26. Sine–Gordon equation *wave propagation with nonlinear forcing*

In view of the remarkable properties of the KdV equation (→ *ref*), it is natural to ask whether other nonlinear PDEs admit soliton solutions or can be analysed by the inverse scattering transform. As it happens, both answers are yes for the very different looking *Sine–Gordon equation*, a semilinear PDE which in one space dimension takes the form

$$u_{tt} - u_{xx} = \sin u. \tag{1}$$

This is simply the wave equation with a nonlinear source term. Though this equation first appeared in Bäcklund's differential geometry investigations during the late nineteenth century, its name is derived by analogy with the Klein–Gordon equation proposed in the 1920s (→ *ref*). The Klein–Gordon equation is linear, however, and nonlinearity adds great richness to the behaviour.

The Sine–Gordon equation is generally posed as an initial value problem on $\mathbb{R}$ for $t \geq 0$. For example, suppose we specify an initial wave profile $u(x, 0)$ with the property that all of its derivatives, though not necessarily u itself, approach 0 as $|x| \to \infty$. For such a problem, there are a variety of possible soliton solutions, solutions with wave structures that emerge with the same shape when they collide with other such structures. These solitons were first spotted in 1962 by Perring and Skyrme, who were conducting numerical simulations of a simple quantum field. The constant solutions $u \equiv 0 (\mathrm{mod}\ 2\pi)$, called *vacuum states*, act as zero-energy barriers which support various types of solitons with specific names. A *kink* is a soliton that begins in one vacuum state, $u = 2k\pi$ as $x \to -\infty$, and ends in the next higher vacuum state, $u = 2(k+1)\pi$ as $x \to \infty$. Similarly, an *antikink* begins in a higher state to the left and ends up in a lower state to the right. Two kinks repel each other when they come in contact (Fig. 1). When a kink and an antikink collide, they temporarily combine, and each emerges in the same shape as before, but at the next lower vacuum state (Fig. 2).

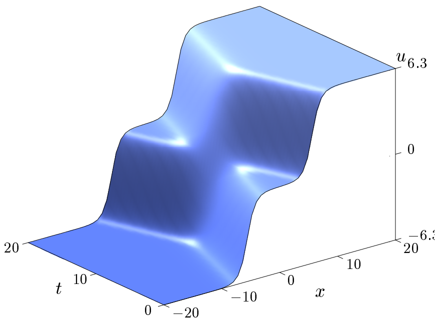

Fig. 1: Two kinks repelling

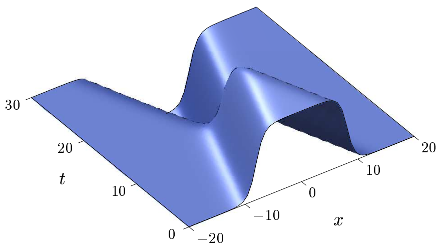

Fig. 2: Kink and antikink

These kink solutions require significant energy. If insufficient energy is present in an initial condition, other solitons will emerge, called *breathers*, which oscillate without entering a vacuum state. They can travel in the x direction or remain stationary (Fig. 3).

The source function $\sin u$ turns out to be rather special. Physical models often suggest other nonlinear source terms for (1), such as $u^3 - u$ (from the ϕ^4 equations of particle physics) or $\sin u + \sin \frac{1}{2}u$ (the *double Sine–Gordon equation*). Though these equations admit non-dissipative kinks as solutions, the kinks do not maintain their shape through collisions and thus are not considered solitons.

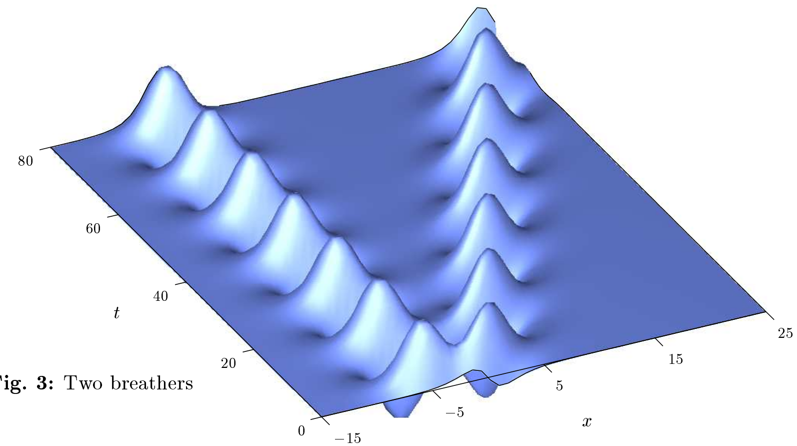

Fig. 3: Two breathers

Like the KdV equation, the Sine–Gordon equation can be solved exactly by analytical techniques. This was an area of vigorous research in the early 1970s, with the first solution produced through direct methods by Hirota in 1972. A year later, Lamb and Ablowitz, Kaup, Newell, and Segur produced inverse scattering methods to solve the equation, reducing the PDE to a solvable eigenvalue problem. The Bäcklund transformation, known for more than a century, can additionally be used to derive a two-parameter family of solutions from a single solution.

For other nonlinear generalisations of the Klein–Gordon equation, neither inverse scattering techniques nor Bäcklund transformations lead to analytical solutions in general, and numerical methods are used instead.

The Sine–Gordon equation arises in diverse applications. It has been proposed to model surfaces of constant negative curvature (1875), displacement in crystals (1939), elementary particles (1958), and propagation of magnetic flux across a Josephson junction (1979). In certain quantum mechanical models, kinks and antikinks correspond to particles and anti-particles.

References

M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Method for solving the Sine–Gordon equation*, Phys. Rev. Lett., **30** (1973), 1262–1264.

R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and nonlinear wave equations*, Academic Press, 1982.

P. G. Drazin and R. S. Johnson, *Solitons: an introduction*, Cambridge University Press, 1989.

L. P. Eisenhart, *A treatise on the differential geometry of curves and surfaces*, Ginn and Company, 1909.

G. L. Lamb, *Bäcklund transformation in nonlinear pulse propagation*, Phys. Lett., **48A** (1974), 73–74.

J. K. Perring and T. H. R. Skyrme, *A model unified field equation*, Nucl. Phys., **31** (1962), 550–555.

G. B. Whitham, *Linear and nonlinear waves*, Wiley, 1974.