The Korteweg–de Vries equation has a colourful history, which began in 1832 with John Scott Russell's celebrated pursuit on horseback of a solitary wave along the Edinburgh–Glasgow canal:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of great agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

This event represents the first known observation of a solitary wave of permanent shape, and it induced Russell to undertake an extensive experimental study of the "great wave of translation". His critical result was a demonstration that the wave speed c increases with amplitude a according to $c = \sqrt{g(h+a)}$, where h is the depth of the channel and g is the acceleration due to gravity. Boussinesq and Lord Rayleigh independently deduced the shape of the solitary wave in the 1870's,

$$a(x,t) = \alpha \operatorname{sech}^2[\beta(x - ct)], \tag{1}$$

and in 1895, Korteweg and de Vries derived the evolution equation for which (1) is a solution,

$$\boxed{u_t + uu_x + u_{xxx} = 0.} \tag{2}$$

To be precise, for arbitrary $\beta \in \mathbb{R}$, a wave of the form (1) satisfies (2) if α and c are given by $\alpha = 12\beta^2$, $c = 4\beta^2$. Note that the wave speed is exactly proportional to the height.

The KdV equation represents the simplest combination of nonlinearity and dispersion in the absence of dissipation. In a solitary wave solution, the steepening effect of the nonlinear term is perfectly balanced by the spreading effect of the dispersive term.

Owing to its nonlinearity, the KdV equation resisted analysis for many years, and it did not come under serious scrutiny until 1965, when Zabusky and Kruskal obtained numerical solutions while investigating the Fermi–Pasta–Ulam problem of masses coupled by weakly nonlinear springs. Hoping to illuminate the phenomenon of recurrence, in which the system evolved to a state close to initial conditions, they discovered that an even more remarkable effect was at play. From a general initial condition, a solution to (2) develops into a series of solitary pulses of the form (1) of varying amplitudes, which pass through one another without modification of shape or speed. The only lingering trace of the strong nonlinear interaction between these so-called *solitons* is a slight forward phase shift for the larger, faster wave and slight backward shift for the smaller, slower one. Soliton solutions have subsequently been discovered for a large number of nonlinear equations, including equations of particle physics, laser physics, and magnetohydrodynamics.

Having yielded a startling new type of wave behaviour, the KdV equation soon stimulated a further mathematical development, when in 1967, Gardner, Greene, Kruskal and Miura introduced the *inverse scattering transform* method for determining the solitons that arise from arbitrary initial conditions. This technique represented a major advance in the mathematical theory of PDEs, as it made it possible to obtain closed-form solutions to nonlinear evolution equations that were previously beyond the reach of analysis. This breakthrough initiated a period of rapid developments both in describing the properties of KdV solitons and in generalising the approach to other nonlinear equations including the sine–Gordon, nonlinear Schrödinger, and Boussinesq equations (→ *ref,ref,ref*).) In a paper that contributed to both endeavours, Lax derived general conditions for the applicability of the inverse scattering transform and demonstrated that the interactions between pairs of KdV solitons fall into three distinct categories depending on the ratio of the amplitudes. In the three cases, the maxima behave differently during the interaction, as illustrated in Figure 1.

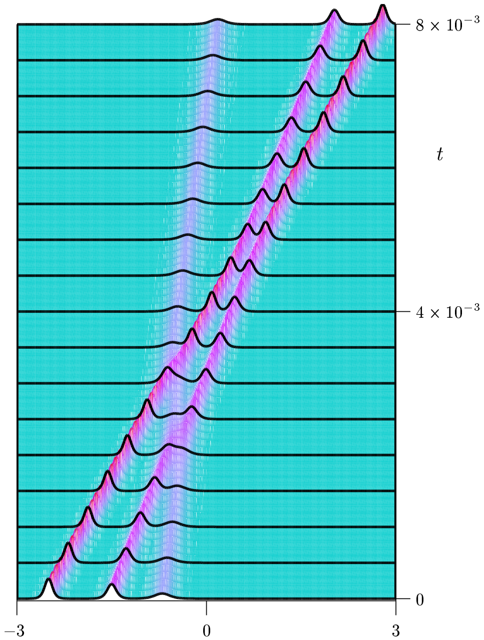

Fig. 1: Soliton interactions

References

M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge University Press, 1991.

P. Drazin and R. Johnson, *Solitons: an introduction*, Cambridge University Press, 1990.

E. Fermi, J. Pasta, and S. Ulam, *Studies of nonlinear problems*, Los Alamos Report LA-1940, (1955).

B. Fornberg and G. Whitham, *A numerical and theoretical study of certain nonlinear wave phenomena*, Phil. Trans. Roy. Soc. A, **289** (1978), 373–404.

C. Gardner, J. Greene, M. Kruskal, and R. Miura, *Method for solving the Korteweg–deVries equation*, Phys. Lett. Rev., **19** (1967), 1095–1097.

D. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag. (Ser. 5), **39** (1895), 422–443.

P. D. Lax, *Integrals of nonlinear equations of evolutionary and solitary waves*, Comm. Pure Appl. Math., **21** (1968), 467–490.

J. Russell, *Report on waves*, Rep. 14th Meet. Brit. Assoc. Adv. Sci., (1844), 311–390.

N. Zabusky and M. Kruskal, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*, Phys. Lett. Rev., **15** (1965), 240–243.