

## Hot gases: The transition from line spectra to thermal radiation

M. Vollmer

Citation: *American Journal of Physics* **73**, 215 (2005); doi: 10.1119/1.1819931

View online: <http://dx.doi.org/10.1119/1.1819931>

View Table of Contents: <http://scitation.aip.org/content/aapt/journal/ajp/73/3?ver=pdfcov>

Published by the [American Association of Physics Teachers](#)

### Articles you may be interested in

[Blackbody Radiation from an Incandescent Lamp](#)

Phys. Teach. **52**, 371 (2014); 10.1119/1.4893096

[Reply to "Comment on 'Hot gases: The transition from line spectra to thermal radiation,'" by M. Vollmer \[Am. J. Phys. 73 \(3\), 215–223 \(2005\)\]](#)

Am. J. Phys. **75**, 949 (2007); 10.1119/1.2735634

[Comment on "Hot gases: The transition from the line spectra to thermal radiation" by M. Vollmer \[Am. J. Phys. 73 \(3\), 215–223 \(2005\)\]](#)

Am. J. Phys. **75**, 947 (2007); 10.1119/1.2735628

[Interactive Instruction on Ideal and "Real" Gases](#)

Phys. Teach. **42**, 92 (2004); 10.1119/1.1646484

[The onset of Rayleigh–Bénard instability in molecular radiating gases](#)

Phys. Fluids **9**, 3858 (1997); 10.1063/1.869486



American Association of **Physics Teachers**

Explore the **AAPT Career Center** –  
access **hundreds of physics education and other STEM teaching jobs** at two-year and four-year colleges and universities.

<http://jobs.aapt.org>



# Hot gases: The transition from line spectra to thermal radiation

M. Vollmer<sup>a)</sup>

Physikalische Technik, University of Applied Sciences Brandenburg, Magdeburger Straße 50,  
14770 Brandenburg, Germany

(Received 13 May 2004; accepted 24 September 2004)

The question of how the line spectrum of atoms is transformed into the continuous spectrum of thermal radiation of a gas by repeated absorption and emission of photons is addressed. We consider a simple model of a hydrogen-like gas in thermal equilibrium where only the Lyman- $\alpha$  transition is allowed and calculate the spectra for increasing thickness of the gas. The transition from line spectra for optically thin samples to the spectra of thermal radiation for optically thick samples is demonstrated. We estimate the emissivities for hot gases and study the line width of spectral lines due to self-absorption processes. © 2005 American Association of Physics Teachers.  
[DOI: 10.1119/1.1819931]

## I. INTRODUCTION

What spectrum will be emitted by  $1 \text{ m}^3$  of atomic hydrogen at 6000 K and the densities of the sun's photosphere? When studying light sources, the distinction often is made between thermal and nonthermal sources. Thermal sources such as black bodies in thermal equilibrium are characterized by a continuous spectrum [see Fig. 1(a)], in contrast to nonthermal sources such as spectral lamps which show a discrete spectrum [see Fig. 1(b)]. To compare these two types of light sources and the underlying physics, we will consider hydrogen atoms in a hydrogen spectral lamp as the nonthermal source and an extended volume of a hot gas of hydrogen atoms as the thermal source. For clarity, structures in the thermal spectra, such as the Fraunhofer lines, are omitted.

The line spectra of hydrogen usually are explained by introducing quantized electronic energy levels. In the simplest case of a two level system, the excited state may be populated by processes such as electron impact in a discharge or thermal excitation. The electron may return to the ground state by spontaneous emission of a photon of energy  $h\nu$ . We can thus easily explain the line spectrum consisting of a single spectral line with just a simple two level system.

Although the continuous spectra of an extended hot gas is due to the same transitions (because it consists of the same kind of atoms with the same electronic energy levels), students eventually realize that thermal radiation is not obviously related to the discrete energy levels and transition frequencies of atomic physics. Rather, textbook discussions on thermal radiation use a totally different explanation. Thermal radiation is usually modeled as a black body. The spectrum results from the radiation field, which is assumed to be in thermal equilibrium with the walls of the enclosure. In the usual derivation of Planck's law,<sup>1</sup> the atoms that form the walls of the cavity are assumed to behave like harmonic oscillators. The energy of the oscillators is quantized, but the frequency of radiation, which may be absorbed and emitted by the oscillators, is continuous. Usually, the nature of the harmonic oscillators is not discussed.<sup>2</sup> Sometimes, there is a reference to Rayleigh-Jeans theory for which the oscillators are thought of as being the electrons in the wall atoms, which accelerate and hence emit radiation.<sup>3</sup>

The more elegant derivation of Planck's law uses Einstein's method with  $A$  and  $B$  coefficients and the concept of detailed balance for a two level system (see, for example, Refs. 4–6). Recent articles on the Planck spectrum also use

this approach but address different topics.<sup>7,8</sup> But although these articles start with discrete energy levels, there is little information on how a continuous spectrum could evolve from them, or how it is related to the nature of the sample that is emitting the radiation at a given temperature.

We could use a plausible argument, noting that Planck's law for thermal radiation is usually derived for cavities where the enclosure, which must oscillate to produce the continuous spectrum, is not a gas, but a solid. It is reasonable that solids, with their complex energy band structures, would have continuous spectra, given the fact that there are many phonons present at high temperatures. For example, it has been argued that "line spectra are produced by matter in the gaseous state in which atoms are far apart and interactions between atoms become negligible..., but in condensed matter...there are strong interactions between atoms. These interactions cause shifts in the energy levels and levels are shifted by different amounts for different atoms. Because of the very large number of atoms, practically any photon energy is possible. Therefore hot condensed matter always emits a spectrum with a continuous distribution of wavelengths, not a line spectrum."<sup>9</sup> Similarly, it was stated that "incandescence occurs when electrical resistive heating creates thermally excited atoms. Some of the thermal kinetic energy is transferred to electronic excitations within the solid. The excited states are relieved by photonic emission...In a solid, there is a near continuum of electron energy levels, resulting in a continuous non-discrete spectrum of radiation."<sup>10</sup>

For most students it is neither plausible nor convincing to explain the absence of the electronic structure of atoms by such an argument, because thermal radiation also may be produced by gases consisting of atoms. And why use an argument based on cavity radiation when considering the thermal radiation of hot gases like those present in the photospheres of stars like the sun? Only a limited number of physics books discuss this problem in detail (see, for example, Ref. 11).

This paper will discuss the apparent discrepancy between the two explanations. In particular, we will demonstrate using a simple pedagogical model that a line spectrum of gaseous atoms will become the continuous spectrum of thermal radiation, the classical blackbody radiation spectrum, as the radiation passes through an increasing thickness. Our goal is to resolve the apparent discrepancy in the explanations of the various light sources by relating the explanations to a common origin. The equations are simple and appropriate for

Wrong because selection rules forbid intraband transitions, so a semi-cond could not emit photons of  $E < \text{gap}$  !?

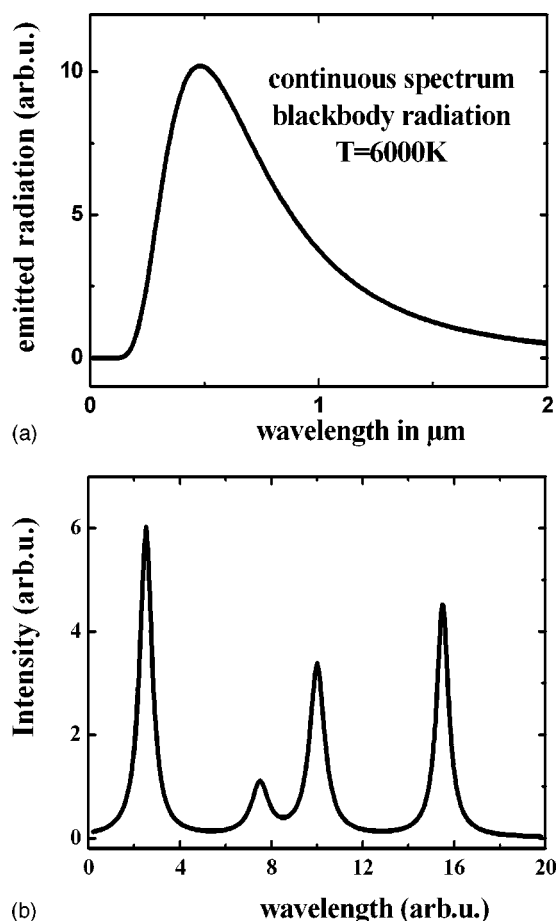


Fig. 1. (a) Typical continuous spectrum. (b) Typical line spectrum.

undergraduates. The model gives numerical results, but due to the simplicity of the model, the results are only qualitative. Applications to real systems such as stellar atmospheres would require a complex model using complete radiation transfer theories, which are well known in astrophysics.<sup>12–15</sup>

## II. ABSORPTION OF LIGHT IN A SIMPLIFIED MODEL SYSTEM

We will introduce a simple model of a homogeneous gas and discuss the absorption of light using a Lorentzian approximation for the absorption cross section. Then the optical thickness and the penetration depth of light in the gas will be discussed. Radiative transfer within the gas will be calculated from a two level system in thermal equilibrium. For simplicity, the spectrum for the radiation within the gas is interpreted as the one emitted by the gas, that is, changes due to the escape of radiation from the boundary of the gas is neglected. Other complications arising in the calculation of the radiative transfer are neglected. For example, inelastic collisions causing ionization or recombination are neglected as are continuum radiative processes (for example, the photo ionization of excited H atoms by Lyman- $\alpha$  photons). Hence, **only transitions between the two energy levels of the system are considered and no free electrons or hydrogen ions are allowed.**

The neglect of many processes is motivated by the simplicity of the model in comparison to more detailed models (for example, Refs. 12–15). The results will provide a clear

qualitative picture of the origin of thermal radiation from a simple line spectrum due to gas atoms. **If we consider only the Lyman- $\alpha$  transition of hydrogen at a wavelength of  $\lambda=121.6$  nm, the peak of the spectrum of the thermal radiation from a gas at a temperature of 6000 K will be either in the visible if the spectrum is plotted as a function of wavelength or the near infrared if plotted versus frequency<sup>16,17</sup> and thus far away from the atomic resonance.**

### A. Describing the absorption of light by atoms

The absorption cross section for light by an atom can be **calculated classically from the model of an electron as a forced damped oscillator.** For atoms at rest, we find Eq. (1) for  $\omega \approx \omega_0 \gg \gamma$ ,<sup>11</sup> where  $\gamma$  is the damping constant related to the natural line width of the transition:

$$\sigma_{abs}(\omega) = \frac{\pi e^2}{2\epsilon_0 mc} \frac{\gamma/2\pi}{(\omega_0 - \omega)^2 + (\gamma/2)^2}. \quad (1)$$

The prefactor in Eq. (1) may be rewritten as

$$\frac{\pi e^2}{2\epsilon_0 mc} = 2\pi^2 r_0 c = 1.67 \times 10^{-5} \text{ m}^2/\text{s}, \quad (2)$$

where  $r_0$  is the classical electron radius  $r_0 = e^2/4\pi\epsilon_0 mc^2 = 2.82 \times 10^{-15} \text{ m}$ .

Equation (1) is slightly modified by the oscillator strength  $f$  if the calculation is done quantum mechanically, because the  $f$ -value of an absorption line is the “...fraction of the integral of the classical total absorption cross section which is to be associated with the given transition.”<sup>11</sup> For a two level system we have

$$\sigma_{abs}(\omega) = f 2\pi^2 r_0 c \frac{\gamma/2\pi}{(\omega_0 - \omega)^2 + (\gamma/2)^2}. \quad (3)$$

Equation (3) yields the **Lorentz profile of a spectral line.**

**In a gas, there are collisions between the gas molecules which lead to pressure broadening, which often leads to a similar line shape.** The main effect is an increased line width, which is due to the contribution from the collisional broadening.<sup>12</sup>

**The thermal motion of the atoms leads to Doppler broadening of the spectral lines,** the line shape given by a Gaussian distribution. The Doppler width is usually larger than the natural width and the collisional width and the line shape is described by an overlap integral of the Lorentzian and the Gaussian distribution, which is known as the Voigt profile (see, for example, Ref. 11). Close to the center, the line shape is similar to the Gaussian profile, whereas in the wings, the line shape can be considered to be Lorentzian.<sup>12</sup> **In our simple model we will use a Lorentzian.** The emphasis will be on explaining the formation of the continuous spectrum in the far wings of the transition.

The justification for the assumption that the Doppler width or collisional width is not important comes from a numerical estimate of the Doppler width and the effects of pressure broadening. For hydrogen at 6000 K, the Doppler width (see, for example, Ref. 11) is  $\Delta\omega_D \approx 8.6 \times 10^{11} \text{ s}^{-1}$ , that is,  $\Delta\lambda_D$  is the order of  $10^{-2} \text{ nm}$ . This width, although being more than a factor of 1000 larger than the natural line width, is still very small compared to the width of a thermal spectrum. The magnitude of collisional broadening may be estimated from the mean free time between collisions for a gas. We use the

Table I. Parameters of the hypothetical two-level system, resembling atomic hydrogen gas.

Gas:	volume of linear dimension $L$ hydrogen-like atoms temperature $T=6000$ K density $n=N/V=2.4\times 10^{17}$ cm $^{-3}$
Energy levels:	resemble $1s$ and $2p$ of hydrogen atoms with $\Delta E \approx 10.2$ eV corresponding to the Lyman- $\alpha$ line center $\lambda=121.6$ nm natural line width: $\gamma=1/\tau=6.26\times 10^8$ s $^{-1}$ (Ref. 11) oscillator strength absorption: $f=0.4162$ (Ref. 11) line shape: Lorentzian according to Eq. (3)

kinetic theory of gases, which gives the mean free path between collisions and the mean velocity of the atoms for a gas of  $T=6000$  K,  $p=2\times 10^4$  Pa (see below), and a diameter of a hydrogen atom of  $10^{-10}$  m. Therefrom the mean time between collisions is estimated to be about  $8\times 10^{-9}$  s, which is only a factor of 5 larger than the natural lifetime. Hence, the collisional width is of the same order of magnitude as the natural line width, that is, it is much smaller than the Doppler width (also see the discussion of the collisional width in the sun in Sec. V).

## B. Optical depth and penetration depth of radiation in matter

If the absorption cross section is known, we may compute the attenuation of radiation in matter according to Lambert and Beer:<sup>18</sup>

$$I(x)=I_0e^{-n\sigma x}, \quad (4)$$

where  $n$  is the number density (assumed to be constant) and  $x$  is the distance traversed by the radiation;  $n$  and  $\sigma$  often are combined to give the absorption coefficient  $\kappa(\omega)$ ,

$$\kappa(\omega)=n\sigma(\omega). \quad (5)$$

To simplify Eq. (4), we introduce the (geometrical) penetration depth  $x_e$ , defined by the  $1/e$ -decrease in intensity

$$x_e(\omega)=\frac{1}{n\sigma(\omega)}=\frac{1}{\kappa(\omega)}. \quad (6)$$

In addition, we may define the optical depth  $\tau(\omega,L)$  for a given distance  $x=L$  as

$$\tau(\omega,L)=\kappa(\omega)L=L/x_e(\omega). \quad (7a)$$

If  $\kappa(\omega)$  also depends on the position, the optical depth is defined as

$$\tau(\omega,L)=\int_0^L \kappa(\omega,x)dx. \quad (7b)$$

Then, Eq. (4) becomes

$$I(\omega,L)=I_0e^{-\tau(\omega,L)}. \quad (8)$$

We distinguish optically thin,  $\tau\ll 1$ , and optically thick,  $\tau\gg 1$ , samples. For  $\tau=1$ , that is, a distance  $L=x_e$ , the intensity of the incident radiation has decreased by a factor of  $e$ . The penetration depth is thus the photon mean free path. This distance is a measure of the depth from which radiation may escape from a sample. Light that is incident from larger depths has a high probability of being absorbed before reaching the surface. Therefore, a sample appears transparent if its dimension  $L$  is much smaller than  $x_e$ , that is,  $\tau\ll 1$ , but is opaque if  $\tau\gg 1$ .

There is an old demonstration experiment for the optical depth that uses a sodium discharge lamp (or a laser tuned to the Na-D-lines) shining into a glass cell with sodium vapor. At a suitable vapor pressure, the path of the light beam is visible via resonance fluorescence. Increasing the vapor pressure will increase the optical depth, that is, decrease the penetration depth. This increase results in a decrease of the beam path showing resonance fluorescence. For very high pressures, the column of resonance fluorescent light may be as small as 1 mm.

## C. Numerical values of the parameters

A hydrogen-like atomic gas was chosen as the model system, that is, a homogeneous gas within a volume of linear dimension  $L$  at a temperature of 6000 K and a density similar to the sun's photosphere. For simplicity only one electronic transition is treated, the Lyman- $\alpha$  transition at  $\lambda=121.6$  nm. Table I summarizes the parameters used in our calculations. The excited state is about 10 eV above the ground state, and

Table II. Absorption cross section of the two-level model system according to Eq. (3) for several selected wavelengths (frequencies) from the resonance ( $\lambda_0/\omega_0$ ) toward the visible spectral range.

Wavelength	Frequency s $^{-1}$	Cross section $\sigma(\omega)$
$\lambda_0=121.60$ nm	$\omega_0=15.4905\times 10^{15}$ s $^{-1}$	$\sigma_{\max}\approx 7.1\times 10^{-11}$ cm $^2$
$\lambda_0+1.5\times 10^{-5}$ nm	$\omega_0+3\gamma$ ( $3\gamma\approx 1.9\times 10^9$ s $^{-1}$ )	$\sigma=1/37\sigma_{\max}=1.9\times 10^{-12}$ cm $^2$
121.7 nm	$15.4778\times 10^{15}$ s $^{-1}$ ( $\Delta\nu\approx 2\times 10^5\gamma$ )	$\sigma\approx 4.3\times 10^{-20}$ cm $^2\approx 6.1\times 10^{-10}\sigma_{\max}$
400 nm	$\omega=4.709\times 10^{15}$ s $^{-1}$	$6.0\times 10^{-26}$ cm $^2\approx 8.4\times 10^{-16}\sigma_{\max}$
550 nm	$\omega=3.425\times 10^{15}$ s $^{-1}$	$4.8\times 10^{-26}$ cm $^2\approx 6.8\times 10^{-16}\sigma_{\max}$
700 nm	$\omega=2.691\times 10^{15}$ s $^{-1}$	$4.2\times 10^{-26}$ cm $^2\approx 5.9\times 10^{-16}\sigma_{\max}$



Table III. Penetration depth of visible light in a hydrogen-like gas with absorption as a function of gas density.

Gas density $n$ (cm <sup>-3</sup> )	Penetration depths		
	$x_e$ (400 nm) (km)	$x_e$ (550 nm) (km)	$x_e$ (700 nm) (km)
$2.4 \times 10^{15}$	70 000	88 000	99 000
$2.4 \times 10^{17}$	700	880	990
$2.4 \times 10^{19}$	7	8.8	9.9

its population is very small even at  $T=6000$  K, because the Boltzmann factor is  $\approx 10^{-9}$ . Equation (3) is used to calculate the absorption cross sections for the spectral range from the near infrared to the ultraviolet (see Table II). The cross sections decrease rapidly with increasing detuning from resonance. In the visible range the suppression is the order of  $10^{-15}$  compared to the value at resonance. Cross sections in the visible depend strongly on the resonance position. For example, the shift of the center wavelength from  $\lambda=121.6$  nm to the near ultraviolet,  $\lambda \approx 365$  nm, would increase the cross section for green light by about a factor of 50.

By using the cross sections from Eq. (3), the penetration depths  $x_e$  are calculated using Eq. (6). Obviously,  $x_e$  depends on the number density  $n$ . Table III gives the values of  $x_e$  for frequencies in the visible range for various densities. Figure 2 shows  $x_e$  from the infrared to the ultraviolet for a mean density of  $2.4 \times 10^{17}$  cm<sup>-3</sup> for our model hydrogen gas, corresponding to a mass density of  $4 \times 10^{-4}$  kg/m<sup>3</sup>. All calculations were performed at this density. If treated as an ideal gas, the density would be given by  $n=p/kT$ . Hence  $n=2.4 \times 10^{17}$  cm<sup>-3</sup> corresponds to  $p=2 \times 10^4$  Pa and  $T=6000$  K.

The penetration depths  $x_e$  for visible light for a given density  $n$  range over several hundred kilometres (see Table III). From the discussion of Eq. (8), the model hot gas will change from completely transparent to completely opaque when moving from the surface ( $x \ll x_e$ ) into the interior to depths of  $x \gg 100$  km.

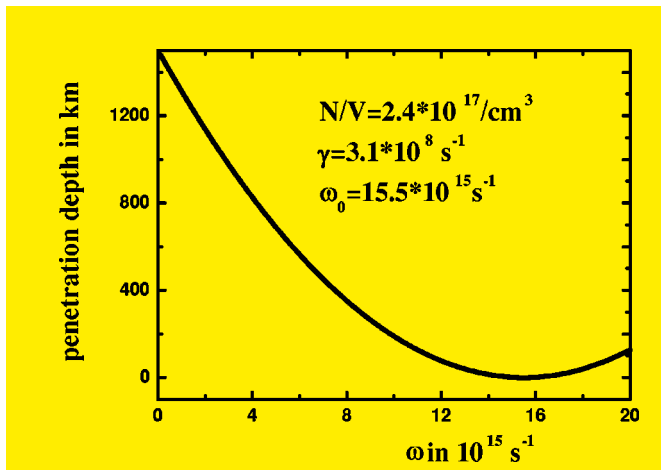


Fig. 2. The penetration depth as a function of frequency for the model system.

### III. RADIATION TRANSFER IN THE SIMPLIFIED MODEL SYSTEM

#### A. Planck's law for thermodynamic equilibrium in an infinite homogeneous hot gas

In the following, we will use Einstein's derivation of Planck's law of blackbody radiation. Thermodynamic equilibrium is characterized by a single value of the temperature  $T$ .

The derivation of Planck's law usually proceeds as follows: The three fundamental processes, absorption, spontaneous emission, and stimulated emission, can be used to set up equations for the respective transition rates  $Z_{ij}$ :

$$Z_{12}^{\text{abs}} = B_{12}N_1\rho(\nu) \quad (\text{absorption}), \quad (9a)$$

$$Z_{21}^{\text{spon}} = A_{21}N_2 \quad (\text{spontaneous emission}), \quad (9b)$$

$$Z_{21}^{\text{stim}} = B_{21}N_2\rho(\nu), \quad (\text{stimulated emission}), \quad (9c)$$

where  $N_1$  and  $N_2$  denote the populations of the two levels,  $\rho(\nu)$  is the energy density of the radiation field, and  $A$  and  $B$  are the Einstein factors. In equilibrium the number of absorptions equals the number of emissions,

$$B_{12}N_1\rho(\nu) = B_{21}N_2\rho(\nu) + A_{21}N_2. \quad (10)$$

If we assume that  $B_{12} = B_{21}$ ,  $A_{21} = B_{21} 8\pi h \nu^3 / c^3$  and  $N_2/N_1 = \exp(-h\nu/kT)$ , then Eq. (10) yields the spectral energy density formula of Planck,

$$\rho(\nu)d\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu. \quad (11)$$

Note we assumed discrete energy levels and transitions between them. But suddenly the result, Eq. (11), is interpreted as a continuous spectrum. Where do all the possible transition energies arise?

The answer is simple: each atom or molecule possesses at least one resonance frequency for absorption. An absorption line extends over the entire spectrum, because it has a finite width, that is, each atom or molecule can absorb radiation at every wavelength, although with different cross sections according to Eq. (3).

Einstein's derivation was applied here for an infinite gas. In the following, we examine what happens if the volume of the gas is finite.

#### B. Radiation from a finite isothermal gas in thermodynamic equilibrium

To calculate the behavior of a finite volume of a hot gas, we follow an argument by Corney<sup>11</sup> which gives the simplest radiation transfer model. A beam of light with intensity  $I$  is incident on the gas. The change of intensity with the distance  $x$  within the gas is due to absorption, and spontaneous and stimulated emission of radiation. We find<sup>11</sup>

$$\frac{dI(\omega, x)}{dx} = \delta(\omega) - \kappa(\omega)I(\omega, x), \quad (12)$$

where  $\delta(\omega)$  describes the spontaneous emission and  $\kappa(\omega)$  the absorption and emission induced by the light field. They are related to the Einstein coefficients by

$$\delta(\omega) \sim A_{21}N_2g(\omega), \quad (13a)$$

$$\kappa(\omega) \sim (B_{12}N_1 - B_{21}N_2)g(\omega), \quad (13b)$$

where  $g(\omega)$  is the line shape factor [for example, the normalized Lorentian, given by the fraction in Eq. (3)]. For a hot gas, the stimulated emission is small compared to the spontaneous emission, and hence  $\kappa(\omega)$  represents absorption alone. Also, we assume that  $\delta(\omega)$  and  $\kappa(\omega)$  are independent of position, which holds for a uniform gas density. In this case, the solution of Eq. (12) for the intensity after passing through the gas from  $x=0$  to  $x=L$  is

$$I(\omega, L) = I_0(\omega) e^{-\tau(\omega, L)} + \frac{\delta(\omega)}{\kappa(\omega)} (1 - e^{-\tau(\omega, L)}). \quad (14)$$

Here  $\tau(\omega, L) = \kappa(\omega)L$  as defined in Eq. (7a), and  $I_0$  is the incident intensity. The first term in Eq. (14) corresponds to the law of Lambert and Beer [Eq. (8)] and describes the attenuation due to absorption. The second term contains the emission processes and is usually called the source function.<sup>11-13</sup> The importance of the source function becomes clear when setting  $I_0=0$ , which will be done in the following. In this case, there is no incident radiation and the resulting radiation is entirely due to radiative processes within the gas itself.

### C. Emission spectra as a function of optical thickness: General case

If we solve Eq. (14) for  $I_0=0$  by using Eq. (13) and setting  $B_{12}=B_{21}$ , we find

$$I(\omega, L) = \frac{A_{21}N_2}{B_{21}(N_1+N_2)} (1 - e^{-\tau(\omega, L)}). \quad (15)$$

Because  $A_{21} = B_{21} 8\pi h \nu^3 / c^3$  and  $N_2/N_1 = \exp(-h\nu/kT)$ , Eq. (15) gives

$$I(\omega, L) \sim \frac{(\hbar \omega)^3}{e^{h\omega/kT} - 1} (1 - e^{-\tau(\omega, L)}). \quad (16)$$

The first term in Eq. (16) resembles Planck's law of black body radiation, and the second factor gives a correction which depends on the optical thickness of the gas.

For optically thin gases, it is convenient to start with Eq. (14) for  $I_0=0$  and write

$$I(\omega, L) = \frac{\delta(\omega)}{\kappa(\omega)} (1 - e^{-\tau(\omega, L)}) \approx \delta(\omega)L \propto A_{21}N_2g(\omega)L. \quad (17)$$

The spectrum of the radiation emitted by the gas is determined by  $g(\omega)$  and  $N_2$ , that is, optically thin hot gases will emit a line spectrum. In contrast, optically thick gases will lead to Planck's law, because the second factor of Eq. (16) becomes unity.

### D. Spectra of hot gases as a function of dimension

In the following, the solutions of Eq. (16) will be discussed for various parameters. Figure 3 shows the optical thickness  $\tau$  for the model from very long wavelengths to very short wavelengths in the ultraviolet ( $\lambda \approx 94$  nm) with the resonance at  $\lambda = 121.6$  nm according to Eqs. (6), (7a), and (3) for dimensions  $L$  of the gas ranging from 1 to  $10^4$  km. The visible range of  $\omega$  is between  $\approx 2.4 \times 10^{15} \text{ s}^{-1}$  and  $\approx 4.7 \times 10^{15} \text{ s}^{-1}$ .

As expected, the optical thickness is very large close to the resonance and rapidly decreases for larger or smaller fre-

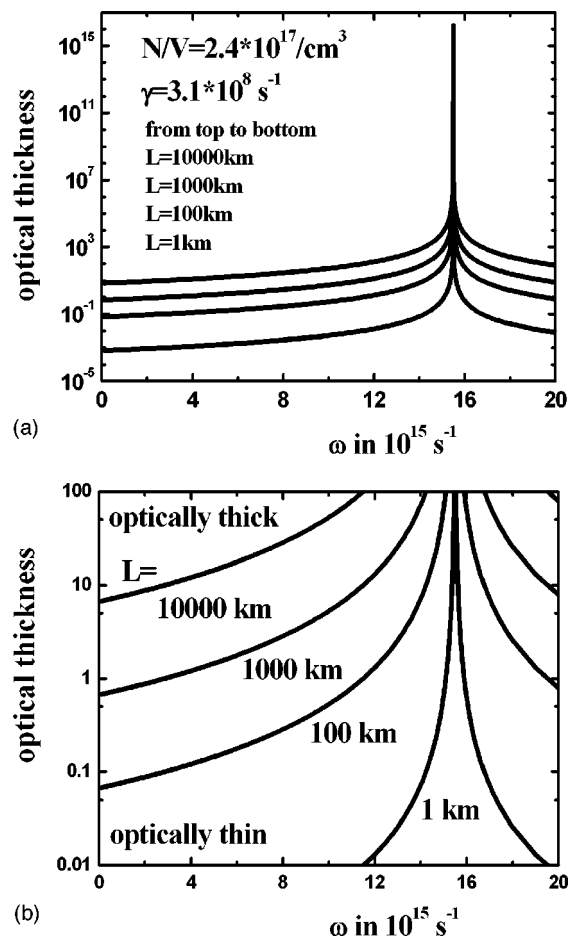


Fig. 3. Optical thicknesses from the model system for different parameters.

quencies. The magnification of the scale in Fig. 3(b) demonstrates that values of  $L$  of  $\approx 1000$  km are sufficient to achieve  $\tau > 1$  in the entire spectral range investigated. However, only  $L = 10^4$  km can be considered as optically thick ( $\tau \gg 1$ ) everywhere. An optically thin system can be reached only for  $L = 1$  km or less.

To estimate the influence of the optical thickness on the spectrum, we plot in Fig. 4 the factor in parentheses in Eq. (16). This factor accounts for deviations from the blackbody spectrum and is interpreted as the emissivity  $\epsilon(\omega)$  of the gas:

$$\epsilon(\omega) = 1 - e^{-\tau(\omega, L)}. \quad (18)$$

For  $L = 10^4$  km, there is no correction to the blackbody spectrum expected and  $\epsilon=1$ . However, for smaller values of  $L$ , significant changes occur.

The top curve of Fig. 5 depicts the first term of Eq. (16), that is, the spectrum of a black body ( $\epsilon=1$ ) at a temperature of  $T=6000$  K. In Fig. 5 the spectrum peaks in the near infrared. As a function of wavelength, the peak would follow Wien's law and be  $\approx 483$  nm. We will use the top curve of Fig. 5 in the following as a reference black body spectrum. All other spectra are calculated with the same parameters and are directly compared to it.

Figure 5 also shows the spectra of the gas for the various optical thicknesses or emissivities shown in Figs. 3 and 4. The curve for  $L = 10^4$  km coincides with the blackbody spectrum. At first glance, it is obvious that although  $\epsilon$  changes appreciably as a function of frequency (see Fig. 4) for  $L$

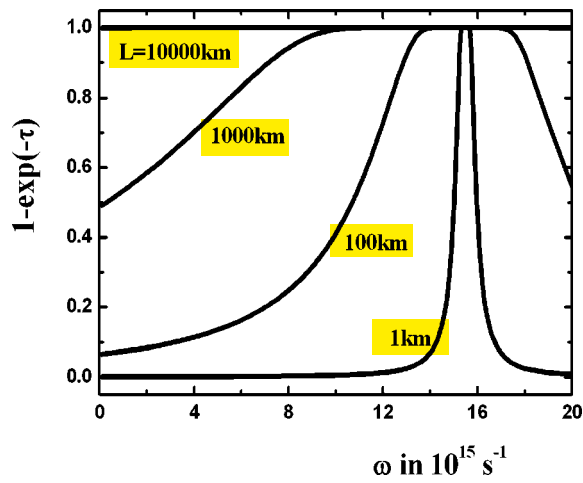


Fig. 4. Optical thickness correction factor for spectrum, as given by the second term of Eq. (16). According to Eq. (18), it also resembles the emissivity  $\epsilon(\omega)$ .

=1000 km or  $L=100$  km, the shape of the spectra looks similar to the blackbody curve. For even smaller values of  $L$ ,  $\epsilon$  becomes more or less constant in the interesting part of the spectrum (see below), that is, the gas behaves like a gray body.

According to Eq. (16), the intensity cannot exceed the value of the Planck function at a given wavelength. Hence, the question of whether the line spectrum or the continuous contribution dominates depends on the suppression of the thermal spectrum, which occurs for optically thin gases. Therefore, even thinner gases were studied.

Figure 6 gives a magnified view of the  $L=100$  m, 10 m and 1 m curves. The scale has been reduced by a factor of  $10^4$ . Because the continuous spectrum is emitted with nearly constant  $\epsilon$  (Fig. 4), we easily deduce a value of  $\epsilon \approx 10^{-4}$  for  $L=100$  m. With decreasing  $L$ ,  $\epsilon$  decreases rapidly and is  $\approx 10^{-5}$  for  $L=10$  m, and  $\approx 10^{-6}$  for  $L=1$  m. As a consequence, the strongly suppressed continuous spectrum is dominated by the line spectrum with its resonance peak at  $\lambda=121.6$  nm ( $\omega=15.5 \times 10^{15} \text{ s}^{-1}$ ) for  $L < 10$  m.

Figure 7 gives a magnified view of the line shape of the resonance transition for  $L=1$  m. The full width at half maxi-

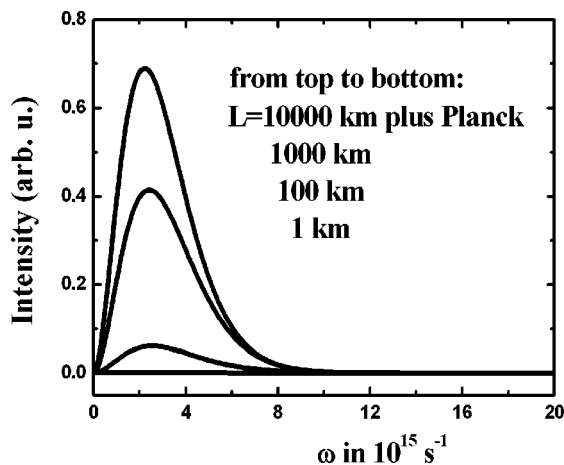


Fig. 5. Spectra of model gas at  $T=6000$  K for different thicknesses  $L$  of the gas.

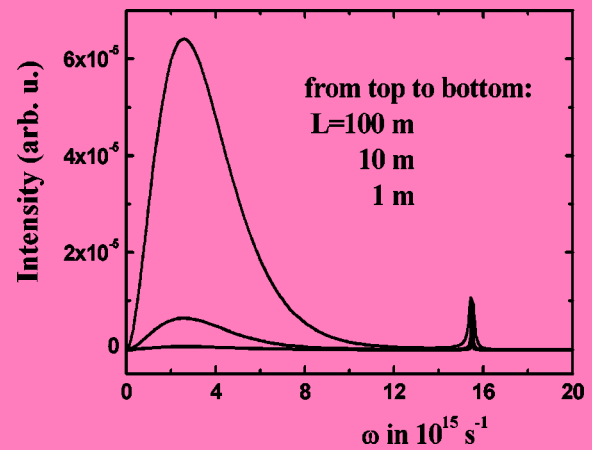


Fig. 6. Spectra of model gas at  $T=6000$  K for small values of  $L$ .

um is  $\Delta\omega \approx 4 \times 10^{13} \text{ s}^{-1}$ , corresponding to  $\Delta\lambda \approx 0.3$  nm. The line is extremely broadened compared to the natural line width of  $\Delta\omega \approx 6.3 \times 10^8 \text{ s}^{-1}$  due to self-absorption which is related to the optical thickness.

#### IV. DISCUSSION

##### A. Qualitative explanation for the origin of the continuous spectrum

Our model calculations show that a hot gas does not only emit the Lyman- $\alpha$ -resonance line, corresponding to the two level hydrogen-like atoms, but a continuous spectrum as well. This behavior also may be understood qualitatively (see Fig. 8). For simplicity, we assume a spherical gas.

Light from the inside of the gas volume can be emitted only from the surface if it originates from a depth that is the order of the penetration depth or less. Light emitted from greater depths will be reabsorbed before reaching the surface. The argument is simple: consider the penetration depth as a function of frequency as calculated from Eqs. (4) and (6) (see Fig. 2). Ultraviolet light emitted from the gas can emerge only from a very thin spherical shell, the shell thickness given approximately by the penetration depth. (See sector 1 in Fig. 8; for simplicity, only photons emitted in the radial direction are plotted.) Light at smaller frequencies has

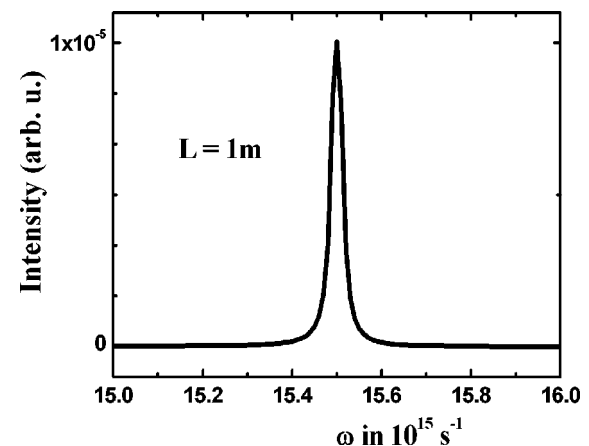


Fig. 7. Resonance transition in the spectrum for  $L=1$  m.

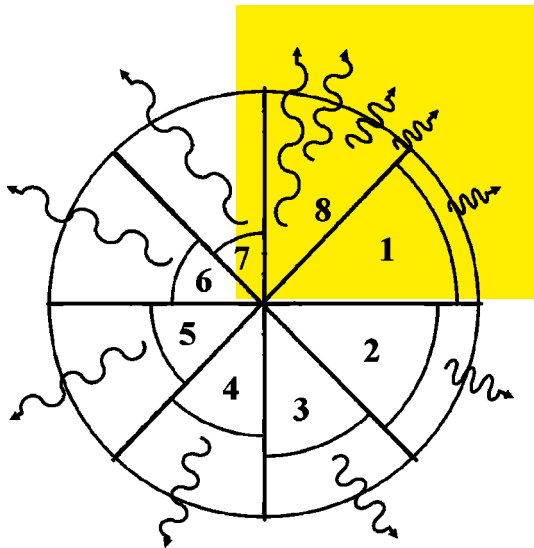


Fig. 8. Light with different wavelengths emerges from different depths of a spherical gas ball, depending on the respective penetration depth  $x_e$ . Although ultraviolet light only emerges from a thin shell, longer wavelengths also are emitted from deeper regions of the gas ( $x_e$  increases from sector 1–7). Sector 8 gives a schematic overview.

larger values of  $x_e$ , and hence may be emitted from every point within the spherical shells of larger thicknesses (sectors 2–7). That is, light of a given frequency will be mainly emitted from a spherical shell with the outer radius being the radius of the gas volume and the thickness being the penetration depth. Consequently, we observe different depths within the gas while changing the observation frequency, a fact that is well known in the spectroscopy of stellar atmospheres.<sup>12–14</sup>

In the simplest explanation the intensity of the emitted radiation is given by the number of atoms that emit the radiation times the probability of emission, which is governed by the line profile, for example, by Eq. (3). For visible light, the line profile is strongly suppressed by, say, a factor of  $10^{15}$  with respect to the ultraviolet light near the line center. However, the total number of light emitting atoms is larger by a similar factor due to the much larger penetration depth, which determines the thickness of the spherical shell. The enormous increase in the number of atoms with increasing penetration depth can compensate for the strong suppression due to the line shape factor [Eq. (3)]. Thus we can understand qualitatively why the continuous spectrum can be comparable or even stronger than the original resonance transition.

## B. Emissivity of hot gases

The correction to Planck's law in Eq. (16) has been identified as the emissivity of the gas [see Eq. (18)]. Hence, the results of Fig. 4 may be interpreted as the emissivity of a hot gas of linear dimension  $L$ . The emissivity is very important in measurements of the temperature using pyrometers or infrared thermal imaging techniques.<sup>19,20</sup> When radiation is measured over extended portions of the spectrum—in infrared imaging we use wavelengths between 3.5 and 5.5  $\mu\text{m}$  or between 8 and 14  $\mu\text{m}$  the analysis by commercial instruments usually assumes gray bodies with constant  $\epsilon$  for the respective ranges of wavelengths. The changes of the emissivity as a function of the wavelength obviously lead to errors in determining the temperature. Problems in quantitative

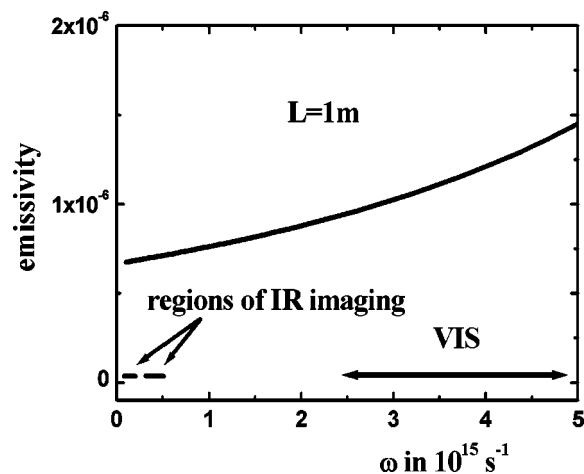


Fig. 9. Emissivity of a hydrogen-like model system with  $T=6000\text{ K}$  and  $n=2.4 \times 10^{17}\text{ cm}^{-3}$  for  $L=1\text{ m}$ .

infrared imaging occur if  $\epsilon$  is very small, which happens for polished metals for which the emissivities may be as low as  $\epsilon=0.01$  and even less for optically thin hot gases. Our analysis offers the possibility of estimating the emissivities of hot gases, provided the resonance frequency or, more generally, the resonance frequencies nearest to the interesting spectral region are known.

Figure 4 shows that for  $L < 10^4\text{ km}$ , all gases should be described as selective emitters, that is, emitters with a frequency dependent emissivity in the visible spectral region.

The  $\epsilon$ -values below  $L=1\text{ km}$  seem to be nearly constant (Fig. 4), and estimates can be made by comparing the peak values in Figs. 5 and 6 with the Planck function, the top curve of Fig. 5. This comparison gives the following order of magnitude results for the emissivity:  $\epsilon(L=1\text{ km}) \approx 10^{-3}$ ;  $\epsilon(L=100\text{ m}) \approx 10^{-4}$ ;  $\epsilon(L=10\text{ m}) \approx 10^{-5}$ ;  $\epsilon(L=1\text{ m}) \approx 10^{-6}$ .

The exact  $\epsilon$ -results from our model for  $L=1\text{ m}$  are shown in Fig. 9;  $\epsilon$  increases by about a factor of 2 between the near infrared and near ultraviolet. In the spectral regions of infrared camera systems,  $\epsilon$  changes only on the order of several percent and may therefore be assumed to be constant.

The smallest  $\epsilon$ -values of commercial systems are usually about  $\epsilon=0.01$ . Therefore infrared imaging techniques would make no sense in determining the temperature for the small densities of our model systems. However, there may be practical examples with different densities and different positions of electronic transitions for which larger  $\epsilon$ -values result.

## C. Broadening of spectral lines due to self-absorption within optically thick sources

The hydrogen Lyman- $\alpha$ -line in our model demonstrates the shift and broadening of spectral lines due to the effects of optical thickness. Figure 10 gives the spectra in the vicinity of the resonance transition at 121.6 nm for different  $L$ -values, that is, different optical thicknesses of our model. The envelope, the Planck function of blackbody radiation, leads to a shift of the resonance for large  $L$ -values. For  $L < 10\text{ m}$ , the position can be regarded as nearly constant. In this case, there is mostly a broadening effect due to the emissivity factor [Eq. (18)] (see Fig. 4), which accounts for the self-absorption in optically thick sources.



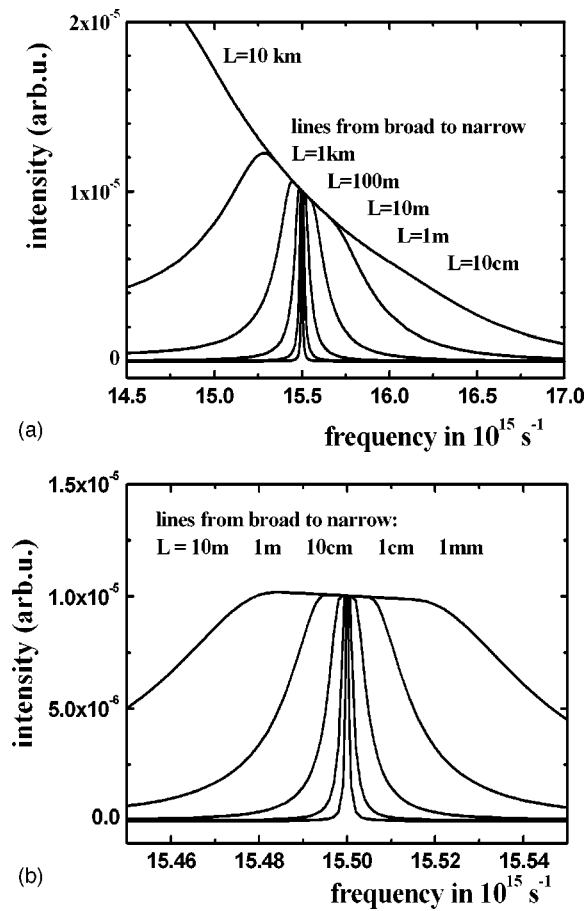


Fig. 10. Broadening of spectral lines due to optical thickness.

For more than one spectral line, this self-absorption effect tends to level out intensity differences between spectral lines. This leveling can cause errors regarding the analysis of oscillator strengths from measurements of the spectra.<sup>11</sup>

The width of the resonances resembles the width of the emissivity curves (similar to Fig. 4). From Eq. (18) with  $\epsilon=0.5$  we find  $\tau=\ln 2$ . From the definition of  $\tau$  [Eqs. (3), (5), and (7a)], we may easily deduce the full width at half maximum due to this effect. The results are shown in Fig. 11. The width follows a more or less straight line in the log-log plot for  $L > 10^{-8} \text{ m}$  and slightly deviates from it below  $10^{-9} \text{ m}$ . For comparison, the limit of the natural line width is shown as a bar. From the simple theory the limit would be reached at  $8.2 \times 10^{-10} \text{ m}$ . However, this limit is meaningless in our simple model, because the spacing of H atoms at the assumed density is at least a factor of 10 larger, that is, a photon would not have a chance to hit a single atom. Furthermore, the model will not be applicable because we have neglected all broadening effects. If we include the Doppler width of about  $8.6 \times 10^{11} \text{ s}^{-1}$ , we would obtain a more reasonable limiting distance of about  $10^{-3} \text{ m}$ .

## V. APPLICABILITY FOR THE SPECTRUM OF THE SUN

The closest resemblance of systems in nature to our model is the stellar atmosphere. Radiative processes within the sun's atmosphere are much more complex than in our model, which is intended to illustrate the basic physics of how a

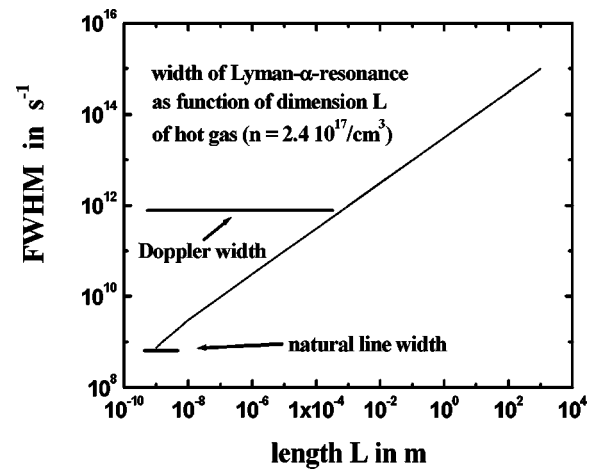


Fig. 11. Full width at half-maximum of the resonance line in the model system as a function of  $L$ .

continuous spectrum can originate from only two discrete electronic energy levels in an atom. Therefore, any quantitative conclusions drawn from our calculations for the sun would be incorrect. Here we briefly summarize some complications within the sun (for details, see Refs. 13–15).

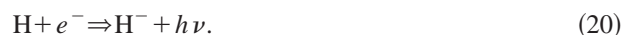
The chemical composition of the sun differs appreciably from a pure hydrogen gas and is about 73%–74% hydrogen and 25% helium; all other elements account for the other 1% or 2%. The spectrum of the sun only very approximately resembles a black body. Therefore, we will find very different values for the blackbody temperature used to approximate the spectrum, values ranging from about 5700 K to 6000 K. The deviations are due to many reasons. In a realistic model of energy transport within the sun, we must include not only radiation but also convection. The opacity of the sun's atmosphere takes all processes into account where a photon is absorbed or scattered by atoms, ions, or electrons. In our model we only assumed bound-bound transitions between discrete energy levels, and neglected the effects of Doppler broadening. However for the sun, we also must include bound-free absorption processes, that is, photoionization and radiative recombination, and free-free absorption processes, where a free electron may absorb a photon (inverse Bremsstrahlung) if a third particle (H atom) is present to account for simultaneous conservation of energy and momentum. Also Thomson scattering of photons by free electrons must be considered. The number densities for the neutral and charged particles result from the Boltzmann and Saha equations. Depending on the height in the atmosphere, we need to consider deviations from local thermodynamic equilibrium. To calculate all the cross sections of the processes, we must assume a solar model with variation of temperature and pressure as a function of height in the atmosphere. For a quiescent sun, the temperature will vary as a function of position within the photosphere by at least a factor of 1.5. The densities of hydrogen are of the order of  $10^{17} \text{ cm}^{-3}$  as assumed in our model, but there also is an appreciable number density of electrons of about  $10^{14} \text{ cm}^{-3}$ . Consequently, the resulting electric fields lead to a Stark broadening of the spectral lines. The resulting collisional width for the Lyman- $\alpha$  line is much larger than the Doppler width and can still be appreciable at  $\lambda=160 \text{ nm}$ . The wave-

length dependence in the line wing is usually not Lorentzian, but is described by an inverse power law with exponent 2.5.<sup>21</sup>

Clearly the sun differs appreciably from our isothermal model. The absorption coefficient in the visible and near infrared, where the continuous Planck spectrum is emitted, is dominated by  $H^-$  bound-free absorption, the continuum extending to 0.745 eV, which is the ionization energy for  $H^-$ :



The reverse process generates light in the visible spectral range:



Toward the ultraviolet, several ionization edges are important such as those from Mg, Al, and Si at wavelengths above the Lyman- $\alpha$  line. As a result, all radiation from the sun is emitted from the photosphere with a thickness of about 300 km. Its spectrum is altered while propagating through the outer layers in the sun's atmosphere, where thousands of Fraunhofer absorption lines are formed. Our assumption of including only Lyman- $\alpha$  absorption, described by a Lorentzian line profile, is much too crude to account for the richness of the sun's spectrum.

## VI. SUMMARY

It has been shown using a simple model how a line spectrum of gaseous atoms with sharp features is transformed into a continuous spectrum of thermal radiation by the repeated absorption and emission of photons. In particular, it was demonstrated that the thermal radiation spectrum can be related to a hydrogen-like isothermal atomic gas with just two discrete energy levels resembling the Lyman- $\alpha$  transition. The parameter that determines whether a line spectrum or a continuous thermal spectrum is emitted is the optical thickness of the gas. For low optical thickness, a line spectrum will dominate. The nature of the spectrum emitted by  $1 \text{ m}^3$  of atomic hydrogen at 6000 K and densities of the sun's photosphere can now be given. The spectrum consists of a weak continuous contribution which can be characterized by the emissivity  $\epsilon \approx 10^{-6}$ , but is dominated by the broadened resonance line.

Our analysis also gives estimates for the emissivities of optically thin hot gases and the line width which is affected by self-absorption in the gas. However, the model is too simple to allow comparison to spectra of stellar atmospheres.

## ACKNOWLEDGMENTS

I wish to thank F. Hermann (Karlsruhe) for inspiring this work, and W. Schneider (Erlangen), R. Neumann (Darmstadt) and U. Kreibitz (Aachen) for helpful discussions.

<sup>a)</sup>Electronic mail: vollmer@fh-brandenburg.de

<sup>1</sup>M. Planck, "Die Ableitung der Strahlungsgesetze (Derivation of the laws of radiation)," Ostwalds Klassiker der Naturwissenschaften Band 206, Harri Deutsch 1995 (reprints of seven original papers on the topic of radiation laws).

<sup>2</sup>D. Haliday, R. Resnick, and K. Krane, *Physics*, 5th ed. (Wiley, New York, 2002), Vol. 2.

<sup>3</sup>R. Eisberg, *Fundamentals of Modern Physics* (Wiley, New York, 1961).

<sup>4</sup>A. Einstein, "Zur Quantentheorie der Strahlung" (Quantum theory of radiation), *Mitteil. Physikal. Gesell. Zürich* **16**, 47–62 (1916); "Strahlungsemission und Absorption nach der Quantentheorie" (Emission and absorption of radiation according to quantum theory), *Verh. Dtsch. Phys. Ges.* **18**, 318–323 (1916); "Zur Quantentheorie der Strahlung" (Quantum theory of radiation), *Phys. Z.* **18**, 121–128 (1917).

<sup>5</sup>F. Munley, "Approach of gas and radiation to equilibrium," *Am. J. Phys.* **58**, 357–362 (1990).

<sup>6</sup>F. S. Crawford, "Using Einstein's method to derive both the Planck and Fermi-Dirac distributions," *Am. J. Phys.* **56**, 883–885 (1988).

<sup>7</sup>T. H. Boyer, "Thermodynamics of the harmonic oscillator: Wien's displacement law and the Planck spectrum," *Am. J. Phys.* **71**, 866–870 (2003).

<sup>8</sup>M. Nauenberg, "The evolution of radiation toward thermal equilibrium: A soluble model that illustrates the foundations of statistical mechanics," *Am. J. Phys.* **72**, 313–323 (2004).

<sup>9</sup>F. W. Sears, M. W. Zemansky, and H. D. Young, *College Physics*, 7th ed. (Addison-Wesley, Reading, MA, 1991).

<sup>10</sup>D. MacIsaac, G. Kanner, and G. Anderson, "Basic physics of the incandescent lamp," *Phys. Teach.* **37**, 520–525 (1999).

<sup>11</sup>A. Corney, *Atomic and Laser Spectroscopy* (Clarendon, Oxford, 1977).

<sup>12</sup>V. V. Ivanov, *Transfer of Radiation in Spectral Lines*, NBS Spec. Publ. 385 (U.S. Government Printing Office, Washington, DC, 1973).

<sup>13</sup>M. Stix, *The Sun, An Introduction*, 2nd ed. (Springer, New York, 2002).

<sup>14</sup>A. Unsöld, *Physik der Sternatmosphären*, 2nd ed. (Springer, New York, 1955).

<sup>15</sup>D. Michalas, *Stellar Atmospheres*, 2nd ed. (Freeman, San Francisco, 1978).

<sup>16</sup>B. H. Soffer and D. K. Lynch, "Some paradoxes, errors, and resolutions concerning the spectral optimization of human vision," *Am. J. Phys.* **67**, 946–953 (1999).

<sup>17</sup>J. A. Overduin, "Eyesight and the solar Wien peak," *Am. J. Phys.* **71**, 216–219 (2003).

<sup>18</sup>*Handbook of Optics*, edited by M. Bass (McGraw-Hill, New York, 1995), Vol. 1.

<sup>19</sup>*The Infrared Handbook*, edited by W. L. Wolfe and G. J. Zissis (The Infrared Information Analysis Center, Environmental Research Institute of Michigan, rev. ed., 4th printing, 1993).

<sup>20</sup>D. Karstädt, K. P. Möllmann, F. Pinno, and M. Vollmer, "There is more to see than eyes can detect," *Phys. Teach.* **39**, 371–376 (2001).

<sup>21</sup>R. Wehrse, Institute of Theoretical Astrophysics, Heidelberg, private communication.