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Energy dissipation of hydraulic jump in gradually expanding channel after free overfall

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This study derived equations for examining the energy dissipation characteristics of a hydraulic jump after free overfall. Comparison of energy dissipation characteristics was further made between hydraulic flows occurring in a prismatic channel of constant cross-sectional width and that in an expanding channel with gradual increase in width along the length. Results calculated using the derived equations show that the conjugate depth ratio (ϕ), relative energy loss ($\Delta E/E_i$), and energy dissipation efficiency (η) are not only influenced by the pre-jump Froude number (F_i) and the relative height of the weir (H), but also by pre- and post-jump cross-sectional width ratio. In addition, energy dissipation of hydraulic jump after free overfall is greater in an expanding channel than in a prismatic channel. Hence, future design of stilling basins should consider having a gradually expanding channel so as to achieve lower level of tailwater.

Keywords: expanding channel; free overfall flow; hydraulic jump

1. Introduction

In hydrotechnical constructions such as check dams or drop structures, stilling basins are designed with considerations of being compact, low cost, and efficient at energy dissipation. A hydraulic jump is produced upon impact of an upstream flow free falling from a weir into a downstream stilling basin. This study analyzed the energy dissipation characteristics of a hydraulic jump of a free overfall flow in an expanding channel. A hydraulic jump occurs when a supercritical flow changes into a subcritical flow, thus producing a sudden rolling motion on the water surface. A large amount of kinetic energy is consumed in a hydraulic jump when the upstream flow impacts upon the downstream channel. Hence, energy dissipation is characteristic of a hydraulic jump.

Compared with a prismatic channel of constant cross-sectional width, an expanding channel with gradual increase in width along the length can contribute to reducing the required length of the stilling basin and the height of the end sill. Arbhabhirama and Abella (1971) assumed that the hydraulic jump profile in an expanding channel is quarter-elliptical in shape. According to Hans and Warren (1991), an optimum hydraulic jump should occur after the free overfall flow has impacted the channel bed. Kolseus and Ahmad (1969) and Khalifa and McCorquodale (1979) studied the length of hydraulic jump channels under different conditions. Hager (1985) calculated energy dissipation after free overfall using the momentum equation. Rajaratnam and Subramanya (1968) investigated the formation of a hydraulic jump downstream. In this study, equations have been derived to examine the energy dissipation efficiency of a hydraulic jump downstream.

2. Theoretical analysis

2.1. *Channel type*

This study examined the total energy dissipation of the hydraulic jump of a free overfall flow in an expanding channel. The cross section of the channel is rectangular in shape but with variations in width along the length. The upstream channel is narrower than the downstream one, thus making it a gradually expanding channel. Figure 1 shows the schematic of the gradual expansion of the flat-bottom channel and that of the hydraulic jump phenomenon. The value of $\mu(r_2 \sin\theta / r_1 \sin\theta = b_i/b_s)$ denotes the ratio of half cross-sectional width before and after the hydraulic jump. $\mu > 1$ indicates an expanding channel with different pre- and post-jump cross-sectional widths; while $\mu = 1$ a prismatic channel with the same cross-sectional width throughout the length of channel.

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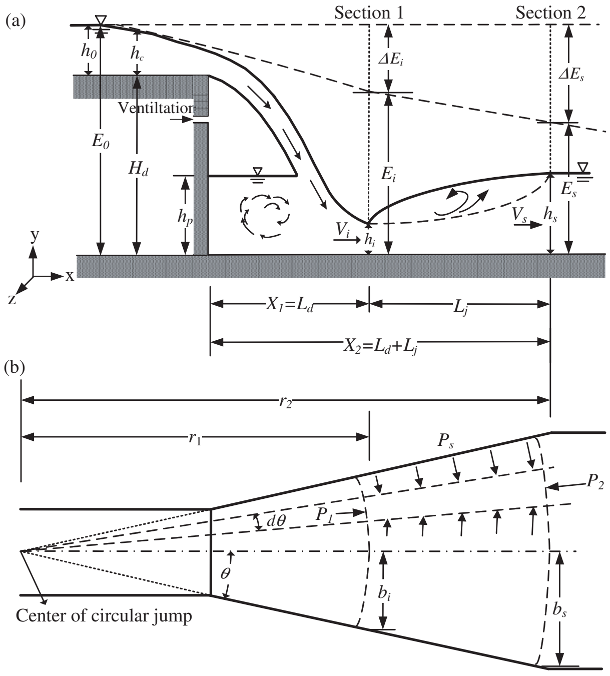

Figure 1. Sketch of hydraulic jump in expanding channel: (a) side view and (b) top view.

2.2. *Derivation of hydraulic jump equation*

A hydraulic jump occurs when the free overfall flow impacts the channel downstream. This study focused on the conjugate depth of the flow before and after the hydraulic jump. According to the control volume approach, the momentum equation before and after the hydraulic jump can be expressed as:

$$P_{sx} + P_{1x} - P_{2x} = \int\int v_x(\rho\vec{V}\cdot \mathrm{d}\vec{A}) \tag{1}$$

where P_{sx} is the side pressure force in x-direction ($P_{sx} = P_s \sin\theta$); P_{1x} and P_{2x} the hydrostatic forces in x-direction at sections 1 and 2, respectively; and $\int\int v_x(\rho\vec{V}\cdot \mathrm{d}\vec{A})$ the rate of efflux of momentum across the control volume in x-direction with $\vec{V}$ being the average flow velocity. The hydrostatic pressure at sections 1 and 2 can be expressed, respectively, as:

$$P_{1x} = \int_0^{\theta} \frac{\gamma h_i}{2} h_i r_1 \cos\theta \quad \mathrm{d}\theta = \frac{\gamma h_i^2 r_1}{2}\sin\theta, \tag{2a}$$

$$P_{2x} = \int_0^{\theta} \frac{\gamma h_s}{2} h_s r_2 \cos\theta \quad \mathrm{d}\theta = \frac{\gamma h_s^2 r_2}{2}\sin\theta, \tag{2b}$$

$$P_s = \int_0^{L} \frac{\gamma y^2}{2}\mathrm{d}x, \tag{2c}$$

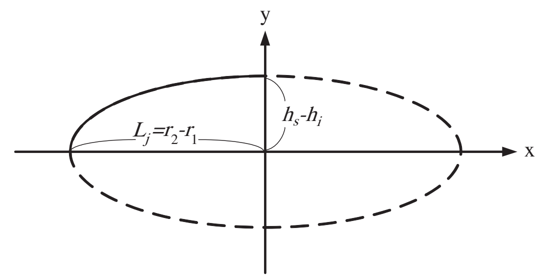

Figure 2. Hypothetical water profile of hydraulic jump.

where r_1 and r_2 stand, respectively, for the radius from the imaginary center to the beginning and the end of the hydraulic jump and h_i and h_s, respectively, the depth of flow before and after the hydraulic jump.

The surface profile of the hydraulic jump is assumed to be quarter-elliptical in form with the horizontal major semi-axis equal to the vertical minor semi-axis (Arbhabhirama and Abella 1971), as shown in Figure 2.

$$\frac{(y-h_i)^2}{(h_s-h_i)^2} + \frac{(x-L_j)^2}{L_j^2} = 1 \tag{3}$$

Substituting Equation (3) into Equation (2c) yields the following:

$$P_{sx} = \gamma(r_2 - r_1)\left(\frac{h_s^2}{3} + 0.118h_s h_i + 0.048h_i^2\right)\sin\theta, \tag{4}$$

$$\int\int v_x(\rho\vec{V}\cdot \mathrm{d}\vec{A}) = \rho V_s^2 h_s r_2 \int_0^{\theta}\cos\theta \mathrm{d}\theta - \rho V_i^2 h_i r_1 \int_0^{\theta}\cos\theta \mathrm{d}\theta = \rho V_s^2 h_s r_2 \sin\theta - \rho V_i^2 h_i r_1 \sin\theta. \tag{5}$$

In addition, the continuity equation is:

$$V_s = \frac{V_i h_i r_1}{h_s r_2}, \tag{6}$$

where V_i and V_s represent, respectively, the flow velocity before and after the hydraulic jump. From the Bernoulli equation, the following parameters can be obtained: the pre- and post-jump cross-sectional width ratio $\mu = r_2/r_1 (= b_s/b_i)$, the conjugate depth ratio $\phi = h_s/h_i$, the pre-jump Froude number $F_i^2 = V_i^2/gh_i$, and the specific weight of water $\gamma = \rho g$.

Substituting the above parameters into Equation (1) and after simplification, we have:

$$2(\mu-1)\left(\frac{\phi^2}{3} + 0.118\phi + 0.048\right) + 1 - \mu\phi^2 = 2F_i^2\left(\frac{1-\mu\phi}{\mu\phi}\right). \tag{7}$$

With the side-pressure correction factor C_p :

$$C_p = \frac{\mu\phi(\mu-1)\left[\mu\left(\frac{\phi^2}{3}+0.118\phi+0.048\right)+\frac{1}{2}\right]}{(\mu\phi-1)}, \quad (8)$$

then $\frac{1}{2}\mu\varphi(1+\mu\varphi) = F_i^2\mu + C_p$.

$$F_i^2 = \frac{1}{\mu}\left[\frac{\mu\phi(1+\mu\phi)}{2} - C_p\right] = \frac{\phi(1+\mu\phi)}{2} - \frac{C_p}{\mu}, \quad (9a)$$

if $\mu = 1,\ b_s = b_i,$ then

$$F_i^2 = \frac{1}{2}[\phi(1+\phi)] \quad (9b)$$

Equation (9a) is transformed into a single variable called an effective Froude number F_e^2

$$F_e^2 = F_i^2\mu + C_p, \quad (10)$$

$$\phi = \frac{1}{2\mu}\left(\sqrt{1+8F_e^2} - 1\right), \quad (11a)$$

if $\mu = 1,$ then

$$\phi = \frac{1}{2}\left(\sqrt{1+8F_i^2} - 1\right). \quad (11b)$$

2.3. *Energy head loss of hydraulic jump in expanding channel*

Applying the energy equations to sections 1 and 2 (Figure 1) gives:

$$\frac{V_i^2}{2g} + h_i = \frac{V_s^2}{2g} + h_s + \Delta E, \quad (12)$$

where ΔE denotes energy dissipated by the hydraulic jump and $\Delta E = E_i - E_s$ where E_i and E_s represent, respectively, the energy before and after the hydraulic jump. From the continuity equation, we have:

$$V_s = V_i y_i r_1/(y_s r_2), \mu = r_2/r_1,$$
$$\phi = h_s/h_i, \text{ and } F_i^2 = V_i^2/gh_i.$$

According to $\Delta E,$ we have the equation of relative energy loss, which is $\Delta E/h_i$:

$$\frac{\Delta E}{h_i} = \frac{F_i^2}{2}\left(\frac{\mu^2\phi^2-1}{\mu^2\phi^2}\right) + (1-\phi), \quad (13a)$$

if $\mu = 1,$ then

$$\frac{\Delta E}{h_i} = \frac{(\phi-1)^3}{4\phi}. \quad (13b)$$

From Equation (10):

$$\frac{\Delta E}{E_i} = \frac{\left(\frac{V_i^2}{2g}+h_i\right) - \left(\frac{V_s^2}{2g}+h_s\right)}{\left(\frac{V_i^2}{2g}+h_i\right)} = 1 - \frac{2\phi + F_i^2\mu^{-2}\phi^{-2}}{2+F_i^2}, \quad (14a)$$

if $\mu = 1,$ then

$$\frac{\Delta E}{E_i} = 1 - \frac{2\phi + F_i^2\phi^{-2}}{2+F_i^2} = \frac{E_i - E_s}{E_i} = 1 - \frac{E_s}{E_i}, \quad (14b)$$

$$\frac{E_s}{E_i} = \frac{2\phi + F_i^2\phi^{-2}\mu^{-2}}{2+F_i^2}; \quad (15a)$$

if $\mu = 1,$ then

$$\frac{E_s}{E_i} = \frac{2\phi + F_i^2\phi^{-2}}{2+F_i^2}. \quad (15b)$$

3. Energy dissipation

Free overfalls from weirs of check dams or drop structures exert strong kinetic energy on the apron, creating supercritical flow. Hydraulic jump occurs when the free overfall flow impacts the apron, affecting the downstream flow of the tailrace. The occurrence of hydraulic jump consumes energy, resulting in energy loss or dissipation. Hence, analyzing the energy dissipation characteristics of hydraulic jumps can shed light on the optimal length of aprons downstream that would have the least impact on tailwater flow. Using the continuity equation and the momentum principles, White (1943) proposed the following:

$$\frac{h_i}{h_c} = \frac{\sqrt{2}}{1.06 + \sqrt{(H_d/h_c) + 1.5}}, \quad (16)$$

where h_i is flow depth before the hydraulic jump, h_c the critical flow depth at the top of the weir, and H_d the weir height.

If the channel downstream is prismatic, then:

$$F_i^2 = \frac{V_i^2}{gh_i} = \left(\frac{h_c}{h_i}\right)^3. \quad (17)$$

$F_i > 1$ denotes a supercritical flow, a condition that would lead to a hydraulic jump.

Moore (1943) further pointed out:

$$h_0 \cong 1.5h_c. \quad (18)$$

where h_0 is flow depth at the top of the weir. If the energy head upstream is $E_0 = h_0 + H_d,$ the relative depth before the hydraulic jump $\psi = h_i/h_c,$ and the relative height of the weir $H = H_d/h_c$, then the energy dissipation efficiency of the hydraulic jump η can be expressed as:

$$\eta = \frac{\Delta E}{E_0} = 1 - \frac{E_s}{E_0} = 1 - \frac{(E_s/E_i)(E_i/h_c)}{(E_0/h_c)}. \qquad (19)$$

E_i/h_c is the relative energy head before the hydraulic jump and can be expressed as:

$$\frac{E_i}{h_c} = \frac{h_i}{h_c} + \frac{1}{2}\left(\frac{h_c}{h_i}\right)^2. \qquad (20a)$$

It can be argued that $E_0 \cong h_0 + H_d$ and $h_0 \cong 1.5h_c$; hence, we have:

$$\frac{E_0}{h_c} = 1.5 + \frac{H_d}{h_c}. \qquad (20b)$$

Substituting Equations (15a), (20a), and (20b) into Equation (19) and after simplification, we have:

$$\eta = 1 - \left(\frac{2\psi + \psi^{-2}}{3 + 2H}\right)\left(\frac{2\phi + F_i^2\phi^{-2}\mu^{-2}}{2 + F_i^2}\right), \qquad (21a)$$

if $\mu = 1$, then

$$\eta = 1 - \left(\frac{2\psi + \psi^{-2}}{3 + 2H}\right)\left(\frac{2\phi + F_i^2\phi^{-2}}{2 + F_i^2}\right). \qquad (21b)$$

4. Analysis

Different parameter values are set for calculation to be made using the equations derived above. The results, thus, obtained are analyzed and discussed in the following to shed light on the energy dissipation characteristics of a hydraulic jump in a gradually expanding channel after a free overfall.

4.1. *Conjugate depth ratio*

Figure 3 shows the variations in conjugate depth ratio ϕ at different pre-jump Froude numbers F_i. As can be observed, there exists a linear relationship between the two; that is, the higher the F_i, the larger the ϕ. On the other hand, ϕ is inversely related to the pre- and post-jump cross-sectional width ratio μ; that is, the higher the μ, the lower the ϕ. In comparison, the conjugate depth ratio of a prismatic channel is smaller than that of non-prismatic ones. In short, post-jump flow depth is affected by the size of the stilling basin. When a hydraulic jump occurs in a non-prismatic channel with increasing channel width ratio, the conjugate depth ratio will be decreasing, meaning that the height of tailwater downstream will also be lower.

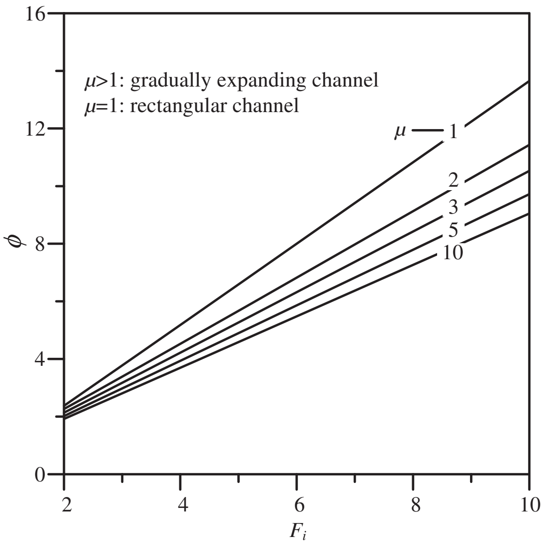

Figure 3. Relationship between ϕ and F_i in different types of channels.

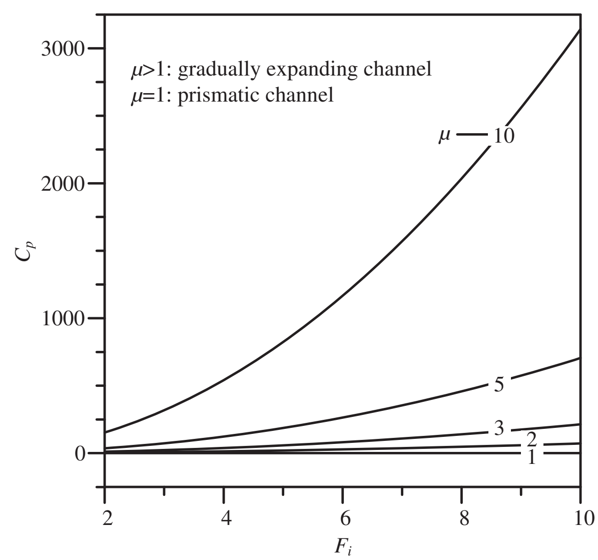

Figure 4. Relationship between C_p and F_i in different types of channels.

Figure 4 shows the variations in side-pressure correction factor C_p at different pre-jump Froude numbers F_i. As can be observed, C_p remains constant ($C_p = 0$) in the scenario of free overfall in a prismatic channel ($\mu = 1$) regardless of changes in F_i. On the contrary, for non-prismatic channels, C_p increases with increasing F_i; and the higher the μ, the larger the increase in C_p. In other words, the side-pressure correction factor increases in an expanding channel; and the larger the channel width ratio, the more

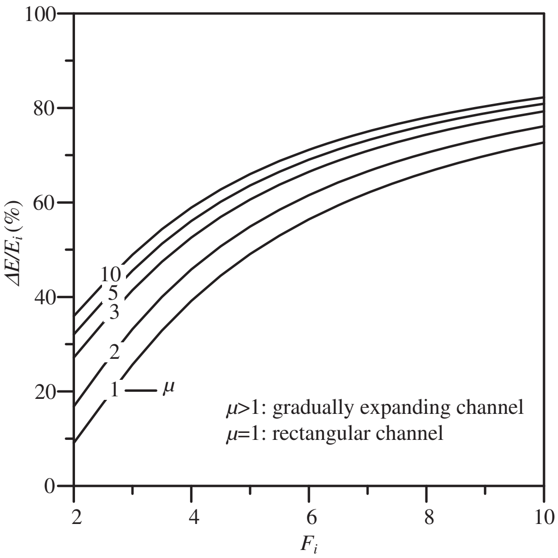

Figure 5. Relationship between $\Delta E/E_i$ and F_i in different types of channels.

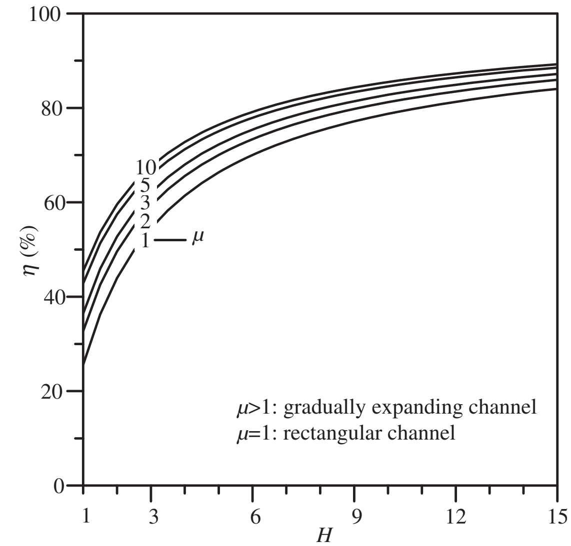

Figure 6. Relationship between η and H in different types of channels.

significant the increase in side-pressure correction factor.

4.2. *Relative energy loss*

Figure 5 shows the variations in relative energy loss $\Delta E/E_i$ at different pre-jump Froude numbers F_i. As can be observed, $\Delta E/E_i$ increases with F_i, and $\Delta E/E_i$ in a prismatic channel is smaller than in non-prismatic ones. Moreover, in expanding channels, the higher the channel width ratio μ, the greater the relative energy loss $\Delta E/E_i$. In other words, more energy is lost or dissipated in a hydraulic jump that occurs in an expanding channel; and the relative energy loss increases with increasing channel width ratio.

4.3. *Energy dissipation of hydraulic jump*

Figure 6 shows the variations in energy dissipation efficiency of the hydraulic jump η at different relative heights of the weir H. As can be observed, η increases with H, and η in a prismatic channel is smaller than that in non-prismatic ones. Moreover, in expanding channels, the higher the channel width ratio μ, the greater the energy dissipation efficiency η. In other words, more energy is dissipated in the hydraulic jump that occurs in an expanding channel; and the energy dissipation efficiency increases with increasing channel width ratio. Indeed, energy is consumed when a hydraulic jump occurs. As expressed in Equation (19), energy is lost or dissipated beginning from upstream to the end of the hydraulic jump; and the total energy loss or dissipation is related to the energy head loss. Hence, it can be concluded that $E_0 \sim h_0 + H_d$ where E_0 is the energy head upstream, h_0 the flow depth at the top of the weir, and H_d the height of the weir (Figure 1).

Table 1. Relationship between H and I in different types of channels.

H	$I_1 = \frac{\eta(\mu=2)}{\eta(\mu=1)}$	$I_2 = \frac{\eta(\mu=3)}{\eta(\mu=1)}$	$I_3 = \frac{\eta(\mu=5)}{\eta(\mu=1)}$	$I_4 = \frac{\eta(\mu=10)}{\eta(\mu=1)}$
2	1.13	1.20	1.31	1.45
3	1.09	1.14	1.21	1.31
4	1.07	1.11	1.16	1.24
6	1.05	1.08	1.11	1.17
8	1.04	1.06	1.09	1.14
10	1.03	1.05	1.07	1.11

Notes: H, relative height of the weir $(=H_d/h_c)$ and I, relative energy dissipation efficiency of the hydraulic jump.

Table 1 presents the relationship between relative height of the weir H and energy dissipation efficiency of the hydraulic jump I. As can be observed, at the same H, I increases with increasing μ. In other words, the relative energy dissipation efficiency is greater at larger channel width ratios.

5. Conclusions

According to the above results, the following conclusions can be drawn.

(1) When a hydraulic jump occurs in a non-prismatic channel with increasing channel

width ratio, the conjugate depth ratio will be decreasing, meaning that the height of tailwater downstream will also be lower.

(2) More energy is lost in the hydraulic jump that occurs in an expanding channel; and the relative energy loss increases with increasing channel width ratio.

(3) The relative energy dissipation efficiency is greater at larger channel width ratios.

(4) Energy dissipation of hydraulic jump after free overfall is higher in an expanding channel compared with a prismatic channel. Hence, future designs of stilling basins can consider having a gradually expanding channel so as to achieve lower level of tailwater.

Acknowledgments

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Nomenclature

A	cross-sectional area of flow (L^2)
b_i	pre-jump half cross-sectional width ($=r_1 \sin\theta$) (L)
b_s	post-jump half cross-sectional width ($=r_2 \sin\theta$) (L)
C_p	side pressure correction factor
E_0	energy head upstream (L)
E_i	pre-jump energy (L)
E_s	post-jump energy (L)
ΔE	energy loss (L)
F_i	pre-jump Froude number
F_e	effective Froude number assuming a quarter-elliptical jump profile
g	gravitational acceleration (LT^{-2})
H	relative height of the weir ($=H_d/h_c$)
H_d	height of the weir (L)
I	relative energy dissipation
h_c	critical depth of flow at the top of the weir (L)
h_i	pre-jump flow depth (L)
h_s	post-jump flow depth (L)
L_d	distance between point of overfall impact and base of weir (L)
L_j	length of the hydraulic jump ($=r_2 - r_1$) (L)
$P_{1x}.P_{2x}$	hydrostatic forces at sections 1 and 2 in x-direction (MLT^{-2})
P_{sx}	side pressure force in x-direction ($P_{sx} = P_s \sin\theta$) (MLT^{-2})
Q	total discharge (L^3T^{-1})
r_1	radius of the beginning of the hydraulic jump from the imaginary center (L)
r_2	radius of the end of the hydraulic jump from the imaginary center (L)
V_i	pre-jump flow velocity (LT^{-1})
V_s	post-jump flow velocity (LT^{-1})
$\bar{V}$	average flow velocity (LT^{-1})
γ	specific weight of water ($ML^{-2}T^{-2}$)
η	energy dissipation efficiency of the hydraulic jump ($=\Delta E/E_0$)
θ	half-angle of divergence
μ	pre-jump and post-jump cross-sectional width ratio ($=r_2 \sin\theta / r_1 \sin\theta$)
ρ	density of water (ML^{-3})
ϕ	conjugate depth ratio ($=h_s/h_i$)
ψ	relative pre-jump depth ($=h_i/h_c$)

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