Cylindrical Waves

Daniel S. Weile

Department of Electrical and Computer Engineering
University of Delaware

ELEG 648—Waves in Cylindrical Coordinates

Outline

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1. Cylindrical Waves
   - Separation of Variables
   - Bessel Functions
   - TE_z and TM_z Modes
2. Guided Waves
   - Cylindrical Waveguides
   - Radial Waveguides
   - Cavities

The Scalar Helmholtz Equation

Just as in Cartesian coordinates, Maxwell's equations in cylindrical coordinates will give rise to a scalar Helmholtz Equation. We study it first.

$$\nabla^2\psi + k^2\psi = 0$$

In cylindrical coordinates, this becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

We will solve this by separating variables:

$$\psi = R(\rho)\Phi(\phi)Z(z)$$

Separation of Variables

Substituting and dividing by ψ, we find

$$\frac{1}{\rho R}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right)+\frac{1}{\rho^2\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2}+k^2=0$$

The third term is independent of ϕ and ρ, so it must be constant:

$$\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2}=-k_z^2$$

This leaves

$$\frac{1}{\rho R}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right)+\frac{1}{\rho^2\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+k^2-k_z^2=0$$

Separation of Variables

Now define the

Radial Wavenumber

$$k_\rho^2 = k^2 - k_z^2$$

and multiply the resulting equation by ρ^2 to find

$$\frac{\rho}{R}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + k_\rho^2\rho^2 = 0$$

The second term is independent of ρ and z, so we let

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -n^2$$

Separation of Variables

This process leaves an ordinary differential equation in ρ alone. We thus have:

The Cylindrical Helmholtz Equation, Separated

$$\begin{aligned} \rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \left[(k_\rho \rho)^2 - n^2\right] R &= 0 \\ \frac{d^2\Phi}{d\phi^2} + n^2\Phi &= 0 \\ \frac{d^2 Z}{dz^2} + k_z^2 Z &= 0 \\ k_\rho^2 + k_z^2 &= k^2 \end{aligned}$$

The first of these equations is called Bessel's Equation; the others are familiar.

The Harmonic Equations

We have already seen equations like those in the z and ϕ directions; the solutions are

- trigonometric, or
- exponential.

The only novelty is that ϕ is periodic or finite; it therefore is always expanded in a series and not an integral.
If there is no limit in the ϕ direction we find

The Periodic Boundary Condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

This implies that $n \in \mathbb{Z}$ if the entire range is included.

Outline

Bessel's Equation For Statics

- The remaining equation to be solved is the radial equation, i.e. Bessel's Equation.
- Note that the problem simplifies considerably if $k_\rho = 0$ (which would be the case if $\rho = 0$.

In this case, we have

Bessel's Equation for Statics

$$\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}R}{\mathrm{d}\rho}\right) - n^2 R = 0$$

To solve it, let

$$\rho = e^x$$

so

$$\frac{\mathrm{d}\rho}{\mathrm{d}x} = e^x = \rho.$$

Bessel's Equation For Statics

This implies that

$$\frac{d}{dx} = \frac{d\rho}{dx}\frac{d}{d\rho} = \rho\frac{d}{d\rho}$$

Our equation therefore becomes

$$\frac{d^2R}{dx^2} - n^2R = 0$$

The solutions to this are

$$R(x) = \begin{cases} A + Bx & n = 0 \\ Ae^{nx} + Be^{-nx} & n \neq 0 \end{cases}$$

and thus the solutions really are

Static Solutions of Bessel's Equation

$$R(\rho) = \begin{cases} A + B\ln\rho & n = 0 \\ A\rho^n + B\rho^{-n} & n \neq 0 \end{cases}$$

Bessel Functions

We are generally more interested in the dynamic case in which we must solve the full Bessel Equation:

$$\xi \frac{d}{d\xi}\left(\xi \frac{dR}{d\xi}\right) + \left[\xi^2 - n^2\right] R = 0$$

(We normalize $k_\rho = 1$, and rewrite the equation in terms of ρ instead of ξ) To solve this equation, we suppose

$$R(\xi) = \xi^\alpha \sum_{m=0}^{\infty} c_m \xi^m$$

Now

$$\frac{dR}{d\xi} = \sum_{m=0}^{\infty} (\alpha + m) c_m \xi^{\alpha+m-1}$$

Bessel Functions

Thus

$$\xi \frac{dR}{d\xi} = \sum_{m=0}^{\infty} (\alpha + m) c_m \xi^{\alpha+m}$$

and

$$\frac{d}{d\xi}\left(\xi \frac{dR}{d\xi}\right) = \sum_{m=0}^{\infty} (\alpha + m)^2 c_m \xi^{\alpha+m-1}$$

so finally

$$\xi \frac{d}{d\xi}\left(\xi \frac{dR}{d\xi}\right) = \sum_{m=0}^{\infty} (\alpha + m)^2 c_m \xi^{\alpha+m}$$

Now, we can plug in...

$$\sum_{m=0}^{\infty} (\alpha + m)^2 c_m \xi^{\alpha+m} + \left[\xi^2 - n^2\right] \sum_{m=0}^{\infty} c_m \xi^{\alpha+m} = 0$$

Bessel Functions

We now have

$$\sum_{m=0}^{\infty}\left[(\alpha+m)^2-n^2\right]c_m\xi^{\alpha+m}+\sum_{m=0}^{\infty}c_m\xi^{\alpha+m+2}=0$$

or

$$\sum_{m=0}^{\infty}\left[(\alpha+m)^2-n^2\right]c_m\xi^{\alpha+m}+\sum_{m=2}^{\infty}c_{m-2}\xi^{\alpha+m}=0$$

We can proceed by forcing the coefficients of each term to vanish. We fix $c_0 \neq 0$ because of the homogeneity of the equation.

Bessel Functions

For ξ^{α}:

$$(\alpha^2 - n^2) = 0$$

since $c_0 \neq 0$ by assumption,

$$\alpha = \pm n.$$

For $\xi^{\alpha+1}$:

$$\left[(\alpha + 1)^2 - n^2\right] c_1 = 0$$

Thus

$$c_1 = 0$$

Bessel Functions

Finally, for all other $\xi^{\alpha+m}$:

$$\left[(\alpha+m)^2-n^2\right]c_m+c_{m-2}=0$$

Assuming $\alpha=n$,

$$c_m=\frac{-1}{m(m+2n)}c_{m-2}$$

Thus, immediately, for $p\in\mathbb{Z}$

$$c_{2p+1}=0$$

Bessel Functions

Given that the odd coefficients vanish, we let $m = 2p$ and let

$$a_p = c_{2p}$$

So...

$$a_p = c_{2p} = \frac{-1}{4p(p+n)} c_{2p-2} = a_{p-1}$$

and

$$\begin{aligned} a_1 &= \frac{-1}{4(n+1)} a_0 \\ a_2 &= \frac{-1}{4(n+2)(2)} \frac{-1}{4(n+1)} a_0 \\ a_3 &= \frac{-1}{4(n+3)(3)} \frac{-1}{4(n+2)(2)} \frac{-1}{4(n+1)} a_0 \end{aligned}$$

Bessel Functions

Thus, in general,

$$a_p = \frac{(-1)^p n!}{4^p p!(n+p)!} a_0$$

Thus, if we choose $2^{-n} n! a_0 = 1$, and recall

$$R(\xi) = \xi^\alpha \sum_{m=0}^{\infty} c_m \xi^m = \sum_{p=0}^{\infty} a_p \xi^{2p+n}$$

we can (finally!) define the

Bessel Function of Order n

$$J_n(\xi) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!} \left(\frac{\xi}{2}\right)^{2p+n}$$

Observations

- This function is entire; it exists and is differentiable for all ξ.
- This is only one solution of the equation:
  - We will get to the other shortly.
  - The other solution is not regular at the origin since the coefficient of the second order derivative vanishes there.
- Note that the solution looks like the corresponding static (ρ^n) solution at the origin.
- Note also that fractional orders are possible, but do not arise as commonly in applications (Why not?)

The Other Solution

Our original equation (normalized) was

$$\xi \frac{d}{d\xi}\left(\xi \frac{du}{d\xi}\right) + \left[\xi^2 - n^2\right] u = 0$$

The other solution, v must solve

$$\xi \frac{d}{d\xi}\left(\xi \frac{dv}{d\xi}\right) + \left[\xi^2 - n^2\right] v = 0$$

Multiply the first equation by v and the second by u, subtract, and divide by ξ:

$$v \frac{d}{d\xi}\left(\xi \frac{du}{d\xi}\right) - u \frac{d}{d\xi}\left(\xi \frac{dv}{d\xi}\right) = 0$$

The Other Solution

Expanding this out

$$\xi(u''v - uv'') + u'v - uv' = 0,$$

or

$$\frac{d}{d\xi}\left[\xi(u'v - uv')\right] = 0$$

It therefore stands to reason that

$$\xi(u'v - uv') = C$$

or

$$\frac{u'v - uv'}{v^2} = \frac{C_2}{\xi v^2}$$

The Other Solution

This of course implies

$$\frac{d}{d\xi}\left(\frac{u}{v}\right) = \frac{C_2}{\xi v^2}$$

This can be integrated to give

$$\frac{u}{v} = C_1 + C_2 \int \frac{d\xi}{\xi v^2}$$

or

$$u(\xi) = C_1 v(\xi) + C_2 v(\xi) \int \frac{d\xi}{\xi v(\xi)^2}$$

The Other Solution

Setting $C_1 = 0$, $v(\xi) = J_n(\xi)$, expanding the series and integrating gives rise to the

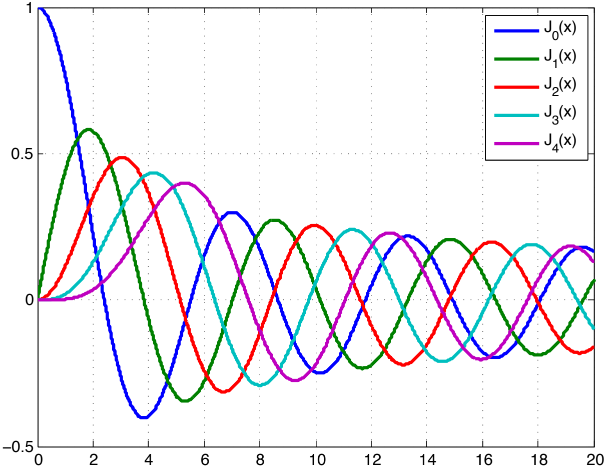
Neumann Function

$$Y_n(\xi) = J_n(\xi) \int \frac{d\xi}{\xi J_n^2(\xi)}$$

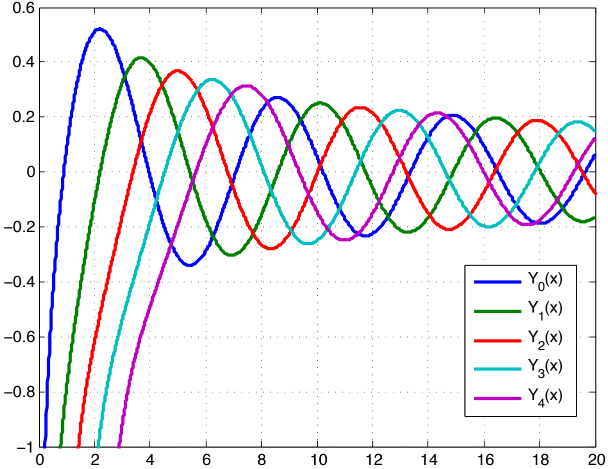
This function

- This function is also called the "Bessel function of the second kind."
- It is sometimes denoted by $N_n(\xi)$.
- This function is not defined for $\xi = 0$.

Graphs of $J_n(x)$



Graphs of $Y_n(x)$



Hankel Functions

- The J_n and Y_n are both real functions for real arguments.
- They must therefore represent standing waves (Why?).

Hankel Functions

- The J_n and Y_n are both real functions for real arguments.
- They must therefore represent standing waves (Why?).
- Hankel functions represent traveling waves.

Traveling waves are represented by

Hankel Functions

$$
\begin{aligned}
H_n^{(1)}(x) &= J_n(x) + jY_n(x) \\
H_n^{(2)}(x) &= J_n(x) - jY_n(x)
\end{aligned}
$$

These are called Hankel functions of the first and second kind, respectively.

Small Argument Behavior

- Suppose $\text{Re}(\nu) > 0$.
- Let $\ln \gamma = 0.5772 \Rightarrow \gamma = 1.781$ (i.e. $\ln \gamma$ is "Euler's constant").

Consider the behavior of the Bessel and Neumann functions as $x \to 0$:

$$
\begin{aligned}
J_0(x) &\to 1 \\
Y_0(x) &\to \frac{2}{\pi} \ln \frac{\gamma x}{2} \\
J_\nu(x) &\to \frac{1}{\nu!} \left(\frac{x}{2}\right)^\nu \\
Y_\nu(x) &\to -\frac{(\nu - 1)!}{\pi} \left(\frac{2}{x}\right)^\nu
\end{aligned}
$$

The only Bessel functions finite at the origin are the $J_n(x)$.

Large Argument Behavior

As $x \to \infty$:

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)$$

$$Y_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)$$

Given the definition of Hankel functions, we must also have

$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{j\pi x}} j^{-\nu} e^{jx}$$

$$H_\nu^{(2)}(x) \to \sqrt{\frac{2j}{\pi x}} j^{\nu} e^{-jx}$$

- The $H_\nu^{(2)}$ represent outward traveling waves.
- Why are these all proportional to $x^{-\frac{1}{2}}$?

Imaginary Arguments

- In applications, we get Bessel functions of dimensionless quantities: $B_n(k_\rho \rho)$.
- If k_ρ becomes imaginary, we have evanescence in the ρ direction.

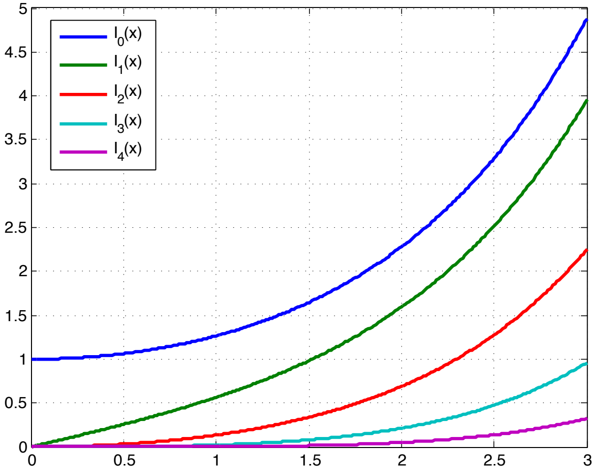
For these applications, we define the

Modified Bessel Functions

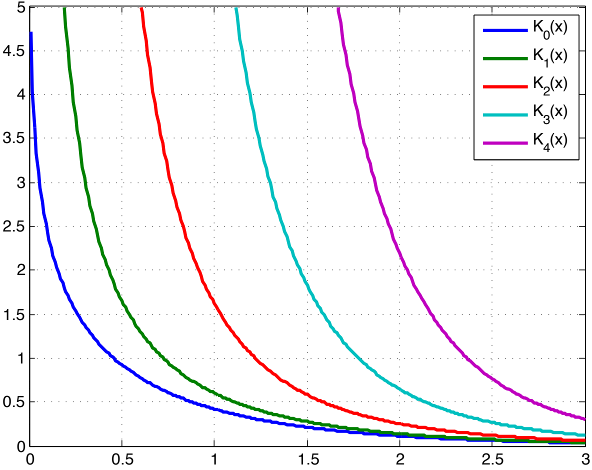
$$\begin{aligned} I_n(x) &= j^n J_n(-jx) \\ K_n(x) &= \frac{\pi}{2}(-j)^{n+1} H_n^{(2)}(-jx) \end{aligned}$$

These are real functions of real arguments.

Graphs of $I_n(x)$



Graphs of $K_n(x)$



Outline

Transverse Magnetic Fields

Let

$$\frac{\mathbf{A}}{\mu} = \mathbf{u}_z \psi$$

for some solution of the Helmholtz equation ψ. Then

$$H_\rho = \frac{1}{\rho}\frac{\partial \psi}{\partial \phi} \qquad E_\rho = \frac{1}{\hat{y}}\frac{\partial^2 \psi}{\partial \rho \partial z}$$

$$H_\phi = -\frac{\partial \psi}{\partial \rho} \qquad E_\phi = \frac{1}{\hat{y}\rho}\frac{\partial^2 \psi}{\partial \phi \partial z}$$

$$H_z = 0 \qquad E_z = \frac{1}{\hat{y}}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

This is a general formula for a TM_z field; $H_z = 0$.

Transverse Electric Fields

Let

$$\frac{\mathbf{F}}{\epsilon} = \mathbf{u}_z \psi$$

for some solution of the Helmholtz equation ψ. Then

$$E_\rho = -\frac{1}{\rho}\frac{\partial \psi}{\partial \phi} \qquad H_\rho = \frac{1}{\hat{z}}\frac{\partial^2 \psi}{\partial \rho \partial z}$$

$$E_\phi = \frac{\partial \psi}{\partial \rho} \qquad H_\phi = \frac{1}{\hat{z}\rho}\frac{\partial^2 \psi}{\partial \phi \partial z}$$

$$E_z = 0 \qquad H_z = \frac{1}{\hat{z}}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

This is a general formula for a TE_z field; $E_z = 0$.

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   - Radial Waveguides
   - Cavities

The Circular Waveguide

- A circular waveguide is a tube of (say) radius *a*.
- The field must be finite at $\rho = 0$, so only J_n are admissible.

Without further ado, the wave function for TM_z waves is

$$\psi = J_n(k_\rho \rho) \left\{ \begin{array}{c} \sin n\phi \\ \cos n\phi \end{array} \right\} e^{-jk_z z}$$

- The azimuthal dependence keeps the transverse fields in phase; either sine or cosine can be chosen.
- For $n = 0$ we obviously choose the cosine.
- The restriction to $n \in \mathbb{Z}$ is required because of the periodic boundary condition. Other boundaries would lead to other results.

Boundary Conditions

Since

$$E_z = \frac{1}{\hat{y}}(k^2 - k_z^2)\psi$$

we need ψ to vanish on the walls. This implies

$$J_n(k_\rho a) = 0$$

giving the

Values for k_ρ

$$k_\rho = \frac{\chi_{np}}{a}$$

where χ_{np} is the pth solution of

$$J_n(\chi_{np}) = 0.$$

Roots of Bessel Functions, χ_{np}

The roots of the Bessel functions are well tabulated.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$p=1$	2.405	3.832	5.136	6.380	7.588	8.771
$p=2$	5.520	7.016	8.417	9.761	11.065	12.339
$p=3$	8.654	10.173	11.620	13.015	14.732	
$p=4$	11.792	13.324	14.796			

For TM_z modes, we have:

$$\psi_{np}^{\text{TM}} = J_n\left(\frac{\chi_{np}\rho}{a}\right)\begin{Bmatrix}\sin n\phi \\ \cos n\phi\end{Bmatrix} e^{-jk_z z}$$

with

$$\left(\frac{\chi_{np}}{a}\right)^2 + k_z^2 = k^2$$

TE_z Modes

The wave function for TE_z waves is

$$\psi = J_n(k_\rho \rho) \left\{ \begin{array}{c} \sin n\phi \\ \cos n\phi \end{array} \right\} e^{-jk_z z}$$

just as for TM_z and for the same reasons. (Here, of course, $\mathbf{F} = \epsilon\psi\mathbf{u}_z$.)
Now

$$E_\phi = \frac{\partial \psi}{\partial \rho},$$

so we need

$$J_n'(k_\rho a) = 0.$$

TE_z Modes

We therefore find the radial wavenumber must be of the form

Values for k_ρ

$$k_\rho = \frac{\chi'_{np}}{a}$$

where χ'_{np} is the p$^{\text{th}}$ solution of

$$J'_n(\chi'_{np}) = 0.$$

Roots of Bessel Function Derivatives, χ'_{np}

The roots of the Bessel function derivatives are also well tabulated.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$p=1$	3.832	1.841	3.054	4.201	5.317	6.416
$p=2$	7.016	5.331	6.706	87.015	9.282	10.520
$p=3$	10.173	8.536	9.969	11.346	12.682	13.987
$p=4$	13.324	11.706	13.170			

For TE_z modes, we have:

$$\psi_{np}^{\text{TE}} = J_n\left(\frac{\chi'_{np}\rho}{a}\right)\left\{\begin{array}{c}\sin n\phi\\ \cos n\phi\end{array}\right\}e^{-jk_z z}$$

with

$$\left(\frac{\chi'_{np}}{a}\right)^2 + k_z^2 = k^2$$

Cutoff

Cutoff occurs when $k_z = 0$ We can thus compute the

Cutoff Wavenumber

$$(k_c)^{TM}_{np} = \frac{\chi_{np}}{a} \qquad (k_c)^{TE}_{np} = \frac{\chi'_{np}}{a}$$

and the

Cutoff Frequencies

$$(f_c)^{TM}_{np} = \frac{\chi_{np}}{2\pi a\sqrt{\mu\epsilon}} \qquad (f_c)^{TE}_{np} = \frac{\chi'_{np}}{2\pi a\sqrt{\mu\epsilon}}$$

- Cutoff frequency is thus proportional to the roots χ_{np} and χ'_{np}.
- The fundamental mode is TE_{11}.

Modal Impedance

The calculation of impedance is just like that in rectangular waveguides. The result is the same too:

$$Z_0^{\mathrm{TE}} = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho} = \frac{\omega\mu}{k_z}$$

and

$$Z_0^{\mathrm{TM}} = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho} = \frac{k_z}{\omega\epsilon}$$

Other cylindrical waveguides can be analyzed similarly; homework will contain examples.

Outline

Radial Waveguides

- In addition to cylindrical waveguides, we can consider guides that carry waves radially.
- The simplest such guide is a parallel plate waveguide, analyzed to consider radial propagation.
- We call the distance between the plates (in the z-direction) a.

Radial Waveguides

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TM_z Wavefunctions for Parallel Plate Guide

$$\psi_{mn}^{\mathrm{TM}} = \cos\left(\frac{m\pi}{a} z\right) \cos n\phi H_n^{(2)}(k_\rho \rho)$$

Radial Wavguides

- The solution on the previous slide chose $\cos n\phi$ for simplicity; $\sin n\phi$ is legal but gives no new information.
- The solution on the previous slide is for outgoing waves, incoming waves are proportional to $H_n^{(1)}(k_\rho \rho)$.
- For TE$_z$ waves, we need the same components to vanish. Here, they are directly proportional to the wavefunction so we find the

TE$_z$ Wavefunctions for Parallel Plate Guide

$$\psi_{mn}^{\mathrm{TE}} = \sin\left(\frac{m\pi}{a} z\right) \cos n\phi H_n^{(2)}(k_\rho \rho)$$

In either case we find that

$$k_\rho = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2}$$

Phase Constant and Velocity: An Aside

In general, any wave can be written in polar form

$$\psi(x, y, z) = A(x, y, z)e^{j\Phi(x,y,z)}$$

where $A, \Phi \in \mathbb{R}$. In the time domain, this wave becomes

$$\text{Re}\left\{A(x, y, z)e^{j\Phi(x,y,z)}e^{j\omega t}\right\}$$

A surface of constant phase thus has the form

$$\omega t + \Phi(x, y, z) = \text{constant}$$

Phase Constant and Velocity: An Aside

This equation can be differentiated with respect to *t* to give

$$\omega + \nabla\Phi \cdot \mathbf{v}_{\mathrm{p}} = 0$$

- This equation cannot be solved *per se*; it is one equation in three unknowns!
- If we choose a direction, we can solve it along that direction (i.e., we can find the speed we need to move to keep the phase fixed.)

Thus, we have the

Phase Velocity in the *x*-Direction

$$v_{\mathrm{p}x} = -\frac{\omega}{\frac{\partial\Phi}{\partial x}}$$

Phase Constant and Velocity: An Aside

We can also define the

Phase Velocity in the Direction of Travel

$$v_{\mathrm{p}} = -\frac{\omega}{|\nabla\Phi|}$$

In any case, we have the

Wavevector

$$\beta = -\nabla\Phi$$

We would like to see how these ideas apply to radial travel.

Phase Velocity of Radial Waves

From the above discussion, we can define

$$\beta_\rho = -\frac{\mathrm{d}}{\mathrm{d}\rho}\tan^{-1}\frac{Y_n(k_\rho\rho)}{J_n(k_\rho\rho)}$$

Let us compute this:

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\rho}\tan^{-1}\frac{Y_n(k_\rho\rho)}{J_n(k_\rho\rho)} &= -\frac{1}{1+\left(\frac{Y_n(k_\rho\rho)}{J_n(k_\rho\rho)}\right)^2}\left[\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{Y_n(k_\rho\rho)}{J_n(k_\rho\rho)}\right] \\
&= -\frac{k_\rho}{1+\left(\frac{Y_n(k_\rho\rho)}{J_n(k_\rho\rho)}\right)^2}\frac{Y_n'(k_\rho\rho)J_n(k_\rho\rho)-Y_n(k_\rho\rho)J_n'(k_\rho\rho)}{J_n^2(k_\rho\rho)}
\end{aligned}$$

Phase Velocity of Radial Waves

The Wronskian

$$J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = \frac{2}{\pi x}.$$

Thus

$$\begin{aligned}\beta_\rho &= \frac{k_\rho}{1+\left(\frac{Y_n(k_\rho \rho)}{J_n(k_\rho \rho)}\right)^2}\frac{1}{J_n^2(k_\rho \rho)}\frac{2}{\pi k_\rho \rho} \\ &= \frac{2}{\pi\rho}\frac{1}{J_n^2(k_\rho \rho)+Y_n^2(k_\rho \rho)}\end{aligned}$$

Phase Velocity of Radial Waves

Now, as $\rho \to \infty$, we can substitute the large argument approximations:

$$\beta_\rho = \frac{2}{\pi\rho}\frac{1}{J_n^2(k_\rho\rho) + Y_n^2(k_\rho\rho)}$$
$$\to \frac{2}{\pi\rho}\left\{\left[\sqrt{\frac{2}{\pi k_\rho}}\cos\left(k_\rho\rho - \frac{\pi}{4} - \frac{n\pi}{2}\right)\right]^2 + \left[\sqrt{\frac{2}{\pi k_\rho}}\sin\left(k_\rho\rho - \frac{\pi}{4} - \frac{n\pi}{2}\right)\right]^2\right\}^{-1} = k_\rho$$

Why would we expect this?

Modal Impedance and Cutoff

Impedance can be computed in the usual manner:

Outward-Travelling Modal Impedance

$$Z_{+\rho^{\mathrm{TM}}} = -\frac{E_z}{H_\phi} = \frac{k_\rho}{j\omega\epsilon}\frac{H_n^{(2)}(k_\rho\rho)}{H_n^{(2)\prime}(k_\rho\rho)}$$

$$Z_{+\rho^{\mathrm{TE}}} = \frac{E_\phi}{H_z} = \frac{j\omega\mu}{k_\rho}\frac{H_n^{(2)}(k_\rho\rho)}{H_n^{(2)\prime}(k_\rho\rho)}$$

Note that this is not purely real. Now, it should be remembered that

$$k_\rho = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2}$$

Modal Impedance and Cutoff

We thus have a purely imaginary radial wavenumber ($-j\alpha$) if $ka < m\pi$. Recall

$$H_n^{(2)}(-j\alpha\rho) = \frac{2}{\pi} j^{n+1} K_n(\alpha\rho)$$

Plugging this in to our expression for TM impedance, we find

$$Z_{+\rho^{\text{TM}}} = \frac{j\alpha}{\omega\epsilon} \frac{K_n(\alpha\rho)}{K_n'(\alpha\rho)}$$

Since the K_n are real functions of real arguments, this expression is purely imaginary and no energy propagates.

TM_{0n} Modes

If

$$a < \frac{\lambda}{2}$$

only $m = 0$ modes propagate. (These are all TM_z.)
In this case we have

TM_{0n} Wavefunctions

$$\psi_{0n}^{\text{TM}} = \cos n\phi H_n^{(2)}(k\rho)$$

- How can large n modes propagate?
- Why is this cause for concern?

Impedance of TM_{0n} Modes

To understand what is happening, we look at the expression

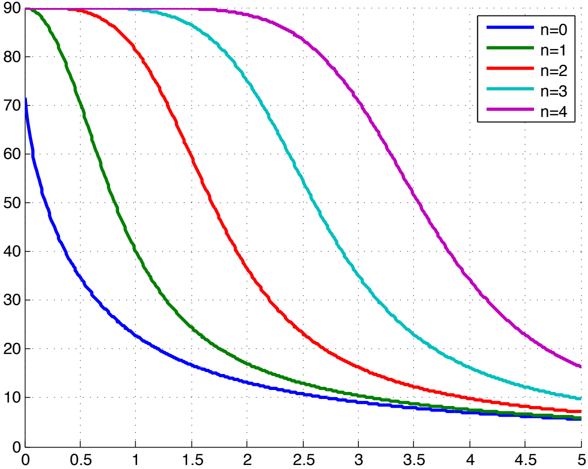
$$Z_{+\rho}^{\mathrm{TM}} = -j\eta \frac{H_n^{(2)}(k\rho)}{H_n^{(2)\prime}(k\rho)}$$

Using a well-known identity (obtained by differentiating and manipulating Bessel's equation) we can write

$$Z_{+\rho}^{\mathrm{TM}} = -j\eta \frac{H_n^{(2)}(k\rho)}{\frac{n}{x} H_n^{(2)}(k\rho) - H_{n+1}^{(2)}(k\rho)}$$

By examining the (absolute) phase angle of the impedance, we can see how efficiently each mode carries energy.

Absolute Impedance Phase vs. $k\rho$



Gradual Cutoff

- The last slide shows that as the frequency increases, each mode carries power more efficiently.
- The transition from storing energy to carrying it occurs at

$$k\rho = n$$

- This is exactly where the radial waveguide is *n* wavelengths in circumference.
- This phenomenon is called gradual cutoff and it is related to the poor radiation ability of small antennas.

The TM_{00} mode

- The dominant mode in the radial parallel plate guide is the TM_{00} mode.
- The outwardly traveling fields are given by

$$\begin{aligned} E_z^+ &= \frac{k^2}{j\omega\epsilon} H_0^{(2)}(k\rho) \\ H_\phi^+ &= k H_1^{(2)}(k\rho) \end{aligned}$$

- This is a TEM mode and can be analyzed with a transmission line analysis if desired.
- The inductance/capacitance per unit length change with distance.

Feynman's Analysis

- The Nobel Laureate Richard P. Feynman used a particularly simple approach to the analysis of cylindrical resonators.
- The approach also demonstrates the evolution from statics to dynamics.
- It also introduces Bessel Functions without partial (or even ordianary) differential equations!

So consider a circular capacitor with a static electric field E_0. If the field begins to oscillate with frequency ω, a magnetic field is created.

Feynman's Analysis

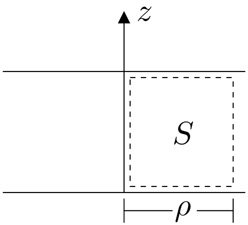
To find the magnetic field, we can apply the Ampere-Maxwell law to a circle C of radius ρ centered on the axis:

$$\begin{aligned}
\oint_{\partial C} \mathbf{H} \cdot d\mathbf{l} &= j\omega\epsilon \iint_C \mathbf{E} \cdot d\mathbf{S} \\
2\pi\rho H_\phi &= j\omega\epsilon\pi\rho^2 E_z \\
H_\phi &= \frac{j\omega\epsilon\rho}{2} E_0 \\
&= \frac{jk\rho}{2\eta} E_0
\end{aligned}$$

Feynman's Analysis

Now, this magnetic field is also oscillating, so the original electric field is also wrong.

- We will call the original field $E_1 = E_0$; the field at the center of the plate.
- The magnetic field we have found is H_1.
- It will give rise to an E_2.
- To find E_2 we use the surface S shown below.



Feynman's Analysis

To find the correction to **E**, we use Faraday's Law.

$$\begin{aligned}
\oint_{\partial S} \mathbf{E}\cdot \mathrm{d}\mathbf{l} &= -j\omega\mu \iint_S \mathbf{H}\cdot \mathrm{d}\mathbf{S} \\
-\int_0^a E_2 \mathrm{d}z &= -j\omega\mu \int_0^a \mathrm{d}z \int_0^\rho \mathrm{d}\rho' H_1 \\
E_2 &= j\omega\mu \int_0^\rho \mathrm{d}\rho' \frac{jk\rho'}{2\eta} E_0 \\
&= -\frac{k^2\rho^2}{2^2} E_0
\end{aligned}$$

Thus, at the moment,

$$E_z = E_1 + E_2 = E_0\left(1 - \frac{k^2\rho^2}{2^2}\right)$$

Feynman's Analysis

We can now correct **H** again:

$$
\begin{aligned}
\oint_{\partial C} \mathbf{H} \cdot d\mathbf{l} &= j\omega\epsilon \iint_C \mathbf{E} \cdot d\mathbf{S} \\
2\pi\rho H_2 &= j\omega\epsilon \int_0^{2\pi} d\phi \int_0^{\rho} \rho d\rho E_2 \\
H_2 &= -\frac{j\omega\epsilon k^2 E_0}{2^2 \rho} \int_0^{\rho} d\rho \rho^3 \\
&= -\frac{jk^3\rho^3}{2^2 \cdot 4} \frac{E_0}{\eta}
\end{aligned}
$$

Feynman's Analysis

And, we can now correct **E** again:

$$\begin{aligned}
\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} &= -j\omega\mu \iint_S \mathbf{H} \cdot \mathrm{d}\mathbf{S} \\
-\int_0^a E_3 \mathrm{d}z &= -j\omega\mu \int_0^a \mathrm{d}z \int_0^\rho \mathrm{d}\rho' H_2 \\
E_3 &= \omega\mu \int_0^\rho \mathrm{d}\rho' \frac{k^3 \rho'^3}{2^2 \cdot 4} \frac{E_0}{\eta} \\
&= \frac{k^4 \rho^4}{2^2 \cdot 4^2} E_0
\end{aligned}$$

Thus, at the moment,

$$E_z = E_1 + E_2 + E_3 = E_0 \left(1 - \frac{k^2 \rho^2}{2^2} + \frac{k^4 \rho^4}{2^2 \cdot 4^2} \right)$$

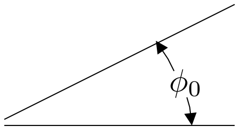
Feynman's Analysis

- The importance of these terms we keep adding is proportional to
  - Frequency
  - Plate size

  In this way, we see how statics naturally morphs into dynamics.
- It is easy to see that the E-field found after continuing the process is

$$E = E_0\left(1 - \frac{k^2\rho^2}{2^2} + \frac{k^4\rho^4}{2^2\cdot 4^2} - \frac{k^4\rho^4}{2^2\cdot 4^2\cdot 6^2} + \cdots\right)$$
$$= E_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{k\rho}{2}\right)^2 = E_0 J_0(k\rho)$$

Wedge Waveguide



- A wedge guide is formed by two half-planes inclined at an angle ϕ_0.
- The problem is independent of z.
- For the TM$_z$ mode, we must have $E_z = 0$ on the plates.

Thus

$$\psi \propto \sin\left(\frac{p\pi}{\phi_0}\phi\right)$$

Wedge Waveguide Wavefunctions

Now, the order of the Bessel function equals the coefficient of ϕ. We thus have the

TM Outwardly Traveling Wavefunctions

$$\psi_p^{\mathrm{TM}} = \sin\left(\frac{p\pi}{\phi_0}\phi\right) H^{(2)}_{\frac{p\pi}{\phi_0}}(k\rho)$$

By the same token, we have the

TE Outwardly Traveling Wavefunctions

$$\psi_p^{\mathrm{TE}} = \cos\left(\frac{p\pi}{\phi_0}\phi\right) H^{(2)}_{\frac{p\pi}{\phi_0}}(k\rho)$$

Observations

- Most of the discussion of the radial parallel plate guide applies here:
  - Wave impedances are given by the same formulas (with fractional orders),
  - Gradual cutoff occurs when $\frac{p\pi}{\phi_0} = k\rho$. Why?
- The dominant mode is TE_0; it is a TEM mode derivable by transmission line theory. Its fields are

TE_0 Mode Fields

$$
\begin{aligned}
E_\phi^+ &= kH_1^{(2)}(k\rho) \\
H_z^+ &= \frac{k^2}{j\omega\mu} H_0^{(2)}(k\rho)
\end{aligned}
$$

Outline

1. Cylindrical Waves
   - Separation of Variables
   - Bessel Functions
   - TE_z and TM_z Modes
2. Guided Waves
   - Cylindrical Waveguides
   - Radial Waveguides
   - Cavities

The Circular Cavity

- The circular cavity is a circular waveguide shorted at both ends.
- We will assume the height of the cavity is denoted by d.

For TM$_z$ modes,

$$E_\rho \propto \frac{\partial^2 \psi}{\partial \rho \partial z}$$

so we must have (assuming $\cos n\phi$ variation)

TM$_z$ Modes

$$\psi_{npq}^{\text{TM}} = J_n\left(\frac{\chi_{np}\rho}{a}\right)\cos n\phi \cos\left(\frac{q\pi z}{d}\right)$$

The Circular Cavity

By the same token

TE_z Modes

$$\psi_{npq}^{\text{TE}} = J_n\left(\frac{\chi'_{np}\rho}{a}\right)\cos n\phi \sin\left(\frac{q\pi z}{d}\right)$$

We can also immediately write

Separation Equations

$$\begin{aligned}\text{TM:} \quad & \left(\frac{\chi_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2 \\ \text{TE:} \quad & \left(\frac{\chi'_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2\end{aligned}$$

Resonant Frequencies

From these equations it is a simply matter to compute

Resonant Frequencies

$$\left(f_r\right)_{npq}^{\text{TM}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\chi_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

$$\left(f_r\right)_{npq}^{\text{TE}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\chi'_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

If $d < 2a$ the TM_{010} mode is dominant, otherwise the TE_{111} mode is.

Quality Factor

- It is also straightforward to compute the Q of the circular cavity.
- We will do so only for the (usually dominant) TM_{010} mode.

Given ψ, it is easy to show that the modal fields are

TM_{010} Mode Fields

$$
\begin{aligned}
E_z &= \frac{k^2}{j\omega\epsilon} J_0\left(\frac{\chi_{01}\rho}{a}\right) \\
H_\phi &= \frac{\chi_{01}}{a} J_1\left(\frac{\chi_{01}\rho}{a}\right)
\end{aligned}
$$

Quality Factor

Now, the total energy stored is

$$W = 2\overline{W_e} = \frac{\epsilon}{2}\int_0^d\int_0^a\int_0^{2\pi}|E|^2\rho d\phi d\rho dz$$
$$= \frac{k^4}{\omega^2\epsilon}\pi d\int_0^a \rho J_0^2\left(\frac{\chi_{01}\rho}{a}\right)d\rho = \frac{\pi k^4 da^2}{2\omega^2\epsilon}J_1^2(\chi_{01})$$

To compute the energy absorbed by the walls, we appeal to the approximate formula

$$\overline{P_d} = \frac{\mathcal{R}}{2}\oiint_{\text{walls}}|H|^2 dS$$

Quality Factor

On the cylindrical side wall of the cavity, the magnetic field is constant, so the value of this integral is

$$\overline{P_\mathrm{d}} = \pi a d \mathcal{R} J_1^2(\chi_{01})$$

On the other two walls together we have

$$\begin{aligned}\overline{P_\mathrm{d}} &= \mathcal{R} \int_0^a \int_0^{2\pi} \left(\frac{\chi_{01}}{a}\right)^2 J_1^2\left(\frac{\chi_{01}\rho}{a}\right) \rho \mathrm{d}\phi \mathrm{d}\rho \\ &= 2\pi\mathcal{R} \left(\frac{\chi_{01}}{a}\right)^2 \int_0^a J_1^2\left(\frac{\chi_{01}\rho}{a}\right) \rho \mathrm{d}\rho \\ &= \pi a^2 \mathcal{R} \left(\frac{\chi_{01}}{a}\right)^2 J_1^2(\chi_{01})\end{aligned}$$

Quality Factor

Plugging into our formula for the quality factor we find

$$Q = \frac{\omega W}{\overline{P}_\mathrm{d}} = \frac{dk^4 a^3}{2\omega\epsilon\mathcal{R}\chi_{01}^2(d+a)}$$

Now, $ka = \chi_{01}$ and $\omega\epsilon = \frac{k}{\eta}$, so we find the final formula for

The Quality Factor

$$Q = \frac{\eta\chi_{01} d}{2\mathcal{R}(a+d)}$$