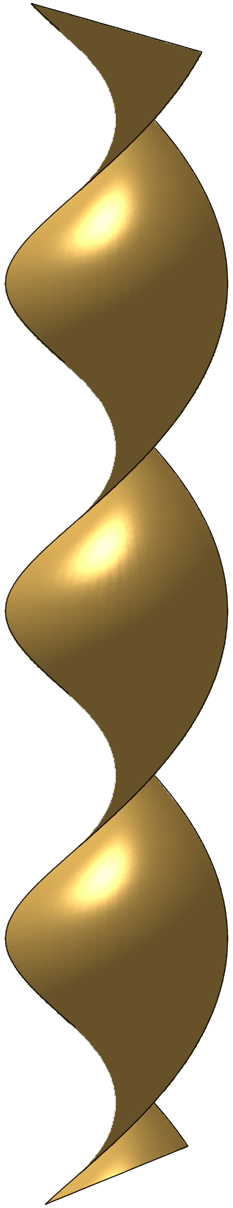

Fig. 1: helicoid

Consider a smooth surface in $\mathbb{R}^{n+1}$ representing the graph of a function $x_{n+1} = u(x_1, \ldots, x_n)$ defined on a bounded open set Ω in $\mathbb{R}^n$. Assuming that u is sufficiently smooth, the area of the surface is given by the nonlinear functional

$$\mathcal{A}(u) = \int_\Omega \left(1 + |\nabla u|^2\right)^{1/2} \mathrm{d}x_1 \ldots \mathrm{d}x_n, \tag{1}$$

where ∇u is the gradient vector $(\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ and $|\nabla u|^2 = (\nabla u) \cdot (\nabla u)$.

The *minimal surface problem* is the problem of minimising $\mathcal{A}(u)$ subject to a prescribed boundary condition $u = g$ on the boundary $\partial\Omega$ of Ω. To do this, we consider the set $\mathcal{U}_g$ of all (sufficiently smooth) functions defined on $\bar{\Omega}$ that are equal to g on $\partial\Omega$. A classical result from the calculus of variations asserts that if u is a minimiser of $\mathcal{A}(u)$ in $\mathcal{U}_g$, then it satisfies the Euler–Lagrange equation

$$\boxed{\nabla \cdot \left(\nabla u \Big/ (1 + |\nabla u|^2)^{1/2}\right) = 0.} \tag{2}$$

This quasi-linear elliptic PDE is known as the *minimal surface equation*.

In the case $n = 2$, (2) can be written as

$$(1 + |u_x|^2)u_{yy} - 2u_x u_y u_{xy} + (1 + |u_y|^2)u_{xx} = 0. \tag{3}$$

Trivially, the plane $u = Ax + By + C$ is a solution to this equation. Non-trivial solutions, including the *helicoid* and the *catenoid*, defined by

$$u = \tan^{-1}(y/x), \qquad u = \frac{1}{a}\cosh^{-1}(a\sqrt{x^2 + y^2})$$

respectively, where a is a constant, were discovered over 200 years ago (Figs. 1 and 2). In fact, the catenoid arose in Euler's treatise *Methodus inveniendi lineas curvas maximi minimive proprietate graduentes* (1774), which laid the foundations of the calculus of variations. The helicoid is the only non-trivial solution to (3) that is a harmonic function, and the catenoid is the only non-trivial solution that is a surface of revolution. The Scherk surface

$$u = \frac{1}{a}\log\frac{\cos ay}{\cos ax},$$

discovered in 1834, where a is again a constant, is the only non-trivial solution to (3) that can be written in the form $u(x, y) = f(x) + g(y)$.

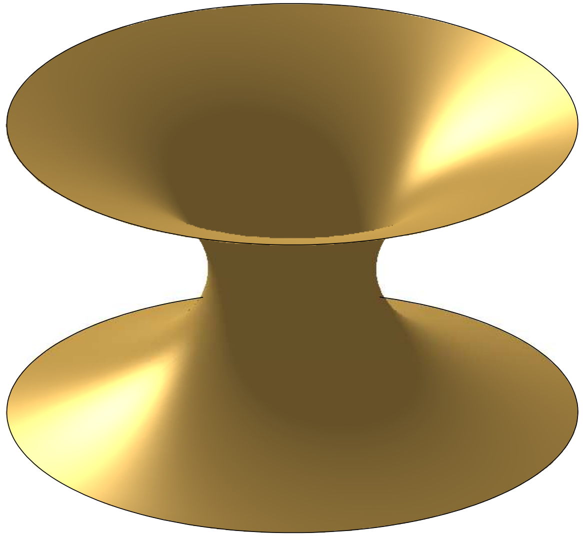

Fig. 2: catenoid

The minimal surface problem is also known as the classical *Plateau problem*, after the Belgian physicist Joseph Antoine Ferdinand Plateau (1801–1883), who experimented by dipping wire contours into solutions of soapy water and glycerine. Plateau did not have the mathematical skills to investigate the problem of existence of a minimal surface theoretically. Subsequently, Weierstrass, Riemann and Schwarz worked on this question, but it was not until the early 1930s that the problem was finally solved, in a much more general form, by Douglas and Radó. In general, there may be one, several or no minimal surfaces spanning a given closed curve σ in space. The theorem of Douglas and Radó asserts that as long as σ is a closed Jordan curve in $\mathbb{R}^{n+1}$ which is the boundary of at least one surface with $\mathcal{A}(u) < \infty$, there exists a solution.

The study of minimal surfaces is an exciting and active field of research which exploits a broad spectrum of mathematical tools from the theory of differential equations, Lie groups and Lie algebras, homologies and cohomologies, bordisms and varifolds as well as a range of computer visualisation and animation techniques. Physically, these surfaces have relevance to a variety of problems in materials science and civil engineering. Of course, they are most familiar as soap films—but not soap bubbles, for a bubble encloses air at a higher pressure than atmospheric, and this changes (2) to the *capillary surface equation* (→ *ref*), with a nonzero right-hand side.

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