ON SLINKY: THE DYNAMICS OF A LOOSE, HEAVY SPRING

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Introduction. A simple but intriguing philosophical toy has recently appeared on the market under the trade-name 'Slinky'. It consists of a single flat-coiled spring of phosphor-bronze, 3 in. in diameter and 2½ in. in length, when closely coiled. According to the directions, the spring is placed upright at the top of a flight of stairs and the upper end is lifted and allowed to fall on to the stair next below. The spring then uncoils itself on to the lower stair, and the end that was at the bottom of the coil flips over on to the stair next but one below. The cycle is repeated and the Slinky walks furtively down the stairs, at a surprisingly steady rate.

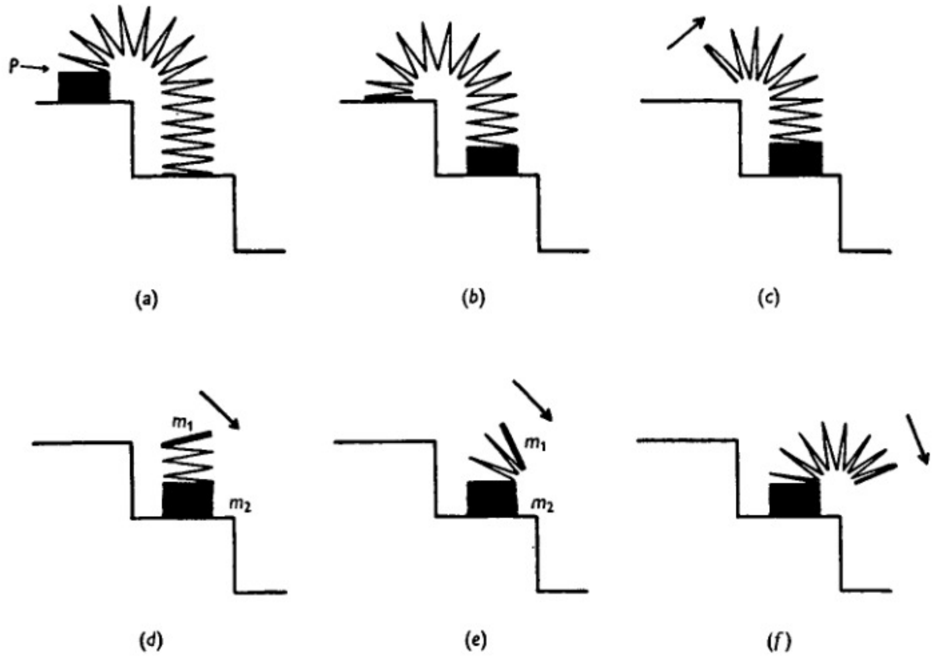

Fig. 1. Illustrating the action of the Slinky.

The present writer was moved by this remarkable behaviour to attempt to calculate the rate of descent of the stairs. In the following note a simple solution is presented. It is assumed that (1) the spring obeys Hooke's law and (2) a 'shock', or discontinuity in the density, forms at the top of the coil on the upper stair. It is then shown that the rate at which the spring uncoils depends only on the characteristics of the spring, and hence that the time taken for one step is independent of the height of the stair. This is confirmed by experiment, and the predicted rate of descent is closely verified.

1. *Idealization of the problem.* For the present purpose we neglect all motion of the spring at right angles to its axis, and all curvature of the axis. The spring is treated as a heavy, elastic fluid flowing along a thin tube. Let

x = distance measured along the axis,
ρ = density, i.e. mass per unit distance along the axis,
u = component of velocity parallel to the axis,
T = component of reaction parallel to the axis.

We assume Hooke's law, namely, that the tension T is proportional to the extension of the spring from its unstressed state. If the spring is unstressed when very closely coiled, the extension of the spring is inversely proportional to its density. Hence Hooke's law can be written

$$T = \frac{C}{\rho}, \tag{1}$$

where C is a constant.

2. *Conditions at a discontinuity.* Suppose that when $x < x_0$ the spring is closely coiled and stationary ($\rho = \infty, u = 0$), and, when $x > x_0$, ρ, u and T are all finite and different from zero (see Fig. 2). To the left of $x = x_0$ the tension vanishes, since $T = C/\rho = 0$, but there may possibly be a reaction R between the coils of the spring. If the spring is not 'sticky' R cannot be negative.

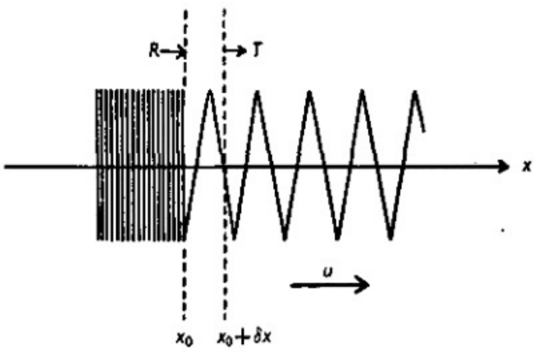

Fig. 2.

In a short interval of time δt a length of spring $\delta x = u\,\delta t$ is unwound from the coil. The momentum imparted to this length depends only on the external forces acting on it. Hence

$$\rho u \,.\, u\,\delta t = (R+T)\,\delta t, \tag{2}$$

and so

$$\rho u^2 = R + T. \tag{3}$$

We shall now show that, if the spring is uncoiling, $R = 0$. For, the loss of energy in the element of spring in time δt is given by

$$\begin{aligned} &\text{work done} - \text{kinetic energy gained} - \text{elastic energy gained} \\ &= T\,\delta x - \tfrac{1}{2}\rho u^2\,\delta x - \tfrac{1}{2}T\,\delta x \\ &= \tfrac{1}{2}(T - \rho u^2)\,\delta x \\ &= -\tfrac{1}{2}Ru\,\delta t. \end{aligned} \tag{4}$$

This cannot be negative. Therefore if u is positive we must have $R \leqslant 0$. But $R \geqslant 0$, and so $R = 0$.

Therefore, from equation (3),
$$\rho u^2 = T, \tag{5}$$
and, since $1/\rho \neq 0$,
$$\rho^2 u^2 = \rho T = C, \tag{6}$$
$$\rho u = C^{\frac{1}{2}}. \tag{7}$$

Thus the rate of uncoiling, ρu, is a constant, characteristic of the spring.

If we imagine a velocity $-u$ to be superposed on the system, we have a stationary spring along which a shock wave is spreading; the front advances or retreats according as u is negative or positive. The rate at which mass is passing through the front is $\pm C^{\frac{1}{2}}$, provided R is zero, and this remains true whatever velocity is superposed.

3. *The staircase problem.* The sequence of events is as shown in Fig. 1. Let us suppose that when the spring is uncoiling there is a discontinuity in the density at the upper coil (at the point P in Fig. 1 (*a*)). Then the rate of uncoiling at that point is $C^{\frac{1}{2}}$, by equation (7). When the coil is unwound (Fig. 1 (*b*)) and the tail end of the coil leaves the upper stair, the tension at the tip falls suddenly to zero. Therefore a shock wave, behind which $T = 0$ and $\rho = \infty$, spreads through the spring, and the rate at which it devours the tail is just $C^{\frac{1}{2}}$. (The tail being free we may assume $R = 0$ in the tip.) Thus if the tail takes a time t_1 between leaving the upper stair and arriving at the lower coil (Fig. 1 (*d*)) its mass m_1 equals $C^{\frac{1}{2}}t_1$. The tail then overturns (Fig. 1 (*e*)), and the lower coil begins to uncoil immediately. If m_2 is the mass of the lower coil, the time t_2 that it takes to uncoil is $m_2/C^{\frac{1}{2}}$. By this time the spring is again as in Fig. 1 (*b*), but a step lower. The total time t taken for a cycle is given by
$$t = t_1 + t_2 = \frac{m_1}{C^{\frac{1}{2}}} + \frac{m_2}{C^{\frac{1}{2}}} = \frac{M}{C^{\frac{1}{2}}}, \tag{8}$$
where M is the total mass. A shock wave has passed right through the spring at a constant rate $C^{\frac{1}{2}}$, starting from the tail at 1 (*b*) and changing its type at the rebound between 1 (*d*) and 1 (*e*).

We may therefore expect that the time taken for the Slinky to descend n steps is simply nt, where t is given by (8). This is independent of the height of each step.

The ratio $M/C^{\frac{1}{2}}$ may be determined independently by a simple experiment. Let the spring be suspended vertically, with its lower end free. If x is measured upwards from the bottom we have
$$\frac{\partial T}{\partial x} = \rho g, \tag{9}$$
where g denotes gravity. Substituting from (1) we have
$$\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = \frac{g}{C}, \tag{10}$$
and so on integration
$$\frac{1}{2\rho^2} = \frac{gx}{C} + \text{constant}. \tag{11}$$
The constant of integration is zero, since $\dfrac{1}{\rho} = \dfrac{T}{C} = 0$ when $x = 0$. Thus
$$\rho = \left(\frac{C}{2gx}\right)^{\frac{1}{2}}, \tag{12}$$

and M, the total mass, is given by

$$M = \int_0^L \rho dx = \left(\frac{2CL}{g}\right)^{\frac{1}{2}}, \tag{13}$$

L being the total length. We have then from (8) and (13)

$$t = \frac{M}{C^{\frac{1}{2}}} = \left(\frac{2L}{g}\right)^{\frac{1}{2}}. \tag{14}$$

4. *Experimental verification.* In determining L it was necessary to avoid stretching the spring beyond its elastic limit. Accordingly, the hanging length of half the spring ($44\frac{1}{2}$ coils out of 89) was first measured, and then multiplied by 4, since, from equation (13), L is proportional to M^2. As a result it was found that

$$L = 4 \times 31{\cdot}0\,\text{in.} = 124\,\text{in.}$$

Assuming $g = 32{\cdot}2\,\text{ft./sec.}^2$ we have from (14)

$$t = 0{\cdot}80\,\text{sec.}$$

as the theoretical time of descent per step.

Table 1. *Observed times of descent*

Flight	Riser (in.)	Tread (in.)	From stair	To stair	n	nt (sec.)	t (sec.)
			(a) One stair at a time				
A	8	8	4	14	10	7·9 (0·1)	0·79
			10	14	4	3·1 (0·1)	0·78
B	7	$7\frac{1}{2}$	4	16	12	9·6 (0·2)	0·80
C	7	$9\frac{3}{4}$	3	14	11	8·7 (0·1)	0·79
			4	14	10	8·0 (0·1)	0·80
			10	14	4	3·3 (0·2)	0·82
D	$6\frac{1}{4}$	$10\frac{1}{2}$	3	11	8	6·7 (0·2)	0·84
E	$5\frac{1}{4}$	12	no steady run				
			(b) Two stairs at a time				
A	8	8	4	14	5	4·1 (0·1)	0·82
B	7	$7\frac{1}{2}$	4	16	6	4·5 (0·1)	0·75
D	$6\frac{1}{4}$	$10\frac{1}{2}$	4	10	3	2·5 (0·1)	0·80

The Slinky was tried out on five different flights of stairs, of various dimensions, in Trinity College, Cambridge. The descent was successfully made on all but one flight, whose gradient appeared to be too gentle; the Slinky came to rest after three or four steps at most. On the other flights it was found possible to persuade the Slinky to descend continuously, either one or two stairs at a time (i.e. per step).

The descent was timed with a stop-watch, and the results are summarized in Table 1. The measurements of the stairs are given in columns 2 and 3; the value for the tread has been corrected by subtracting the width of the nosing, or overhang, which was about $\frac{1}{2}$ in. The stop-watch was usually started at the third or fourth stair from the start (see column 4), by which time the rate of descent appeared to be steady.

To check this, the lower part of the run was timed on two occasions (see Table 1) and it was found that the mean rate of descent was not significantly different from that over the whole run. The observed times are given in the seventh column, where each figure represents the average of five observations. The figures placed afterwards in brackets are the standard deviations. Finally, the mean times of descent per step are given in the last column.

The observations are surprisingly consistent. The observed time t varies very little, if at all, with the height of the step. Although there is perhaps a tendency in group (a) for t to increase as the height decreases, this tendency is reversed in group (b). More significant is the fact that the times are almost the same for the second group as for the first, in spite of the height of each step being, in effect, doubled. The mean of all the observed times in the last column is 0·80 sec. in close agreement with the predicted time. None of the observed times differs from this by more than 0·05 sec.

I am indebted to A. M. Binnie for introducing me to this interesting toy, and for his assistance with the final experiments.

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