + \beta^2 r^2}A''d\zeta. \tag{11.65}$$

It is easy to see that the equation and upstream conditions are satisfied under the conditions that $R(0) = 0$. For the boundary condition we have

$$\psi'(z,0) = -\frac{\rho_0 U_0}{2\pi}\int_0^z (z-\zeta)A''d\zeta = -\frac{\rho_0 U_0}{2\pi}\int_0^z A'd\zeta = -\rho_0\frac{U_0}{2}r^2. \tag{11.66}$$

Now the drag is given by

$$D = \int_0^l p'A'(\zeta)d\zeta, \tag{11.67}$$

where the linear theory gives $p' \approx -\rho_0 U_0 u_z$. However it turns out that the u_r velocity components become sufficiently large near the body to make a leading order contribution. Thus we have

$$p' \approx -\rho_0 U_0 u_z - \frac{1}{2}u_r^2 + \dots. \tag{11.68}$$

We note that

$$u_z = \frac{1}{\rho r}\frac{\partial\psi}{\partial r} \approx \frac{1}{\rho_0 r}\frac{\partial\psi'}{\partial r} - \frac{U_0}{\rho_0 c_0^2}p', \tag{11.69}$$

from which we have

$$-\beta^2 u_z = \frac{1}{\rho_0 r}\frac{\partial\psi'}{\partial r}. \tag{11.70}$$

Thus

$$u_z = -\frac{U_0}{2\pi}\int_0^{z-\beta r}\frac{A''(\zeta)}{\sqrt{(z-\zeta)^2+\beta^2 r^2}}d\zeta. \tag{11.71}$$

Similarly

$$u_r = \frac{U_0}{2\pi r}\int_0^{z-\beta r}(z-\zeta)\frac{A''(\zeta)}{\sqrt{(z-\zeta)^2+\beta^2 r^2}}d\zeta. \tag{11.72}$$

We see by letting $r \to R \approx 0$ in (11.72) that

$$u_r \approx \frac{U_0}{2\pi}A'(z)/R(z). \tag{11.73}$$

Also

$$u_z \approx -\frac{U_0}{2\pi}\Big[A''(z-\beta r)\cosh^{-1}\frac{z}{\beta r} + \int_0^{z-\beta r}\frac{A''(\zeta)-A''(z-\beta r)}{\sqrt{(z-\zeta)^2+\beta^2 r^2}}d\zeta\Big] \tag{11.74}$$

$$\approx -\frac{U_0}{2\pi}\Big[A''\log\frac{2z}{\beta R} - \int_0^z \frac{A''(z)-A''(\zeta)}{z-\zeta}d\zeta\Big]. \tag{11.75}$$

We are now in a position to compute D:

$$D = \frac{\rho_0 U_0^2}{2\pi}\int_0^l A'(z)\int_0^{z-\beta R}\Big[A''(z)\log\frac{2z}{\beta r} - \int_0^z\frac{A''(z)-A''(\zeta)}{z-\zeta}d\zeta - \frac{1}{4}A^{-1}(z)(A')^2(z)\Big]dz. \tag{11.76}$$

After an integration by parts and a cancelation we have

$$D = \frac{\rho_0 U_0^2}{2\pi}\int_0^l A'(z)\Big[A''(z)\log z - \int_0^z\frac{A''(z)-A''(\zeta)}{z-\zeta}d\zeta\Big]dz. \tag{11.77}$$

Our last step is to show that (11.77) agrees with (11.58). Now if $B(z) = A'(z)$,

$$\int_0^l\int_0^l B'(z)B'(\zeta)\log|z-\zeta|d\zeta dz = 2\int_0^l B'(z)\int_0^z B'(\zeta)\log|z-\zeta|d\zeta dz$$

$$= 2\int_0^l B'(z)B(z)\log z dz + 2\int_0^l B'(z)\int_0^z\frac{B(z)-B(\zeta)}{z-\zeta}d\zeta dz$$

$$= -2\int_0^l B'(z)B(z)\log z dz - 2\int_0^l B(z)\int_0^z\frac{B'(z)-B'(\zeta)}{z-\zeta}d\zeta dz, \tag{11.78}$$

which proves the agreement of the two expressions. Here we have used

$$\frac{d}{dz}\int_0^z\frac{B(z)-B(\zeta)}{z-\zeta}d\zeta = \frac{B(z)-B(\zeta)}{z-\zeta}\Big|_\zeta = z + \int_0^z\frac{(z-\zeta)B'(z)-B(z)+B(\zeta)}{(z-\zeta)^2}d\zeta$$

$$= B'(z) + \Big[\frac{(z-\zeta)b'(z)-B(z)+B(\zeta)}{(z-\zeta)^2}\Big]_0^z + \int_0^z\frac{B'(z)-B(\zeta)}{z-\zeta}d\zeta$$

$$= \int_0^z\frac{B'(z)-B'(\zeta)}{z-\zeta}d\zeta + B(z)/z. \tag{11.79}$$

Chapter 12

Shock waves

12.1 Scalar case

We have seen that the equation $u_t + uu_x = 0$ with a initial condition $u(x, 0) = 1 - x$ on the segment $0 < x < 1$ produces a family of characteristics

$$x = (1 - x_0)t + x_0. \tag{12.1}$$

This family of lines intersects at $(x, t) = (1, 1)$. If the initial condition is extended as

$$u(x, 0) = \begin{cases} 1, & \text{if } x < 0, \\ 0, & \text{if } x > 1, \end{cases} \tag{12.2}$$

we see that at $t = 1$ a discontinuity develops in u as a function of x. We thus need to study how such discontinuities propagate for later times as *shock waves*. We study first the general scalar wave equation in conservation form, $u_t + (F(u))_x = 0$. This equation is assumed to come from a conservation law of the form

$$\frac{d}{dt}\int_a^b u dx = F(u(a, t)) - F(u(b, t)). \tag{12.3}$$

Suppose now that in fact there is a discontinuity present at position $\xi(t) \in (a, b)$. Then we study the conservation law by breaking up the interval so that differentiation under the integral sign is permitted:

$$\frac{d}{dt}\Big[\int_a^\xi u dx + \int_\xi^b u dx = F(u(a, t)) - F(u(b, t)). \tag{12.4}$$

Now differentiating under the integral and using the wave equation to eliminate the time derivatives of u we obtain

$$\frac{d\xi}{dt}[u(\xi+, t) - u(\xi-, t)] = F(u(\xi+, t)) - F(u(\xi-, t)). \tag{12.5}$$

Thus we have an expression for the propagation velocity of the shock wave:

$$\frac{d\xi}{dt} = \frac{[F]_{x=\xi}}{[u]_{x=\xi}}, \tag{12.6}$$

where here $[\cdot]$ means "jump in". The direction you take the jump is immaterial provided that you do the same in numerator and denominator.

Example: Let $u_t + uu_x = 0$,

$$u(x,0) = \begin{cases} 0, & \text{if } x < -1, \\ 1+x, & \text{if } -1 < x < 0, \\ 1-2x, & \text{if } 0 < x < 1/2 \\ ,0, & \text{if } x > 1/2. \end{cases} \tag{12.7}$$

The characteristic family associated with the interval $-1 < x < 0$ is $x = (1+x_0)t + x_0$, while that of $0 < x < 1/2$ is $x = (1-2x_0)t + x_0$. The shock first occurs at $(x,t) = (1/2, 1/2)$. To the right of the shock $u = 0$, while to the left the former family gives

$$u = 1 + x_0(x,t) = 1 + \frac{x-t}{1+t} = \frac{1+x}{1+t}. \tag{12.8}$$

Then

$$\frac{d\xi}{dt}\frac{1}{2}[u(\xi-,t) + u(\xi+,t)] = \frac{1}{2}\frac{\xi+1}{1+t}, \quad \xi(1/2) = 1/2. \tag{12.9}$$

Thus

$$\xi(t) = \sqrt{3/2}\sqrt{1+t} - 1. \tag{12.10}$$

We show the x,t-diagram for this in figure 12.1.

12.1.1 A cautionary note

One peculiarity of shock propagation theory is that it is strongly tied to the physics of the problem. Suppose that $u \geq 0$ solves $u_t + uu_x = 0$. Then it will also solve $v_t + [G(v)]_x = 0$ where $G = \frac{2}{3}v^{3/2}$ and $v = u^2$. In the firmer case the shock wave propagation speed is $\frac{1}{2}(u_+ + u_-)$, while in the latter it is $\frac{2}{3}(u_+^2 + u_+u_- + u_-^2)/(u_+ + u_-)$, which is different. What's going on??

The point is that $u_t + uu_x = 0$ is based fundamentally on a conservation law involving $F(u) = u^2/2$. In actual physical problems the conservation laws will be known and have to be respected. Another way to say this is that equivalent partial differential equations can arise from different conservation laws. It is the conservation law that determines the relevant shock velocity however.

We illustrate this with a simple example from the continuum theory of traffic flow. Consider a single-lane highway with $n(x,t)$ cars per mile as the traffic density. The cars are assumed to move at a speed determined by the local density, equal to $u = U(1 - n/n_0$ where U is the maximum velocity and n_0 is the density of full packing and zero speed. The flux of cars is the $F(n) = nu = Un(1 - n/n_0)$, and the corresponding conservation of car number yields

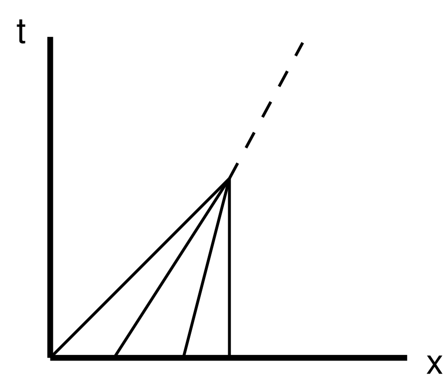

Figure 12.1: Example of shock formation and propagation.

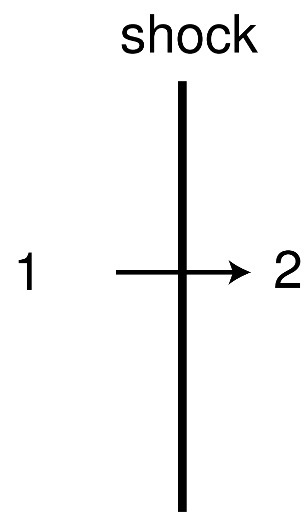

Figure 12.2: The stationary normal shock wave.

the PDE $n_t + [F(n)]_x = 0$. This is equivalent to $v_t + [G(v)]_x = 0$ where $v = n^2$ and $G = U[v - \frac{2}{3n_0} v^{3/2}$. However the conservation law associated with v, G makes no physical sense. We know how the speed of the cars depends upon n, and conservation of number (if indeed that is what happens) dictates the former conservation law. Not that if the square of density was somehow what was important in the conservation of mass, we would end up with a conservation of mass equation $\frac{\partial \rho^2}{\partial t} + \nabla \cdot (\rho^2 \mathbf{u}) = 0$.

12.2 The stationary normal shock wave

We now want to consider a stationary planar shock in gas dynamics, without viscosity or heat conduction. We assume that constant conditions prevail on either side of the shock denoted by the subscripts $1, 2$, see figure 12.2.

We have the following conservation laws:

$$\text{Mass}: \quad \rho_1 u_1 = \rho_2 u_2. \tag{12.11}$$

$$\text{Momentum}: \quad p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2. \tag{12.12}$$

Recall the following form of the energy equation:

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) = -p \nabla \cdot \mathbf{u}. \tag{12.13}$$

From conservation of momentum we also have

$$\frac{D}{Dt} \frac{1}{2} \rho u^2 = -\mathbf{u} \cdot \nabla p - \frac{1}{2} \rho u^2 \nabla \cdot \mathbf{u}. \tag{12.14}$$

Combining these to equation we have

$$\frac{\partial E}{\partial t} + \nabla \cdot (E+p)\mathbf{u} = 0, \quad E = \rho(e + \frac{1}{2}u^2). \tag{12.15}$$

At the shock we must therefore require continuity of $(\rho u(e+\frac{p}{\rho}+\frac{1}{2}u^2)$, and since ρu is continuous we have that $e + \frac{p}{\rho} + \frac{1}{2}u^2 = h + \frac{1}{2}u^2$ is continuous:

$$\text{Energy}: \quad h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2. \tag{12.16}$$

Let us write $m = \rho u$ as the constant mass flux, and let $v = 1/\rho$. Then the conservation of energy may be rewritten

$$h_2 - h_1 = \frac{1}{2}m^2(v_1^2 - v_2^2). \tag{12.17}$$

Also conservation of momentum can be rewritten

$$m^2 = \frac{p_1 - p_2}{v_2 - v_1} \tag{12.18}$$

Thus

$$h_1 - h_2 = \frac{1}{2}(v_1 + v_2)m^2(v_2 - v_1), \tag{12.19}$$

which is equivalent to

$$h_1 - h_2 = \frac{1}{2}(v_1 + v_2)(p_1 - p_2). \tag{12.20}$$

Written out, this means

$$e_1 - e_2 + p_1 v_1 - p_2 v_2 = \frac{1}{2}(v_1 + v_2))p_1 - p_2) \tag{12.21}$$

or

$$e_1 - e_2 = \frac{1}{2}(v_2 - v_1)(p_1 + p_2). \tag{12.22}$$

The relations (12.20),(12.22) involving the values of the primitive thermodynamic quantities on either side of the shock are called the *Rankine-Hugoniot* relations.

For a polytropic gas we have

$$e = \frac{1}{\gamma - 1}pv. \tag{12.23}$$

This allows us to write (12.22) as

$$\frac{2}{\gamma - 1}(p_1 v_1 - p_2 v_2) + (v_1 - v_2)(p_1 + p_2) = 0, \tag{12.24}$$

or

$$\frac{\gamma + 1}{\gamma - 1}(p_1 v_1 - p_2 v_2) - v_2 p_1 + p_2 v_1 = 0. \tag{12.25}$$

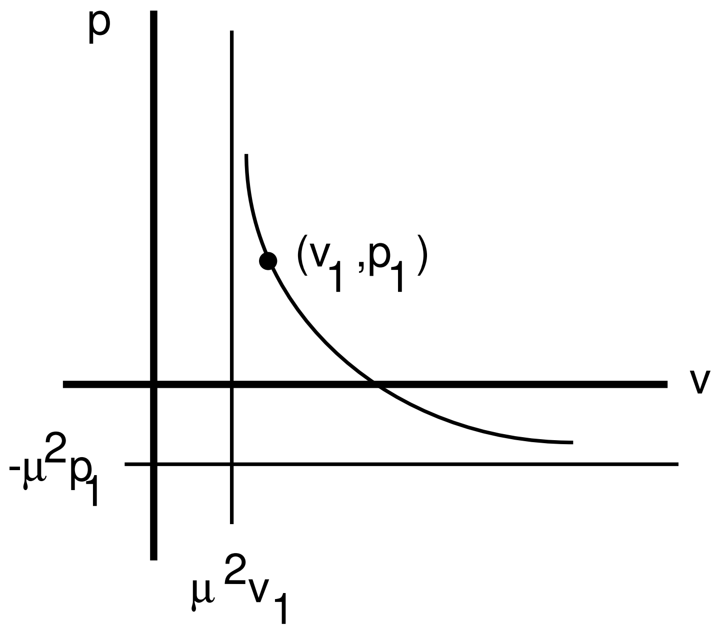

Figure 12.3: The Hugoniot of the stationary normal shock wave.

We now introduce notation from Courant and Friedrichs: Let $\nu^2 = \frac{\gamma-1}{\gamma+1}$. ($\mu$ has no relation whatsoever to viscosity.)According to (12.25), if the state p_1, v_1 exists upstream of a shock, the possible downstream states p, v satisfy

$$-p_1 v_1 + pv + \mu^2 v p_1 - \mu^2 p v_1 = 0, \tag{12.26}$$

or

$$(p + \mu^2 p_1)(v - \mu^2 v_1) + (\mu^4 - 1)p_1 v_1 = 0. \tag{12.27}$$

We shall late see that the only allowed transition states are upward along the Hugoniot from the point (v_1, p_1), as indicated by the arrow, corresponding to an increase of entropy across the shock.

12.2.1 Prandtl's relation

For a polytropic gas the energy conservation may use

$$h = \frac{\gamma}{gamma - 1}\frac{p}{\rho} = \frac{1-\mu^2}{2\mu^2}c^2. \tag{12.28}$$

Then conservation of energy across a shock becomes

$$(1-\mu^2)c_1^2 + \mu^2 u_1^2 = (1-\mu^2)c_2^2 + \mu^2 u_2^2 \equiv c_*^2. \tag{12.29}$$

Note that then constancy of $(1-\mu^2)c^2 + \mu^2 u^2$ implies that $(1-\mu^2)(u^2-c^2) = u^2 - c_*^2$. Since $\mu < 1$, this last relation shows that $u > c_*$ iff $u > c$ and $u < c_*$ iff $u < c$.

Prandtl's relation asserts that

$$u_1 u_2 = c_*^2. \tag{12.30}$$

This implies that the on one side of the shock $u > c_*$ and hence $u > c$, i.e. the flow is supersonic relative to the shock position, and on the other side it is subsonic. Since density increases as u decreases, the direction of transition on the Hugoniot indicates that the transition must be from supersonic to subsonic as the shock is crossed.

To prove Prandtl's relation, note that $(1+\mu^2)p = \rho(1-\mu^2)c^2$ since $\mu^2 = \frac{\gamma-1}{\gamma+1}$. Then, if $P = \rho u^2 + p$,

$$\mu^2 P + p_1 = \mu^2 u_1^2 \rho_1 + (1+\mu^2)p_1 = \rho_1[\mu^2 u_1^2 + (1-\mu^2)c_1^2] = \rho_1 c_*^2. \tag{12.31}$$

Similarly $\mu^2 P + p_2 = c_*^2 \rho_2$. Thus

$$p_1 - p_2 = c_*^2(\rho_1 - \rho_2),$$

or

$$c_*^2 = \frac{p_1 - p_2}{\rho_1 - \rho_2} = \frac{p_1 - p_2}{\frac{1}{\rho_2} - \frac{1}{\rho_1}} \frac{1}{\rho_2 \rho_1} = \frac{m^2}{\rho_1 \rho_2} = u_1 u_2' \tag{12.32}$$

where we have used (12.18).

12.2.2 An example of shock fitting: the piston problem

Suppose that a piston is driven through a tube containing polytropic gas at a velocity u_p. We seek to see under what conditions a shock will be formed. Let the shock speed be U. In going to a moving shock our relations for the stationary shock remain valid provided that $u - U$ replaces u. Thus Prandtl's relation becomes

$$(u_1 - U)(u_2 - U) = c_*^2 = \mu^2(u_1 - U)^2 + (1-\mu^2)c_1^2, \tag{12.33}$$

where the gas velocities are relative to the laboratory, not the shock. Rearranging, we have

$$(1-\mu^2)(u_1 - U)^2 + (u_1 - U)(u_2 - u_1) = (1-\mu^2)c_1^2. \tag{12.34}$$

Consider now the flow as shown in figure 12.4.

The gas ahead of the chock is at rest, $u_1 = 0$, with ambient sound speed $c_1 = c_0$. Behind the shock the gas moves with the piston speed, $u_2 = u_p$. Thus we have a quadratic for U:

$$(1-\mu^2)U^2 - u_p U = (1-\mu^2)c_0^2. \tag{12.35}$$

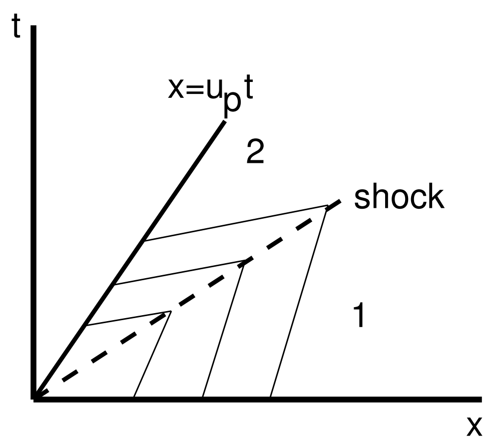

Figure 12.4: Shock fitting in the piston problem.

Thus

$$U = \frac{u_p + \sqrt{u_p^2 + 4(1-\mu^2)c_0^2}}{2(1-\mu^2)}. \tag{12.36}$$

We see that a shock forms for any piston speed. If $u+p$ is small compared to c_0, the shock speed is approximately c_0, but slightly faster, as we expect. To get the density ρ_p behind the shock in terms of that ρ_0 of the ambient air, we note that mass conservation gives $(u_p - U)\rho_p = -U\rho_0$ or

$$\rho_p = \frac{U}{U - u_p}\rho_0. \tag{12.37}$$