58. Equations of fluid mechanics — *half the world of PDEs*

It is probably fair to say that half the subject of PDEs has its roots in fluid mechanics. Almost every feature that a PDE may possess can be illustrated by some fluid mechanical problem or other, including nonlinearity (convection), small parameters (viscosity), higher order terms (creeping flow), lower order terms (Coriolis forces), shocks (sonic booms), dispersion (water waves) solitons (more water waves), and chaos (turbulence).

One could not compile a book like this one without paying special attention to fluid mechanics, and as mentioned in the Introduction, one of our models has been Van Dyke's classic, *An Album of Fluid Motion.* At the same time one must remember that in principle, the subject of PDEs is much wider than this particular area of application, however large and important it may be. We hope we have achieved a good balance.

The fundamental equations of fluid mechanics are the Navier–Stokes equations. In full generality, these are a nonlinear system of five equations describing viscous, compressible Newtonian flow in three space dimensions and time. In principle, much of fluid mechanics might be derived directly from the Navier–Stokes equations, but in practice, for most problems, simplifications are made. In particular there are four axes along which the equations are routinely varied:

- 1D or 2D or 3D
- Inviscid (Euler) or viscous (Navier–Stokes)
- incompressible or compressible
- steady-state or time-dependent

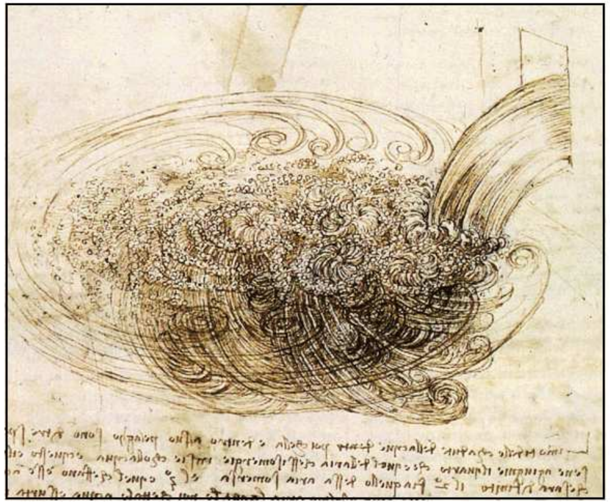

Fig. 1: Sketch by Leonardo, before PDEs were invented

Of course, to say that a fluid is incompressible, say, does not mean that the compressibility is zero, merely that it is small enough to neglect for the application at hand. Air, for example, is effectively incompressible for many applications at low Mach numbers (speeds much less than that of sound). Similarly, both air and water are effectively inviscid for many applications in portions of the domain well separated from boundaries, a fact that many calculations take advantage of by using the Navier–Stokes equations in the boundary layer and the Euler equations outside.

We have just identified $3 \times 2 \times 2 \times 2 = 24$ sets of PDEs derived from the Navier–Stokes equations. In the *PDE Coffee Table Book*, 6 of the 24, all time-dependent, have entries of their own:

Inviscid, incompressible

1D	—
2D	XX: 2D incompressible Euler eqs. . . . vortex dynamics, nonuniqueness, Kutta condition
3D	—

Inviscid, compressible

1D	23: Euler eqs. in 1D . . . a hyperbolic system of conservation laws
2D	XX: 2D compressible Euler eqs. . . . transonic bubbles
3D	—

Viscous, incompressible

1D	—
2D	XX: 2D incompressible Navier–Stokes eqs. . . . boundary layers and separation
3D	XX: 3D incompressible Navier–Stokes eqs. . . . turbulence

Viscous, compressible

1D	—
2D	XX: 2D compressible Navier–Stokes eqs. . . . shock/boundary layer interactions
3D	—

These represent the "main sequence" of fluids-related PDEs, but many other PDEs also arise from fluid mechanics when additional effects are taken into account such as surface tension or non-Newtonian rheology, or when special configuations are at issue such as inclined planes or narrow gaps, or when a special limit is taken such as nearly-zero speed or nearly-zero perturbation from a base flow. Examples in this book include the Boussinesq, Buckley–Leverett, Burgers, Darcy, Hele–Shaw, KdV, Laplace, porous medium, shallow water, thin film and Trouton equations.

References

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