7. Laplace equation — *the basis of potential theory*

The Laplace equation is so important that functions that satisfy it have a special name: they are said to be *harmonic*. The equation is

$$\boxed{\Delta u = 0,} \qquad (1)$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ is the operator known, inevitably, as the *Laplacian*. Other standard notations for the Laplacian of u are $\nabla^2 u, \nabla\cdot\nabla u$, and $\operatorname{div}\operatorname{grad} u$. The Laplacian operator and thus the Laplace equation are *isotropic*, that is, invariant with respect to rotations of space.

There is even a name for the field of study of Laplace's equation—*potential theory*—and this name gives a hint as why the equation is so important. Throughout the sciences, a *potential* is a scalar function of space whose gradient, a vector, represents a field that is divergence- and curl-free. As a consequence, Laplace's equation arises in the description of all kinds of conservative physical systems in equilibrium. For example, if u is temperature, then ∇u is the temperature gradient, which is associated with flow of heat; if $\Delta u = 0$, then energy is conserved and the heat fluxes balance, so u is independent of time. For another example, if u is gravitational (or electrostatic) potential, then ∇u is the gravitational (electric) field, and the equation $\Delta u = 0$ expresses the condition of conservation of energy for a massive (charged) particle in a region free of other masses (charges).

Laplace investigated his eponymous differential equation in several papers in the 1780s and made it famous with his treatise *Mécanique Céleste* some years later. The equation had already been studied by Euler in 1752 and Lagrange and others, however, and the essential ideas of potential theory have their roots with Newton in the 17th century.

A *fundamental solution* of Laplace's equation is a function u that satisfies (1) everywhere in space except at a single point, where the behaviour is that of a delta function, $\Delta u = \delta$. For any dimension $d \ge 3$, the fundamental solution with singularity at the origin is

$$u(r) = C_d r^{2-d}, \qquad (2)$$

where $r = (x_1^2 + \cdots + x_d^2)^{1/2}$ and $C_d = \pi^{-d/2}\Gamma(d/2)/(4-2d)$. (For $d = 2$ it is $u(r) = (\log r)/2\pi$.) In particular, the potential associated with a point mass or charge has the form r^{-1} in 3D. The nucleus of an atom and the sun in our solar system are perhaps the two most familiar electrostatic and gravitational examples, respectively.

Laplace's equation is the classic example of an *elliptic* PDE. This means that it has no characteristics, and one typically encounters the problem of satisfying (1) in the interior of a domain subject to one boundary condition at each point along the boundary. If the boundary data are function values, this is a *Dirichlet problem*, and if they are normal derivatives, it is a *Neumann problem*. Solutions to such problems are unique and infinitely differentiable—in fact, real analytic. Provided the domain is connected, the solution depends at every point on all of the boundary data.

For certain domains of special forms, solutions to Laplace's equation can be obtained by separation of variables. In the three-dimensional box $[0,\pi]^2$, for example, one solution is $\sin(mx)\sin(ny)\exp(kz)$ for any integers m and n and $k^2 = m^2 + n^2$, and general boundary data can be handled by expansion in series of these and similar functions. In a spherical ball in three dimensions, separation of variables leads to solutions known as *solid harmonics*, and their restrictions to the sphere are known as *spherical harmonics*, illustrated in Figure 1.

Laplace's equation is also the classic example of a PDE whose solutions satisfy a variational principle. Given a function u defined in a domain Ω in d-space and satisfying Dirichlet boundary conditions on the boundary, the associated *Dirichlet integral* is

$$\int_\Omega (\nabla u)\cdot(\nabla u)\mathrm{d}x = \int_\Omega (\frac{\partial u}{\partial x_1})^2 + \cdots + (\frac{\partial u}{\partial x_d})^2 \,\mathrm{d}x_1 \ldots \mathrm{d}x_d. \qquad (3)$$

Under reasonable assumptions, it can be shown that there is a unique u that minimises (3), and it is precisely the solution of Laplace's equation with the given boundary data. This method of construction, known as the *Dirichlet principle*, is the starting point of the *finite element method*, the preeminent technique for solving elliptic PDE numerically.

Harmonic functions satisfy the *maximum principle*: the maximum of any harmonic function u in a domain Ω is achieved on the boundary of Ω. (Symmetry gives us also a *minimum principle*.) If the maximum is also achieved in the interior, then u must be constant throughout Ω, and according to *Liouville's theorem*, if a function harmonic on all of $\mathbb{R}^d$ is bounded, then it must be constant.

Harmonic functions also satisfy the *mean value property*: the value at any point is equal to the mean of all the values on any sphere centered there. (Conversely, a function that satisfies the mean value property must be harmonic.) The value at a point inside a sphere that is not the center can be obtained by an integral known as *Poisson's formula*. By consideration of appropriately weighted means, this idea can be generalised to means over non-spherical surfaces, and this is the starting point of the subject of *integral equations*. Further pursuit of connections between behaviour of potentials in a volume and on a boundary surface leads to *Gauss's Law*, *Stokes' Theorem*, *Green's theorems*, and much more.

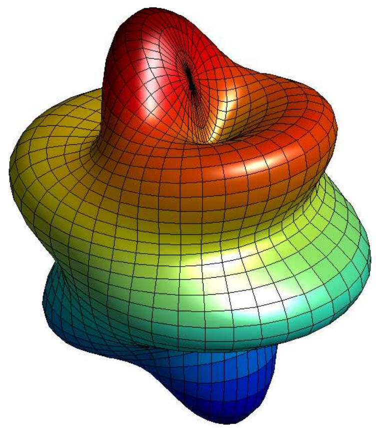

Fig. 1: The $(6,1)$ spherical harmonic $Y_{1,6} = (\cos\phi)P_6^1(\cos\theta)$

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