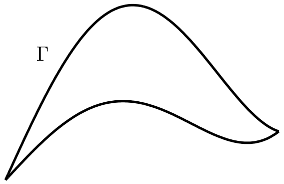
42. Eigenfunctions of the Laplace equation — *can one hear the shape of a drum?*

Earlier we alluded to the eigenfunctions of the Laplace operator in a box or a ball ($\rightarrow$ *ref*). There is a famous converse to this kind of investigation. Suppose we are given an infinite sequence of eigenvalues of the Laplace operator on an unknown two-dimensional region bounded by a curve Γ.



$$\boxed{u_{xx} + u_{yy} = \lambda u} \qquad (1)$$

Is it possible in principle to determine Γ, apart from symmetries such as reflections and rotations?

This is a question of uniqueness. Must two distinct shapes have distinct sets of eigenvalues? That is to say, *Can one hear the shape of a drum?* This memorable problem was posed in 1966 in a paper by Mark Kac in the *American Mathematical Monthly*. In 1968 the paper won the Chauvenet Prize of the American Mathematical Society, and the problem became famous. Wide-ranging research ideas were generated, but no solution.

Finally, in 1992, a solution to Kac's problem was found. The answer is no, one cannot hear the shape of a drum. This was proved by Carolyn Gordon, David Webb, and Scott Wolpert by arguments that, after some consolidation, turned out to be surprisingly elementary. They found counterexamples of great simplicity—distinct polygons with as few as eight sides that could be proved to have identical eigenvalues. Two such regions are said to be *isospectral.* Later authors have established numerous extensions of these results—including simple derivations of isospectral domains via paper-folding. Figure 1 shows a few of these examples, taken from a paper of Buser, Conway, Doyle and Semmler.

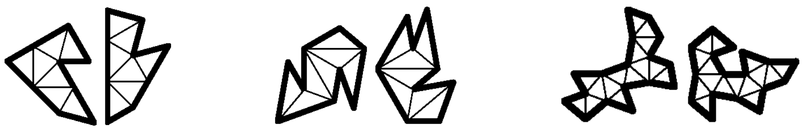


Fig. 1: Some isospectral pairs of domains devised by Buser et al.

Figure 2 illustrates the isospectral property of the simplest original example of the Gordon–Webb–Wolpert "drums" by actually plotting one of the eigenfunctions. Though the isospectrality of these regions can be proved on the back of an envelope, determining the eigenfunctions to high accuracy is a nontrivial numerical problem. The twelve-figure results underlying the upper half this figure are due to Driscoll, who used an algorithm adapted from earlier work of Descloux and Tolley. The trick is to handle the corner singularities exactly and to subdivide the domain according to a kind of domain decomposition; either idea alone is incapable of achieving high accuracy. The lower half of the figure is based on measurements by Sridhar and Kudrolli in microwave cavities carefully engineered to have the cross-sectional shapes of these drums. If the scale in this figure is set so that each of the shortest horizontal or vertical segments is of length 2, then the corresponding eigenvalue is 5.17555935622.

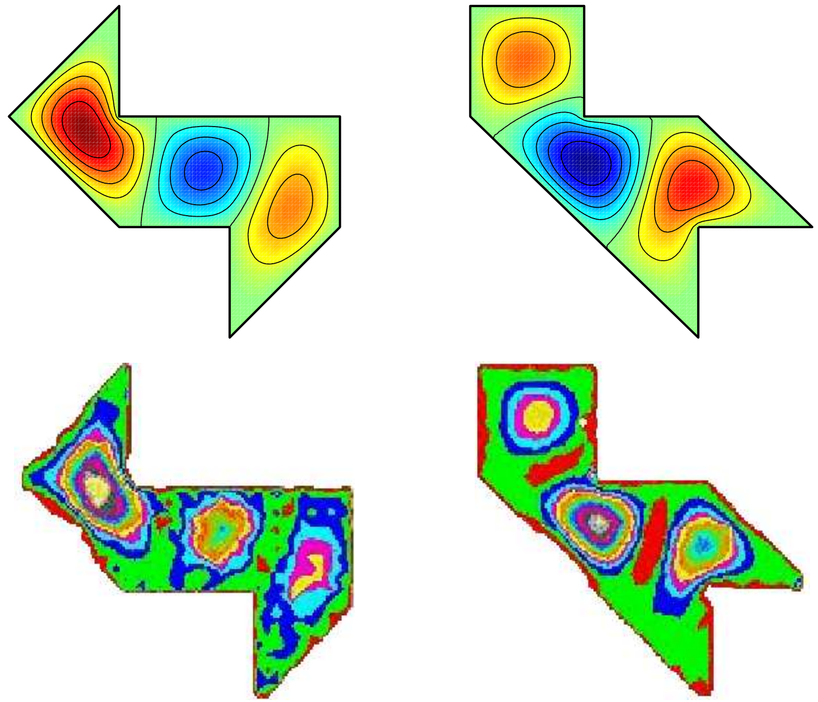


Fig. 2: Third eigenmodes of the Gordon–Webb–Wolpert "isospectral drums", exact and experimental

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