3. Backward heat equation — *ill-posed problems and regularisation*

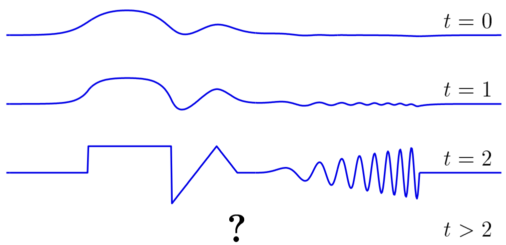
Some physical processes are reversible in the sense that changing the direction of time makes no essential difference. According to the Second Law of Thermodynamics, on the other hand, processes involving randomness generally gain entropy with time and are therefore *irreversible*. The canonical PDE that arises from random processes is the heat equation ($\rightarrow$ *ref*). Thus we can expect that the behaviour will be very different for the *backward heat equation*,

$$\boxed{u_t = -\Delta u.} \qquad (1)$$

The usual way in which (1) may arise in applications is if one is faced with a *terminal value problem* for the ordinary heat equation, in which data are specified at some time t_f and the solution is desired for $t < t_f$. Such problems appear commonly in image enhancement, and t may correspond to a distance as well as a time. What sharp original gave rise to an image degraded by, say, erosion or atmospheric noise or blurred optics? The change of variables $t \rightarrow -t$ converts such a question to an initial-value problem of the form (1)—a heat equation with *negative diffusion*.

Since the heat equation makes a function smoother as t increases, the backward heat equation must make it less smooth. Fig. 1 shows an example. In the figure, we see that the broad hump to the left steepens slowly, then sharpens abruptly to a square pulse at $t = 2$. The narrower hump to its right becomes a triangular pulse at the same moment. Further to the right, an initial small oscillation, barely visible to the eye at $t = 0$, grows by $t = 2$ to a considerable amplitude, with the fastest growth in the higher wave numbers.

Fig. 1: Loss of smoothness



What happens for $t > 2$? Certainly no solution in the ordinary sense is possible, for this would imply that the solution had to have been smooth at the earlier time $t = 2$.

We can quantify these observations by Fourier analysis. For definiteness, consider the 1D problem $u_t = -u_{xx}$ with initial data $u_0(x) \in L^2(\mathbb{R})$ (i.e., $u_0(x)$ is square-integrable). As described in ($\rightarrow$ *ref*), the Fourier transform decomposes u_0 into its components at various wave numbers k,

$$u_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\hat{u}_0(k)\,dk.$$

Each component e^{ikx} then grows independently under (1) at the rate $e^{k^2 t}$, so by superposition, it would seem that we must have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+k^2 t}\,\hat{u}_0(k)\,dk. \qquad (2)$$

Now so long as the integrand of (2) remains in $L^2(\mathbb{R})$, this procedure makes sense and we have indeed constructed a solution to (1). For certain very smooth functions $u_0(x)$, with $\hat{u}_0(k) = O(e^{|k|^{-\alpha}})$ for some $\alpha > 2$, for example, we get a solution in this way for all $t > 0$. More typically, however, the exponentially growing factor $e^{k^2 t}$ renders the integrand of (2) unintegrable after some finite time t_c, and for $t > t_c$, no classical solution exists.

According to a formulation dating to Hadamard, a PDE is *well-posed* if for all initial data in a prescribed space, a unique solution exists and depends continuously on that data. Otherwise it is *ill-posed*. It is clear that the backward heat equation must be ill-posed in $L^2(\mathbb{R})$, since solutions do not exist for all data. Moreover, even if $u_0(x)$ is smooth enough that a solution exists for all $t > 0$, there is no continuous dependence on the initial data. An arbitrarily small perturbation of $u_0(x)$—the addition of any term whose Fourier transform decays more slowly than e^{-Ck^2}—will preclude the existence of a solution for any time $t > 0$. In a word, the backward heat equation is ill-posed because all solutions are instantly swamped by high-frequency noise.

Is the backward heat equation then just a mathematical curiosity? Certainly not. This ill-posed equation arises so naturally in problems of image enhancement, or more generally of undoing the effects of diffusion—*deconvolution*, in the language of signal processing—that it must have scientific meaning if only we tread carefully enough. In fact, there is a thriving field of the study of ill-posed differential and integral equations, and a key technique in this field, going back to Tikhonov, is *regularisation*. If a problem is ill-posed, the idea is to attach certain extra conditions that exclude pathological solutions and render it well-posed. Typically a regularisation parameter is involved—an adjustable notion of "pathological"—and throughout medicine and engineering, in applications as important as ultrasound, MRI, CAT, and PET imaging, these ideas are well developed.

Fig. 2: What did she look like before diffusion?

Alternatively, one may go back and design a different PDE that has the main properties needed for applications but is well-posed from the start, like the Perona-Malik equation ($\rightarrow$ *ref*).

References

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