Remarks on the Fundamental Postulates on Field Singularities in Electromagnetic Theory

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Abstract

By removing the constraints on field singularities in the divergence and Stokes' theorems, it is demonstrated that the universal boundary relations in electromagnetic theory can be obtained by a postulate on the integral form of Maxwell's equations. In that context, the present postulate is the complement of, and identical with regard to its consequences to, Idemen's original postulate, which he developed in 1990 for Maxwell's equations in differential (point) form.

Keywords: Maxwell equations; electromagnetic fields; boundary value problems; boundary conditions; distribution theory

1. Introduction

In electromagnetic theory, as in other branches of physics, the theoretical explanation of any phenomenon that can usually be cast into a mixed boundary value problem strictly depends on the proper mathematical description of the conditions that accompany the fundamental field equations in space-time. These include:

1. *the boundary/transition conditions*, which describe field behavior on a discontinuity surface;

2. *the initial conditions*, which describe field behavior at the instant the observation starts,

3. *the radiation condition* for open-region problems;

4. *the edge conditions*, which describe field behavior in the vicinity of physical and/or geometrical discontinuities; and

5. *other conditions* specific to the problem, such as symmetry or periodicity.

The superiority of one mathematical description of a phenomenon over another can be measured through the capability or flexibility of the tools employed in covering as many special cases as possible. And the ultimate purpose of different formulations is, naturally, to obtain the most general – *universal* – representation of the field equations and the accompanying conditions listed above. The everlasting efforts for the construction of a unified field theory in modern physics may provide an example for the former, while the search for the most powerful postulates on field equations may provide an example for the latter. Of course, it is equally important that the postulates are experimentally verifiable, and that the existence and uniqueness of the solution of the related mixed boundary-value problem are proven before attacking the problem.

The discussions in this paper are limited to some of the available postulates on the boundary relations, since the mathematical descriptions of the rest of the items in the list are rather less controversial or rigorous, in many cases. In what follows, a number of different techniques for deriving the boundary relations will be reviewed. A universal link between the differential- and integral-form postulates of Maxwell's equations will be established by generalizing the divergence and Stokes' theorems of vector calculus, in the sense of distributions.

2. The Differential and Integral Forms of Maxwell's Equations

The first discussion with regard to the postulates of boundary relations in electromagnetic theory deals with whether the differential- or integral-form representation of Maxwell's equations given below is better suited to the fundamental laws of electricity.

2.1 The Differential Form of Maxwell's Equations

$$curl\bar{B} + \frac{\partial \bar{E}}{\partial t} = -\bar{J}_g^m, \qquad (1a)$$

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$$curl\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}^e_\vartheta, \quad (1b)$$

$$div\vec{D} = \rho^e_\vartheta, \quad (1c)$$

$$div\vec{B} = \rho^m_\vartheta. \quad (1d)$$

2.2 The Integral Form of Maxwell's Equations

$$\oint_{C=\partial S} \vec{E}\cdot d\vec{c} + \iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} = -\iint_S \vec{J}^m_\vartheta \cdot d\vec{S}, \quad (2a)$$

$$\oint_{C=\partial S} \vec{H}\cdot d\vec{c} - \iint_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{S} = \iint_S \vec{J}^e_\vartheta \cdot d\vec{S}, \quad (2b)$$

$$\oiint_{S=\partial \vartheta} \vec{D}\cdot d\vec{S} = \iiint_\vartheta \rho^e_\vartheta \, d\vartheta, \quad (2c)$$

$$\oiint_{S=\partial \vartheta} \vec{B}\cdot d\vec{S} = \iiint_\vartheta \rho^m_\vartheta \, d\vartheta. \quad (2d)$$

In Equations (2a) and (2b), S denotes an arbitrary, regular, open, two-sided surface, while in Equations (2c) and (2d), ϑ stands for an arbitrary regular region.

One can intuitively claim that two different formulations of the same phenomenon need to be equally informative (though they are not supposed to be equally practical, in every case) as long as both sets are *postulated properly*. By equal information, we imply that any mathematical relation derived through one set of Maxwell's equations should also be derivable through the other set.

Maxwell himself had postulated the law of electricity in differential form [1, Chapter 9]. However, Schelkunoff stated in [2, Section 5] that Maxwell's equations were required to be postulated in integral form in parallel to all the experimental evidence until Maxwell introduced the displacement-current concept, since the partial derivatives of field terms can take infinite values on a surface of discontinuity, which makes the point-form representation impractical or meaningless. Tai also supported the integral-form postulation in [3, Section 3.3], concluding that Maxwell's equations are more informative in integral form where the boundary relations are concerned. In his books, Jones emphasized (see [4, p. 46], [5, p. 44]) that the integral form of Maxwell's equations can be used in deriving the boundary relations only when they are *assumed* to hold in the presence of field singularities. It is seen in the literature that both forms of Maxwell's equations are employed for postulating the boundary relations, and the validity of the mathematical tools in passing from one postulate to another has always been subject to debate. In the next section, we shall give a brief account of these postulates.

3. Review of Some Integral- and Differential-Form Postulates on Maxwell's Equations

3.1. Integral-Form Postulates

There are two techniques widely used in the literature for deriving the boundary relations through the integral-form postulate of Maxwell's equations. These are:

> I. The integration of fields and sources in a volume and along a narrow strip that straddles the surface of discontinuity, Σ; and
>
> II. The integration of fields in regions on adjacent sides of Σ and joining these relations over Σ in the Cauchy sense.

The first approach is embedded in the textbooks. However, the great majority of authors hardly comment on the validity of the divergence and Stokes' theorems in these applications, especially when the fields are piecewise continuous (as on the interface between two simple media), or when they possess a first-order singular term (as is the case for a double layer on the interface (cf. [6, Section 1.4], [7]).

The widely known restriction on the vector field in the divergence/Stokes' theorems, as met in basic calculus books, is that the field and its partial derivatives must be continuous at all points inside the regular domain of integration (volume/open, two-sided surface) and on its enclosure (or boundary).

The first extension principle serves to lighten the field constraints on the divergence and Stokes' theorems to a certain extent:

> The divergence and Stokes' theorems still hold as long as the field is continuous inside the integration domain and on its enclosure and *piecewise continuously differentiable* in the interiors of a finite number of regions, the sum of which constitute the integration domain. [8, Chapter 4].
>
> The integrals in such theorems need to be understood to be improper integrals when/if the partial derivatives of field components become infinite at the boundaries between the regions. [9, p. 488].

It should be noted that in the presence of sources expressed by first- and higher-order Dirac delta distributions, the fields possess higher-order singularities than covered by the first extension principle. Besides, the mathematical steps of this approach are not rigorous, and the integrations need to be done very carefully.

The second approach, on the other hand, relies on the application of the divergence and Stokes' theorems for smooth functions, and is therefore incapable of representing sources expressed by first- and higher-order Dirac delta distributions. And this limits the applicability to the mathematical interface between two arbitrary media (cf. [10, pp. 113-125], [11, pp. 98-107], [12, pp. 4-8] for a demonstration of the method).

3.2. Differential Form Postulates

The boundary relations on a mathematical interface can be obtained rigorously when the field components are *assumed* to possess a jump discontinuity on the interface. Examples are available in many books and papers, including the works of Panicali [13] and Namias [14]. The methodology is quite simple and powerful, based on two key points:

> I. The source quantities are expressed in terms of Dirac delta distributions of a given order, and so are the fields on adjacent sides of the discontinuity surface of one order less.
>
> II. There is one-to-one correspondence between the field equations that must hold, and the coefficients of the singular terms (Dirac delta distributions) at the two sides of the field equations must be equal when the field and source representations are entered.

The postulate on Maxwell's equations in differential form that covers all types (orders) of polarization mechanisms was presented by İdemen in 1973 [15]: "*The Maxwell's equations are always valid in the sense of distributions.*"

According to this postulate, any vector and scalar functions (say $\vec{A}$ and V) met in Maxwell's equations in differential form are described in the general form

$$\vec{A}(\vec{r};t) = \left\{\vec{A}(\vec{r};t)\right\} + \sum_{k=0}^{\infty} \vec{A}_k(\vec{r};t)\delta_\Sigma^{(k)}, \tag{3}$$

$$V(\vec{r};t) = \left\{V(\vec{r};t)\right\} + \sum_{k=0}^{\infty} V_k(\vec{r};t)\delta_\Sigma^{(k)}, \tag{4}$$

and assumed to satisfy the set of Equations (1a)-(1d) at any point, including the surface of discontinuity, Σ. In the standard terminology of generalized functions, the terms in curly brackets are called the "regular part" of the quantity (the part defined at points *other than* on the discontinuity surface, Σ). The sum term with the Dirac delta distributions is called the "singular part" of the quantity (the part defined *on* the discontinuity surface, Σ). $\delta_\Sigma^{(k)}$ denotes kth derivative of the Dirac delta distribution. $\vec{A}_k(\vec{r};t)\delta_\Sigma^{(k)}$ and $V_k(\vec{r};t)\delta_\Sigma^{(k)}$ correspond to the $(k+1)$th terms in the singular part of the field quantities.

While the postulate was implemented only for a mathematical boundary in the 1973 paper, a very general implementation for a planar material boundary is also available in İdemen's 1990 paper [16] (see also [17-20] for an entire list of İdemen's work on the topic).

The representation of scalar and vector sources in terms of distributions has been widely used since the beginning of the last century, and has a solid physical correspondence. For instance, when V corresponds to the volume charge density, the zeroth-order distribution $V_0\delta_\Sigma$ denotes the surface charge density on Σ, while $V_1\delta_\Sigma^{(1)}$ and $V_2\delta_\Sigma^{(2)}$ signify a layer of dipoles and quadrupoles on Σ, and so on (cf. the works of Namias [21-22] for the incorporation of the Dirac delta distribution in representing first-and higher-order polarization mechanisms).

On the other hand, such a representation of sources also requires the field terms to be expressed in a similar fashion, for compatibility. However, where field quantities are concerned one should not relate $\vec{A}_k$ to $\vec{A}$ and V_k to V physically, since they have different units. Actually, $\vec{A}_k$ and V_k are not supposed to have a physical correspondence by themselves, at all.

The application of the divergence and curl operators on $\vec{A}$ in Equation (3) and the steps of the solution shall also be reviewed, for completeness:

Assuming that the only surface of discontinuity in ϑ is Σ, and that there are no sources inside regions ϑ_1 and ϑ_2 that constitute ϑ through $\vartheta = \vartheta_1 \cup \Sigma \cup \vartheta_2$, one may express the regular part of $\vec{A}$ as

$$\left\{\vec{A}\right\} = \begin{cases} \vec{A}^I, & \vec{r} \in \vartheta_1 \\ \vec{A}^{II}, & \vec{r} \in \vartheta_2 \end{cases}, \tag{5}$$

and the application of the divergence and curl operators to $\left\{\vec{A}\right\}$ gives

$$div\vec{A} = div\left\{\vec{A}\right\} + \sum_{k=0}^{\infty} div\left[\vec{A}_k\delta_\Sigma^{(k)}\right], \tag{6a}$$

where

$$div\left\{\vec{A}\right\} = \left\{div\vec{A}\right\} + \hat{\Sigma}\cdot\Delta\left[\vec{A}\right]\delta_\Sigma \tag{6b}$$

with

$$\left\{div\vec{A}\right\} = \begin{cases} div\vec{A}^I, & \vec{r} \in \vartheta_1 \\ div\vec{A}^{II}, & \vec{r} \in \vartheta_2 \end{cases}, \tag{6c}$$

$$\Delta\left[\vec{A}\right] = \vec{A}^I - \vec{A}^{II}, \tag{6d}$$

$$div\left[\vec{A}_k\delta_\Sigma^{(k)}\right] = div\vec{A}_k\delta_\Sigma^{(k)} + \vec{A}_k \cdot grad\delta_\Sigma^{(k)}, \tag{6e}$$

and

$$curl\vec{A} = curl\left\{\vec{A}\right\} + \sum_{k=0}^{\infty} curl\left[\vec{A}_k\delta_\Sigma^{(k)}\right], \tag{7a}$$

where

$$curl\left\{\vec{A}\right\} = \left\{curl\vec{A}\right\} + \hat{\Sigma}\wedge\Delta\left[\vec{A}\right]\delta_\Sigma, \tag{7b}$$

with

$$\left\{curl\vec{A}\right\} = \begin{cases} curl\vec{A}^I, & \vec{r} \in \vartheta_1 \\ curl\vec{A}^{II}, & \vec{r} \in \vartheta_2 \end{cases}, \tag{7c}$$

$$curl\left[\bar{A}_k\delta_\Sigma^{(k)}\right] = curl\bar{A}_k\,\delta_\Sigma^{(k)} + grad\,\delta_\Sigma^{(k)} \wedge \bar{A}_k\,. \tag{7d}$$

In Equations (6b), (6d), (6e), (7b) and (7d), $\hat{\Sigma}$ denotes the normal of the surface Σ, and is assumed to be directed into ϑ_1. Proofs of the relations of Equations (6b)-(6d), (7b), and (7c) are available in [15] and [23, Chapter 1].

Next, one inserts all sources and field quantities expressed by Equations (3)-(7) into Equations (1a)-(1d) to get four sets of equations, which include regular parts and singular terms with the Dirac delta distributions of every order. The constitutive relations are not needed to be included for our purposes.

Now there are only two steps ahead in order to reach the universal boundary relations. The first one is due to Item II given in this section. This yields two sets of solutions: one indicates that the regular parts of the field and source terms satisfy the set of Equations (1a)-(1d), and the other signifies an infinite number of equations for $k = 0,1,2,\ldots$ to hold on the interface. As can easily be seen from Equations (6b) and Equations (7b) and (7c), the equations for $k = 0$ give the amount of discontinuity of the normal and tangential components of field quantities on the interface. They are therefore called the universal boundary conditions. The remaining (infinite number) equations for $k = 1,2,\ldots$ are called the compatibility equations. The next and final step in the solution is the incorporation of the fact that the singularity that a source or field quantity can possess on a discontinuity surface is supposed to be of finite order, i.e., there is a finite number N for which

$$\bar{A}_k(\bar{r};t) \equiv \bar{0}\,,\ \ k \geq N+1, \tag{8a}$$

$$V_k(\bar{r};t) \equiv 0\,,\ \ k \geq N+1. \tag{8b}$$

By substituting the conditions of Equations (8a) and (8b) into the compatibility conditions, one can reach the resultant form of the boundary conditions in a straightforward manner. It should be noted that the conditions of Equations (8a) and (8b) bring no restrictions on the universal nature of the postulate. The explicit expressions of the final set of boundary relations for a planar discontinuity surface are available in the work of İdemen, and are beyond the scope of this paper.

4. The Missing Link Between the Integral- and Differential-Form Postulates

As mentioned in Section 2, one may expect the differential- and integral-form postulates to be equally informative. However, It has been the lack of associated mathematical tools that have led to comparisons of these two sets in the presence of field singularities. A universal link between them can be established *if* one can demonstrate that the universal boundary relations of İdemen can also be obtained when the *integral* (rather than differential!) forms of Maxwell's equations are postulated to be valid in the sense of distributions.

We will next prove this assertion in two steps. First, we shall consider the equivalence of the set of Equations (1a)-(1d) to its formal integral form, Equations (9a)-(9d), given below:

$$\iint_S curl\bar{E}\cdot d\bar{S} + \iint_S \frac{\partial\bar{B}}{\partial t}\cdot d\bar{S} = -\iint_S \bar{J}_\vartheta^m \cdot d\bar{S}\,, \tag{9a}$$

$$\iint_S curl\bar{H}\cdot d\bar{S} - \iint_S \frac{\partial\bar{D}}{\partial t}\cdot d\bar{S} = \iint_S \bar{J}_\vartheta^e \cdot d\bar{S}\,, \tag{9b}$$

$$\iiint_\vartheta div\bar{D}\,d\vartheta = \iiint_\vartheta \rho_\vartheta^e\,d\vartheta\,, \tag{9c}$$

$$\iiint_\vartheta div\bar{B}\,d\vartheta = \iiint_\vartheta \rho_\vartheta^m\,d\vartheta\,. \tag{9d}$$

The equivalence of Equations (1a)-(1d) to Equations (9a)-(9d) can be expressed as follows. If an equation is known (or assumed) to hold at all points in space, then the integration of both sides of this equation over a regular region – or a regular, open, two-sided surface – is supposed hold as well. And, similarly, if an integral relation is seen to hold for *every* arbitrary regular integration domain, then one can infer from the basic theorems of integration that this can be provided *if and only if* the integrand functions at both sides of the relation are equal, or differ at most by a null function. It is an easy task to show that this general one-to-one correspondence applies in the presence of integrands expressed in terms of generalized functions (distributions) as well as smooth functions, provided the integrals are bounded. An example of this formal equivalence between the integral and point form of an equation is available in [24, p. 3] for the divergence theorem as a lemma with proof.

Next, we shall compare Equations (9a)-(9d) with Equations (2a)-(2d). It is seen that the equivalence of Equations (9a)-(9d) to Equations (2a)-(2d) (and therefore to Equations (1a)-(1d)) strictly depends on the validity of the divergence and Stokes' theorems in the presence of singular fields given in the general form of Equation (3). On the other hand, this property of the divergence and Stokes' theorems is proved in the Appendix, through the generalization of the extension principle.

Therefore, one can conclude that *the postulate of İdemen, which was intended for the differential form of Maxwell's equations, applies equally to the same set in integral form.*

5. Concluding Remarks

I. The validity of Maxwell's equations (in differential and integral form) in the sense of distributions yields the result that the relations that evolve from them (such as the continuity condition and energy relations) are also valid in the sense of distributions.

II. The Lorentz force equation, as one of the main field equations, should also be postulated in the sense of distributions for completeness.

III. It is expected that the procedure presented in the Appendix for removing the field requirements in the divergence and Stokes' theorems can be applied to a large number of integral theorems in vector calculus, and can find direct application in other branches of physics as well, such as acoustics, fluid mechanics, and elasticity.

IV. It should be noted that the initial conditions in electromagnetic theory have also been postulated by İdemen in 1993 [20], based on a representation similar to that applied for the boundary relations. The formal equivalence between the differential and integral forms of Maxwell's equations reveals that the universal initial relations given in [20] for the set of Equations (1a)-(1d) can also be reached through Maxwell's equation in integral form with respect to the time variable.

V. The postulate of İdemen based on an infinite-sum representation of field singularities has also been applied for a comprehensive treatment of impedance boundary relations [16, 20] and edge conditions [25, 26].

VI. The postulate of İdemen for Maxwell's equations is also valid in the presence of moving boundaries. It has been shown by the author that a combination of the methodologies given in [14] and [16] can be applied directly to yield universal initial and boundary relations in the presence of moving boundaries [27].

VII. Regarding education, the books written by İdemen (in Turkish) have been used in Istanbul Technical University in both undergraduate and graduate electromagnetics courses since 1973, in which the initial and boundary relations have been presented in parallel to his contributions in the area. With the help of an earlier math course on vector calculus, and introduction of the basic properties of the Dirac delta distribution (which includes the relations of Equations (6b) and (7b) without proof) within the electromagnetics course, the theories have always been fully grasped by sophomore and junior students.

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7. Appendix

7.1 Generalized Extension Principle

The validity of a general class of integral theorems relating volume to surface integrals requires that the following relation hold:

$$\iiint_{\vartheta} \frac{\partial f}{\partial x_i}(\bar{r})\,d\vartheta = \oiint_{S=\partial\vartheta} f(\bar{r})(\hat{x}_i \cdot \hat{n})\,dS\,,\ i=1,2,3, \qquad \text{(A1)}$$

where $\bar{r} = (x_1, x_2, x_3)$; ϑ is an arbitrary regular region enclosed by a regular surface, S; and $\hat{n}$ is the outer normal of S. In general, the region ϑ is assumed to be convex with no holes. However, it is an easy task to remove such constraints in the way explained in many textbooks (cf. [28], pp. 138-139, Remarks 2, 4).

The first extension principle and the remarks by Van Bladel as presented in Section 3.1 lighten the field requirements to a certain extent. However, for removing the requirements on $f(\bar{r})$ *totally*, one may attempt to generalize the first extension principle as follows:

The relations of Equation (A1) are always valid in the sense of distributions.

7.2 Proof

Due to the singular nature of the scalar field $f(\bar{r})$, the relation of Equation (A1) describes a distribution in the form

$$\left\langle \frac{\partial f}{\partial x_i}, 1 \right\rangle_{\vartheta} = \left\langle f\delta_S(\hat{x}_i \cdot \hat{n}), 1 \right\rangle_S,\ i=1,2,3, \qquad \text{(A2)}$$

where the inner-product operation between a distribution, g, and a test function, ϕ, is given by

$$\langle g, \phi \rangle_{\vartheta_a} = \iiint_{\vartheta_a} g(\bar{r})\phi(\bar{r})\,d\vartheta,$$

and the surface distribution, δ_S, is described by

$$\langle \delta_S, \phi \rangle_{S_a} = \iint_{S_a} \phi(\bar{r})\,dS.$$

The subscripts ϑ_a and S_a for the angular brackets indicate the domain of integration, for clarity.

For the derivation of many (differential-form) properties of distributions, it is generally assumed that $\phi(\bar{r})$ is in class C^{∞} and has compact support, while no restriction is required on the support of the distribution $g(\bar{r})$. For the proof of an integral relation in the sense of distributions, one can alternatively define $g(\bar{r})$ to be identically zero outside ϑ_a and take $\phi(\bar{r}) = 1$, as is done in [29, Section 5.4]. The validity of the latter approach is plausible, since the inner products still yield finite values.

As a rule of thumb, the distribution, f, in terms of its regular ($\{f\}$) and singular (f_Σ) parts may be written as

$$f = \{f\} + f_\Sigma, \qquad \text{(A3)}$$

where Σ is the surface of discontinuity straddling the region ϑ. The regular part of f is expressed by

$$\{f\} = \begin{cases} f^I, & \bar{r} \in \vartheta_1 \\ f^{II}, & \bar{r} \in \vartheta_2 \end{cases}, \qquad \text{(A4)}$$

as in Equation (5).

The proofs of the present and the following theorems are limited to the special case of Dirac delta distributions, i.e., f_Σ is considered to have the special form

$$f_\Sigma = \sum_{k=0}^{\infty} f_k \delta_\Sigma^{(k)}, \qquad \text{(A5)}$$

where the f_k, $\forall k$, are assumed to be smooth functions on the surface Σ. In this case, the representation of Equation (A1) in angular-bracket notation, as in Equation (A2), does not yield more informative or general results than can be obtained through the integral representation Equation (A1) itself, and therefore this does not need to be carried in the following steps of the proof.

Substituting Equation (A3) into Equation (A1) yields

$$\iiint_{\vartheta}\left(\frac{\partial\{f\}}{\partial x_i}+\frac{\partial f_\Sigma}{\partial x_i}\right)d\vartheta = \oiint_{S=\partial\vartheta}\left(\{f\}+f_\Sigma\right)\left(\hat{x}_i\cdot\hat{n}\right)dS,\quad i=1,2,3. \tag{A6}$$

Since the regular and singular components of f have nonintersecting regions of support by definition, our assertion requires the following two relations:

$$\iiint_{\vartheta}\frac{\partial\{f\}}{\partial x_i}d\vartheta = \oiint_{S=\partial\vartheta}\{f\}\left(\hat{x}_i\cdot\hat{n}\right)dS,\quad i=1,2,3, \tag{A7}$$

$$\iiint_{\vartheta}\frac{\partial f_\Sigma}{\partial x_i}d\vartheta = \oiint_{S=\partial\vartheta}f_\Sigma\left(\hat{x}_i\cdot\hat{n}\right)dS,\quad i=1,2,3, \tag{A8}$$

to hold separately, under the assumption that all four of the inner products in Equations (A7) and (A8) yield finite values.

Regarding Equation (A7), its validity is obvious through the standard theorems of vector calculus when $\{f\}$ and its first partial derivatives are smooth functions (i.e., in the case where they do not possess jump discontinuities). When $\{f\}$ is discontinuous on Σ, its derivative is expressed by (cf. [30, pp. 118-119] for proof)

$$\frac{\partial\{f\}}{\partial x_i}=\left\{\frac{\partial f}{\partial x_i}\right\}+\Delta[f]\left(\hat{x}_i\cdot\hat{\Sigma}\right)\delta_\Sigma,\quad i=1,2,3, \tag{A9}$$

with $\Delta[f]=f^I-f^{II}$. In this case, the left-hand side of Equation (A7) still yields a finite value, since the support of $\frac{\partial\{f\}}{\partial x_i}$ is again limited to the region ϑ and its singular term is integrable.

Substituting Equation (A9) into Equation (A7) yields

$$\iiint_{\vartheta_1+\vartheta_2}\left\{\frac{\partial f}{\partial x_i}\right\}d\vartheta = \iint_{S_1+S_2}\{f\}\left(\hat{x}_i\cdot\hat{n}\right)dS-\iint_{\Sigma}\Delta[f]\left(\hat{x}_i\cdot\hat{\Sigma}\right)dS,\quad i=1,2,3, \tag{A10}$$

where the $S_{1,2}$ denote the parts of the closed surface S that remain in regions $\vartheta_{1,2}$.

The proof of Equation (A10) can be reached upon the summation in the Cauchy sense of the following two integral relations, well known for regular functions:

$$\iiint_{\vartheta_1}\frac{\partial f^I}{\partial x_i}d\vartheta = \iint_{S_1}f^I\left(\hat{x}_i\cdot\hat{n}_1\right)dS+\iint_{\Sigma_1}f^I\left(\hat{x}_i\cdot\hat{n}_1\right)dS,\quad i=1,2,3, \tag{A11}$$

$$\iiint_{\vartheta_2}\frac{\partial f^{II}}{\partial x_i}d\vartheta = \iint_{S_2}f^{II}\left(\hat{x}_i\cdot\hat{n}_2\right)dS+\iint_{\Sigma_2}f^{II}\left(\hat{x}_i\cdot\hat{n}_2\right)dS,\quad i=1,2,3, \tag{A12}$$

with

$$\hat{n}_1=\begin{cases}\hat{n}, \bar{r}\in S_1\\ -\hat{\Sigma}, \bar{r}\in\Sigma_1\end{cases},$$

$$\hat{n}_2=\begin{cases}\hat{n}, \bar{r}\in S_2\\ +\hat{\Sigma}, \bar{r}\in\Sigma_2\end{cases}.$$

In Equations (A11) and (A12), the $\Sigma_{1,2}$ denote surfaces residing in regions $\vartheta_{1,2}$, and parallel to and coinciding with the interface Σ in the limiting case.

Regarding Equation (A8), the regions of support of $\frac{\partial f_\Sigma}{\partial x_i}$ and f_Σ are the interface Σ. However, the formal integration on the right-hand side is *always* reduced to the left-hand side in Equation (A8), based on the fact that both integrals in Equation (A8) have finite values due to the basic properties of the Dirac delta distribution, such as

$$f(x)\delta^{(k)}(x-x_0)=(-1)^k f^{(k)}(x_0)\delta(x-x_0) + \text{higher order terms}$$

and

$$\int_{x_1}^{x_2}\delta^{(k)}(x-x_0)\,dx=\begin{cases}H(x-x_0)\Big|_{x_1}^{x_2}=1, & k=0\\ \delta^{(k)}(x-x_0)\Big|_{x_1}^{x_2}=0, & k=1,2,\ldots\end{cases}$$

for $x_1<x_0<x_2$, where H denotes the Heaviside unit step function.

It should be noted that the integrands in the standard integral theorems of vector calculus are smooth functions, and the integrations are therefore defined in the Riemann sense. However, formal integrations, such as in Equation (A8), also apply in the generalized sense, as long as the basic properties of the Dirac delta distributions are invoked properly.

This concludes the proof.

7.3 Generalized Divergence Theorem

The divergence theorem is always valid in the sense of distributions.

7.4 Proof

It is required to show that the integral relation

$$\iiint_{\vartheta}div\vec{f}(\bar{r})\,d\vartheta = \oiint_{S=\partial\vartheta}\vec{f}(\bar{r})\cdot\hat{n}\,dS \tag{A13}$$

holds in an arbitrary regular region ϑ, enclosed by a regular surface S, when the vector field $\vec{f}(\bar{r})$

I. possesses a singular part of *any order* on an arbitrary regular surface Σ straddling the region ϑ, and

II. its partial derivatives are continuous inside the region $\vartheta - \Sigma$ and on its boundary surface, S.

For $\vec{f}(\vec{r}) = \sum_{i=1}^{3} f_i(\vec{r})\hat{x}_i$, the proof of Equation (A13) reduces to the requirement that the following three constituents,

$$\iiint_{\vartheta} \frac{\partial f_i}{\partial x_i}(\vec{r})\, d\vartheta = \oiint_{S=\partial\vartheta} f_i(\vec{r})(\hat{x}_i \cdot \hat{n})\, dS, \; i = 1,2,3, \qquad \text{(A14)}$$

of Equation (A13) hold in the sense of distributions. This is nothing but the already-proved generalized extension principle with $f = f_i$.

This concludes the proof.

7.5 Generalized Stokes' Theorem

Stokes' theorem is always valid in the sense of distributions.

7.6 Proof

It is required to show that the integral relation

$$\iint_S curl\vec{f}(\vec{r}) \cdot d\vec{S} = \oint_{C=\partial S} \vec{f}(\vec{r}) \cdot d\vec{c} \qquad \text{(A15)}$$

holds on an arbitrary, regular, open, two-sided surface S, enclosed by a regular contour C, when the vector field $\vec{f}(\vec{r})$

I. possesses a singular part of *any order* on an arbitrary, regular, open, two-sided surface Σ straddled by S, and

II. its partial derivatives are continuous on the surface $S - C_\Sigma$ and along its boundary, C.

Here, C is traversed in the direction such that S appears to the left of an observer moving along C, and $C_\Sigma = S \cap \Sigma$ is the intersection line of the surfaces S and Σ.

This time, we need to prove the relation

$$\iint_S curl\left[\left\{\vec{f}(\vec{r})\right\} + \vec{f}_\Sigma(\vec{r})\right] \cdot d\vec{S} = \oint_{C=\partial S} \left[\left\{\vec{f}(\vec{r})\right\} + \vec{f}_\Sigma(\vec{r})\right] \cdot d\vec{c}. \qquad \text{(A16)}$$

It suffices to show that the following two relations, which constitute Equation (A16), hold separately:

$$\iint_S curl\left\{\vec{f}(\vec{r})\right\} \cdot d\vec{S} = \oint_{C=\partial S} \left\{\vec{f}(\vec{r})\right\} \cdot d\vec{c} \qquad \text{(A17)}$$

and

$$\iint_S curl\left[\vec{f}_\Sigma(\vec{r})\right] \cdot d\vec{S} = \oint_{C=\partial S} \left[\vec{f}_\Sigma(\vec{r})\right] \cdot d\vec{c}. \qquad \text{(A18)}$$

Since $\left\{\vec{f}\right\}$ may in general be discontinuous and have discontinuous partial derivatives on Σ, we can construct Equation (A17) as the sum of the following two relations:

$$\begin{aligned}\iint_S curl\left\{\vec{f}\right\} \cdot d\vec{S} &= \iint_{S_1+S_2} \left\{curl\vec{f}\right\} \cdot d\vec{S} + \iint_S \left[\hat{\Sigma} \wedge \Delta\left[\vec{f}\right]\delta_\Sigma\right] \cdot d\vec{S} \\ &= \iint_{S_1+S_2} \left\{curl\vec{f}\right\} \cdot d\vec{S} + \int_{C_\Sigma} \left[\hat{\Sigma} \wedge \Delta\left[\vec{f}\right]\right] \cdot d\vec{c},\end{aligned} \qquad \text{(A19)}$$

and

$$\iint_{S_1+S_2} \left\{curl\vec{f}\right\} \cdot d\vec{S} = \oint_{C=\partial S} \left\{\vec{f}\right\} \cdot d\vec{c} - \int_{C_\Sigma} \left[\hat{\Sigma} \wedge \Delta\left[\vec{f}\right]\right] \cdot d\vec{c}. \qquad \text{(A20)}$$

Equation (A19) is the relation of Equation (7b) in integral form, and Equation (A20) can be obtained through the summation of the following applications of Stokes' theorem for regular functions:

$$\iint_{S_1} \left\{curl\vec{f}\right\} \cdot d\vec{S} = \oint_{\partial S_1 - C_{\Sigma 1}} \left\{\vec{f}\right\} \cdot d\vec{c} + \int_{C_{\Sigma 1}} \left\{\vec{f}\right\} \cdot d\vec{c}, \qquad \text{(A21)}$$

$$\iint_{S_2} \left\{curl\vec{f}\right\} \cdot d\vec{S} = \oint_{\partial S_2 - C_{\Sigma 2}} \left\{\vec{f}\right\} \cdot d\vec{c} + \int_{C_{\Sigma 2}} \left\{\vec{f}\right\} \cdot d\vec{c}, \qquad \text{(A22)}$$

where $C_{1\Sigma}$ and $C_{2\Sigma}$ are parts of the enclosures of S_1 and S_2 parallel to and coinciding with C_Σ in the limiting case (see the previously given references ([10, pp. 113-125], [11, pp. 98-107], [12, pp. 4-8]) for the details of the derivation).

Regarding Equation (A18), the procedure followed in the proof of the standard Stokes' theorem yields the desired relation, since the integrations of both sides *always* have finite values, due to the basic properties of the Dirac delta distribution mentioned earlier.

This concludes the proof.

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