Université Pierre et Marie Curie
M1 Mathématiques
MM26E Numerical Approximation of PDEs
Quick final exam correction

Exercise I

a. By definition, $\mathscr{P}$ is spanned by a family of vectors, it is thus a vector space. It contains all polynomials of the form $\sum_i \alpha_i\lambda_i$, that is all P_1 polynomials, hence $P_1 \subset \mathscr{P}$. The degree of λ_i is 1 and the degree of $\lambda_1\lambda_2\lambda_3$ is 3, thus any polynomial in $\mathscr{P}$ has degree less than 3, i.e., $\mathscr{P} \subset P_3$.
To show that $\dim \mathscr{P} = 4$, let us find a basis. We already have a generating family, by definition, $\{\lambda_1,\lambda_2,\lambda_3,\lambda_1\lambda_2\lambda_3\}$. We show that it is free. Let us be given α_i, $i = 1,\dots,4$ such that

$$0 = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 + \alpha_4\lambda_1\lambda_2\lambda_3.$$

Evaluating the above equality at A^i yields $\alpha_i = 0$ for $i = 1,\dots,3$. We are thus left with $0 = \alpha_4\lambda_1\lambda_2\lambda_3$, which we evaluate at $G = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$, so that $\frac{\alpha_4}{27} = 0$ or $\alpha_4 = 0$.

b. We first construct $p_1 \in \mathscr{P}$ such that $p_1(A^i) = \delta_{1i}$ and $p_1(G) = 0$. By the previous question, we look for

$$p_1 = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 + \alpha_4\lambda_1\lambda_2\lambda_3.$$

Evaluating again at A^i, we get $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, so that

$$p_1 = \lambda_1 + \alpha_4\lambda_1\lambda_2\lambda_3.$$

Evaluating at G, we get

$$0 = \frac{1}{3} + \alpha_4\frac{1}{27},$$

so that $\alpha_4 = -9$. The polynomial

$$p_1 = \lambda_1 - 9\lambda_1\lambda_2\lambda_3$$

is thus ok. By permutation of indices

$$p_2 = \lambda_2 - 9\lambda_1\lambda_2\lambda_3, \quad p_3 = \lambda_3 - 9\lambda_1\lambda_2\lambda_3.$$

We proceed in the same way for p_4 and obtain $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = 27$, so

$$p_4 = 27\lambda_1\lambda_2\lambda_3.$$

We have constructed four basis polynomial that satisfy the duality conditions with the degrees of freedom, hence the finite element is unisolvent.

c. This follows directly from unisolvence, see class notes.

d. Interpolation values at the internal nodes and centers of gravity define a piecewise-$\mathscr{P}$ function. The only question is whether this function is continuous and satisfies the boundary conditions. The key here is to notice that $\lambda_1\lambda_2\lambda_3$ vanishes on the three edges of its triangle. Thus $\mathscr{P}$-functions are affine on the edges of the triangles (but not inside). So the reasoning used for P_1 Lagrange elements applies here as well.

e. Same as in class: **c.** and **d.** make it possible to construct hat-functions in V_h that vanish at all internal vertices except one and at all centers of gravity, and bubble functions that vanish at all internal vertices and at all centers of gravity except one. This is a basis of V_h (same proof as always) and the dimension is thus $N_t + N_s$ (one center of gravity per triangle).

f. ...

Exercise II

a. We have $A\xi\cdot\xi = 2\xi_1^2 + 2\xi_1\xi_2 + 2\xi_2^2 = (\xi_1+\xi_2)^2 + \xi_1^2 + \xi_2^2 \geq \xi_1^2 + \xi_2^2$.

b. Apply Lax-Milgram. $H^1(\Omega)$ is a notorious Hilbert space. The bilinear form is continuous, since

$$|a(u,v)| \leq \int_\Omega 2\Big(\Big|\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}\Big| + \Big|\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_1}\Big| + \Big|\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_2}\Big| + 2\Big|\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\Big|\Big)\,dx \\ + \int_\Omega |uv|\,dx + \int_{\partial\Omega} |\gamma_0(u)\gamma_0(v)|\,d\sigma.$$

Now by Cauchy-Schwarz

$$\int_\Omega \Big|\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\Big|\,dx \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

for all i, j,

$$\int_\Omega |uv|\,dx \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

and

$$\int_{\partial\Omega} |\gamma_0(u)\gamma_0(v)|\,d\sigma \leq \|\gamma_0(u)\|_{L^2(\partial\Omega)}\|\gamma_0(v)\|_{L^2(\partial\Omega)} \leq C_{\gamma_0}^2\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

where C_{γ_0} is the continuity constant of the trace mapping $\gamma_0\colon H^1(\Omega)\to L^2(\partial\Omega)$. Finally

$$|a(u,v)| \leq (7 + C_{\gamma_0}^2)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

The bilinear form is $H^1(\Omega)$-elliptic, since

$$a(v,v) = \int_\Omega A\nabla v\cdot\nabla v\,dx + \int_\Omega v^2\,dx + \int_{\partial\Omega}\gamma_0(v)^2\,d\sigma$$
$$\geq \int_\Omega \nabla v\cdot\nabla v\,dx + \int_\Omega v^2\,dx = \|v\|^2_{H^1(\Omega)}.$$

The linear form is continuous, since

$$\ell(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} g\gamma_0(v)\,d\sigma$$

and

$$|\ell(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\|\gamma_0(v)\|_{L^2(\partial\Omega)}$$
$$\leq (\|f\|_{L^2(\Omega)} + C_{\gamma_0}\|g\|_{L^2(\partial\Omega)})\|v\|_{H^1(\Omega)}.$$

c. Take first $v = \varphi \in \mathscr{D}(\Omega) \subset H^1(\Omega)$. There are no boundary terms and by IPP

$$a(u,\varphi) = \int_\Omega \Big(-2\Delta u - 2\frac{\partial^2 u}{\partial x_1\partial x_2} + u\Big)\varphi\,dx,$$

hence the PDE

$$-2\Delta u - 2\frac{\partial^2 u}{\partial x_1\partial x_2} + u = f.$$

Taking now the boundary terms into account by using a general $v \in H^1(\Omega)$, we obtain

$$2\frac{\partial u}{\partial n} + \frac{\partial u}{\partial x_1}n_2 + \frac{\partial u}{\partial x_2}n_1 + u = g$$

on $\partial\Omega$.

Exercise III

We work in one space dimension with $\Omega =]0,1[$. Let c be a constant such that $c > -\pi^2$. We consider problem (P)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) + cu(x,t) = 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x), \end{cases}$$

for $x \in \Omega$ and $t \in [0,T]$, u_0 and T being given.

a. Let a_k be the Fourier coefficients of u_0 (extended to be odd on $[-1,1]$), then $u(x,0) = u_0(x)$. Moreover, the series of all partial derivatives converge normally on $[0,1] \times [t_0, +\infty[$ for all $t_0 > 0$. Thus the series is of class C^∞ on $[0,1] \times]0, +\infty[$ and we can differentiate term by term. Now

$$\frac{\partial^2}{\partial x^2}(\sin(k\pi x)e^{-(k^2\pi^2+c)t}) = -k^2\pi^2 \sin(k\pi x)e^{-(k^2\pi^2+c)t}$$

and

$$\frac{\partial}{\partial t}(\sin(k\pi x)e^{-(k^2\pi^2+c)t}) = -(k^2\pi^2 + c)\sin(k\pi x)e^{-(k^2\pi^2+c)t}.$$

b. The scheme is implicit since we cannot obtain U^{j+1} without solving a non-trivial linear system.

c. The scheme reads

$$u_i^{j+1} - k\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + kcu_i^{j+1} = u_i^j,$$

or

$$(((1+ck)I_h + kA_h)U^{j+1})_i = (U^j)_i$$

so that

$$U^{j+1} = ((1+ck)I_h + kA_h)^{-1}U^j,$$

assuming the matrix in parentheses is invertible (back to that later). Since it is symmetric, its inverse is symmetric and symmetric matrices are obviously normal.

d. Routine calculation with Taylor-Lagrange expansions. The order is one in time and two in space.

e. The eigenvalues of $(1+ck)I_h + kA_h$ are $\lambda_n = 1 + ck + \frac{4k}{h^2}\sin^2\left(\frac{n\pi}{2(N+1)}\right)$, $n = 1, \ldots, N$. The smallest eigenvalue is thus $\lambda_1 = 1 + ck + \frac{4k}{h^2}\sin^2\left(\frac{h\pi}{2}\right)$. We can rewrite it as

$$\begin{aligned}\lambda_1 &= 1 + ck + \frac{2k}{h^2}\big(1 - \cos(h\pi)\big) \\ &= 1 + ck + \frac{2k}{h^2}\Big(1 - 1 + \frac{h^2\pi^2}{2} - \frac{h^4\pi^4}{24} + O(h^6)\Big) \\ &= 1 + ck + k\Big(\pi^2 - \frac{h^2\pi^4}{12} + O(h^4)\Big)\end{aligned}$$

Since $c > -\pi^2$, we thus have

$$\lambda_1 \geq 1 - k\Big(\frac{h^2\pi^4}{12} + O(h^4)\Big).$$

Now for h small, we have $\frac{h^2\pi^4}{12} + O(h^4) \leq 1$, thus

$$\lambda_1 \geq 1 - k.$$

Hence $\mathscr{A}_h$ is invertible and

$$\rho(\mathscr{A}_h) = \lambda_1^{-1} \leq \frac{1}{1-k} = 1 + \frac{k}{1-k} \leq 1 + 2k$$

for k small: unconditional stability in the $2,h$ norms.

f. Apply the Lax theorem.