29. Compacton equations ——— *approximate solitons with compact support*

The KdV equation $u_t = (u^2)_x + u_{xxx}$ ($\to$ *ref*) mixes nonlinear convection with linear dispersion. Its solutions include the famous solitary waves, which pass through one another as solitons with no lasting effect other than a phase shift. These solitary waves are localised in space in the sense that they decay exponentially.

In the early 1990s Rosenau and Hyman, motivated by the formation of patterns in liquid drops ($\to$ *ref*), investigated generalisations of the KdV equation in which the dispersion too is nonlinear. For $m > 0$ and $1 < n \leq 3$, they defined the *compacton equation* $K(m,n)$ by

$$\boxed{u_t + (u^m)_x + (u^n)_{xxx} \;=\; 0.} \tag{1}$$

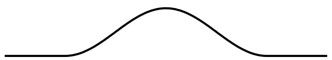
The equation $K(2,2)$, for example, is $u_t + (u^2)_x + (u^2)_{xxx} = 0$. Qualitatively, we can guess what the effect of the restriction $n > 1$ will be. We now have *amplitude-dependent dispersion* which shuts off as $|u| \to 0$. This will tend to eliminate exponentially decaying tails, for in such a tail, the dispersion term that makes the signal spread would be negligible. Thus the possibility arises of traveling wave solutions with *compact support*, i.e., exactly zero value outside a bounded interval.

To calculate a traveling wave solution of (1), we make the substitution $u(x,t) = U(s)$ with $s = x - ct$ for some wave velocity c. After some manipulations the PDE reduces to

$$(U_s)^2 + \frac{U^m}{m+2} - \frac{1}{3}cU - CU^{-2} \;=\; D$$

for arbitrary constants C and D. If we choose $C = D = 0$ we find that there are solutions with compact support for any $c \in R$, known as *compactons*. Here are three of the simplest cases:

$K(2,2): \quad U(s) = (4c/3)\cos^2(s/4), \qquad |s| \leq 2\pi$



$K(3,2): \quad U(s) = (37.5c - s^2)/30, \qquad |s| \leq \sqrt{37.5c}$

$K(3,3): \quad U(s) = \pm\sqrt{3c/2}\cos(s/3), \quad |s| \leq \frac{3}{2}\pi$

Note that as with a KdV soliton, the amplitude of a compacton varies with its velocity. In cases (2,2) and (3,3), however, the width is fixed.

The use of the word "compacton" suggests that these traveling wave solutions to $K(m,n)$ behave like solitons, interacting as particles in the sense that the long-term effect of an interaction is a phase shift and nothing more. Approximately speaking, this is what Rosenau and Hyman discovered numerically. For certain values of m and n, a fast compacton will overtake a slow one, pass through it, and emerge approximately unchanged. More general initial data typically separates into a train of two or more compactons that travel at the appropriate individual speeds. In Figure 1, for example, note how the initial signal, with global support, separates into at least three waves of different amplitudes but the same widths, each with compact support.

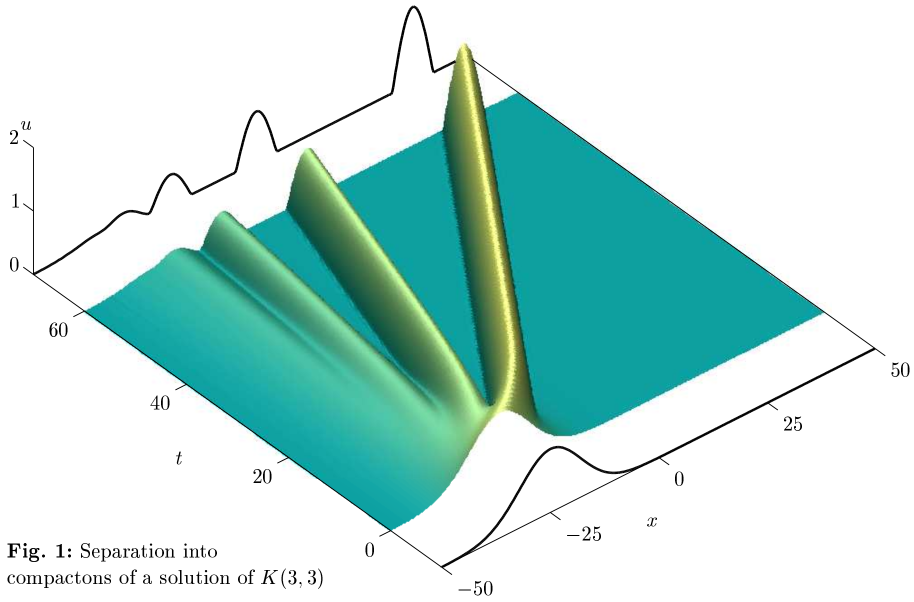


Fig. 1: Separation into compactons of a solution of $K(3,3)$

The pictures are suggestive, but in fact, Rosenau and Hyman found that compacton interactions are not mathematically perfect. Though it is not visible in our figure, a small residue consisting of low amplitude and hence low-velocity "compacton-anticompacton pairs" is left behind after a compacton interaction; the behaviour of compactons as particles is only approximate. This is probably related to the fact that unlike the KdV equation, compacton equations apparently do not possess an infinite set of conservation laws.

Compacton equations are among the hardest one-dimensional PDEs to solve numerically, because they contain terms that are simultaneously nonlinear and of high order. Nevertheless, most of what is known about them is based on numerical experiments; there is little mathematical theory.

References

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