23. Gray–Scott equations — *stripes and spots, waves and rolls*

Patterns are everywhere in nature. Examples include spots on butterflies, stripes on zebras, triangles on seashells, ripples on sand dunes, clouds in the sky, waves in the ocean, chemical waves, fingerprints, convection cells in fluids, and pulses in a neuron. Patterns may be regular or quasi-regular or random; they may be stationary or quasi-stationary or rapidly moving. Where do these structures come from? In many cases, they arise as solutions to *reaction-diffusion equations,* that is, systems of PDEs that combine linear diffusion with nonlinear interactions.

In this book we consider seven pattern-forming equations individually: the Fisher, Allen–Cahn, Cahn–Hilliard, Field–Noyes, Fitzhugh–Nagumo, Hodgkin–Huxley, and XXX equations (→ *refs*). In each case we discuss the details of how diffusion and reaction interact to generate certain behaviours. Yet in examining these special cases, there is a risk of losing sight of the great variety of further behaviours that reaction-diffusion equations may also give rise to. With this in mind, we present here first a relatively complicated example in which the point is precisely to exhibit variety.

The *Gray–Scott equations* were formulated originally by Gray and Scott in 1983; we shall not discuss their original chemical motivation:

$$\boxed{u_t = \epsilon_1 \Delta u - uv^2 + F(1-u), \qquad v_t = \epsilon_2 \Delta v + uv^2 - (k+F)v.} \tag{1}$$

These are a coupled pair of equations in two independent variables, u and v. Each variable diffuses independently, and we fix the diffusion constants at $\epsilon_1 = 2\times 10^{-5}$ and $\epsilon_2 = 10^{-5}$. Each also grows or decays independently according to the linear term $F(1-u)$ or $-(k+F)v$. The coupling between the two equations is provided by the third term, $\pm uv^2$, which transfers energy nonlinearly from u to v. The parameters k and F are arbitrary positive numbers that we will adjust.

The constant state $u=1$, $v=0$ is a steady solution to (1), and for any $k, F > 0$, this state is stable with respect to infinitesimal perturbations. A back-of-the-envelope calculation shows that there is also a second constant steady state $u = F/(k+2F)$, $v = \sqrt{k+F}$, which is stable for some values of k and F and unstable for others. The question is, how do non-constant solutions evolve? The most interesting answers appear for parameter choices where the second steady solution is close to neutrally stable.

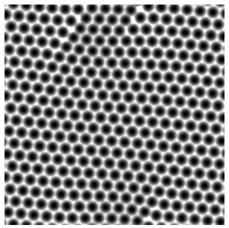

Fig. 1a. $k = 0.65$, $F = 0.3$

Figure 1a illustrates a typical pattern generated by (1) for one choice of parameters. The snapshot shows values of u, with white corresponding to $u = 1$. We see regular dots in a hexagonal array interrupted by occasional dislocations. In Figure 1b, with different parameters, the favoured structures are interlocking black and white channels. Both of these patterns move in time.

Fig. 1b. $k = 0.65$, $F = 0.6$

Figure 2 shows a whole range of parameter choices all at once, indicating the typical patterns exhibited for each choice. This beautiful image was generated by Roy Williams of the California Institute of Technology with the aid of many hours of supercomputer time, based on a 1993 paper of Pearson. One can interpret it as a map of the zoo, showing what fauna inhabit each region of k-F space. Mathematically, it is a solution at large t of the equation (1), except that k and F are chosen to vary gradually in space. The red and yellow dots correspond to still and movie images available online at Williams' 'Xmorphia' Web site.

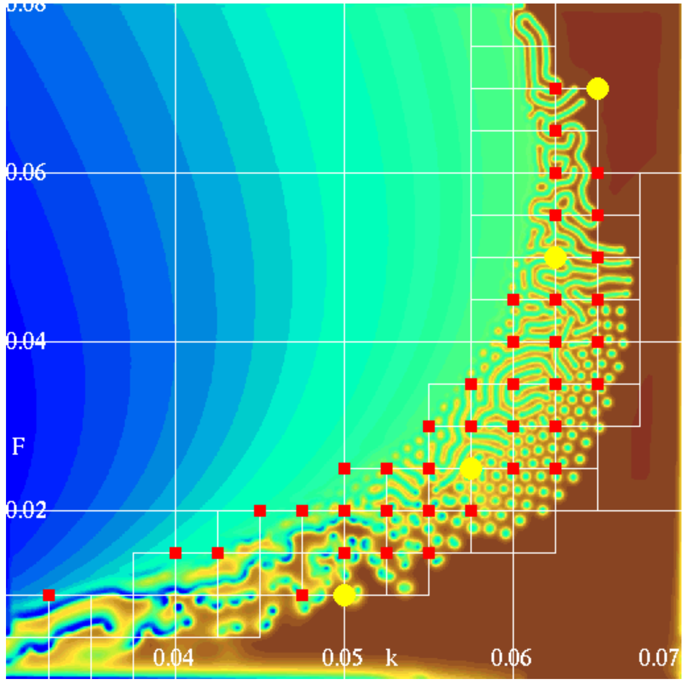

Fig. 2. Dependence on k and F.

To change the structures generated by a PDE, you may not have to change the form of the PDE, just the parameters in it. Evolution by natural selection takes advantage of this principle.

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