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Inverted oscillations of a driven pendulum

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Inverted oscillations of a parametrically driven planar pendulum are considered, together with their relationship to the inverted solution. In particular, a horseshoe structure of the associated manifolds is identified which explains the similarity between the bifurcations of the inverted position and the hanging position. This allows us to apply a large body of existing knowledge to the dynamics enabling a lower bound on the forcing required to achieve inverted oscillations to be established.

Keywords: parametrically forced pendulum; nodding oscillations; subharmonic oscillations; manifolds; horseshoe; braids

1. Introduction

It has been known for some time that a simple pendulum can be stabilized in the upside down position by applying an oscillating vertical force to the pivot point (Stephenson 1908). This has been demonstrated both experimentally and by numerical simulation (Kalmus 1970; Pippard 1987; Smith & Blackburn 1992), and more recently the concept has been extended to multiple pendulums (Acheson & Mullin 1993). The forcing parameters which bound the regions of stability can be found approximately by analytical techniques based on linearization (Acheson 1993), or the bifurcations that bound the stable regions can be followed by numerical methods (Parker & Chua 1989). Comparison is made between the two approaches by Bryant & Miles 1990). It is less well known that the pendulum can be made to oscillate around the inverted position in periodic limit cycles. This behaviour was noted by Acheson (1995), and the stable solutions were termed 'multiple-nodding oscillations' since the pendulum 'nods' either side of the inverted vertical position. If we take θ to measure the angle that a pendulum makes with the downward, hanging position then, as considered by Acheson, these solutions have three basic characteristics (i) they oscillate about the inverted position $\theta = \pi$, (ii) the angle θ remains in the range $\pi/2 < \theta < 3\pi/2$ for all time, and (iii) their velocity changes sign at least three times on one side of the upright position. Multiple-nodding solutions form a subset of (generally) subharmonic solutions which may be of various periods and undergo different numbers of changes in velocity during the periodic time. Many distinct 'nodding' oscillations are possible, with different periods and different numbers of nods. However, the origin of these solutions and their role in the bifurcations which determine the stability of the inverted position have until now not been fully investigated.

2. Equation of motion and numerical simulations

If we consider the pendulum to be a light rod of length, l, with a point mass, m, while the pivot is vertically driven, then it is possible to write the equation of motion in the form,

$$l^2\theta'' + \frac{d}{m}\theta' + l(z'' + g)\sin\theta = 0, \tag{2.1}$$

in which the pendulum is subject to periodic displacement $z(\tau)$, and has linear damping d, where a dash denotes differentiation with respect to time, τ. By writing

$$z(\tau) = -Z\cos\Omega\tau, \qquad t = \omega_0\tau, \qquad \omega_0 = \sqrt{\frac{g}{l}}, \tag{2.2}$$

equation (2.1) reduces to

$$\ddot{\theta} + c\dot{\theta} + (1 = p\cos\omega t)\sin\theta = 0, \tag{2.3}$$

where

$$c = \frac{d}{\omega_0 m l^2}, \qquad p = \frac{Z\Omega^2}{g}, \qquad \omega = \frac{\Omega}{\omega_0}, \tag{2.4}$$

in which θ is the angle of rotation measured from the downwards vertical, c is a damping constant taken here as 0.1 throughout, p is the scaled parametric excitation amplitude, ω is the scaled frequency of excitation, and a dot represents differentiation with respect to the scaled time, t (Capecchi & Bishop 1994). The behaviour of the parametrically excited pendulum has been the subject of considerable recent research (Mullin 1993; Bishop & Clifford 1994, 1996a; Clifford & Bishop 1995a, b). In these previous studies, the bifurcations of the hanging solutions ($|\theta(t)| < \pi\ \forall t$) have been determined by numerical and analytical techniques. It has also been shown that the pendulum possesses a countable infinity of (unstable) periodic orbits which populate the well-known tumbling chaotic attractor. The bifurcations at which these periodic orbits are created or destroyed have been considered in terms of the formation of a particular Smale horseshoe (Clifford & Bishop 1993, 1994).

The dynamics of the inverted position can equally be examined by using similar techniques to those in the works cited. Solutions paths have been numerically detected by following stable and unstable solutions located around the inverted $\theta = \pi$ solution. For convenience, to illustrate their loci, a schematic bifurcation diagram showing the effect of increasing the amplitude of parametric excitation, p is shown in figure 1. At $p = p_n$, the inverted solution stabilizes, as two mirror image unstable period-1 solutions collide with the unstable inverted solution at a pitchfork bifurcation. The inverted solution then becomes unstable at a supercritical bifurcation ($p = p_f$) leaving a symmetric period-2 solution. The symmetric period-2 solution in turn undergoes a symmetry-breaking bifurcation at $p = p_s$ only one of which is indicated on the figure, and the subsequent period-2 mirror image solutions period-double repeatedly to possibly chaotic attractors before disappearing at a catastrophic bifurcation (crisis) at $p = p_e$ similar to those discussed by Stewart (1987). This bifurcation sequence is identical to that of the hanging pendulum determined by Clifford & Bishop (1993) with two exceptions. The initial bifurcation which stabilizes the inverted position has no equivalent for the hanging solution, and the bifurcation at $p = p_f$ may be subcritical in the hanging case. When $\omega = 2$, the bifurcations occur

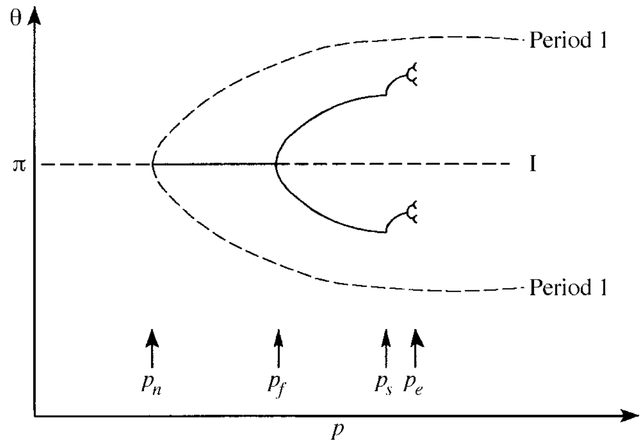

Figure 1. Schematic bifurcation diagram for the inverted parametrically excited pendulum. Solid (dashed) lines represent stable (unstable) solutions. From left to right, the bifurcations are as follows: at $p = p_n$ the inverted solution becomes stable as two period-1 unstable solutions collide with an unstable inverted solution (labelled I). At $p = p_f$ the inverted solution becomes unstable at a supercritical pitchfork bifurcation leaving a stable symmetric period-2 solution. This period-2 solution then undergoes a symmetry-breaking bifurcation at $p = p_s$ leaving two mirror image asymmetric stable solutions, only one of which is shown. These period-2 orbits then rapidly period-double to possibly chaotic attractors before a catastrophic bifurcation at $p = p_e$.

when $p_n = 3.12$, $p_f = 3.42$, $p_s = 3.55$, $p_e = 3.37$. The bifurcations of the hanging solution corresponded to the formation of a three-striped Smale horseshoe (termed a 3-shoe) in the invariant manifolds of the inverted unstable solutions (Smale 1967; Clifford & Bishop 1995*b*). We propose that a similar horseshoe exists contained in the invariant manifolds of the two symmetric (unstable) period-1 solutions, which accounts for the almost identical behaviour. In effect, these two unstable period-1 solutions are analogous to the unstable inverted saddles (Bishop & Clifford 1996*b*).

Rather than determining the invariant manifolds of the inverted unstable period-1 solutions directly, which would involve locating the unstable solutions with a great deal of accuracy, we can see the formation of horseshoe dynamics by numerically integrating a range of initial conditions forwards and backwards in time. Figure 2 shows the results of integrating the range of initial conditions around the inverted state forwards and backwards through one cycle of forcing for $\omega = 2, p = 4$. The intersection between these two sets contains orbits that will remain around the inverted position for all time. Hence, the hatched regions contain a countable infinity of (unstable) inverted oscillatory orbits, which can be located individually by numerical methods. As an example, we show the position of a period-4 orbit in the figure. This topological structure is identical to the 3-shoe which governed the dynamics of the hanging pendulum.

3. Subharmonic oscillations

The origin of the multiple-nodding or other subharmonic oscillations can now be explained in terms of the formation of the 3-shoe. While many of the inverted oscillations may come from the bifurcations of the inverted state, it is clear that many

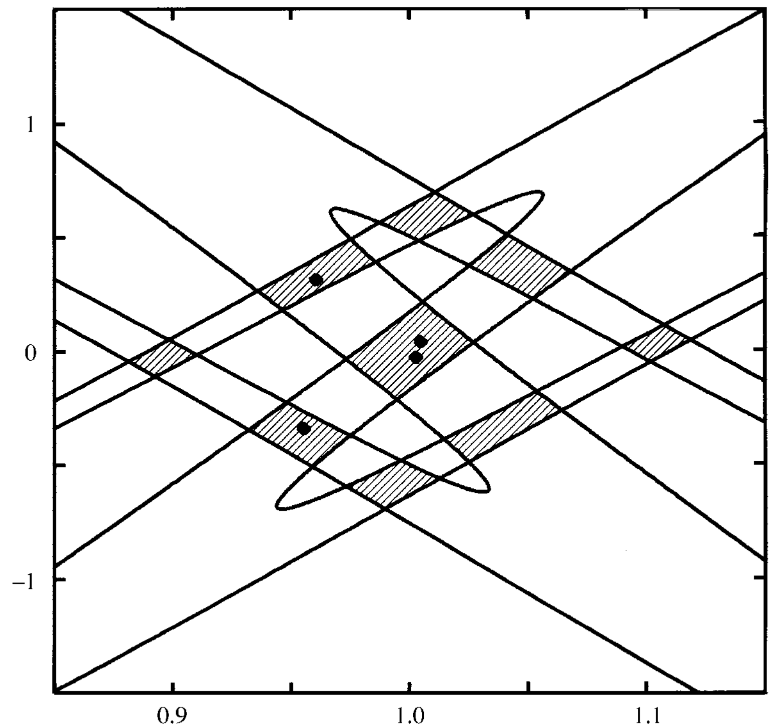

Figure 2. Intersection of forward and backward integration of a range initial conditions which remain around the inverted position $\theta = \pi$ plotted in $(\dot{\theta}, \theta/\pi)$ space. Unstable orbits that oscillate about the unstable inverted position are located in the hatched regions by numerical methods. The position of a period-4 orbit is also shown.

other oscillations, in particular those with odd periods, cannot come from this source. The same was true for the hanging position, and many other orbits were successfully located by applying braid and knot theory to the sequence of events which surrounded the 3-shoe creation process (Clifford & Bishop 1995*a*; Bishop & Clifford 1996). In particular, stable oscillations with odd period were located that had been overlooked by earlier efforts. These additional orbits originate at subharmonic saddle-node bifurcations, and are typically only stable over a narrow range of parameters. Of particular note, in the hanging case, subharmonic solutions of a 'nodding' type were located beyond the value of forcing amplitude for which the hanging state (or bifurcations from it) was stable, akin to p_e for the inverted case (Clifford & Bishop 1995*a*). The same is true for the inverted pendulum since the dynamics are governed by the same topological process. Using the invariant manifolds, and a symbolic approach to locate periodic orbits, an approximation to their location can be found. These solutions may be subsequently pin-pointed via a Newton–Raphson scheme. In the problem under consideration, two period-4 subharmonic inverted oscillations were located. Initially, these oscillations are unstable, but subsequently stabilize at saddle-node bifurcations as the control parameter, p is varied. Time histories of the two inverted oscillations are shown in figure 3 for $\omega = 2, p = 4$, well beyond p_e. At first glance the orbits appear to be similar: both perform similar oscillating motions close to $\theta = \pi$ before a larger negative movement ($\theta < \pi$). However, closer inspection

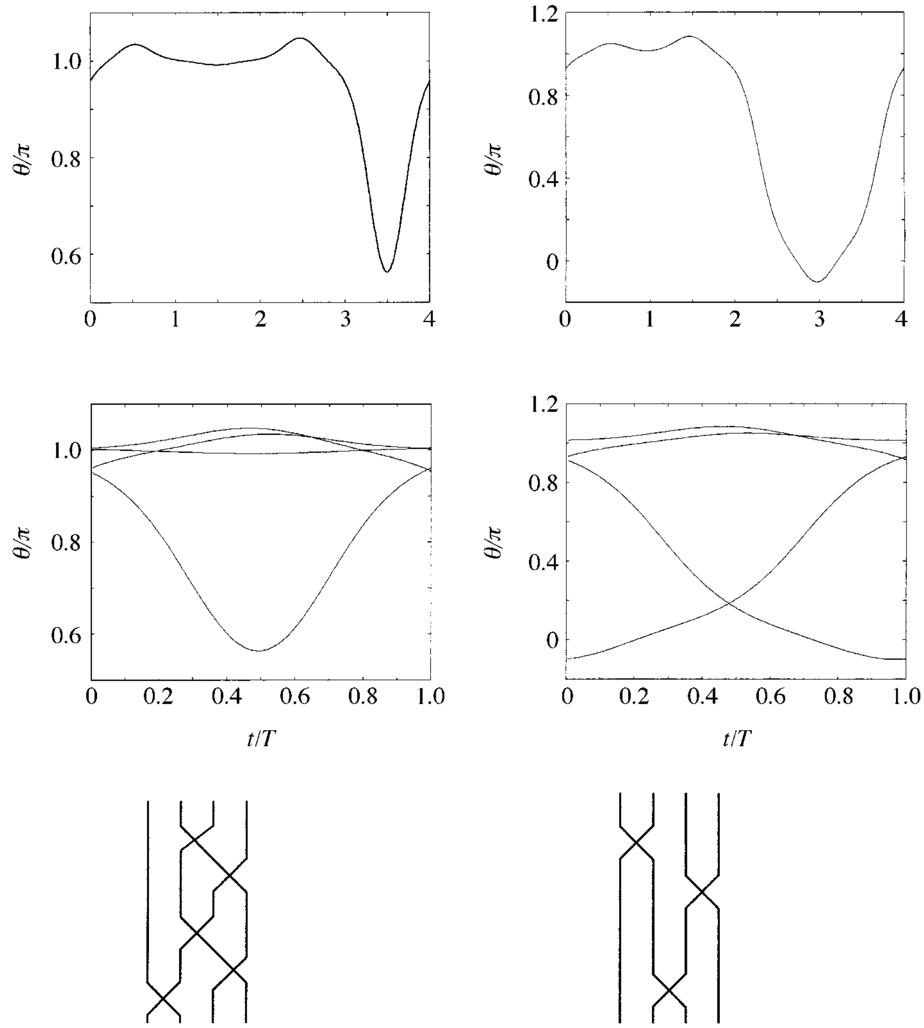

Figure 3. Construction of braids from two period-4 subharmonic orbits. The time histories of the two orbits appear to be similar (top two pictures), but if we construct braid diagrams by first plotting the time histories with time modulo T (second two pictures), and subsequently rotating the pictures through 90° clockwise and straightening out the individual strands, we can easily see that the left-hand orbit has five crossings, while the right-hand orbit has only three crossings. It should be noted that all crossings are positive (left over right).

reveals that the second solution passes the $\theta = 0$ hanging position while the first solution does not proceed beyond the horizontal ($\theta = \pi/2$). Thus, although these are not multiple-nodding orbits as defined by Acheson, they show a similar 'nodding' behaviour, and exist beyond p_e. Furthermore, if the solutions are displayed with time plotted modulo T, then the solutions begin to look more distinct. Indeed, if we construct braid diagrams (Birman 1974; McRobie & Thompson 1993) from the time histories, it is apparent that there is a difference in the number of crossings: the first solution has five crossings while the second has only three. This has many important consequences when it comes to determining the possible bifurcations of different pairs of orbits (McRobie 1992), and is a much stronger topological invariant than number of nods.

One very simple conclusion that can be drawn directly from the analogy to a 3-shoe is that no subharmonic inverted oscillation can exist (stable or unstable) before the initial bifurcation that stabilizes the inverted position takes place (Clifford & Bishop 1994). Or to put it more simply, the inverted subharmonics cannot exist if the inverted state has not been stabilized. The physical significance of this is that the pendulum cannot be stabilized in an inverted oscillation with less excitation amplitude, p, than it would take to stabilize the inverted state. However, these oscillations can continue to exist beyond the symmetry-breaking bifurcation where the inverted state again becomes unstable. The same analogy indicates that subharmonic solutions, possibly including those of multiple-nodding type, can occur beyond p_e, the value beyond which inverted motions do not exist in stable form.

4. Conclusions

We have determined the bifurcations of the inverted pendulum by numerical methods, and have observed the creation of a similar horseshoe structure for the inverted pendulum to the hanging pendulum. We have also successfully located subharmonic oscillations, though for complete understanding a topological consideration is required, including their braid type. More significantly, we have shown that by analogy to the 3-shoe, the pendulum cannot be stabilized in a subharmonic inverted oscillation with less excitation amplitude, p, than it would take to stabilize the inverted state.

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