3. Constant coefficient linear equations

Fourier analysis and boundedness

The most basic of all problems involving partial differential equations are linear PDEs with constant coefficients posed on unbounded domains. Such problems are translation-invariant, and as a result, their solutions can be found by the Fourier transform.

For example, here are three linear constant-coefficient equations in one space variable:

$$u_t = u_x, \qquad u_t = -u_{xx} - u_{xxxx}, \qquad u_t = u_{xxxx}. \tag{1}$$

Inserting the ansatz $u(x,t) = \exp(\mathrm{i}kx + f(k)t)$ gives a relation between k and $f(k)$—the *dispersion relation*,

$$f(k) = \mathrm{i}k, \qquad f(k) = k^2 - k^4, \qquad f(k) = k^4.$$

The corresponding solutions for real k are

$$u(x,t) = e^{\mathrm{i}kx + \mathrm{i}kt}, \qquad u(x,t) = e^{\mathrm{i}kx + (k^2-k^4)t}, \qquad u(x,t) = e^{\mathrm{i}kx + k^4 t}. \tag{2}$$

Fourier analysis tells us that in the space L^2 defined by the norm $\|u\| = (\int_{-\infty}^{\infty} |u(x)|^2 \,\mathrm{d}x)^{1/2}$, all solutions to (1) can be obtained as superpositions of the solutions (2):

$$u(x,t) \;=\; \int_{-\infty}^{\infty} \hat{u}(k,t) e^{\mathrm{i}kx}\,\mathrm{d}k \;=\; \int_{-\infty}^{\infty} \hat{u}(k,0) e^{\mathrm{i}kx + f(k)t}\,\mathrm{d}k, \tag{3}$$

where $\hat{u}(k,t)$ denotes the Fourier transform of $u(x,t)$ with respect to x. In other words, $\hat{u}(k,t)$ evolves for each k according to the trivial ordinary differential equation $\hat{u}_t = f(k)\hat{u}$ with solution $\hat{u}(k,t) = \exp(f(k)t)\hat{u}(k,0)$. Thus we see that for linear equations with constant coefficients on unbounded domains, when we take the Fourier transform,

- *Differential operators become polynomials in k*, and
- *The PDE becomes an uncoupled system of ODEs, one ODE for each k.*

In various entries of this book, we will consider the significance of dispersion relations for wave propagation (→ *refs*). Here, instead, we consider the even more basic issue of *boundedness*. Given a linear constant-coefficient PDE of the form (1), does there exist a constant C such that

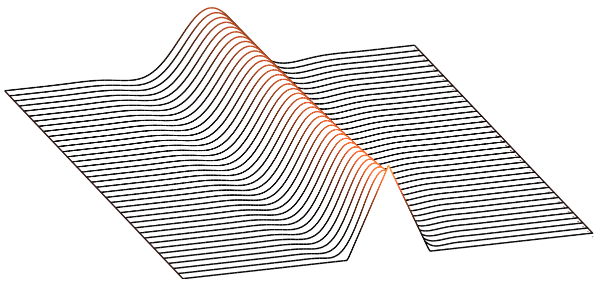

Fig. 1: $u_t = -u_{xxxx}$

$$\|u(t)\| \;\le C\,\|u(0)\| \tag{4}$$

uniformly for all initial data $u(0) = u(x,0)$ and all $t > 0$?

For the examples it is clear how to answer this question. Since $|\exp(\mathrm{i}kt)| = 1$ for all $k \in \mathbb{R}$, the equation $u_t = u_x$ has $\|u(t)\| = \|u(0)\|$ for all $t > 0$. Its solutions $u(x,t) = u_0(t+x)$ satisfy (4) with $C = 1$. The solution $\exp(\mathrm{i}kx + (k^2 - k^4)t)$ of the second equation of (1), on the other hand, grows unboundedly for $0 < |k| < 1$. The maximum growth rate is $\exp(t/4)$, attained with $|k| = 1/\sqrt{2}$, and thus $\|u(t)\| \le \exp(t/4)\|u(0)\|$. The third equation is more explosively unstable. Now, the solutions $u(x,t) = \exp(\mathrm{i}kx + k^4 t)$ not only grow unboundedly but do so unboundedly fast as $|k| \to \infty$. Thus $u_t = u_{xxxx}$ is *ill-posed*, for it lacks the well-posedness property that unique solutions exist for any initial data and depend continuously on that data.

All this carries over to equations in several space variables. For example, the PDEs

$$u_t = u_x + u_y, \qquad u_t = u_{xx} + u_{yy}, \qquad u_t = u_{xx} + u_{xy}$$

have Fourier transforms

$$\hat{u}_t = (\mathrm{i}k_x + \mathrm{i}k_y)\hat{u}, \qquad \hat{u}_t = (-k_x^2 - k_y^2)\hat{u}, \qquad \hat{u}_t = (-k_x^2 - k_x k_y)\hat{u}.$$

Are their solutions bounded in the sense of (4), where the L^2 norm is now defined by an integral over x and y? By considering all values $k_x, k_y \in \mathbb{R}$ we see that the answer is yes for the first two, with $C = 1$, but no for the third, since $-k_x k_y > 0$ when k_x and k_y have opposite signs.

Now at last we can write down the general equation that is the subject of this page of *The PDE Coffee Table Book.* On the domain $\mathbb{R}^n$, the equation is

$$\boxed{u_t = p(D)u,} \tag{5}$$

where $p(D)$ denotes a linear constant-coefficient differential operator with respect to the variables $x_1, \ldots, x_n$. The Fourier transform of (5) is the $\mathbf{k}$-dependent system of ODEs

$$\hat{u}_t \;=\; f(\mathbf{k})\hat{u} \;=\; p(i\mathbf{k})\hat{u}. \tag{6}$$

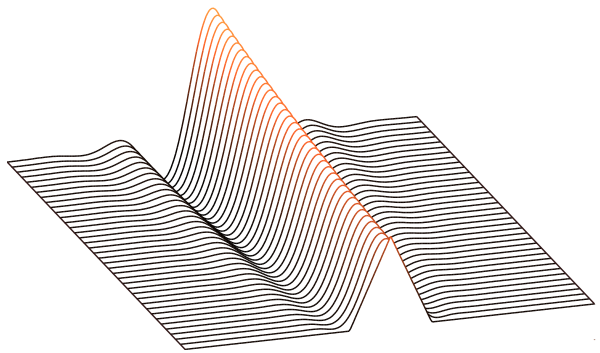

Fig. 2: $u_t = -u_{xx} - u_{xxxx}$

The function p is a polynomial in n variables; for example we might have $p(D) = D_1 D_2^2 - 2D_3^5$, corresponding to $p(D)u = u_x u_{yy} - 2u_{zzzzz}$ and $p(i\mathbf{k}) = -ik_1k_2^2 - 2ik_3^5$. The criterion for bounded solutions becomes $\mathrm{Re}\,p(i\mathbf{k}) \le 0$. In other words, *solutions to (5) satisfy (4) if and only if $p(i\mathbf{k})$ maps $\mathbb{R}^n$ into the closed left half of the complex plane.*

References

F. John, *Partial Differential Equations*, 4th ed., Springer-Verlag, 1982.

J. Rauch, *Partial Differential Equations*, Springer-Verlag, 1991.