The Klein–Gordon equation dates to the legendary early period of development of quantum mechanics in the 1920s. Though physically problematic, it is mathematically intriguing as an equation that is both hyperbolic and dispersive.

The equation originated with attempts to combine relativity theory and quantum mechanics for the description of high energy particles. Toward this end, Oskar Klein and Walter Gordon independently proposed a model for motion of a charged spinless particle in an electromagnetic field. Start with the energy relation of special relativity, $E^2 = m^2 + p^2$, where E is energy, m is mass, and p is momentum. Following the first postulate of quantum mechanics, make the operator substitutions $E \to \mathrm{i}\partial/\partial t$, $p \to -\mathrm{i}\nabla$, and set $m = 1$ for simplicity. The result is the *Klein–Gordon equation*,

$$\boxed{u_{tt} = \nabla^2 u - u,} \tag{1}$$

where $u = u(x, t)$ is the quantum mechanical state function (usually denoted ψ).

In the limit of vanishing mass, the undifferentiated term of (1) would be zero and we would have the standard wave equation ($\to$ *ref*). Even for nonzero mass, the Klein–Gordon equation can be tackled by a number of analytical techniques. It is a linear second-order hyperbolic equation, and it can be solved explicitly through separation of variables or Fourier integrals, and Green's function methods lead to solutions in appropriate geometries in terms of Bessel functions. Such approaches generally do not lead to closed-form solutions, however, and asymptotic or numerical techniques must be used to glean an understanding of the equation's behaviour.

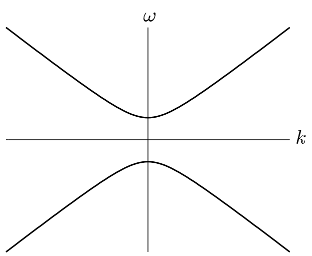

Fig. 1: Dispersion relation

For simplicity let us restrict our attention to one space dimension. Then (1) becomes

$$u_{tt} = u_{xx} - u, \tag{2}$$

and Fourier analysis shows that for any $k \in \mathbb{R}$, the equation admits the wave solution $\exp(i(\omega t + kx))$ provided that ω satisfies

$$\omega = \pm\sqrt{1 + k^2}. \tag{3}$$

This *dispersion relation* is illustrated in Figure 1, to be compared with Figure 3 of ($\to$ *Wave1D*). In the limit $k \to \infty$, the behaviour is that of the wave equation, but for finite k the behaviour is dispersive, with wave crests traveling at a *phase velocity* that depends on k,

$$c = -\frac{\omega}{k} = \pm\sqrt{1 + k^{-2}}.$$

Note that as $k \to 0$, the phase velocity c diverges to ∞. How can this be, since the equation is hyperbolic, with characteristics at slopes ± 1 and thus no information propagating faster than speed 1?

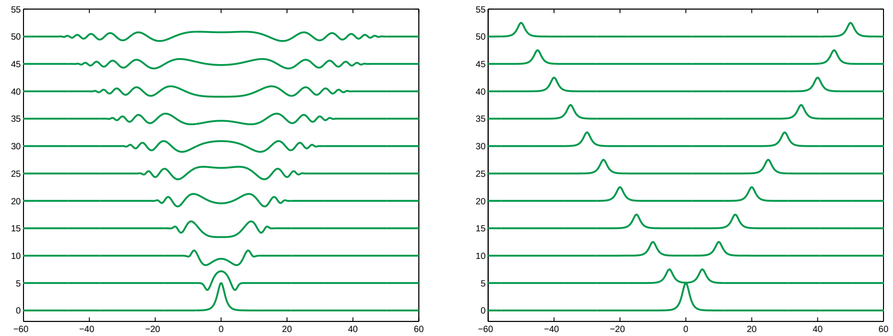

Fig. 1: Klein–Gordon (left) and wave equations (right)

The answer is that under a dispersive wave equation, the energy or information associated with each wave number k propagates not at the phase velocity but at the *group velocity*, defined by

$$c_g = -\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\pm 1}{\sqrt{1 + k^{-2}}}.$$

This formula shows that like c, c_g depends on k, but it is never greater than 1. The significance of c_g can be seen in Figure 1, which compares the evolution of an initial pulse $u(x, 0) = \mathrm{sech}(x)$ under the Klein–Gordon equation to the evolution of the same pulse under the wave equation. The high wave numbers propagate at velocities $\approx \pm 1$, but lower wave numbers move more slowly. For example, along the lines corresponding to $c_g = \pm 1/2$ we find wave forms with local wave number $k = \pm 1/\sqrt{3}$.

Though the Klein–Gordon equation turned out not to play the central role in quantum mechanics that its originators had hoped for, it appears occasionally in other applications. For example, it can be utilised as a model of a wave moving in an elastic medium—a string embedded in a thin rubber sheet, for instance. The dispersive behaviour on the left half of Figure 1 can be interpreted in this light.

References

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