Causality in the Coulomb Gauge

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A theoretical description and comparison of causal relations in the Lorentz and Coulomb gauges is presented with special emphasis upon retardation in the Coulomb gauge. It is shown that the transverse character of the current density in the Coulomb gauge compensates for the apparent instantaneous Coulomb interaction, which appears as a formal consequence of the subsidiary condition imposed upon the vector and scalar potentials.

INTRODUCTION

THE introduction of potentials is a common procedure in dealing with problems in electrodynamics. In this way Maxwell's equations can be written in forms which are formally simple and which permit one to draw upon a variety of formal techniques of analysis in treating the problem. There is a certain lack of uniqueness in the definition of the potentials, moreover, which allows one conveniently to impose conditions on the potentials without affecting the results of measurements made on the system being studied. Such choices are commonly called gauges, and perhaps the most common of these are the Lorentz gauge and the Coulomb gauge.[1] A prominent feature of the Lorentz gauge is that the finite speed of propagation of signals originating at the locations of the charge and current-density sources is emphasized throughout within the form of the field equations. On the other hand, the Coulomb gauge is characterized by an instantaneous Coulomb interaction which is clearly a formal result originating in the conditions one has imposed upon the potentials. Although the instantaneous propagation of signals is not possible in nature, any physically observable quantity calculated within the framework of the Coulomb gauge is independent of the choice of the gauge. Obviously, the description of propagation with finite velocity of the signals in the Coulomb gauge must be included within the formalism of the Coulomb gauge. It is the purpose of the present paper to compare the Coulomb gauge with the Lorentz gauge and to show how the Coulomb gauge properly describes the propagation with finite velocity of the electromagnetic signals due to the presence of charge and current sources. The discussion is carried out within the framework of the classical theory, but one can easily generalize the treatment to the quantized theory by recognizing the correspondence of the various Green's functions to the commutation relations for the quantum field operators in the quantized theory.[2,3]

I. FORMAL DESCRIPTION OF THE LORENTZ AND COULOMB GAUGES

In the Lorentz gauge, Maxwell's equations are[4]

$$\Box\, \mathbf{A}(\mathbf{r},t) = -4\pi \mathbf{J}(\mathbf{r},t) \qquad (1)$$

and

$$\Box\, \psi(\mathbf{r},t) = -4\pi\rho(\mathbf{r},t),$$

where

$$\Box = \nabla^2 - (\partial^2/\partial t^2)\ (c=1),$$

and the subsidiary condition imposed upon the potentials $\mathbf{A}(\mathbf{r},t)$ and $\psi(r,t)$ is

$$\nabla\cdot\mathbf{A}(\mathbf{r},t) + \partial\psi(\mathbf{r},t)/\partial t = 0.$$

[1] J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, Inc., New York, 1962), Chap. 6, p. 179.

[2] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., Inc., New York, 1949), 1st ed., Chap. 14, p. 373.

[3] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Ill., 1961), Chap. 9, p. 242.

[4] Ref. 1, p. 181

These equations have so-called advanced and retarded solutions. Without loss of generality we restrict ourselves to the consideration of the retarded solutions, which are

$$\mathbf{A}(\mathbf{r},t) = 4\pi \int d^3r'\, dt'\, D_R(\mathbf{r}-\mathbf{r}', t-t')\, \mathbf{J}(\mathbf{r}',t') \quad (2)$$

and

$$\psi(\mathbf{r},t) = 4\pi \int d^3r'\, dt'\, D_R(\mathbf{r}-\mathbf{r}', t-t')\, \rho(\mathbf{r}',t').$$

$D_R(\mathbf{r},t)$ is the retarded Green's function for the inhomogeneous D'Alembert's equation, and is related to the homogeneous Green's function $D(\mathbf{r},t)$ by the relation[5]

$$\theta(t)\, D(\mathbf{r},t) = -D_R(\mathbf{r},t), \quad (3)$$

where

$$\theta(t) = \begin{cases} 1 & t > 0 \\ \tfrac{1}{2} & t = 0 \\ 0 & t < 0. \end{cases}$$

Useful forms of $D(\mathbf{r},t)$ are

$$D(\mathbf{r},t) = -\frac{1}{(2\pi)^3} \int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{r}} \sin kt}{k} \quad (4)$$

and

$$D(\mathbf{r},t) = 1/4\pi r\, [\delta(r+t) - \delta(r-t)] = D_A(\mathbf{r},t) - D_R(\mathbf{r},t). \quad (5)$$

Here $k = |\mathbf{k}|$ and $r = |\mathbf{r}|$, and $D_A(\mathbf{r},t)$ is the Green's function for the inhomogeneous D'Alembert's equation corresponding to advanced solutions. $D_A(\mathbf{r},t)$ and $D_R(\mathbf{r},t)$ may be written

$$D_A(\mathbf{r},t) = \delta(r+t)/4\pi r, D_R(\mathbf{r},t) = \delta(r-t)/4\pi r. \quad (6)$$

Using Eq. (4), one can show that

$$\Box\, D_R(\mathbf{r},t) = -\delta^3(\mathbf{r})\delta(t). \quad (7)$$

Equations (2) may be written in the form

$$\mathbf{A}(\mathbf{r},t) = \int d^3r' \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \quad (8)$$

and

$$\psi(\mathbf{r},t) = \int d^3r' \frac{\rho(\mathbf{r}'\ t - |\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}.$$

[5] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publ. Co., Inc., Cambridge, Mass., 1955), Appendix A1, p. 419.

In the Lorentz gauge one then sees that the potentials are treated symmetrically and that both the vector potential $\mathbf{A}(\mathbf{r},t)$ and the scalar potential $\psi(\mathbf{r},t)$ are properly retarded, describing the propagation of source signals originating from the charge and current sources present at all the points $(\mathbf{r}',t - |\mathbf{r}-\mathbf{r}'|)$ and arriving at the point $\mathbf{r}$ at the later time t.

In the Coulomb gauge, Maxwell's equations are[6]

$$\Box\, \mathbf{A}_\perp(\mathbf{r},t) = -4\pi \mathbf{J}_\perp(\mathbf{r},t) \quad (9)$$

and

$$\nabla^2\phi(\mathbf{r},t) = -4\pi\rho(\mathbf{r},t) \quad (10)$$

with the subsidiary condition $\nabla\cdot\mathbf{A}_\perp(\mathbf{r},t) = 0$. It follows that $\nabla\cdot\mathbf{J}_\perp(\mathbf{r},t) = 0$. $\mathbf{J}_\perp(\mathbf{r},t)$ is the transverse part of $\mathbf{J}(\mathbf{r},t)$ in the sense to be described later.

Equation (10) has the solution

$$\phi(r,t) = \int d^3r' \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \quad (11)$$

while Eq. (9) has the retarded solution

$$\mathbf{A}_\perp(\mathbf{r},t) = 4\pi \int d^3r'\, dt'\, D_R(\mathbf{r}-\mathbf{r}'\ t-t')\, \mathbf{J}_\perp(\mathbf{r}',t)$$

or

$$\mathbf{A}_\perp(\mathbf{r},t) = \int d^3r' \frac{\mathbf{J}_\perp(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \quad (12)$$

Thus $\mathbf{A}_\perp(\mathbf{r},t)$ is made up of contributions from the transverse current $\mathbf{J}_\perp(\mathbf{r}',t)$ at the point $\mathbf{r}'$ originating at an earlier time $t - |\mathbf{r}-\mathbf{r}'|$ and propagating with the speed of light to the point $\mathbf{r}$ at the time t. On the other hand, from Eq. (11), one notes that the scalar potential $\phi(\mathbf{r},t)$ is built up of signals from the charge-density source $\rho(\mathbf{r}',t)$ at all points $\mathbf{r}'$ but at the same time t. That is, $\phi(\mathbf{r},t)$ describes an *instantaneous* Coulomb interaction. One notes also that the potentials are not treated symmetrically in the Coulomb gauge. Physically observable quantities, such as the electric intensity $\mathbf{E}(\mathbf{r},t)$, must be independent of the choice of the gauge, however.

[6] Ref. 1, p. 182.

In the Lorentz gauge,

$$\mathbf{E}^L(\mathbf{r},t) = -\partial \mathbf{A}(\mathbf{r},t)/\partial t - \nabla\psi(\mathbf{r},t).$$

Assuming that $\mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},\omega)e^{i\omega t}$ and $\rho(\mathbf{r},t) = \rho(\mathbf{r},\omega)e^{i\omega t}$, one obtains

$$\mathbf{E}^L(\mathbf{r},t) = i\omega \int d^3r' \frac{\mathbf{J}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} - \nabla \int d^3r' \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \quad (13)$$

Similarly in the Coulomb gauge, assuming that $\mathbf{J}_\perp(\mathbf{r},t) = \mathbf{J}_\perp(\mathbf{r},\omega)e^{-i\omega t}$ and $\rho(\mathbf{r},t) = \rho(\mathbf{r},\omega)e^{i\omega t}$, and using the fact that $\mathbf{E}^C(\mathbf{r},t) = -\partial\mathbf{A}_\perp(\mathbf{r},t)/\partial t - \nabla\phi(\mathbf{r},t)$,
one obtains

$$\mathbf{E}^C(\mathbf{r},t) = i\omega \int d^3r' \frac{\mathbf{J}_\perp(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} - \nabla \int d^3r' \frac{\rho(\mathbf{r}'\ t)}{|\mathbf{r}-\mathbf{r}'|}. \quad (14)$$

For a given $\mathbf{r}$ and t, Eqs. (13) and (14) must be equal. Moreover, one requires that Eq. (14) for $\mathbf{E}^C(\mathbf{r},t)$, when properly analyzed, must describe the propagation with finite velocity of the contributions to $\mathbf{E}^C(\mathbf{r},t)$ from the sources $\mathbf{J}(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$ at points on the light cone in the past. The superficial difference between Eq. (13) and Eq. (14) is clearly related to the transversality of $\mathbf{A}_\perp(\mathbf{r},t)$ or, equivalently, to the difference in the subsidiary conditions for the gauges in each case. The properly retarded properties of propagation can be demonstrated in the Coulomb gauge.

II. RETARDATION IN THE COULOMB GAUGE

Defining the transverse projection operator by the relation

$$\mathsf{P}_{ij}(\mathbf{r},\mathbf{r}') = \left[\delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}') - \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x'_j}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right], \quad (15)$$

and using the fact that

$$\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r}), \quad (16)$$

one sees easily that

$$\partial \mathsf{P}_{ij}(\mathbf{r},\mathbf{r}')/\partial x_i = 0. \quad (17)$$

Repeated indices will be summed over. Similarly, the quantity

$$\mathsf{T}_{ij}(\mathbf{r},\mathbf{r}') = \frac{\partial^2}{\partial x_i \partial x'_j}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \quad (18)$$

can be written

$$\mathsf{T}_{ij}(\mathbf{r},\mathbf{r}') = \left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{3(x_i - x'_i)(x_j - x'_j)}{|\mathbf{r}-\mathbf{r}'|^5} + \frac{4\pi}{3}\delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}')\right]. \quad (19)$$

An arbitrary vector $\mathbf{S}(\mathbf{r})$ can be written as the sum of a longitudinal vector $\mathbf{S}_L(\mathbf{r})$ and a transverse vector $\mathbf{S}_T(\mathbf{r})$. Noting that

$$\mathbf{S}(\mathbf{r}) = \frac{-\nabla^2}{4\pi}\int d^3r' \frac{\mathbf{S}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \quad (20)$$

and using the operator identity,

$$\nabla^2 = \nabla(\nabla\cdot - \nabla\times(\nabla\times \quad (21)$$

one obtains, after some manipulation,

$$\mathbf{S}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r'\,\mathsf{T}_{ij}(\mathbf{r},\mathbf{r}')\,S_i(\mathbf{r}')\,\hat{e}_j + \int d^3r'\,\mathsf{P}_{ij}(\mathbf{r},\mathbf{r}')\,S_j(\mathbf{r}')\,\hat{e} \quad (22)$$

The identification is made that

$$\mathbf{S}_L(\mathbf{r}) = \frac{1}{4\pi}\int d^3r'\,\mathsf{T}_{ij}(\mathbf{r},\mathbf{r}')\,S_i(\mathbf{r}')\,\hat{e}_j \quad (23)$$

and

$$\mathbf{S}_T(\mathbf{r}) = \int d^3r'\,\mathsf{P}_{ij}(\mathbf{r},\mathbf{r}')\,S_j(\mathbf{r}')\,\hat{e}_i. \quad (24)$$

Here use has been made of Eq. (13) in the second integral of Eq. (20). Equations (23) and (24) may be written in more convenient form as

$$\mathbf{S}_T = \mathsf{P}\mathbf{S},\ \mathbf{S}_L = (1/4\pi)\,\mathsf{T}\mathbf{S}. \quad (25)$$

When convenient, such functional relations as Eq. (23) and Eq. (24) will be written in this obvious notation. The transverse current $\mathbf{J}_\perp(\mathbf{r},t)$ is related to the current $\mathbf{J}(\mathbf{r},t)$ by the relation $\mathbf{J}_\perp = \mathsf{P}\mathbf{J}$, as may be seen by carrying out the derivation of Eqs. (1), (9), and (10) starting from Maxwell's equations written in their forms involving the electric- and magnetic-intensity field vectors. In addition, one can show that

$$[\Box, \mathsf{P}] = 0 \text{ and } [\mathsf{P}, D_R] = 0. \quad (26)$$

Using this fact, one then notes that

$$\mathsf{P}\Box\mathbf{A} = \Box\mathsf{P}\mathbf{A} = \Box\mathbf{A}_{\perp} = -4\pi(\mathsf{P}\mathbf{J}) = -4\pi\mathbf{J}_{\perp}. \quad (27)$$

Thus $(\mathsf{P}\mathbf{A} - \mathbf{A}_{\perp})$ is a solution to the homogeneous wave equation and only the case where $(\mathsf{P}\mathbf{A} - \mathbf{A}_{\perp}) = 0$ need be studied.

Now

$$\mathbf{A} = 4\pi D_R \mathbf{J}. \quad (28)$$

It follows that

$$\mathbf{A}_{\perp} = 4\pi \mathsf{P} D_R \mathbf{J} = 4\pi D_R \mathbf{J}_{\perp}. \quad (29)$$

At this point one is able to see the relationships between the Lorentz gauge and the Coulomb gauge. Consider the equation

$$\mathbf{A}_{\perp} = 4\, \mathsf{P} D_R \mathbf{J} = \mathsf{P}\mathbf{A}.$$

In explicit form,

$$\mathbf{A}_{\perp}(\mathbf{r}, t) = 4\pi \int d^3r'' \left[\int d^3r'\, dt''\, \mathsf{P}_{ij}(\mathbf{r}, \mathbf{r}') D_R(\mathbf{r}'\ \mathbf{r}'', t - t'') \right] J_j(\mathbf{r}'', t'')\, \hat{e}_i \quad (30)$$

or

$$\mathbf{A}_{\perp}(\mathbf{r}, t) = 4\pi \int d^3r' \left[\int d^3r''\, dt''\, \mathsf{P}_{ij}(\mathbf{r}, \mathbf{r}') D_R(\mathbf{r}', \mathbf{r}''\ t - t'') \right] J_j(\mathbf{r}'', t'')\, \hat{e}_i. \quad (31)$$

Some of the details of the composition of $\mathbf{A}_{\perp}(\mathbf{r},t)$ from signals propagating from the source currents, as described by these equations, are demonstrated schematically in Fig. 1. $\mathbf{A}(\mathbf{r}',t)$ at the point 2 with the space-time coordinates $(\mathbf{r}',t)$ is built up of signals from elements of the current $\mathbf{J}(\mathbf{r}'',t'')$ at the point 3 with space-time coordinates $(\mathbf{r}'',t'')$. Point 3 is located on the light cone the apex of which lies at point 2. One notes that $\mathbf{A}(\mathbf{r}',t)$ is the vector field one would obtain in the Lorentz gauge at the point $(\mathbf{r}',t)$ and that the signals from the current at point 3 propagate to point 2 with the speed of light. ($\mathbf{A}_{\perp}(\mathbf{r},t)$, the vector potential one obtains in the Coulomb gauge, is then seen to be the transverse part of $\mathbf{A}(\mathbf{r},t)$ or, in the framework of the present discussion. the result of the instantaneous propagation of $\mathbf{A}(\mathbf{r}',t)$ to the point $(\mathbf{r},t)$ through the transverse projection operator $\mathsf{P}_{ij}(\mathbf{r},\mathbf{r}')$. In Eq. (31) one sees that $\mathbf{A}_{\perp}(\mathbf{r},t)$ is the sum of such contributions taken over all times $t'' \leq t$, and all space points $\mathbf{r}''$ and $\mathbf{r}'$, with the restriction that $(\mathbf{r}',t)$ and $(\mathbf{r}'',t'')$ lie on the light cone as indicated above. In a similar way, recalling that $\mathbf{J}_{\perp} = \mathsf{P}\mathbf{J}$, one may write Eq. (12) in an explicit form:

$$\mathbf{A}_{\perp}(\mathbf{r},t) = 4\pi \int d^3r'\, dt''\, d^3r'' \{D_R(\mathbf{r},\mathbf{r}', t - t'')\, \mathsf{P}_{ij}(\mathbf{r}',\mathbf{r}'') J_j(\mathbf{r}'',t'')\, \hat{e}_i\}. \quad (32)$$

Figure 2 shows how $\mathbf{J}_{\perp}(\mathbf{r}',t'')$ is the transverse part of $\mathbf{J}(\mathbf{r}'',t'')$, or the instantaneous propagation of $\mathbf{J}(\mathbf{r}'',t'')$ to the point $(\mathbf{r}',t'')$ through the transverse projection $\mathsf{P}_{ij}(\mathbf{r}',\mathbf{r}'')$. The potential $\mathbf{A}_{\perp}(\mathbf{r},t)$ is then seen to be the properly retarded composition of signals originating with elements of the current density $\mathbf{J}_{\perp}(\mathbf{r}',t'')$ at all points in

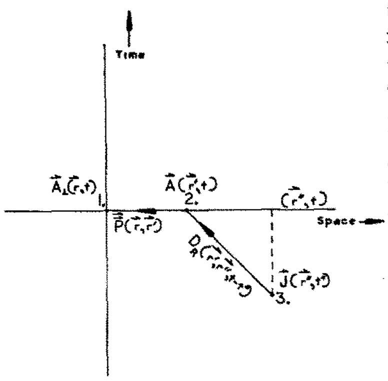

FIG. 1. $\mathbf{A}(\mathbf{r}',t)$, the vector potential in the Lorentz gauge, is built up from contributions of the current density $\mathbf{J}(\mathbf{r}'',t'')$ which propagate with the speed of light from the points $(\mathbf{r}'',t'')$. $\mathbf{A}_{\perp}(\mathbf{r},t)$, the vector potential in the Coulomb gauge, is the formal result of an instantaneous propagation of $\mathbf{A}(\mathbf{r}',t)$ from the points $(\mathbf{r}',t)$ to $(\mathbf{r},t)$.

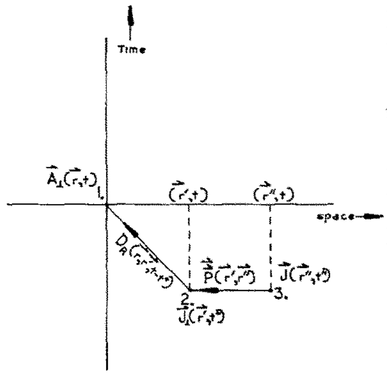

FIG. 2. $\mathbf{A}_{\perp}(\mathbf{r},t)$, the vector potential in the Coulomb gauge, is built up from contributions of the transverse current density $\mathbf{J}_{\perp}(\mathbf{r}',t'')$ which propagate with the speed of light from the points $(\mathbf{r}',t'')$. $\mathbf{J}_{\perp}(\mathbf{r}',t'')$ is the formal result of an instantaneous propagation of $\mathbf{J}(\mathbf{r}'',t'')$ from the points $(\mathbf{r}'',t'')$ to $(\mathbf{r}',t'')$.

the past on the light cone with its apex at $(\mathbf{r},t)$.

Returning to the question involving the reconciliation of the instantaneous Coulomb interaction in the Coulomb gauge with the fact that all interactions in physics must propagate with a finite velocity which must not be greater than the vacuum speed of light, one can now see that the instantaneous Coulomb interaction is a formal result and that the interactions in the Coulomb gauge do have the proper propagation features. The retarded propagation of the interactions is somewhat disguised in the form of $\mathbf{A}_{\perp}(\mathbf{r},t)$ as given by equivalent forms in the Eqs. (31) and (32). One notes in Fig. 1 and 2 that, in each case, a propagation along the light cone is involved in the composition of $\mathbf{A}_{\perp}(\mathbf{r},t)$ in addition to an instantaneous propagation in each case corresponding to the operation of taking the transverse part of $\mathbf{A}(\mathbf{r}',t)$ and $\mathbf{J}(\mathbf{r}',t'')$, respectively.

With these comparisons in mind, one can show that the terms arising from taking the transverse parts of these quantities include a contribution which exactly cancels the instantaneous Coulomb interaction term in Eq. (14) for $\mathbf{E}^C(\mathbf{r},t)$ and contributes another term which results in the properly retarded Coulomb interaction demonstrating the equivalence of Eqs. (13) and (14).

III. DEMONSTRATION OF THE EQUIVALENCE OF $\mathbf{E}^L(\mathbf{r},t)$ $\mathbf{E}^C(\mathbf{r},t)$

In particular, one sees that the key to recognizing the equivalence of Eq. (13) and Eq. (14) lies in the transversality of $\mathbf{J}_{\perp}(\mathbf{r},t)$. Substituting $\mathbf{J}_{\perp}=\mathsf{P}\mathbf{J}$ into Eq. (14) and rearranging terms, one obtains

$$E_i^C(\mathbf{r},\omega)=i\omega\int d^3r''\frac{J_i(\mathbf{r}'',\omega)e^{i\omega|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}''|}-\frac{\partial}{\partial x_i}\int d^3r'\frac{\rho(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|}$$
$$-\frac{i\omega}{4\pi}\int d^3r''\,d^3r'\frac{J_j(\mathbf{r}'',\omega)e^{i\omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial^2}{\partial x_j''\,\partial x_i'}\left(\frac{1}{|\mathbf{r}'-\mathbf{r}''|}\right). \quad (33)$$

After a partial integration with respect to $\mathbf{r}''$, the third term of this expression may be written in the form

$$III=\frac{i\omega}{4\pi}\int d^3r'\frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\int d^3r''\frac{\partial}{\partial x_i'}\left(\frac{1}{|\mathbf{r}'-\mathbf{r}''|}\right)\frac{\partial}{\partial x_j''}J_j(\mathbf{r}'',\omega).$$

Noting that

$$(\nabla^2+\omega^2)\frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}=-4\pi\delta^3(\mathbf{r}-\mathbf{r}'), \quad (34)$$

and using the equation for conservation of charge

$$\partial/\partial x_j''\,J_j(\mathbf{r}'',\omega)-i\omega\rho(\mathbf{r}'',\omega)=0, \quad (35)$$

one can write

$$III=\frac{1}{4\pi}\int d^3r'\left\{\nabla^2\frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right\}\left\{\frac{\partial}{\partial x_i'}\int d^3r''\frac{\rho(\mathbf{r}'',\omega)}{|\mathbf{r}'-\mathbf{r}''|}\right\}+\frac{\partial}{\partial x_i}\int d^3r''\frac{\rho(\mathbf{r}'',\omega)}{|\mathbf{r}-\mathbf{r}''|}. \quad (36)$$

Adding this result to the second term of Eq. (33), one sees that the terms involving $\frac{\partial}{\partial x_i}\int d^3r''\,\rho(\mathbf{r}'',\omega)\,(|\mathbf{r}-\mathbf{r}''|)^{-1}$ cancel. That is, the transverse current contains a term which results in the cancellation of the instantaneous Coulomb interaction term. The remaining terms are

$$\mathbf{E}_i^C(\mathbf{r},\omega)=i\omega\int d^3r''\frac{J_i(\mathbf{r}'',\omega)e^{i\omega|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}''|}+\frac{1}{4\pi}\int d^3r\left[\nabla^2\frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right]\left[\frac{\partial}{\partial x_i'}\int d^3r''\frac{\rho(\mathbf{r}'',\omega)}{|\mathbf{r}'-\mathbf{r}''|}\right].$$

In the second term of this expression one may replace ∇^2 by ∇'^2, then integrate by parts with respect to $\mathbf{r}'$, to obtain

$$\mathbf{E}_i^C(\mathbf{r},\omega)=i\omega\int d^3r''\frac{J_i(\mathbf{r}'',\omega)e^{i\omega|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}''|}+\frac{1}{4\pi}\int d^3r'\frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\left[\frac{\partial}{\partial x_i'}\int d^3r''\,\rho(\mathbf{r}'',\omega)\,\nabla'^2\left(\frac{1}{|\mathbf{r}'-\mathbf{r}''|}\right)\right].$$

Because of Eq. (16), this becomes

$$E_i^C(\mathbf{r},\omega) = i\omega \int d^3r'' \frac{J_i(\mathbf{r}'',\omega)\, e^{i\omega|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}''|} - \frac{\partial}{\partial x_i} \int d^3r' \frac{\rho(\mathbf{r}'',\omega)\, e^{i\omega|\mathbf{r}-\mathbf{r}''|}}{|\mathbf{r}-\mathbf{r}'|}.$$

Reinserting the time dependence $e^{-i\omega t}$, one then sees that

$$\mathbf{E}^C(\mathbf{r},t) = \mathbf{E}^L(\mathbf{r},t).$$

IV. DISCUSSION

A description and comparison of causal relations in the Lorentz and Coulomb gauges has been made. The instantaneous Coulomb interaction has been shown to be a formal consequence of the choice of subsidiary condition imposed upon the potentials introduced into Maxwell's equations. Moreover, the necessary result that the electric intensity calculated within the framework of each gauge must be the same has been derived and it has been shown that the transverse character of the current density $\mathbf{J}_\perp(\mathbf{r},t)$ compensates for the apparent instantaneous Coulomb interaction appearing in the expression for $\mathbf{E}^C(\mathbf{r},t)$.