Master 1 Lecture Notes 2011–2012
Numerical Approximation of PDEs

Hervé Le Dret

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Contents

Chapter 1

Mathematical Modeling and PDEs

In this chapter, we will consider several concrete situations stemming from various areas of applications, the mathematical modeling of which involves partial differential equation (PDE) problems. We will not be rigorous mathematically speaking. There will be many often rather brutal approximations, not always convincingly justified. This is however the price to be paid if we want to be able to derive mathematical models that aim to describe the complex phenomena we will be dealing with in a way that remains manageable. At a later stage, we will study some of these models with all required mathematical rigor.

The simplest examples arise in mechanics. Let us start with the simplest example of all.

1.1 The elastic string

Let us first consider the situation depicted in Figure 1 below.

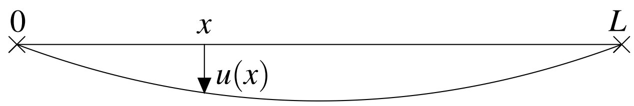

Figure 1. An elastic string stretched between two points and pulled by some vertical force. The point initially located at x moves by a displacement $u(x)$.

What is an elastic string in real life? It can be several different objects, such as a stretched rubber band, a musical instrument string made of nylon or steel, or again a cabin lift cable. Up to a certain level of approximation, all these objects are modeled in the same way. What they all have in common is a very small aspect ratio: they are three-dimensional and much thinner in two directions than in the third. Thus, the first step toward a simple mathematical model is to simply declare them to be one-dimensional. Points in a string will be labeled by a single real-valued variable x belonging to a segment $[0,L]$, embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$. Another implicit assumption used here is that of continuum mechanics. We assume that matter is a continuum which can be divided indefinitely. This is obviously untrue, but it is true enough at the macroscopic scale to be an extremely effective modeling hypothesis.

We assume that the string is stretched with a tension $T > 0$. The tension is just the force that is applied at both extremities 0 and L in order to make the string taut, for instance by working the tuners of a guitar or by passing the string on a wheel and suspending a weight at the end. If the only force acting on the string is the tension, then the string settles in an equilibrium position which is none but the segment $[0,L]$.

We now perform a thought experiment by cutting the string at two points of $[0,L]$, located at abscissae x and y. If the piece of string is going to stay in the same place, it is quite clear that we need to apply horizontal forces $T_-(x) < 0$ and $T_+(y) > 0$ at the two newly created extremities, in order to compensate for the disappearance of the rest of the string.

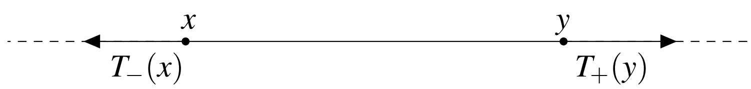

Figure 2. A piece of string kept in equilibrium.

As a matter of fact, $T_-(x)$ is the force that was exerted by the $[0,x]$ part of the string on the segment $[x,y]$ at point x before the string was cut, and $T_+(y)$ is likewise the force formerly exerted by the $[y,L]$ part of the string at point y. The action-reaction principle immediately shows that we have $T_-(x) = -T_+(x)$. Let us thus just set $T(x) = T_+(x)$.

Since the cut piece of string stays in equilibrium and the only forces acting on it are the above two tensions, Newton's law implies that the resultant force vanishes, that is to say that

$$T(y) - T(x) = 0.$$

Now this holds true for all x and y, therefore the tension $T(x)$ inside the string is constant. Taking $x = 0$, we see that this constant is equal to T, the string tension applied at the extremities. This is an important, even if obvious, result, because it holds true irrespective of the physical nature of the string. Whether it is made of rubber, nylon or steel, the tension inside a string is constant and equal to the applied tension. This is quite remarkable.

From now on, we will work with planar deformations, that is to say, we assume that the string lives in $\mathbb{R}^2$. Of course, a similar model can be derived in three dimensions. Let us apply other forces to the string, for example its weight or the weight of an object that is suspended to the string. For simplicity, we assume that this extra force is perpendicular to the segment—we will say that it is vertical whereas the segment is considered to be horizontal—and described by a lineic density. This means that we are given a function $f\colon [0,L] \to \mathbb{R}$ such that the vertical component of the force applied to a portion $[a,b]$ of the string is equal to the integral $\int_a^b f(x)\,dx$. Such is the case of the weight of the string. Assume the string is homogeneous, then its weight is represented by the function $f(x) = -\rho g$ where ρ is the string mass per unit length and g is the gravitational acceleration. If we suspend a weight P to a device occupying a segment $[\alpha,\beta]$ of the string, we may take $f(x) = -P\mathbf{1}_{[\alpha,\beta]}(x)$, where $\mathbf{1}_E$ denotes the characteristic function of a set E: $\mathbf{1}_E(x) = 1$ if $x \in E$, $\mathbf{1}_E(x) = 0$ otherwise.

Due to the extra applied force, the string deforms and settles in a new, unknown equilibrium position that we wish to determine. We assume that a point initially situated at $(x,0)$ moves vertically and reaches an equilibrium position $(x,u(x))$. This is again a *modeling hypothesis*. It is not strictly speaking true. In reality, the point in question also moves a little bit to the left or to the right. However, this hypothesis is reasonable when the force is vertical and the displacement is small. In this case, it can be justified, and we just admit it here, that the horizontal displacements are negligible in comparison with the vertical displacement $u(x)$. If the force was slanted, or the displacement large, it would be an entirely different story.

The deformed string is at this point described by a parametric curve in $\mathbb{R}^2$, $x \mapsto (x,u(x))$ and u is an unknown function. We now make another modeling hypothesis, which is to only be interested in those situations where the derivative $u'(x)$ has small absolute value. In this case, the length element of the deformed string satisfies

$$\sqrt{1+u'(x)^2} \approx 1 + \frac{1}{2}u'(x)^2 \approx 1,$$

since if $u'(x)$ is small, then $u'(x)^2$ is negligible.[1] We are thus dealing with situations in which the string is approximately inextensional, *i.e.*, there are no length changes.

[1]As a general rule, we neglect all terms of order strictly higher than one with respect to $u'(x)$. This leads to a simplified *linearized* model. A model that would take into account such higher order terms would be by nature *nonlinear*, and thus a lot more difficult to study from the point of view of mathematics.

Let us pick up our thought experiment scissors again and cut the string between abscissae x and $x+\Delta x$. This time, the string is no longer straight. When we think about the forces exerted by the rest of the string on the cut part, it appears reasonable that these forces should be tangent to the deformed string at the cut points, see Figure 3 below. This is yet another modeling hypothesis, which can be justified by a more refined mechanical analysis.

The scissors wielding thought experimenter thus applies a tension force of the form $-T(x)\tau(x)$ at point $(x,u(x))$, and a tension force $T(x+\Delta x)\tau(x+\Delta x)$ at point point $(x+\Delta x, u(x+\Delta x))$, where τ is the unit tangent vector

$$\tau(x) = \frac{1}{\sqrt{1+u'(x)^2}}\begin{pmatrix}1\\ u'(x)\end{pmatrix} \approx \begin{pmatrix}1\\ u'(x)\end{pmatrix},$$

in order to keep the cut piece in equilibrium. The above approximation of the tangent vector is legitimate in view of our decision to neglect terms $u'(x)^2$ and higher.

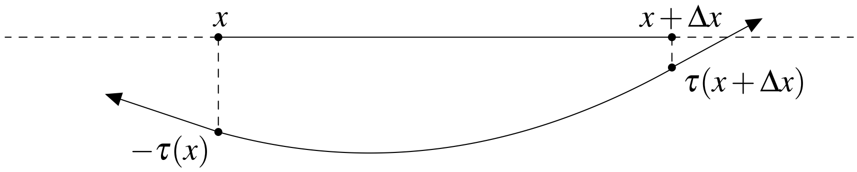

Figure 3. Cutting a piece of the deformed string.

We apply Newton's law again, which yields the vector equation

$$T(x+\Delta x)\tau(x+\Delta x) - T(x)\tau(x) + \begin{pmatrix}0\\ \int_x^{x+\Delta x} f(s)\,ds\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

The horizontal component of the equation implies that $T(x) = T$ is the same constant as before. The vertical component then reads

$$T\big(u'(x+\Delta x) - u'(x)\big) + \int_x^{x+\Delta x} f(s)\,ds = 0,$$

using the above approximation for the tangent vector. Now faced with such an equation, one should feel an irrepressible urge to divide everything by Δx, whence

$$-T\frac{u'(x+\Delta x)-u'(x)}{\Delta x} = \frac{1}{\Delta x}\int_x^{x+\Delta x} f(s)\,ds,$$

which should normally trigger an irrepressible urge to let Δx tend to 0, since the left-hand side is a differential quotient, and the right-hand side is an average over a small interval. We thus obtain in the limit $\Delta x \to 0$,

$$-Tu''(x) = f(x),$$

which can be rewritten as the first equation of the following *string problem:*

$$\begin{cases} -u''(x) = \frac{1}{T} f(x) \text{ in }]0,L[, \\ u(0) = u(L) = 0. \end{cases} \tag{1.1}$$

The second line of (1.1) expresses the fact that the string is fixed at the endpoints $x = 0$ and L. These points never move and the displacement is zero there. This condition is called a *boundary condition*. Problem (1.1) consists of a differential equation in an open set (here an ordinary differential equation, since we are in dimension one), together with a condition on the boundary of the open set. This type of problem is called a *boundary value problem*, and we will see more of them.

If we are somehow capable of solving this problem, then we will have determined the deformed shape of the string under the action of the applied forces. Indeed, it is easily checked by following the computations backward, that any solution of problem (1.1) yields an equilibrium position for the string.

Remark 1.1.1 In order to appease natural suspicions that it does not feel right to neglect terms before differentiating them, we can note that

$$\Big(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\Big)' = \frac{u''(x)}{\sqrt{1+u'(x)^2}} + \frac{u'(x)^2 u''(x)}{(1+u'(x)^2)^{3/2}} \approx u''(x),$$

therefore it was not so bad, a posteriori. □

Remark 1.1.2 Let us admit for the time being that problem (1.1) has a unique solution for given f and T. If we consider the same string subjected to the same force, but with different tensions, we see that the displacement is inversely proportional to the tension: the tauter the string, the more rigid it is and conversely. This is in agreement with day-to-day experience.

Let us emphasize again that the string model is independent of the physical nature of the string, which can be counterintuitive. A rubber string and a steel string stretched with the same tension behave the same insofar is the model is concerned. □

It is important to understand that, even though we have here an ordinary differential equation because we are in dimension one, a boundary value problem such as problem (1.1) has *strictly nothing to do* with the Cauchy problem for the same ordinary differential equation, either in terms of theory or in terms of numerical approximation.

In particular, the numerical schemes used for the Cauchy problem, such as the forward and backward Euler methods or the Runge-Kutta method, are of no use to compute approximations of the solution of problem (1.1) (apart from their use in the shooting method).

To illustrate this, let us introduce a slightly generalized version of the string problem. We thus consider the following boundary value problem:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) \text{ in }]0,L[, \\ u(0) = A, u(L) = B, \end{cases} \tag{1.2}$$

where f and c are two given functions defined on $]0,L[$ and A, B are two constants. The function c has no specific mechanical interpretation in the context of the elastic string. It just adds generality without costing any extra complexity. The boundary condition in (1.2) is called a *Dirichlet boundary condition*. In the case when $A = B = 0$, it is called a *homogeneous* Dirichlet condition.

Let us now see some of the fundamental differences between a boundary value problem and a seemingly similar Cauchy problem. The Cauchy problem consists in replacing the boundary conditions in (1.2) by initial conditions of the form $u(0) = \alpha$, $u'(0) = \beta$. Clearly, the Cauchy problem always has one and only one solution. The boundary value problem (1.2) may however not have any solution at all!

We take the following apparently innocuous example: $L = 1$, $A = B = 0$, $c(x) = -\pi^2$ and $f(x) = 1$. Assume that the problem

$$\begin{cases} -u''(x) - \pi^2 u(x) = 1 \text{ in }]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

has a solution. We multiply the differential equation by $\sin(\pi x)$, which yields

$$-u''(x)\sin(\pi x) - \pi^2 u(x)\sin(\pi x) = \sin(\pi x).$$

We now integrate this equality between 0 and 1. We obtain

$$-\int_0^1 u''(x)\sin(\pi x)\,dx - \pi^2\int_0^1 u(x)\sin(\pi x)\,dx = \int_0^1 \sin(\pi x)\,dx = \frac{2}{\pi}. \tag{1.3}$$

Now, if we integrate the first integral by parts twice, we see that

$$\int_0^1 u''(x)\sin(\pi x)\,dx = [u'(x)\sin(\pi x)]_0^1 - \pi[u(x)\cos(\pi x)]_0^1 - \pi^2\int_0^1 u(x)\sin(\pi x)\,dx.$$

The first two terms vanish because the sine function vanishes for the first one, and u vanishes for the second one by the homogeneous Dirichlet condition. The remaining integral cancels out with the second integral in equation (1.3). Finally, we find that

$$\frac{2}{\pi} = 0,$$

which is absurd. This is a contradiction, hence the problem cannot have any solution. $\square$

The above trick of multiplying the equation by certain well-chosen functions and integrating the result by parts will be at the heart of the existence and uniqueness theory using variational formulations, as well as the basis of such variational approximation methods as the finite element method that we will encounter later on.

We can already prove a uniqueness result. The problem in the previous example is that the function c is negative (and take the specific value $-\pi^2$).

Theorem 1.1.1 *If c is a continuous, nonnegative function, then problem* (1.2) *has at most one solution of class C^2.*

Proof. Let u_1 and u_2 be two solutions of class C^2, and set $w = u_2 - u_1$. It is easily checked that w solves the homogeneous boundary value problem:

$$\begin{cases} -w''(x) + c(x)w(x) = 0 \text{ in }]0,L[, \\ w(0) = w(L) = 0. \end{cases}$$

Let $v \in C^2([0,L])$ be a function such that $v(0) = v(L) = 0$.[2] We multiply the differential equation by v as before and integrate between 0 and L. This yields

$$-\int_0^L w''(x)v(x)\,dx + \int_0^L c(x)w(x)v(x)\,dx = 0.$$

Integrating the first term by parts once, we obtain

$$\int_0^L [w'(x)v'(x) + c(x)w(x)v(x)]\,dx = 0,$$

since $[w'v]_0^L = 0$ given the boundary conditions satisfied by v. In particular, we may choose $v = w$. With this choice, we get

$$\int_0^L [w'(x)^2 + c(x)w(x)^2]\,dx = 0.$$

The integrand is a continuous function which is nonnegative due to the sign hypothesis for c. Its integral is zero, hence it is identically zero. In particular, $w'(x) = 0$, which implies $w(x) = w(0) = 0$ for all x, hence $u_1 = u_2$, which is the uniqueness result. □

Such boundary problems as (1.2) have an important property called the *maximum principle*. As we will see shortly, the proof is banal in dimension one, but this is a profound property in dimensions strictly greater than one.

[2] Such functions are called *test-functions*, since they are used to test the equation in a sense.

Theorem 1.1.2 (Maximum Principle) *Assume that $c \geq 0$ and that problem* (1.2) *has a solution u of class C^2. If $f \geq 0$ in $]0,L[$, $A \geq 0$ and $B \geq 0$, then we have $u \geq 0$ in $]0,L[$.*

Proof. We argue by contradiction by assuming that there exists a point x_0 such that $u(x_0) < 0$. Since $u(0) = A \geq 0$ and $u(L) = B \geq 0$, it follows that $x_0 \in]0,L[$. Now u is continuous, therefore there is an interval $[\alpha,\beta]$ such that $x_0 \in [\alpha,\beta] \subset [0,L]$ and $u \leq 0$ on $[\alpha,\beta]$. We may assume that $u(\alpha) = u(\beta) = 0$ by the intermediate value theorem.

On the interval $[\alpha,\beta]$, c and f are positive and u is nonpositive, therefore

$$u''(x) = c(x)u(x) - f(x) \leq 0.$$

We deduce from this that the function u is concave on $[\alpha,\beta]$.

Now as $x_0 \in [\alpha,\beta]$, there exists $\lambda \in [0,1]$ such that $x_0 = \lambda\alpha + (1-\lambda)\beta$. Consequently, the concavity of u implies that

$$u(x_0) \geq \lambda u(\alpha) + (1-\lambda)u(\beta) = 0,$$

which is a contradiction. □

Remark 1.1.3 Under the form given above, it is a little hard to see where the maximum of the principle is… because it is hiding. Anyway, the right way to understand Theorem 1.1.2 is to see it as a monotonicity result. Indeed, the function that maps the data triple (f,A,B) to the solution u is monotone. Thus, in the case of the elastic string, when $A = B = 0$ and $f \geq 0$, in other words when we pull upwards on the string, then $u \geq 0$, which means that the string bends upwards too. So we see a very natural physical interpretation of the maximum principle that is in agreement with our intuition. This is also the reason why, in mathematics, we prefer the operator $-u''$, or more generally $-\Delta u$ in higher dimension, to the operator u'', which has the opposite behavior. □

1.2 The elastic beam

Our second example is also an example taken from mechanics. However, the mathematical modeling of this example is considerably more complicated than that of the string, and we will not explain it here. We are again dealing with essentially one-dimensional objects that are a lot more rigid than the previous ones, such as a metal rod, a concrete pillar or a wooden beam. Such objects exhibit a strong resistance to bending, as opposed to strings.

If we assume that our beam is clamped in a rigid wall at both ends, see Figure 4 below, then the following boundary value problem is found for the vertical displacement u:

$$\begin{cases} Eu''''(x) = f(x) \text{ in }]0,L[, \\ u(0) = u'(0) = u(L) = u'(L) = 0, \end{cases} \tag{1.4}$$

where f is again the density of the applied vertical force and $E > 0$ is a coefficient which is characteristic of the material of the beam[3], the higher the coefficient, the more rigid the material. This is in striking contrast with the string model in which the nature of the string material plays no role whatsoever.

The differential equation is a fourth order equation, as opposed to a second order equation in the case of the string, and accordingly, the Dirichlet boundary conditions involve both u and u'.

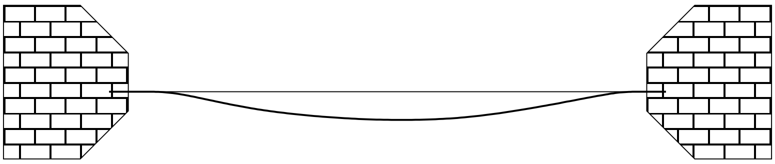

Figure 4. A beam clamped at both ends.

We can generalize in the same spirit as before by considering the boundary value problem

$$\begin{cases} u''''(x) - (a(x)u'(x))' + c(x)u(x) = f(x) \text{ in }]0,L[, \\ u(0) = u'(0) = u(L) = u'(L) = 0, \end{cases} \tag{1.5}$$

where the given functions a and c still have no particular mechanical meaning. We also have uniqueness of C^4 solutions when a and c are nonnegative. Indeed, if $w = u_2 - u_1$ is the difference between two solutions, multiplying by w the differential equation satisfied by w, which is (1.5) with zero right-hand side, and integrating by parts as many times as needed, we obtain

$$\int_0^L [(w''(x))^2 + a(x)(w'(x))^2 + c(x)w(x)^2]\,dx = 0,$$

whence $w''(x) = 0$. Consequently, w is an affine function of the form $w(x) = \alpha x + \beta$. Since it vanishes at both ends, we deduce that $w = 0$.

Remark 1.2.1 A word of warning: there is no maximum principle for such problems as (1.5) in general. The maximum principle is a property of second order boundary value problems that does not extend to fourth order problems. $\square$

[3]The coefficient E is called the Young modulus of the material. It is measured in units of pressure.

1.3 The elastic membrane

Let us now switch to real PDEs in more than one dimension. The first example is still taken form mechanics. It is the two-dimensional version of the elastic string, and it is called the elastic membrane. As we will see, many of the characteristics of the elastic string carry over to the elastic membrane.

To get a feeling for what an elastic membrane is, think of saran wrap suitable for food contact, that you can find in your favorite supermarket. Stretch the film up to the sides of some container in order to seal it before you store it in the fridge. In the beginning, the stretched part of the plastic film is planar. Then, as the temperature of the air inside the container goes down, the inside air pressure diminishes. At the same time, the atmospheric pressure inside the fridge remains more or less constant (you are bound to open the door every once in a while). The pressure differential thus created pushes on the film which bends inwards. We wish to determine the final shape of the film in three-dimensional space.

This kitchen example above is by far not the only one. There are many instances of elastic membranes around: the skin of a drum, a biological cell membrane, a boat sail, a party balloon, and so on.

To model this situation, let us be given an open set Ω of $\mathbb{R}^2$, whose boundary $\partial\Omega$ represents the sides of the container. Each point x of the closure $\bar{\Omega}$ of Ω represents a material point of the membrane when it is stretched without any other applied force. Again, we identify a small aspect ratio, three-dimensional object with a two-dimensional object filling $\bar{\Omega}$.

We now subject the membrane to a force density, such as the above pressure differential, which is orthogonal to its plane, and is represented by a given function $f\colon \Omega \to \mathbb{R}$. This time, f is a surfacic force density and the resultant force applied to a part ω of Ω is given by the integral $\int_\omega f(x_1,x_2)\,dx_1dx_2$.

As in the case of the elastic string, we make the reasonable albeit approximative hypothesis that point x is displaced of a quantity $u(x)$ perpendicularly to the membrane (vertically in the picture below). The displacement u is thus now a function in two variables $u\colon \bar{\Omega} \to \mathbb{R}$, and the shape of the membrane at equilibrium is a parametric surface in $\mathbb{R}^3$ given by $(x_1,x_2) \mapsto (x_1,x_2,u(x_1,x_2))$ for all (x_1,x_2) in $\bar{\Omega}$.

Since we assume that the membrane sticks to the sides of the container, we get at once a homogeneous Dirichlet boundary condition

$$u(x) = 0 \text{ on } \partial\Omega. \tag{1.6}$$

which is the exact analogue of its one-dimensional counterpart for the elastic string. We next need to obtain an equation that will determine the function u in Ω, and based on our previous one-dimensional experience, we can expect partial differential equations to play the leading role here.

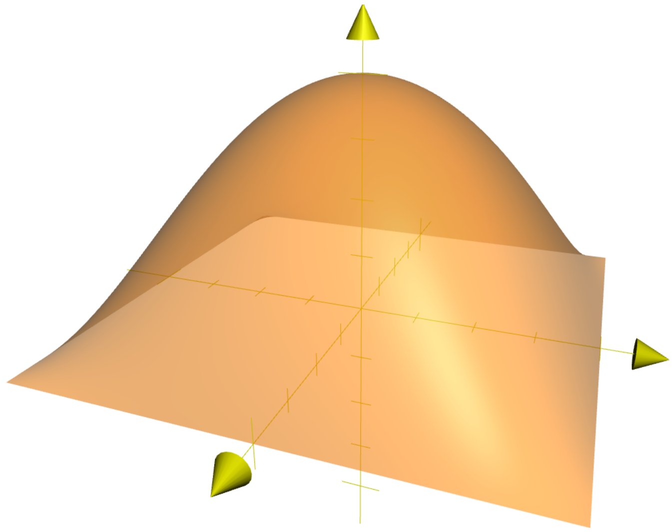

Figure 5. An elastic membrane stretched with tension $T = 1$ on the square $\Omega =]-1,1[^2$, subjected to a vertical force of density $f(x_1,x_2) = 4-2(x_1^2+x_2^2)$ and homogeneous Dirichlet conditions on the boundary of the square.

In Figure 5 above, the drawn surface represents the graph of the function u. The two vectors

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial u}{\partial x_1}(x_1,x_2) \end{pmatrix} \text{ and } a_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial u}{\partial x_2}(x_1,x_2) \end{pmatrix}$$

form a basis of the tangent plane to the surface at point $(x_1,x_2,u(x_1,x_2))$.

As in the case of the elastic string, we will only consider situations in which $\|\nabla u\| = \sqrt{\big(\frac{\partial u}{\partial x_1}\big)^2 + \big(\frac{\partial u}{\partial x_2}\big)^2}$ is small (which is not exactly the case in Figure 5!). This hypothesis leads us to neglect all quantities that are at least quadratic in the partial derivatives of u. In particular, when we normalize the above tangent basis vectors, we obtain the approximation

$$\frac{a_i}{\|a_i\|} = \frac{1}{\sqrt{1+\big(\frac{\partial u}{\partial x_i}\big)^2}} a_i \approx a_i,$$

which is analogous to the normalization of the tangent vector to the deformed elastic string used earlier.

Let us now explain what the word tension means in the case of a membrane. Because we are in a two-dimensional setting, the situation is a bit more complicated than for the elastic string. The general principle remains however the same. Let us consider a part A of the membrane and isolate this part as if it was cut out of the membrane. Just like the cut piece of string before, what keeps the part A in place must be forces exerted by the rest of the membrane. It seems reasonable to assume that these forces are exerted exactly on the boundary Γ_A of A relative to the membrane, since the membrane cannot act at a distance. Now the boundary in question is a curve, so that the force in question must be given by a lineic density distributed on Γ_A, the resultant force being the integral of the density on Γ_A. This is general for all two-dimensional continuum mechanics models.

In the case of an elastic membrane, as in the case of a string, we assume that the above force density lies in the tangent plane to the deformed surface, and furthermore, that it is normal to Γ_A in the tangent plane and pointing outwards, see Figure 6 below. Actually, this assumption can be seen as the very definition of an elastic membrane. The tension $T > 0$ is the norm of this force density—we admit here for simplicity that this norm is a constant, independent of the point[4]—it measures the tautness of the membrane. The physical unit for T is the N/m.

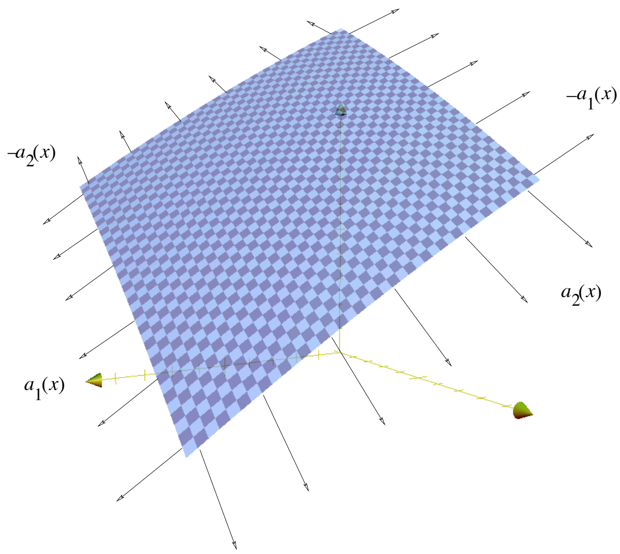

Figure 6. Magnified view of a small square cut out of the membrane. A few of the normal vectors are drawn. The force density exerted by the rest of the membrane is equal to T times these vectors. We can see it pulling to stretch the piece of membrane.

Let us thus take the scissors out again and cut out a small square in the membrane around an arbitrary point $(x_0, u(x_0))$. More precisely, we consider the square

$$C_{x_0,\Delta x} =]x_{0,1} - \Delta x, x_{0,1} + \Delta x[\times]x_{0,2} - \Delta x, x_{0,2} + \Delta x[,$$

in $\mathbb{R}^2$, and cut out its image by the mapping $x \mapsto (x, u(x))$ in $\mathbb{R}^3$, see Figure 6. We also make no distinction between the boundary of the image of $C_{x_0,\Delta x}$ in $\mathbb{R}^3$ and

[4]This can of course be proved with a little more work.

its boundary as a subset of $\mathbb{R}^2$ in the computation of the integrals. This is because $\|\nabla u\|$ is assumed to be small. We already made this approximation in the case of the string, without mentioning it... It can be fun to perform the exact computations to make sure that this approximation is really justified.

In order to compute the integral, we number the four sides of the square counterclockwise: $\gamma^1_{x_0,\Delta x} =]x_{0,1}-\Delta x, x_{0,1}+\Delta x[\times \{x_{0,2}-\Delta x\}$, $\gamma^2_{x_0,\Delta x} = \{x_{0,1}+\Delta x\} \times]x_{0,2}-\Delta x, x_{0,2}+\Delta x[$, and so on for $\gamma^3_{x_0,\Delta x}$ and $\gamma^4_{x_0,\Delta x}$. According to the normal vectors depicted in Figure 6, Newton's law for the vertical force component then reads

$$T\Big[\int_{\gamma^1_{x_0,\Delta x}} -[a_2(x)]_3\,d\gamma + \int_{\gamma^2_{x_0,\Delta x}} [a_1(x)]_3\,d\gamma + \int_{\gamma^3_{x_0,\Delta x}} [a_2(x)]_3\,d\gamma + \int_{\gamma^4_{x_0,\Delta x}} -[a_1(x)]_3\,d\gamma\Big] + \int_{C_{x_0,\Delta x}} f(x)\,dx = 0, \quad (1.7)$$

where $[z]_3$ denotes the vertical component of vector z. It is a simple exercise to check that the horizontal components already satisfy Newton's law. Let us write each integral separately. We have

$$\int_{\gamma^1_{x_0,\Delta x}} [a_2(x)]_3\,d\gamma = \int_{x_{0,1}-\Delta x}^{x_{0,1}+\Delta x} \frac{\partial u}{\partial x_2}(s, x_{0,2}-\Delta x)\,ds,$$
$$\int_{\gamma^2_{x_0,\Delta x}} [a_1(x)]_3\,d\gamma = \int_{x_{0,2}-\Delta x}^{x_{0,2}+\Delta x} \frac{\partial u}{\partial x_1}(x_{0,1}+\Delta x, s)\,ds,$$
$$\int_{\gamma^3_{x_0,\Delta x}} [a_2(x)]_3\,d\gamma = \int_{x_{0,1}-\Delta x}^{x_{0,1}+\Delta x} \frac{\partial u}{\partial x_2}(s, x_{0,2}+\Delta x)\,ds,$$
$$\int_{\gamma^4_{x_0,\Delta x}} [a_1(x)]_3\,d\gamma = \int_{x_{0,2}-\Delta x}^{x_{0,2}+\Delta x} \frac{\partial u}{\partial x_1}(x_{0,1}-\Delta x, s)\,ds.$$

Formula (1.7) can thus be rewritten as

$$T\Big[\int_{x_{0,1}-\Delta x}^{x_{0,1}+\Delta x} \Big(\frac{\partial u}{\partial x_2}(s, x_{0,2}+\Delta x) - \frac{\partial u}{\partial x_2}(s, x_{0,2}-\Delta x)\Big)\,ds + \int_{x_{0,2}-\Delta x}^{x_{0,2}+\Delta x} \Big(\frac{\partial u}{\partial x_1}(x_{0,1}+\Delta x, s) - \frac{\partial u}{\partial x_1}(x_{0,1}-\Delta x, s)\Big)\,ds\Big] + \int_{C_{x_0,\Delta x}} f(x)\,dx = 0. \quad (1.8)$$

The situation is less transparent than in dimension one, but the idea is the same. We divide everything by $4(\Delta x)^2$,

$$-T\frac{1}{2\Delta x}\Big[\int_{x_{0,1}-\Delta x}^{x_{0,1}+\Delta x}\frac{\frac{\partial u}{\partial x_2}(s,x_{0,2}+\Delta x)-\frac{\partial u}{\partial x_2}(s,x_{0,2}-\Delta x)}{2\Delta x}\,ds$$
$$+\int_{x_{0,2}-\Delta x}^{x_{0,2}+\Delta x}\frac{\frac{\partial u}{\partial x_1}(x_{0,1}+\Delta x,s)-\frac{\partial u}{\partial x_1}(x_{0,1}-\Delta x,s)}{2\Delta x}\,ds\Big]$$
$$=\frac{1}{4(\Delta x)^2}\int_{C_{x_0,\Delta x}}f(x)\,dx. \quad (1.9)$$

Now the length of each of the segments on which differential quotients of the partial derivatives $\partial u/\partial x_i$ are integrated is exactly $2\Delta x$, and the area of the square is exactly $4(\Delta x)^2$. We thus see that all the above quantities are averages over small segments or squares, which is good in view of letting Δx tend to 0 later.

Let us assume that u is of class C^2. We can thus write the following Taylor-Lagrange expansion at $x=(s,t)$

$$\frac{\partial u}{\partial x_2}(x)=\frac{\partial u}{\partial x_2}(x_0)+\frac{\partial^2 u}{\partial x_2\partial x_1}(x_0)(s-x_{0,1})+\frac{\partial^2 u}{\partial x_2^2}(x_0)(t-x_{0,2})+r(x)$$

where $r(x)/\|x-x_0\|\to 0$ when $\|x-x_0\|\to 0$. Therefore

$$\frac{\frac{\partial u}{\partial x_2}(s,x_{0,2}+\Delta x)-\frac{\partial u}{\partial x_2}(s,x_{0,2}-\Delta x)}{2\Delta x}=\frac{\partial^2 u}{\partial x_2^2}(x_0)+r'(s,\Delta x)$$

where $r'(s,\Delta x)\to 0$ when $\|(s,x_{0,2}+\Delta x)-x_0\|\to 0$. Integrating with respect to s, we obtain

$$\frac{1}{2\Delta x}\int_{x_{0,1}-\Delta x}^{x_{0,1}+\Delta x}\frac{\frac{\partial u}{\partial x_2}(s,x_{0,2}+\Delta x)-\frac{\partial u}{\partial x_2}(s,x_{0,2}-\Delta x)}{2\Delta x}\,ds=\frac{\partial^2 u}{\partial x_2^2}(x_0)+r''(\Delta x)$$

where $r''(\Delta x)\to 0$ when $\Delta x\to 0$.

We treat the remaining integrals in the same fashion and obtain in the $\Delta x\to 0$ limit

$$\forall x_0\in\Omega,\quad -\Delta u(x_0)=\frac{1}{T}f(x_0). \quad (1.10)$$

The differential operator $\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$ is called the *Laplacian*. Equation (1.10) is called the *elastic membrane equation*. It must naturally be complemented by some boundary conditions, such as the homogeneous Dirichlet condition (1.6).

The mechanical remarks made in the case of the elastic string also apply to the elastic membrane and we will not repeat them.

More generally, the boundary value problem in any dimension, $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.11}$$

with $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$, is called the *Poisson equation*. The Poisson equation shows up in a surprising number of different areas of mathematics and its applications. For example, for $n = 3$, if f represents the density of electrical charge present in Ω and the boundary of Ω is covered by a perfectly conducting material, then $-u$ is the electric potential[5] inside Ω. The gradient of $-u$ is the electric field. More generally, the Poisson equation is central in all questions relating to the Newtonian potential.

There are many other interpretations. Thus, if f represents a density of heat sources in Ω, say the distribution of radiators in a room and how much heat they give off, then u is the equilibrium temperature in Ω when the walls of the room $\partial\Omega$ are somehow kept at temperature $0°$. This is why the Poisson equation is sometimes referred to as the diffusion equation, as it also models the diffusion of heat (and of other things that may want to diffuse).

There is also a probabilistic interpretation for the Poisson equation, not unrelated to the diffusion interpretation. For $f = 2$, $u(x)$ is the expectation of the first exit time from Ω of a standard Brownian motion starting from point x. Roughly speaking, a particle moving randomly in $\mathbb{R}^n$ and starting from x will reach $\partial\Omega$ in an average time $u(x)$.

Finally, when $f = 0$, the equation is known as the *Laplace equation* whose solutions are the harmonic functions (it is clearly better to impose a nonzero boundary condition to have $u \neq 0$, or no condition at all). Harmonic functions are of course extremely important.

Let us close this section by rapidly mentioning the plate equation. A plate is to a membrane what a beam is to a string: sheet iron, concrete wall, wood plank. The clamped plate problem reads

$$\begin{cases} \Delta^2 u = f \text{ in } \Omega, \\ \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.12}$$

where the operator $\Delta^2 = \Delta \circ \Delta = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}$ is called the *bilaplacian* and $\frac{\partial u}{\partial n} = \nabla u \cdot n = \frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2$ is the *normal derivative* of u on the boundary, n denotes the unit exterior normal vector (we will go back to this later). This is a fourth order boundary value problem.

[5]The minus sign is due to the physical convention that goes contrary to the mathematical convention in this case.

All the problems considered up to now are stationary problems in which time plays no role and only model equilibrium situations. Let us now talk about problems where time intervenes, that is to say evolution problems.

1.4 The transport equation

Let us imagine a kind of gas composed of particles moving along an axis. Instead of tracking each particle individually, which would be impossible in practice due to their huge number, we can describe the gas by using a function $u\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, where $u(x,t)$ measures the quantity of particles, or rather their density at point x and instant t. This is called a *kinetic description*. The initial density of particles at $t=0$ is denoted $u_0(x)=u(x,0)$. We assume it to be given, it is an *initial condition*.

Now the question is how does the gas evolve in time? We clearly need to make hypotheses on the individual motions of particles in order to answer this question. For maximum simplicity, we assume here that all the particles move at the same constant speed $c\in\mathbb{R}$ which is given. If $c>0$, they all move to the right, if $c<0$, they all move to the left, and if $c=0$, they do not move at all.

Let us count the total number of particles in a section $[y,y+\Delta y]$ of the gas. We disregard the fact that this number should be an integer. In fact, we consider cases in which this integer is so large as to appear like a continuous quantity at the macroscopic scale. Think of the Avogadro number and the fact that quantities of matter are actually measured in moles. By definition of a density, at time t, this quantity is equal to $Q(y,\Delta y,t)=\int_y^{y+\Delta y}u(s,t)\,ds$.

Since all particles move as a group at speed c, all the particles that were situated between y and $y+\Delta y$ at time 0, are going to be located between $y+ct$ and $y+\Delta y+ct$ at time t, and no other particle will be there at the same time. Therefore, we have a *conservation law*: for all y, Δy and t

$$Q(y+ct,\Delta y,t)=Q(y,\Delta y,0). \tag{1.13}$$

Let us differentiate relation (1.13) with respect to t. We obtain

$$\begin{aligned}0=\frac{d}{dt}Q(y+ct,\Delta y,t)&=\frac{d}{dt}\Big(\int_{y+ct}^{y+\Delta y+ct}u(s,t)\,ds\Big)\\&=c[u(y+\Delta y+ct,t)-u(y+ct,t)]+\int_{y+ct}^{y+\Delta y+ct}\frac{\partial u}{\partial t}(s,t)\,dx.\end{aligned} \tag{1.14}$$

Here again we find a relation that begs to be divided by Δy. So we oblige and let Δy tend to 0 so that

$$\frac{\partial u}{\partial t}(y+ct,t)+c\frac{\partial u}{\partial x}(y+ct,t)=0. \tag{1.15}$$

Now y and t are arbitrary, therefore we can perform the change of variables $x = y + ct$ and obtain the following PDE problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + c\dfrac{\partial u}{\partial x}(x,t) = 0 \text{ for } (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = u_0(x) \text{ for } x \in \mathbb{R}. \end{cases} \tag{1.16}$$

The PDE above is the *transport equation* (at velocity c), together with an initial condition. The conjunction of the two form an *initial value problem*. There is no boundary condition here since the space variable x ranges over the whole of $\mathbb{R}$.

Let us now proceed to solve the transport equation. Since the particles all move at the same velocity c, we can look at the variation of u on the trajectory of one particle $t \mapsto x + ct$ with x fixed. We thus compute the derivative

$$\frac{d}{dt}u(x+ct,t) = c\frac{\partial u}{\partial x}(x+ct,t) + \frac{\partial u}{\partial t}(x+ct,t) = 0.$$

In other words, u is constant on the trajectories. In particular,

$$u(x+ct,t) = u(x,0) = u_0(x). \tag{1.17}$$

The curves $t \mapsto (x+ct,t)$ in space-time $\mathbb{R} \times \mathbb{R}$—which are here straight lines—are called the *characteristics* of the equation, and their use to solve the equation is accordingly called the *method of characteristics*. They are often drawn in a space-time diagram as follows:

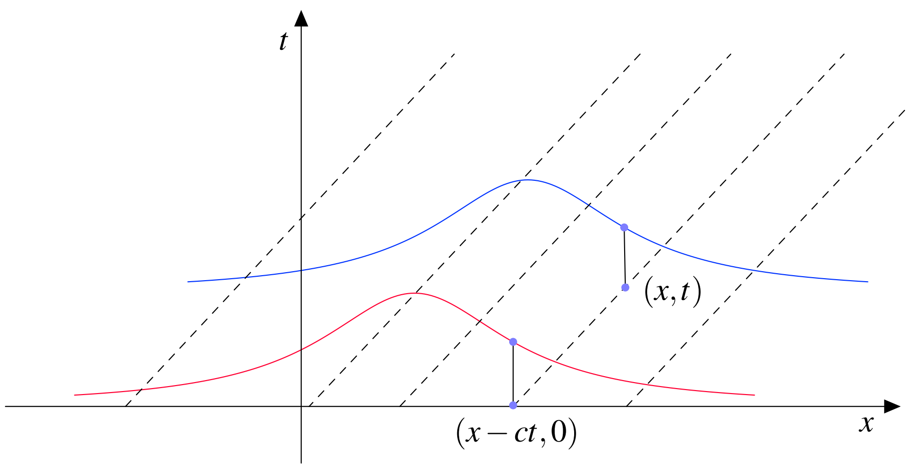

Figure 7. The characteristics are the dashed straight lines with slope $1/c$. If $c = 0$, they are vertical and there is no propagation.

To determine the value of u at a point (x,t) in space-time, it is enough to look at the unique characteristic going through this point, take the point where it intersects the $t = 0$ axis and take the value of u_0 at that point, see Figure 7. This construction simply amounts to rewriting formula (1.17) in the form

$$u(x,t) = u_0(x - ct), \tag{1.18}$$

which proves the uniqueness of the solution, due to an *explicit* formula![6]

We have established uniqueness of the solution, but have not yet established its existence. Fortunately, we have an explicit formula, therefore we just need to check that it actually is a solution. Let us compute the partial derivatives of u given by formula (1.18), assuming u_0 smooth enough. We have

$$\frac{\partial u}{\partial x}(x,t) = u_0'(x - ct) \text{ and } \frac{\partial u}{\partial t}(x,t) = -cu_0'(x - ct),$$

where u_0' is the ordinary derivative of u_0. The PDE is thus clearly satisfied. Moreover, the initial condition is also trivially satisfied by setting $t = 0$ in formula (1.18). Hence, we have found the unique solution.

It is apparent that u propagates or transports the initial data at constant speed c, hence the name of the equation.

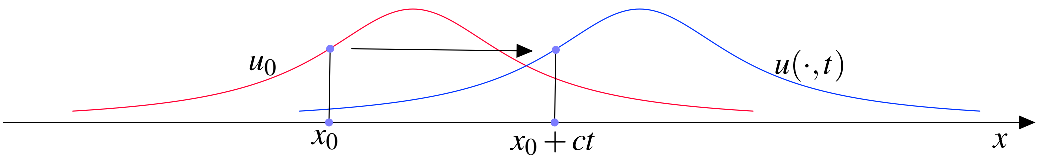

Figure 8. Propagation of the initial data u_0.

If it was possible to animate Figure 8 on paper, the blue curve would be seen to glide to the right at a steady pace ($c > 0$ in the picture) without changing shape, after having coincided with the red curve at $t = 0$.

The transport equation has higher dimensional versions, which are much more complicated than the one-dimensional version. It can also be set in open sets of $\mathbb{R}^n$ instead of on the whole of $\mathbb{R}^n$. In this case, boundary conditions must be added in addition to the initial condition, which makes it an initial-boundary value problem. The boundary value question is delicate depending on whether the transport velocity, which is then a vector, points inwards or outwards of the open set. The transport equation is relevant in many areas, whenever a spatially distributed quantity u_0 is transported by a velocity field, think of a concentration of pollutants carried away by the wind. A diffusion term is often added, yielding convection-diffusion problems.

[6]Explicit solutions are very rare in PDE problems.

1.5 The vibrating string problem

Let us return to the elastic string in the context of dynamics. The displacement u of the string is now a function of space x and time t. The analysis of applied forces is exactly the same as in the static case, except that Newton's law says that the resultant of the applied forces is equal to the time derivative of the momentum for each cut piece of the string. There is no point in going through all the detail again—it is actually a good exercise—and the result is

$$T\Big(\frac{\partial u}{\partial x}(x+\Delta x,t)-\frac{\partial u}{\partial x}(x,t)\Big)+\int_x^{x+\Delta x} f(s,t)\,ds=\int_x^{x+\Delta x}\rho\frac{\partial^2 u}{\partial t^2}(s,t)\,ds,$$

where T is still the constant tension, ρ is the mass of the string per unit length, and $\frac{\partial^2 u}{\partial t^2}(x,t)$ is the acceleration of the string at point x and time t. Note that the applied force f can now depend on time as well. Dividing by Δx and letting Δx tend to 0, we obtain

$$T\frac{\partial^2 u}{\partial x^2}(x,t)+f(x,t)=\rho\frac{\partial^2 u}{\partial t^2}(x,t),$$

which is best rewritten as

$$\frac{\partial^2 u}{\partial t^2}(x,t)-c^2\frac{\partial^2 u}{\partial x^2}(x,t)=\frac{1}{\rho}f(x,t), \tag{1.19}$$

with $c=\sqrt{\frac{T}{\rho}}$. This partial differential equation, which is also called the one-dimensional wave equation, is attributed to Jean le Rond d'Alembert. The constant c is the propagation speed. We will see later that this equation propagates waves to the right at speed c and to the left at speed $-c$. This is easily seen experimentally on a long rope held by two persons. In fact, the vibrating string differential operator is a composition of two transport operators

$$\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}=\Big(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big)=\Big(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big),$$

hence the two propagation directions. Note that the propagation speed increases with the tension and decreases with the mass of the string.

Equation (1.19) must be complemented by initial conditions that prescribe the initial shape and initial velocity of the string (this is a problem of the second order in time)

$$u(x,0)=u_0(x),\frac{\partial u}{\partial t}(x,0)=u_1(x)\text{ for all }x\in\,]0,L[, \tag{1.20}$$

and by boundary conditions, meaning here that the string is fixed at both endpoints

$$u(0,t)=u(L,t)=0,\ \text{for all } t\in\mathbb{R}. \tag{1.21}$$

It should be noted that if a regular solution is expected, then a certain compatibility between initial data (1.20) and boundary conditions (1.21) must be imposed

$$u_0(0) = u_0(L) = 0 \text{ and } u_1(0) = u_1(L) = 0,$$

otherwise a discontinuity in the displacement or velocity will arise at $t = 0$.

We will return later to a more in-depth study of the wave equation. For the time being, let us consider a particular case: *harmonic vibrations*. We are looking for solutions to equation (1.19) with right-hand side $f = 0$ and by separation of variables, *i.e.*, solutions of the special form $u(x,t) = \phi(x)\psi(t)$, non identically zero et satisfying the boundary condition (1.21). Obviously, in the case of harmonic vibrations, we cannot impose an arbitrary initial condition. It fact, it will soon be clear that no initial condition is needed.

Let us rewrite the problem in this setting:

$$\begin{cases} \phi(x)\psi''(t) - c^2\phi''(x)\psi(t) = 0 \text{ for all } x \in \,]0,L[, t \in \mathbb{R}, \\ \phi(0)\psi(t) = \phi(L)\psi(t) = 0, \text{ for all } t \in \mathbb{R}. \end{cases}$$

Naturally, if $\psi = 0$ then $u = 0$ which is not a very interesting solution. We thus assume that there exists t_0 such that $\psi(t_0) \neq 0$. It is therefore legal to divide by $\psi(t_0)$, so that

$$\begin{cases} -\phi''(x) + \dfrac{\psi''(t_0)}{c^2\psi(t_0)}\phi(x) = 0 \text{ for all } x \in \,]0,L[, \\ \phi(0) = \phi(L) = 0. \end{cases}$$

This a boundary value problem in the variable x of a kind we have already encountered, and we know that if $\frac{\psi''(t_0)}{c^2\psi(t_0)} \geq 0$, then $\phi = 0$ is the unique solution. This again means that $u = 0$, which is definitely not interesting. Let us thus consider the case when $\lambda = -\frac{\psi''(t_0)}{c^2\psi(t_0)} > 0$ and see under which conditions there could exist a nonzero solution.

Forgetting the boundary conditions for an instant, we recognize a second order linear differential equation with constant coefficients, the general solution of which is of the form

$$\phi(x) = A\sin\bigl(\sqrt{\lambda}x\bigr) + B\cos\bigl(\sqrt{\lambda}x\bigr).$$

The boundary condition $\phi(0) = 0$ requires $B = 0$. The boundary condition $\phi(L) = 0$ then either imposes $A = 0$, but then we are back to $\phi = 0$, hence a boring $u = 0$, or $\sin(\sqrt{\lambda}L) = 0$, that is to say

$$\sqrt{\lambda}L = k\pi \text{ for some } k \in \mathbb{Z},$$

or again

$$\lambda = \frac{k^2\pi^2}{L^2} \text{ and } \phi(x) = A\sin\Bigl(\frac{k\pi}{L}x\Bigr),$$

where k is an integer. Now this is interesting at last!

Without loss of generality, we take $A = 1$ and plug $u(x,t) = \sin\left(\frac{k\pi}{L}x\right)\psi(t)$ back into the original wave equation, which gives an equation for ψ

$$\psi''(t) + c^2 \frac{k^2\pi^2}{L^2}\psi(t) = 0,$$

that we solve immediately

$$\psi(t) = \alpha \sin\left(\frac{ck\pi}{L}t\right) + \beta \cos\left(\frac{ck\pi}{L}t\right),$$

where α and β are arbitrary constants. Finally, we have found

$$u(x,t) = \left[\alpha \sin\left(\frac{ck\pi}{L}t\right) + \beta \cos\left(\frac{ck\pi}{L}t\right)\right] \sin\left(\frac{k\pi}{L}x\right),$$

and it is easily checked that all these functions solve the wave equation with the homogeneous Dirichlet condition. We thus have found all the separated variable solutions.

These solutions are harmonic vibrations of frequency $\nu_k = \frac{ck}{2L} = \sqrt{\frac{T}{\rho}}\frac{k}{2L}$ indexed by the integer k. The lowest possible frequency is obtained for $k = 1$. It is called the fundamental and is the note that is heard from that string. The following integers correspond to the harmonics of this note: $k = 2$ double frequency, one octave above the fundamental, $k = 3$, $k = 4$ two octaves above the fundamental, etc. Naturally, the actual vibration of a musical string is never a separated variable solution, but a superposition of harmonics. This superposition gives the note its timbre. From the point of view of mathematics, this is a question of Fourier series, but we will not pursue this angle here.

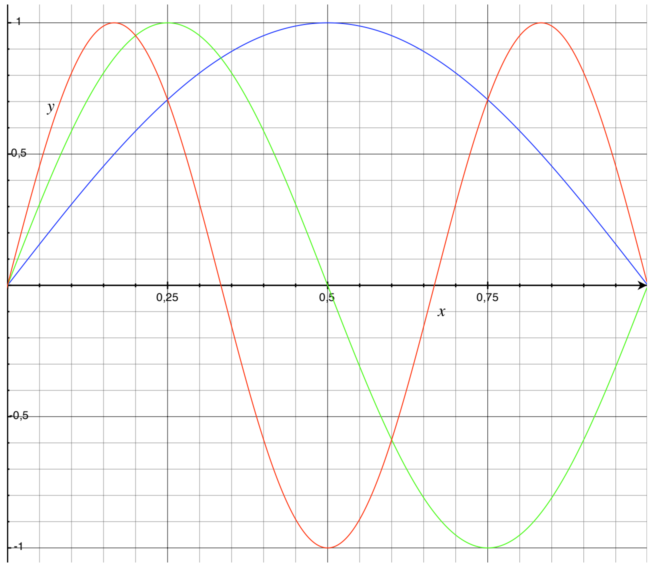

Figure 9. Three successive harmonics: functions ϕ for $k = 1,2,3$.

To close this section, we deduce from the formula for the frequency that a longer string will ring a lower note, hence the relative lengths of the necks of a guitar and a bass and the different notes played on the same string on the frets, that a heavier string also rings a lower note, hence the mass differences between the strings of a guitar or violin, and that a higher tension yields a higher note.

1.6 The wave equation

This is the higher dimensional analogue of the vibrating string equation. If we consider a vibrating membrane in dimension two, we easily obtain the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - c^2 \Delta u(x,t) = f(x,t) \text{ in } \Omega \times \mathbb{R}, \\ u(x,t) = 0 \text{ on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \text{ in } \Omega, \end{cases} \tag{1.22}$$

with $c = \sqrt{\frac{T}{\rho}}$, T is the tension and ρ the membrane mass per unit area. Note that compatibility conditions between the boundary and initial conditions must again be imposed if we expect a regular solution.

The harmonic vibration problem consists in looking for a solution of the form $u(x,t) = e^{i\lambda t}\phi(x)$ (we no longer need to pretend that we do not know what $\psi(t)$ must be...), hence the problem

$$\begin{cases} -\Delta\phi(x) = \dfrac{\lambda^2}{c^2}\phi(x) \text{ in } \Omega, \\ \phi(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.23}$$

with $\phi \neq 0$.

Problem (1.23) is an eigenvalue problem for the linear operator $-\Delta$, that is to say an infinite dimensional spectral problem. This was already the case in dimension one, but there was no need for the whole apparatus of self-adjoint compact operator spectral theory since everything could be done by hand.

What we need to know for now is that the eigenvalues, *i.e.*, the possible values for $\frac{\lambda^2}{c^2}$, form an infinite increasing sequence $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$, with $\mu_k \to +\infty$ when $k \to +\infty$, which depends on the shape of Ω. The situation is thus a lot more complex than in dimension one, where the shape of Ω is just characterized by its length L and we have an explicit formula for the eigenvalues. In particular, the vibration frequencies $\frac{\lambda_k}{2\pi} = c\frac{\sqrt{\mu_k}}{2\pi}$ are no longer proportional to successive integers. If the first eigenvalue still gives the fundamental note, the following harmonics are not in rational proportion to each other, and the timbre of the sound is entirely different. This explains why a drum produces a sound that has nothing in common with the sound produced by a guitar. It is all a question of dimensionality.

A classical problem that was only solved fairly recently was formulated as "Can you hear the shape of a drum?" The meaning of the question was to know whether the knowledge of the spectrum, that is to say of the entire sequence $(\mu_k)_{k\in\mathbb{N}^*}$ made it possible to determine Ω up to a rigid motion. The answer is negative. There are open sets in $\mathbb{R}^2$ of different shapes with exactly the same spectrum. Drums of these shapes would thus sound the same.

In higher dimensions, the wave equation is used to model the propagation of sound waves in the air, the propagation of light waves in the void (the wave equation in this case is deduced from Maxwell's equations, the PDEs of electromagnetism). There are all sorts of different kinds of waves, such as seismic waves or oceanic waves, the propagation of which is more complex than the wave equation.

1.7 The heat equation

The heat equation is yet another evolution equation, of a totally different nature as the previous ones. For example, time is reversible in the wave equation: changing t to $-t$ does not change it. The heat equation describes the evolution of temperature. It thus have a connection with thermodynamics and time can only flow from the past to the future. From the point of view of mathematics, changing t to $-t$ modifies the equation and leads to problems with no solution in general.

The heat equation is as follows:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,t) \text{ in } \Omega \times \mathbb{R}_+, \\ u(x,t) = 0 \text{ on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x) \text{ in } \Omega. \end{cases} \tag{1.24}$$

We have set all physical constants to the value 1, as is customary in mathematics. The equation is first order in time and second order in space with a boundary condition (of Dirichlet type here) and an initial condition. The heat equation was discovered by Fourier, based on arguments of exchange of heat between smaller and smaller, closer and closer balls. Actually these arguments are pretty close to the finite difference method that we will see later.

When $f = 0$, the effect of the heat equation is to diffuse the initial condition.

1.8 The Schrödinger equation

The Schrödinger equation is another evolution equation of again totally different nature. This time, u is a wave function in the sense of quantum mechanics. It is complex-valued. The domain is the whole of $\mathbb{R}^3$. The equation reads

$$i\frac{\partial u}{\partial t}(x,t) + \Delta u(x,t) = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}_+. \tag{1.25}$$

The Schrödinger equation is the basic equation of quantum mechanics that governs the evolution of the wave function of one particle in the absence of any potential, that is in the void. Physical constants are missing (set to 1), such as Planck's constant $\hbar$ and the mass of the particle. Also missing is the initial condition.

Since the square of the module of the wave function is interpreted as a presence probability, we need to impose

$$\int_{\mathbb{R}^3} |u|^2\,dx = 1.$$

Actually, if the initial condition satisfies this normalization condition, then the solution satisfies it automatically at all times.

Even though the Schrödinger equation presents a formal similarity with the heat equation—first order in time, second order in space—the presence of the imaginary factor i gives it radically different properties. In particular, the Schrödinger equation propagates waves, also not at all in the same way as the wave equation, whereas the heat equation does not propagate waves (heat waves notwithstanding!).

Let us note that in the Schrödinger equation for a system of N particles, the variable x must belong to $\mathbb{R}^{3N}$, which becomes rapidly difficult for practical purposes when N is large…

As a general rule, physics is a nearly inexhaustible source of partial differential equations problems. Let us cite the Dirac equation, a first order equation and relativistic version of the Schrödinger equation; Einstein's equations of general relativity, a system of nonlinear PDEs; the Boltzmann equation for the kinetic description of gases, all the equations of fluid mechanics, Euler, Stokes, Navier-Stokes, and so on, and so forth.

1.9 The Black and Scholes equation

Physics is not by far the only source of PDEs. PDEs are also playing an increasing role in diverse areas, such as biology, chemistry, material science, road traffic modeling, crowd movement modeling, economy, finance, among many others. Let us give a famous example in the latter area, the Black and Scholes equation.

The question is to set the price of a call option. A call option is a contract between a seller and a buyer, drawn at time $t = 0$. The contract gives the buyer the right to buy an asset belonging to the seller, not right away but later and at a price K, the strike, that is agreed on in advance. The contract has a price paid by the buyer to the seller at $t = 0$, otherwise the seller would have no real reason to agree to it. For the buyer, it is an insurance against stock fluctuations.

The price C must be computed in such a way that the game should be fair on average, or at least seem to be fair… The possibility of option pricing hinges on a modeling of the market and on a hypothesis called no arbitrage opportunity (no free lunch) meaning that it is impossible to make sure gains without taking risks.

To make things a little more precise, the price of the asset at instant t is denoted S_t. It is a continuous time stochastic process. In the case of an american call, the buyer acquires the right to exercise the option, that is to say to buy the asset for the price K, at any moment $t \in [0,T]$, where T is an expiration date agreed on in advance. The buyer is under no obligation to do so, and after time T, the option disappears.

Of course, the buyer has no interest in exercising the option at time t if $S_t < K$. In this case, it is better to buy at the market price or not buy at all. On the other hand, the buyer could also have invested the amount C at a fixed interest rate r without risk. Therefore, a profit would only made by exercising the option if $S_t > e^{rt}C + K$, which is the decision criterion. The buyer bets this situation will occur before time T, in which case he or she buys the stock for a price K and sells it back immediately on the market at price S_t, thus pocketing the difference $S_t - K$. The global balance of the operation is either $-C$ if the option is not exercised or $s_t - K - C$ if it is exercised.

The seller always gains C and loses $S_t - K$ if the buyer exercises the option, so the bet is that the buyer will not exercise the option. The seller must also seek to cover losses in case the buyer exercises the option through the price C.

The option price C is naturally a function of the asset price, which is represented by a variable $x \in \mathbb{R}_+$, because a price is nonnegative. It is also useful to introduce the price at instant t, that is to say, the price the option would have is it was bought at instant t with the same strike K and expiry date T. The option price is thus a function in two variables $C(x,t)$ (let us emphasize again that the space variable x is actually a price). We want to determine $C(x,0)$ in order to define the terms of the contract, since at $t = 0$, the price of the asset S_0 is known. The price of the option at $t = T$ is obviously $C(x,T) = (x-K)_+$ since the option is exercised at T only if the price of the asset is larger than K, and there is no time left to invest $C(x,T)$ at a fixed interest rate.

At this point, stochastic modeling is needed in order to describe the evolution of asset prices to ensure a viable game. We will refrain from going into the details since they are far beyond our meager probabilistic skills. Anyway, hypotheses are made concerning the S_t process. As recent world events have made quite clear, such hypotheses are not always satisfied in real life, but let us proceed anyway. At the end of this stochastic modeling phase, we end up with a deterministic PDE for the function $C(x,t)$

$$\frac{\partial C}{\partial t}(x,t)+\frac{1}{2}\sigma^2x^2\frac{\partial^2 C}{\partial x^2}(x,t)+\mu x\frac{\partial C}{\partial x}(x,t)-rC(x,t)=0 \text{ in } \mathbb{R}_+\times[0,T], \quad (1.26)$$

with the final condition

$$C(x,T)=(x-K)_+. \quad (1.27)$$

This is the Black and Scholes equation. It has a final condition and not an initial condition because of modeling reasons, as we have seen, in fact the initial value is the unknown quantity of interest. Another reason is that the principal part of the differential operator is basically a backward heat equation. We have seen that the heat equation is incapable of going back in time. Therefore, a backward heat equation needs a final condition in order to be well-posed. There is an additional difficulty since the coefficients of the space derivatives are functions of the space variables that vanish for $x=0$. There is thus a degeneracy at the boundary and it is not so clear what boundary conditions are in order. The constant σ is called the asset volatility, a measure of the more or less erratic behavior of the asset price, and μ is the trend, a sort of average growth rate.

These oddities of the Black and Scholes equation are mostly corrected by a simple change of variable. Let us set $u(y,\tau)=C(e^y,T-\tau)$, then

$$\frac{\partial u}{\partial \tau}-\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2}-\Big(\mu-\frac{1}{2}\sigma^2\Big)\frac{\partial u}{\partial y}+ru=0 \text{ in } \mathbb{R}\times[0,T], \quad (1.28)$$

with the initial (since time has been reversed) condition

$$u(y,0)=(e^y-K)_+. \quad (1.29)$$

We are comfortably back with an ordinary heat equation with the right time direction, whose effect is to diffuse the price, corrected by a transport term whose effect is to make the price drift (in backward time) at speed $-(\mu-\frac{1}{2}\sigma^2)$. The ru term is an updating term with respect to the interest rate. Further changes of variables can make the equation even simpler.

The degeneracy at $x=0$ is gone, replaced by a difficulty at $y\to-\infty$ where a boundary condition is missing, as well as at $+\infty$.

To conclude, let us remark that the Black and Scholes equation for one asset is a two dimensional equation, one space dimension and one time dimension. The analogous equation for a portfolio of N assets is in $N+1$ dimensions, which is a source of difficulty for numerical approximation.

1.10 A rough classification of PDEs

We give a rather informal classification of PDEs which is neither very precise, nor exhaustive, but which has the advantage of giving a general idea of their properties.

Let us start with the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, and replace $\frac{\partial}{\partial x_i}$ by multiplication by a variable ξ_i (which is more or less what the Fourier transform does). The equation $\Delta u = f$ is thus replaced by $\|\xi\|^2 = g$ which the equation of a circle in $\mathbb{R}^2$, a special case of an ellipse. We say that the Poisson equation is *elliptic*. More generally, if we repeat the same operation on a general second order operator $L = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i x_j}$, we obtain $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j = g$. If this yields the equation of an ellipsoid in $\mathbb{R}^n$, then we say that the equation is elliptic. This is the case if the matrix (a_{ij}) is positive definite.

The same game played on the heat equation, replacing $\partial/\partial t$ by ξ_0, yields $\xi_0 - \xi_1^2 = g$, which is the equation of a parabola, or a paraboloid in higher dimension. We say that the heat equation is *parabolic*.

Finally, in the case of the wave equation, we obtain $\xi_0^2 - \xi_1^2 = g$, the equation of a hyperbola. We say that the wave equation is *hyperbolic*.

It is possible to give more precise definitions, but this is not useful here. The important idea is that an elliptic equation has more or less the same properties as the Poisson equation, a parabolic equation has more or less the same properties as the heat equation and a hyperbolic equation has more or less the same properties as the wave equation. The transport equation is considered to be hyperbolic.