Three-dimensional solitons[a)]

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Using an inverse Radon transform we generate an integro-differential evolution equation in three space dimensions that has soliton solutions which vanish at large distances in all directions. The equation is of second order in time and invariant under translations and rotations. The shapes of the solitons are generally changed by their nonlocal interactions, but their number and velocities are conserved. The method lends itself to other equations.

1. INTRODUCTION

The existence of nonlinear evolution equations with soliton solutions is of great physical interest and has many mathematically fascinating aspects.[1,2] These equations have been more or less confined to one spatial dimension and none but a few examples of two- and three-dimensional equations have been found,[3] whose solitons, moreover, always extend to infinity in some directions. If such equations are to have a significant bearing on the physics of elementary particles it would seem to be important to find evolution equations in three space dimensions which have not merely solitary-wave solutions that vanish at large distances in all directions but produce *solitons* with their remarkable stability properties under collisions.[4] Such equations have not yet been found.

Since there appear to be major difficulties in discovering soliton generating differential equations in three space dimensions, it seems worthwhile to search for more general equations with such solutions. A suitable class may be that of integrodifferential equations, in which one may think of the soliton—soliton interaction as nonlocal. The present paper represents an attempt in that direction. From a known one-dimensional evolution equation we generate a three-dimensional one that is not restricted to a line. The equation is rotationally invariant and the asymptotic directions of motion of the solitons are determined by the initial conditions. Our method is, in principle, applicable to other evolution equations and to any dimension of space, but in this paper we restrict ourselves to one three-dimensional equation.

The tool we will use is the *Radon transform*.[5] Since this integral transform is not widely known among physicists we define it and derive the results we need in Sec. 2. In Sec. 3 we apply its inverse to the iterated Korteweg—de Vries equation. This equation is of second order in the time and does not have the unidirectional character of the KdV equation. Its inverse Radon transform is an integrodifferential equation that is rotationally and translationally invariant. We show that it has N-soliton solutions in which the solitons vanish asymptotically in all spatial direction and move with largely arbitrary (both in magnitude and direction) velocities. Their collisions preserve their number and velocities but, in general, will alter their shapes.

Whether the three-dimensional soliton-generating equation (3.6) has any physical application is not known. This question may depend on whether there are physical phenomena that are describable by its solutions, and it remains to be investigated. The method used here can easily be transferred to other equations to generate intergrodifferential equations in higher dimensions with soliton solutions from one-dimensional differential equations that are known to have them. It does not work for the nonlinear Schrödinger equation, because there the one-dimensional solitons are not really solitary waves.[6] While their magnitudes are of the traveling-wave form, their complex phases oscillate. As a result the inverse Radon transform washes them out in the asymptotic region and they disappear.

There is an appendix which contains a direct verification, without use of the inverse-scattering machinery, of the pure N-soliton solution of the KdV equation.

2. THE RADON TRANSFORM

Let us consider two functions, $f(\hat{n},x)$ and $\hat{f}(\mathbf{z})$, where $x\in \mathbb{R}^1$, $\mathbf{z}\in\mathbb{R}^3$, and $\hat{n}\in\mathbb{R}^3$ with $|\hat{n}|=1$. We shall not, in this paper, specify exactly in what spaces f and $\hat{f}$ should be. (The reader may consult Ref. 5.) Let $\tilde{f}(\mathbf{k})$ be the three-dimensional Fourier transform of $\hat{f}(\mathbf{z})$:

$$\tilde{f}(\mathbf{k})=(F_3\hat{f})(\mathbf{k})=\int d^3\mathbf{z}\,\exp(i\mathbf{k}\cdot\mathbf{z})\hat{f}(\mathbf{z}),\tag{2.1}$$

and let $\mathring{f}(\hat{n},k)$ be the one-dimensional Fourier transform of $f(\hat{n},x)$ as a function of x for fixed $\hat{n}$:

$$\mathring{f}(\hat{n},k)=(F_1 f)(\hat{n},k)=\int_{-\infty}^{\infty}dx\,\exp(ikx)f(\hat{n},x).\tag{2.2}$$

We consider $\hat{f}(\mathbf{z})$ to be the R transform of $f(\hat{n},x)$,

$$\hat{f}(\mathbf{z})=(Rf)(\mathbf{z})\tag{2.3}$$

if

$$\tilde{f}(\mathbf{k})=\mathring{f}(\hat{\mathbf{k}},k),\tag{2.4}$$

where $\mathbf{k}=\hat{\mathbf{k}}k$. For consistency it will be necessary to require that $f(\hat{n},x)$ have the symmetry property

$$f(-\hat{n},-x)=f(\hat{n},x)\tag{2.5}$$

so that

$$\mathring{f}(-\hat{n},-k)=\mathring{f}(\hat{n},k).\tag{2.6}$$

It should be noted that if we required $\tilde{f}(\mathbf{k})$ to be analytic as a function of each component of k at $k=0$ then (2.4) would imply a specific $\hat{n}$-dependence of $\mathring{f}(\hat{n},0)$

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and of each derivative of $\mathring{f}(\hat{n},k)$ with respect to k at $k=0$.[7] For example it would imply that

$$\mathring{f}(\hat{n},0)=\int_{-\infty}^{\infty} dx\, f(\hat{n},x)$$

be independent of $\hat{n}$. More generally, it would require that

$$\int d\hat{n}\, Y_l^m(\hat{n})\int_{-\infty}^{\infty} dx\, x^p f(\hat{n},x)=0$$

for all $l>p$, if Y_l^m is a spherical harmonic. *We shall however, make no such demands* on $f(\hat{n},x)$ and as a result we end up with more general functions $\tilde{f}(\mathbf{k})$, and hence a larger class of functions $\hat{f}(\mathbf{z})$.

Insertion of (2.1) and use of (2.4) leads to the direct relation between $\hat{f}(\mathbf{z})$ and $f(\hat{n},x)$,

$$\begin{aligned}(Rf)(\mathbf{z})=\hat{f}(z)&=\frac{1}{(2\pi)^3}\int d\hat{k}\int_0^\infty dk\, k^2\exp(-ik\hat{k}\cdot\mathbf{z})\mathring{f}(\hat{k},k)\\ &=-\Delta\,\frac{1}{16\pi^3}\int d\hat{k}\int_{-\infty}^\infty dk\exp(-ik\hat{k}\cdot\mathbf{z})\mathring{f}(\hat{k},k)\\ &=-\frac{1}{8\pi^2}\int d\hat{n}\, f''(\hat{n},\hat{n}\cdot\mathbf{z}),\end{aligned}\tag{2.7}$$

because of (2.6) and inversion of (2.2). Here $f''(\hat{n},x)\equiv\partial^2 f(\hat{n},x)/\partial x^2$. The relation R between $f(\hat{n},x)$ and $f(\mathbf{z})$ expressed by (2.7) is the *inverse Radon transform*. We shall refer to it as the R transform.

We can also formally express $f(\hat{n},x)$ in terms of $\hat{f}(\mathbf{z})$ and thereby invert the R transform. By inverting (2.2) and using (2.1) we obtain

$$f(\hat{n},x)=(R^{-1}\hat{f})(\hat{n},x)=\int d^3\mathbf{z}\, f(\mathbf{z})\delta(x-\hat{n}\cdot\mathbf{z}).\tag{2.8}$$

This is the Radon transform. [However, it has to be realized that this transform is not uniquely defined,[5] and the result of (2.8) is not necessarily equal to the function $f(\hat{n},x)$ on the right-hand side of (2.7).]

The important properties of R under differentiation immediately follow from (2.4) or (2.7),

$$\nabla\hat{f}(\mathbf{z})=-\frac{1}{8\pi^2}\int d\hat{n}\, f'''(\hat{n},\hat{n}\cdot\mathbf{z})\hat{n},$$

which means

$$R\left(\hat{n}\,\frac{\partial f}{\partial x}\right)=\nabla(Rf),\tag{2.9}$$

and, by repetition,

$$R\left(\frac{\partial^2 f}{\partial x^2}\right)=\Delta(Rf).\tag{2.10}$$

Let us now look at the R transform of a product. Using (2.4) we get

$$R(fg)(\mathbf{z})=\int d^3\mathbf{x}d^3\mathbf{y}\,\Gamma(\mathbf{z},\mathbf{x},\mathbf{y})f(\mathbf{x})g(\mathbf{y}),\tag{2.11}$$

where

$$\begin{aligned}\Gamma(\mathbf{z},\mathbf{x},\mathbf{y})&=(2\pi)^{-4}\int d^3\mathbf{k}\int_{-\infty}^\infty d\alpha\,\exp[i\alpha\hat{k}\cdot(\mathbf{x}-\mathbf{y})\\ &\qquad+i\mathbf{k}\cdot(\mathbf{y}-\mathbf{z})]\\ &=-\Delta_z\gamma(\mathbf{x},\mathbf{y},\mathbf{z}),\end{aligned}\tag{2.12}$$

where Δ_z is the Laplacian with respect to $\mathbf{z}$, and

$$\begin{aligned}\gamma(\mathbf{x},\mathbf{y},\mathbf{z})&=\tfrac{1}{2}(2\pi)^{-4}\int d\hat{k}\int_{-\infty}^\infty d\alpha\int_{-\infty}^\infty d\beta\exp[i\alpha\hat{k}\cdot(\mathbf{x}-\mathbf{y})]\\ &\qquad\times\exp[(i\beta\hat{k}\cdot(\mathbf{y}-\mathbf{z})]\\ &=\frac{1}{8\pi^2}\int d\hat{k}\,\delta[\hat{k}\cdot(\mathbf{x}-\mathbf{y})]\delta[\hat{k}\cdot(\mathbf{y}-\mathbf{z})]\\ &=\frac{1}{4\pi^2}\,\frac{1}{|(\mathbf{x}-\mathbf{y})\times(\mathbf{y}-\mathbf{z})|}\\ &=\frac{1}{4\pi^2}\,\frac{1}{|\mathbf{x}\times\mathbf{y}+\mathbf{y}\times\mathbf{z}+\mathbf{z}\times\mathbf{x}|}.\end{aligned}\tag{2.13}$$

Thus γ is symmetric in all three variables,

$$\gamma(\mathbf{x},\mathbf{y},\mathbf{z})=\gamma(\mathbf{x},\mathbf{z},\mathbf{y})=\gamma(\mathbf{y},\mathbf{x},\mathbf{z}).\tag{2.14}$$

Furthermore it is invariant under simultaneous translation, rotation, or reflection of all three variables.

If we define the symbol $\circ$ by

$$\begin{aligned}(f\circ g)(\mathbf{x})&\equiv-\Delta\int d^3\mathbf{y}d^3\mathbf{z}\,f(\mathbf{y})g(\mathbf{z})\gamma(\mathbf{x},\mathbf{y},\mathbf{z})\\ &=-\Delta\int d^3\mathbf{y}d^3\mathbf{z}\,\frac{f(\mathbf{y}+\mathbf{x})g(\mathbf{z}+\mathbf{x})}{|\mathbf{y}\times\mathbf{z}|},\end{aligned}\tag{2.15}$$

then we can write our result (2.11) in the simple form

$$R(fg)=(Rf)\circ(Rg).\tag{2.16}$$

Since the left-hand product is commutative and associative, so is the $\circ$-product.

In Fourier-transform language we have

$$F_3(f\circ g)(\mathbf{k})=(2\pi)^{-1}\int_{-\infty}^\infty d\alpha\,\tilde{f}(\alpha\mathbf{k})\tilde{g}[(|\mathbf{k}|-\alpha)\hat{k}]\tag{2.17}$$

if $\tilde{f}=F_3 f$ and $\tilde{g}=F_3 g$.

Consider now a family of functions $f(\hat{n},x,t)$ that form solitary waves[8] traveling along x with velocities c which depend on $\hat{n}$ in such a way that there exists a fixed vector $\mathbf{v}$ so that $c=\hat{n}\cdot\mathbf{v}$,

$$f(\hat{n},x,t)=f(\hat{n},x-\hat{n}\cdot\mathbf{v}t,0).\tag{2.18}$$

If $f(\hat{n},x,0)$ satisfies the symmetry (2.5), then so does $f(\hat{n},x,t)$. The R transform of the family is then given by

$$\hat{f}(\mathbf{z},t)=-\frac{1}{8\pi^2}\int d\hat{n}\,f''(\hat{n},\hat{n}\cdot\mathbf{z}-\hat{n}\cdot\mathbf{v}t,0)=\hat{f}(\mathbf{z}-\mathbf{v}t,0)$$

and hence $\hat{f}(\mathbf{z},t)$ forms a solitary wave moving with the velocity $\mathbf{v}$. What is more, if for all $\hat{n}$

$$\lim_{x\to\pm\infty}f''(\hat{n},x,0)=0,\tag{2.19}$$

then it follows that

$$\lim_{|\mathbf{z}|\to\infty}\hat{f}(\mathbf{z},0)=0.\tag{2.20}$$

This is easily proved by writing

$$\begin{aligned}g(\mathbf{z})&=\int d\hat{n}\,f''(\hat{n},\hat{n}\cdot\mathbf{z},0)=2\int_0^{2\pi}d\phi\int_0^1 d\alpha\,f''(\alpha,\phi,|\mathbf{z}|\alpha,0)\\ &=\int_0^1 d\alpha\,h(\alpha,|\mathbf{z}|\alpha)=\left(\int_0^a+\int_a^1\right)d\alpha\,h(\alpha,|\mathbf{z}|\alpha)\end{aligned}$$

by (2.5) and setting

$$2\int_0^{2\pi} d\phi f''(\alpha,\phi,|\mathbf{z}|\alpha,0) \equiv h(\alpha,|\mathbf{z}|\alpha).$$

For any given $\epsilon > 0$ we choose a so that

$$\left|\int_0^a d\alpha h\right| < \tfrac{1}{2}\epsilon$$

and then, using (2.19), we choose R so that for all $|\mathbf{z}| > R$

$$\left|\int_a^1 d\alpha h\right| < \tfrac{1}{2}\epsilon.$$

Thus (2.20) follows. Therefore, if $f(\hat{\mathrm{n}},x,t)$ is a family of solitary waves for which not only f but also f'' vanishes for large $|x|$ (and each $\hat{\mathrm{n}}$), then $\hat{f}(\mathbf{z},t)$ is a solitary wave that vanishes for large $|\mathbf{z}|$ in all directions.

3. A NONLINEAR EVOLUTION EQUATION

The KdV equation

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{3.1}$$

has soliton solutions which move from right to left with velocities c_n that are related to the (discrete) eigenvalues of the associated one-dimensional Schrödinger equation

$$\frac{\partial^2\psi_n}{\partial x^2} + u\psi_n = \lambda_n\psi_n \tag{3.2}$$

by $c_n = 4\lambda_n$. The undirectional nature of the motion of the solitons of (3.1) is the result of the odd character of the KdV equation, and the fact that the discrete eigenvalues of (3.2) are all nonnegative. Before we transfer (3.1) to three dimensions it will be useful to remove its unidirectional character by iteration.

Differentiating (3.1) with respect to t and eliminating first t-derivatives by means of (3.1) yields

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left(12u^3 + 9u\frac{\partial^2 u}{\partial x^2} + \frac{3}{2}\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}\right). \tag{3.3}$$

We may call this the iterated Korteweg—deVries (IKdV) equation. Any solution of (3.1) or of the equation obtained from (3.1) by changing the sign of t, is a solution of (3.3). Thus (3.3) has soliton solutions in which all solitons move to the left, and also soliton solutions in which they all move to the right. Whether it has solutions in which some solitons move to the right and some to the left is unknown. [A pure N-soliton solution of (3.1) in which some but not all of the velocities are reversed does not solve (3.3); see the Appendix.]

We translate the IKdV equation (3.3) into three dimensions by considering $u(x,t)$ as a function of two angle parameters, or a unit vector $\hat{\mathrm{n}}$, is such a way that

$$u(\hat{\mathrm{n}},x,t) = u(-\hat{\mathrm{n}},-x;t), \tag{3.4}$$

and then subjecting it to the R transformation (2.7),

$$\Phi(\mathbf{z},t) = (Ru)(\mathbf{z},t). \tag{3.5}$$

According to (2.16) and (2.9), (3.3) implies that Φ satisfies the integrodifferential equation

$$\frac{\partial^2\Phi}{\partial t^2} = \Delta[12\Phi\circ\Phi\circ\Phi + 9\Phi\circ\Delta\Phi + \tfrac{3}{2}\Delta(\Phi\circ\Phi) + \Delta\Delta\Phi], \tag{3.6}$$

where the $\circ$-product is defined by (2.15). Let us now look at the soliton solutions of (3.3).

To start with,

$$u(c,x,t) = \sigma(|c|, x-ct),$$

$$\sigma(|c|,y) = \tfrac{1}{2}|c|\operatorname{sech}^2(\tfrac{1}{2}|c|^{1/2}y), \tag{3.7}$$

is a solitary wave solution of (3.3). The velocity c may be taken positive or negative. We choose an arbitrary vector v so that $|\mathrm{v}| \geq |c|$ and write $c = \hat{\mathrm{n}}\circ\mathrm{v}$. As we vary $\hat{\mathrm{n}}$, keeping v fixed, this generates a family of solutions

$$u(\mathrm{v},\hat{\mathrm{n}},x,t) \equiv u(\hat{\mathrm{n}}\circ\mathrm{v},x,t) = \sigma(|\hat{\mathrm{n}}\circ\mathrm{v}|, x-\hat{\mathrm{n}}\circ\mathrm{v}t) \tag{3.8}$$

of (3.3) depending on $\hat{\mathrm{n}}$. Because (3.7) is even in y, it follows that

$$u(\mathrm{v},-\hat{\mathrm{n}},-x,t) = u(\mathrm{v},\hat{\mathrm{n}},x,t) \tag{3.9}$$

and we may take its R transform as in (3.5). According to our discussion at the end of Sec. 2, then Φ is a solitary wave of velocity v,

$$\Phi(\mathrm{v},\mathbf{z},t) = \Phi(\mathrm{v},\mathbf{z}-\mathrm{v}t,0). \tag{3.10}$$

Thus (3.6) has solitary wave solutions whose velocities have arbitrary directions and magnitudes.

In fact, the family of solitary wave solutions of (3.6) is larger than the R transforms of the functions given by (3.8). For $\hat{\mathrm{n}}$ in some set of directions, say $\hat{\mathrm{n}}\in\Omega$, we may choose u to be zero,

$$u(\mathrm{v},\hat{\mathrm{n}},x,t) = \begin{cases} 0, & \hat{\mathrm{n}}\in\Omega, \\ \sigma(|\hat{\mathrm{n}}\circ\mathrm{v}|, x-\hat{\mathrm{n}}\circ\mathrm{v}t), & \hat{\mathrm{n}}\notin\Omega. \end{cases} \tag{3.11}$$

If Ω is such that whenever $\hat{\mathrm{n}}\in\Omega$ then $-\hat{\mathrm{n}}\in\Omega$, it follows that u satisfies (3.9) and its R transform is a solitary wave. Of course, the function (3.11) will generally have steplike discontinuities as a function of $\hat{\mathrm{n}}$, but that does not prevent Φ from being continuous and differentiable.

An even larger class of solitrary waves is generated by using the translational invariance of the KdV equation. We may choose, for $\hat{\mathrm{n}}\notin\Omega$,

$$u(\hat{\mathrm{n}}\circ\mathrm{v},x,t) = \sigma(|\hat{\mathrm{n}}\circ\mathrm{v}|, x-\mathrm{n}\circ\mathrm{v}t-\delta) \tag{3.11'}$$

and make δ an arbitrary function of $\hat{\mathrm{n}}$. While this changes only the position of the one-dimensional solitary wave, it generally changes the *shape* of its R transform.

The next step is to consider an N soliton solution of (3.3),

$$u(c_1,\ldots,c_N;x,t) = \sigma_N(c_1,\ldots,c_N;x,t), \tag{3.12}$$

where[9]

$$\sigma_N(c_1,\ldots,c_N;x,t) = 2\frac{\partial^2}{\partial x^2}\log\det M. \tag{3.13}$$

The $N\times N$ matrix M is given by

$$M_{ij} = \delta_{ij} + \frac{2\kappa_i}{\kappa_i+\kappa_j}\exp[\kappa_i(c_it - x + x_i)], \tag{3.14}$$

where the x_i are arbitrary constants, and $\kappa_i = |c_i|^{1/2}$. The velocities c_i must all be of the same sign. Suppose that all $c_i > 0$.

The asymptotic form of σ_N is given by

$$\sigma_N \underset{t\to\pm\infty}{\approx} \sum_{i=1}^{N} \sigma(|c_i|, x - c_i t - \delta_i^\pm) \tag{3.15}$$

in the sense that for each i

$$\lim_{t\to\pm\infty} \sigma_N(c_1, \ldots, c_N; x + c_i t, t) = \sigma(|c_i|, x - \delta_i^\pm), \tag{3.16}$$

where σ is defined by (3.7), and[10]

$$\delta_i^- = x_i,$$

$$\delta_i^+ = x_i + \frac{1}{\kappa_i}\left[\sum_{j<i} \log\left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j}\right) + \sum_{j>i} \log\left(\frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i}\right)\right], \tag{3.17}$$

with the understanding that $0 < \kappa_1 < \cdots < \kappa_N$.

We choose an arbitrary set of N vectors $\mathbf{v}_i$, $i = 1, \ldots, N$, and a unit vector $\hat{\mathbf{n}}$ such that

$$c_i = \hat{\mathbf{n}} \cdot \mathbf{v}_i, \quad i = 1, \ldots, N.$$

Now vary $\hat{\mathbf{n}}$, keeping the $\mathbf{v}_i$ fixed. Let Ω be a set of directions $\hat{\mathbf{n}}$ that includes all those for which not all $\hat{\mathbf{n}} \cdot \mathbf{v}_i$, $i = 1, \ldots, N$, have the same sign. (Ω may contain directions for which all $\hat{\mathbf{n}} \cdot \mathbf{v}_i$ have the same sign, but its complement must have positive Lebesgue measure.[11]) Furthermore, let Ω be such that if $\hat{\mathbf{n}} \in \Omega$, then $-\hat{\mathbf{n}} \in \Omega$. Let Ω^+ (Ω^-) be the set of directions $\hat{\mathbf{n}} \notin \Omega$ such that all $\hat{\mathbf{n}} \cdot \mathbf{v}_i > 0$ ($\hat{\mathbf{n}} \cdot \mathbf{v}_i < 0$).

We now define a family of functions

$$u(\hat{\mathbf{n}}, s, t) \equiv \begin{cases} \sigma_N(\hat{\mathbf{n}} \cdot \mathbf{v}_1, \ldots, \hat{\mathbf{n}} \cdot \mathbf{v}_N, x, t), & \text{if } \hat{\mathbf{n}} \in \Omega^+, \\ 0, & \text{if } \hat{\mathbf{n}} \in \Omega, \\ \sigma_N(\hat{\mathbf{n}} \cdot \mathbf{v}_1, \ldots, \hat{\mathbf{n}} \cdot \mathbf{v}_N, -x, -t, & \hat{\mathbf{n}} \in \Omega^-, \end{cases} \tag{3.18}$$

which, for each $\hat{\mathbf{n}}$, satisfy (3.3). It is clear from (3.14) that

$$u(-\hat{\mathbf{n}}, -x, t) = u(\hat{\mathbf{n}}, x, t)$$

for all t, $\hat{\mathbf{n}}$, x. We may therefore take its R transform as in (3.5), and the resulting function $\Phi(\mathbf{z}, t)$ will satisfy (3.6).

We have (indicating x-derivatives by primes)

$$\begin{aligned}\Phi(\mathbf{z}, t) &= -\frac{1}{8\pi^2}\int d\hat{\mathbf{n}}\, u''(\hat{\mathbf{n}}, \hat{\mathbf{n}} \cdot \mathbf{z}, t) \\ &= -\frac{1}{8\pi^2}\int_{\Omega^+} d\hat{\mathbf{n}}\, \sigma''(\hat{\mathbf{n}} \cdot \mathbf{v}_1, \ldots, \hat{\mathbf{n}} \cdot \mathbf{v}_N; \hat{\mathbf{n}} \cdot \mathbf{z}, t) \\ &\quad - \frac{1}{8\pi^2}\int_{\Omega^-} d\hat{\mathbf{n}}\, \sigma_N''(\hat{\mathbf{n}} \cdot \mathbf{v}_1, \ldots, \hat{\mathbf{n}} \cdot \mathbf{v}_N; -\hat{\mathbf{n}} \cdot \mathbf{z}, -t) \\ &= -\frac{1}{8\pi^2}\int_{\Omega^+} d\hat{\mathbf{n}} [\sigma_N''(\hat{\mathbf{n}} \cdot \mathbf{v}_1, \ldots, \hat{\mathbf{n}} \cdot \mathbf{v}_N, \hat{\mathbf{n}} \cdot \mathbf{z}, t) \\ &\quad + \sigma_N''(-\hat{\mathbf{n}} \cdot \mathbf{v}_1, \ldots, -\hat{\mathbf{n}} \cdot \mathbf{v}_N; \hat{\mathbf{n}} \cdot \mathbf{z}, -t)] \\ &= -\frac{1}{4\pi^2}\int_{\Omega^+} d\hat{\mathbf{n}}\, \sigma_N''(\hat{\mathbf{n}} \cdot \mathbf{v}_1, \ldots, \hat{\mathbf{n}} \cdot \mathbf{v}_N; \hat{\mathbf{n}} \cdot \mathbf{z}, t) \end{aligned} \tag{3.19}$$

because (3.14) shows that σ_N is invariant under a simultaneous sign change of t and all c_i. The asymptotic form of Φ is now determined from (3.16),

$$\Phi(\mathbf{z} + \mathbf{v}_i t, t) = -\frac{1}{4\pi^2}\int_{\Omega^+} d\hat{\mathbf{n}}\, \sigma''(\hat{\mathbf{n}} \cdot \mathbf{v}_i, \hat{\mathbf{n}} \cdot \mathbf{z} - \delta_i^\pm) + o(1) \tag{3.20}$$

as $t \to \pm\infty$. In this sense, then, we have

$$\Phi(\mathbf{z}, t) \underset{t\to\pm\infty}{\approx} \sum_{i=1}^{n} \eta_i^\pm(\mathbf{v}_i, \mathbf{z} - \mathbf{v}_i t), \tag{3.21}$$

where the solitons are given by

$$\eta_i^\pm(\mathbf{v}_i, \mathbf{z}) = -\frac{1}{4\pi^2}\int_{\Omega^+} d\hat{\mathbf{n}}\, \sigma''(\hat{\mathbf{n}} \cdot \mathbf{v}_i, \hat{\mathbf{n}} \cdot \mathbf{z} - \delta_i^\pm). \tag{3.22}$$

Since (3.17) shows that $\delta_i^+ - \delta_i^-$ depends on the c_j, it depends on $\hat{\mathbf{n}}$. Consequently the shapes of the solitons in the infinite past generally are different from those in the infinite future. In this case the collisions produce not only positional shifts of the solitons, but changes in their shapes as well.

Equation (3.22) can be written in the form

$$\eta_i^\pm(\mathbf{v}, z) = Ru(\mathbf{v}_i, \hat{\mathbf{n}}, x - \delta_i^\pm)$$

in terms of the function u defined in (3.11). But it follows from (2.4) that

$$\int d^3\mathbf{z}\, |Rf|^2 = 2\pi^2 \int d\hat{\mathbf{n}} \int_{-\infty}^{\infty} dx\, |f'|^2, \tag{3.23}$$

and hence

$$\begin{aligned}\int d^3\mathbf{z}\, |\eta_i^\pm(\mathbf{v}_i, \mathbf{z})|^2 &= (2\pi)^2 \int_{\hat{\mathbf{n}}\in\Omega^+} d\hat{\mathbf{n}} \int_{-\infty}^{\infty} dx\, |u'(\mathbf{v}_i, \hat{\mathbf{n}}, x - \delta_i^\pm)|^2 \\ &= (2\pi)^2 \int_{\hat{\mathbf{n}}\in\Omega^+} d\hat{\mathbf{n}} \int_{-\infty}^{\infty} dx\, |u'(\mathbf{v}_i, \hat{\mathbf{n}}, x)|^2.\end{aligned}$$

Since the right-hand side has the same value for + and −, it follows that in spite of its changed shape, the volume of the square of each individual soliton at $t \to -\infty$ equals the volume of the square of the corresponding one at $t \to +\infty$. This guarantees, specifically, that no solitons can disappear and all solitons present at $t \to -\infty$ are again present, with equal "strength" but generally altered shape, at $t \to +\infty$.

We have thus demonstrated that the integrodifferential equation (3.6) has solutions which break up into solitons in the infinite past and future. Moreover, each soliton is confined in the sense that it vanishes at large distances in all directions. The initial number of solitons and their velocities are equal to the final ones, but their shapes are generally changed. These solutions, moreover, are expressible by quadrature, being the R transforms, (3.19), of functions explicitly given by (3.13) and (3.14). Whether (3.6) has other soliton solutions is not known. Since we do not have a general solution of the initial-value problem for the IKdV equation (3.3), except for the special cases in which u satisfies either (3.1) or its time reversed, we do not have a general solution of the initial-value problem for (3.6). Furthermore it is not obvious that the Radon transforms of all solutions of (3.6) must solve (3.3).

The important question of possible physical applications of (3.6) has not yet been investigated.

APPENDIX

We want to present here a simple, direct, algebraic verification of the N-soliton solution of the KdV equation that does not use any of the inverse scattering machinery.

We write the matrix M of (3.14) in matrix form

$$M = 1 + EL, \tag{A1}$$

where E is the diagonal matrix

$$E = 2K \exp(K^3 t - Kx + X); \tag{A2}$$

in terms of the matrices X and K,

$$X_{ij} = \delta_{ij} x_i, \quad K_{ij} = \delta_{ij} K_i \tag{A3}$$

and

$$L_{ij} = L_{ji} = (K_i + K_j)^{-1}. \tag{A4}$$

We have

$$KL + LK = Q, \tag{A5}$$

where Q is the matrix $Q_{ij} = 1$, and

$$\frac{\partial E}{\partial x} = -KE, \quad \frac{\partial E}{\partial t} = K^3 E. \tag{A6}$$

Now, by (A6), and writing $N \equiv M^{-1}$,[12]

$$\psi \equiv 2\frac{\partial}{\partial x}\log\det M = 2\operatorname{tr}\left(\frac{\partial M}{\partial x} N\right) \tag{A7}$$

$$= -2\operatorname{tr}(KELN) = -2(ELKN)$$

since EL commutes with N. Also[13]

$$\psi = \operatorname{tr}\left(\frac{\partial \tilde{M}}{\partial x}\tilde{N}\right) = -2\operatorname{tr}(LEK\tilde{N}) = -2\operatorname{tr}(EKLN)$$

and hence

$$\psi = \operatorname{tr} QF = \sum_{ij} F_{ij}, \tag{A8}$$

where

$$F = -NE = -(E^{-1} + L)^{-1}. \tag{A9}$$

Now using (A6), we get

$$\frac{\partial N}{\partial x} = NK(1-N), \quad \frac{\partial N}{\partial t} = NK^3(N-1),$$

$$\frac{\partial F}{\partial x} = -NKF, \quad \frac{\partial F}{\partial t} = NK^3 F,$$

$$\frac{\partial^2 F}{\partial x^2} = NK(2N-1)KF,$$

$$\frac{\partial^3 F}{\partial x^3} = NK(3NK + 3KN - 6NKN - K)KF,$$

and, using (A5),

$$\frac{\partial F}{\partial x} Q \frac{\partial F}{\partial x} = -NK(NK + KN - 2NKN)KF.$$

Consequently the matrix F satisfies the equation

$$-\frac{\partial F}{\partial t} = 3\frac{\partial F}{\partial x} Q \frac{\partial F}{\partial x} + \frac{\partial^3 F}{\partial x^3}. \tag{A10}$$

But since Q is such that for any A and B

$$\operatorname{tr}(QAQB) = \operatorname{tr}(QA)\operatorname{tr}(QB), \tag{A11}$$

multiplication of (A10) by Q, taking traces, and using (A8) gives

$$-\frac{\partial \psi}{\partial t} = 3\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{\partial^3 \psi}{\partial x^3}. \tag{A12}$$

Differentiation with respect to x and setting $u = \partial\psi/\partial x$ finally shows that u satisfies the KdV equation

$$-\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{A13}$$

which is the time-reversed version of (3.1). Thus u satisfies (3.3). It is of some interest that (A13) is the result of the fact that the symmetric matrix F of (A9) satisfies (A10), or more explicitly

$$-\frac{\partial F_{ij}}{\partial t} = 3\sum_{k,l}\frac{\partial F_{ik}}{\partial x}\frac{\partial F_{jl}}{\partial x} + \frac{\partial^3 F_{ij}}{\partial x^3}.$$

We may use this technique to check whether the function defined by (3.13) and (3.14), but with the c_i not all of the same sign, satisfies (3.3). For that case, instead of (A2),

$$E = 2K\exp(K^3 St - Kx + X),$$

where S is a diagonal matrix with some of its diagonal entries -1, and the others, $+1$. Since the x-deriviatives do not involve S, it is only necessary to check if S disappears from $\partial^2\psi/\partial t^2$. One readily finds that

$$\frac{\partial^2 \psi}{\partial t^2} = -\operatorname{tr}[QNSK^3(2N-1)SK^3NE]$$

in which S disappears only if $S = 1$ or $S = -1$. Thus the pure soliton function (3.13), with some solitons coming in from the left and some from the right, does not satisfy the IKdV equation.

[1]C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, Phys. Rev. Lett. **19**, 1095 (1967); P. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).

[2]A.C. Scott, F.Y.F. Chu, and D.W. McLaughlin, Proc. IEEE **61**, 1443 (1973); R.M. Miura, SIAM Rev. **18**, 412 (1976); *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications*, edited by R.M. Miura (Springer, New York, 1976); and references in these papers.

[3]See, for example, R. Hirota, J. Phys. Soc. Jpn. **35**, 1566 (1973), pp. 40–68, in *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications*, Ref. 1; M.J. Ablowitz and R. Haberman, Phys. Rev. Lett. **35**, 1185 (1975); V.S. Dryuma, Zh. Eksp. Teor. Fiz. Pis'ma Red. **19**, 753 (1974) [JETP Lett. **19**, 387 (1974)]; V.E. Zhakharov and A.B. Shabat, Funkts. Anal. i Ego Prilosh. **8**, 43 (1974) [Func. Anal. Appl. **8**, 226 (1974)].

[4]In this paper we shall confine the name *soliton* to solutions that tend to zero at infinity in all directions of space and that have certain stability properties under collisions.

[5]For a detailed treatment see, for example, I.M. Gel'fand, M.I. Graev, and N.Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. 5.

[6]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) and **64**, 1627 (1973) [Sov. Phys. JETP **34**, 62 (1972); 37, 823 (1974)].
[7]These requirements correspond to condition 4 of Ref. 5.
[8]We shall use the name *solitary wave* for any travelling wave that preserves its shape and vanishes at infinity in all directions.
[9]R. Hirota, Phys. Rev. Lett. **27**, 1192 (1971); M. Wadati and M. Toda, J. Phys. Soc. Jpn. **32**, 1403 (1972); C. S. Gardner *et al.*, Commun. Pure Appl. Math. **27**, 97 (1974).
[10]Wadati and Toda, Ref. 6; S. Tanaka, Kyoto Univ. Publ. Res. Inst. Math. Sci. **8**, 419 (1972/73).
[11]This implies a minor restriction on the $\mathbf{v}_i$. For example, two of them must not point in exactly opposite directions.
[12]tr stands for *trace*.
[13]Here, and only here, the tilde indicates *transpose* of a matrix.