The Kuramoto–Sivashinsky equation in one space dimension is given in two standard forms. The first is

$$v_t + \tfrac{1}{2}(v_x)^2 = -v_{xx} - v_{xxxx}, \qquad (1)$$

with an L-periodic initial condition commonly prescribed for some $L > 0$. The second is obtained by differentiating (1) with respect to x and setting $u = v_x$. This gives

$$\boxed{u_t + uu_x = -u_{xx} - u_{xxxx}.} \qquad (2)$$

Since u is the derivative of a periodic function, the initial condition for (2) is customarily taken to satisfy

$$\int_0^L u(x,0)\,\mathrm{d}x = 0. \qquad (3)$$

The Kuramoto–Sivashinsky equation has similarities with Burgers' equation (→ *ref*) but its behaviour is far more complicated and interesting because of the presence of both second and fourth order spatial derivatives. The sign of the second derivative term is such that it acts as an energy source and thus has a destabilising effect. The nonlinear term uu_x, however, transfers energy from low to high wave numbers where the stabilising fourth derivative term dominates. This crucial distinction between small and large wave numbers is illustrated by the dispersion relation for the linear part of the Kuramoto–Sivashinsky equation, $\mathrm{i}\omega = k^2 - k^4$, shown in Figure 1.

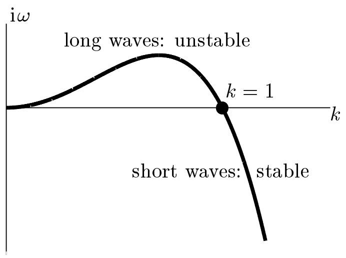

Fig. 1: Dispersion relation for the linear part of (2)

The Kuramoto–Sivashinsky equation dates to the mid-1970s. The first derivation was by Kuramoto in the study of reaction-diffusion equations modelling the Belousov–Zabotinskii reaction (→ *ref*). The equation was also developed by Sivashinsky in higher space dimensions in modelling small thermal diffusive instabilities in laminar flame fronts and in small perturbations from a reference Poiseuille flow of a film layer on an inclined plane. In one space dimension it is also used as a model for the problem of Bénard convection in an elongated box, and it may be used to describe long waves on the interface between two viscous fluids and unstable drift waves in plasmas.

It has been proven that for all L-periodic initial data, a unique solution to (2) exists and remains bounded as $t \to \infty$; the size of the solution is known to be no greater than $O(L^{8/5})$. The details of the behaviour of solutions are highly varied, often exhibiting temporal chaos, depending on the amplitude of the initial data and on L. Various families of travelling wave solutions exist, including periodic, quasi-periodic, solitary and shock-like solutions. Even though the solutions are quite complicated, the long-time behaviour has a great deal of order, tending to feature structures on a length scale $O(2\pi)$ since that is the scale of the crossover between unstable long waves and stable short ones. For equation (2) the existence of an asymptotically complete inertial manifold attracting all initial data in the limit as $t \to \infty$ has been proven. This means that in principle, as $t \to \infty$, the behaviour of solutions on an interval of length L becomes effectively finite-dimensional; it is known that a dimension $O(L^{2.89})$ is sufficiently large.

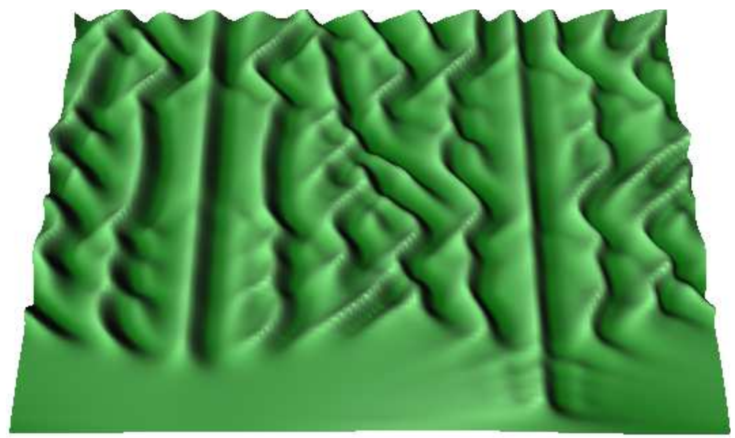

Fig. 2: Chaotic structure emerging from smooth initial data

Figure 2 is a plot in the (x,t) plane of a periodic solution to (2) on an interval of length $L = 32\pi$. The initial function was a sinusoidal wave,

$$u(x,0) = \cos(x/16)\,(1 + \sin(x/16))\,.$$

The plot demonstrates how this simple initial data evolves into something much more complicated. Solutions are attracted exponentially towards the inertial manifold, and a characteristic pattern soon emerges. The solution shown, although computed numerically, is fully converged and unaffected, to plotting accuracy, by rounding or discretisation errors.

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