The collapse of a quantum state as a signal analytic sleight of hand

Peter Morgan

Physics Department, Yale University, New Haven, CT 06520, USA.

Abstract

The collapse of a quantum state is a mathematical sleight of hand that allows the construction of a joint probability density even for operators that are noncommutative. We can formalize that construction as a non-commutative, non-associative *collapse product* that is nonlinear in its left operand as a model for joint measurements at time-like separation, in part inspired by the *sequential product* for positive semi-definite operators. A collapse picture, in which a quantum state collapses after each measurement, can therefore be equivalent to a no-collapse picture that uses a different state and different, mutually commutative, Quantum Non-Demolition operators to construct the same joint probability density for consecutive measurements. The collapse of a state is a mathematical construction that can also be a useful tool when using the Koopman formalism for classical mechanics and signal analysis.

Keywords: Quantum Mechanics, the measurement problem, Signal Analysis, Quantum Non-Demolition measurement, Classical Mechanics

- ▶ Quantum state collapse uses noncommutative operators to construct a joint probability
- ▶ Such joint probabilities can also be constructed using mutually commutative operators
- ▶ The construction suggests an alternative approach to the quantum-classical transition
- ▶ Collapse+noncommutativity $\longleftrightarrow$ no-collapse+commutativity
- ▶ We can think of collapse as applied to subsequent measurements, not to the state

1. Introduction

Linear operators are used in quantum mechanics as models for measurement, with the set of eigenvalues of a self-adjoint operator $\hat{A}$ corresponding to the sample space of measurement results, together with a *state* ρ that models the average results of past measurements or that fixes what we expect the average results of future measurements will be, which we can write as the expression $\rho(\hat{A})$. For the algebraic conditions satisfied by a state in a quantum mechanical setting, see Appendix A and [1, §III.2.2][2, §3.2.1.3]; for accounts of the algebraic approach to quantum mechanics, see [2, Ch. 3] and [3, 4]. As pointed out in "An algebraic approach to Koopman classical mechanics"[5], an algebraic approach to modeling measurements and their results, including noncommutativity, can be used in classical mechanics and in signal analysis as effectively as it is in quantum mechanics: the Wigner function, for example, is widely used in a nontrivial way in classical signal analysis[6]. It has generally been understood that "collapse of the wave function", which we will here call *collapse of the state*, is particularly a feature of quantum mechanics, however as a mathematical tool that allows the construction of joint probability densities even when using noncommutative operators it can be more-or-less useful in any formalism that uses operators as models for measurement, as an effective way to give information about the joint statistics of the results of an experiment.

The rest of this section establishes notation and describes collapse of the state in a form that commonly appears in the literature, then §2 proposes an algebraic approach using what will be called here a *collapse product* and §3 exhibits various equivalences of several different collapse+noncommutativity and no-collapse+commutativity approaches to

Email address: peter.w.morgan@yale.edu (Peter Morgan)

modeling the same joint measurements. Adopting no-collapse+commutativity is close to the idea of a Quantum Mechanics Free Subsystem introduced by Tsang&Caves[7], is somewhat prefigured by Belavkin[8], and the signal analysis aspect is approached by Anastopoulos[9], however the relationship with Koopman's Hilbert space formalism for classical mechanics deserves to be revisited. Adopting no-collapse+commutativity also suggests an alternative approach to the quantum-classical transition[10, 11], however §4 closes by pointing out that in classical mechanics and signal analysis as well as in quantum mechanics some measurement results are not commeasurable, so that a joint probability density that has the required marginal probability densities cannot be constructed and we use noncommutative operators without using the collapse mechanism to model such cases. See Landsman[12, Ch. 11] for a broader discussion of the measurement problem.

In general, sample spaces associated with actually recorded measurement results for real experiments can always be taken to be discrete sets, because each actual record is always encoded in a finite number of bits. If we introduce an idealized operator $\hat{A}$ that has a continuous sample space, we can discretize it using the Heaviside function, so that, for example, the operator $\hat{A}_d = \Theta(\hat{A}-1) + \Theta(\hat{A}-2) + \Theta(\hat{A}-3)$ has the sample space $\{0,1,2,3\}$, corresponding, perhaps, to an actual two bit record. It is fortunate that an effective instrumental discretization will always be available in practice, so that extension of the constructions in this article to operators that have a continuous spectrum is not necessary, because the spectral theorem in the continuous spectrum case does not naturally admit the square root construction that is used here. Indeed much of the literature on the measurement problem restricts itself to the finite sample space case. §2.1, however, briefly suggests a few approaches to working with idealized measurements that have a continuous sample space.

We take a self-adjoint *Moment Generating Operator* $\hat{A}^\dagger = \hat{A}$ to have a discrete sample space $\{\alpha_i\}$, so it can be written as a weighted sum of a complete set of mutually orthogonal projection operators $\hat{A} = \sum_i \alpha_i \hat{P}_i^{(A)}$, where $\hat{P}_i^{(A)}\hat{P}_j^{(A)} = \delta_{i,j}\hat{P}_i^{(A)}$ and $\sum_i \hat{P}_i^{(A)} = \hat{1}$ [13, Ch. II]. From this, we can construct

$$\text{a } \textit{Probability Density Generating Operator}, \dot{\hat{A}}_u = \delta(\hat{A}-u) = \sum_i \delta(u-\alpha_i)\hat{P}_i^{(A)},\ u \in \mathbb{R},$$

and its fourier transform,

$$\text{a } \textit{Characteristic Function Generating Operator}, \mathring{\hat{A}}_\lambda = \mathrm{e}^{\mathrm{j}\lambda\hat{A}} = \sum_i \mathrm{e}^{\mathrm{j}\lambda\alpha_i}\hat{P}_i,\ \lambda \in \mathbb{R},$$

which is a 1-parameter group of unitary operators. With these, a state generates moments, a probability density, and a characteristic function

$$\rho(\hat{A}^n) = \sum_i \alpha_i^n \rho(\hat{P}_i^{(A)}), \quad p(u) = \rho(\dot{\hat{A}}_u) = \sum_i \delta(u-\alpha_i)\rho(\hat{P}_i^{(A)}), \quad \text{and} \quad \tilde{p}(\lambda) = \rho(\mathring{\hat{A}}_\lambda) = \sum_i \mathrm{e}^{\mathrm{j}\lambda\alpha_i}\rho(\hat{P}_i^{(A)}),$$

for which the probability density is nonzero only for values in the sample space of the measurement. As pointed out by Cohen[14], characteristic functions are a natural way to work with probability densities in a Hilbert space setting. In such a setting, the complex structure j can be taken to be introduced, pragmatically, as an effective way to manage the "sine" and "cosine" components of the fourier transform of a probability density. A more abstract mathematical interest is that it is also helpful that $\mathring{\hat{A}}_\lambda$ is a group of bounded operators.

When two operators commute, which they always do when they are models for measurements that are space-like separated from each other, we can construct a joint probability density straightforwardly, using the ordinary multiplication, as

$$p(u,v) = \rho(\dot{\hat{A}}_u \cdot \dot{\hat{B}}_v) = \sum_{i,j} \delta(u-\alpha_i)\delta(v-\beta_j)\rho(\hat{P}_i^{(A)}\hat{P}_j^{(B)}), \qquad \text{where } \hat{B} = \sum_j \beta_j \hat{P}_j^{(B)},$$

which extends to any number of commuting operators. For quantum mechanics, however, operators in general will not commute when they are models for measurements that are time-like separated from each other, in which case $\hat{P}_i^{(A)}\hat{P}_j^{(B)}$ may not be a positive operator and we have to use a different construction as a model for the results of sequential measurements, even though they are joint measurements that we can model using joint probability densities. Following a measurement result α_i, we say that the state ρ *"collapses"* to the state $\rho_i(\hat{X}) = \dfrac{\rho(\hat{P}_i^{(A)}\hat{X}\hat{P}_i^{(A)})}{\rho(\hat{P}_i^{(A)})}$, giving the expected value for any operator $\hat{X}$, using the Lüders operation corresponding to the i'th eigenvalue[13, §II.3.1]. ρ_i satisfies the four conditions required for it to be a state (see Appendix A). A measurement modeled by an operator

$\hat{B}$ then gives a probability density $\rho_i(\dot{B}_v)$. The joint measurement probability density after the whole process of a first measurement, collapse of the state, and a second measurement is therefore

$$p_{A,\text{collapse},B}(u,v) = \sum_i \delta(u-\alpha_i)\rho(\hat{P}_i^{(A)})\frac{\rho(\hat{P}_i^{(A)}\dot{B}_v\hat{P}_i^{(A)})}{\rho(\hat{P}_i^{(A)})} = \sum_{i,j}\delta(u-\alpha_i)\delta(v-\beta_j)\rho(\hat{P}_i^{(A)}\hat{P}_j^{(B)}\hat{P}_i^{(A)}).$$

The sample space $\{(\alpha_i,\beta_j)\}$ of this joint measurement is notably the same as the sample space associated with two commuting operators with sample spaces $\{\alpha_i\}$, $\{\beta_j\}$. $p_{A,\text{collapse},B}(u,v)$ is positive semi-definite by construction and is normalized, as it must be, $\int p_{A,\text{collapse},B}(u,v)\mathrm{d}v\mathrm{d}u = 1$.

2. A *collapse product* of probability density and characteristic function generating operators

Given the construction above, it is quite natural to use the *sequential product* for positive semi-definite operators that is described by Gudder and Greechie[15], $\hat{X}\circ\hat{Y} = \sqrt{\hat{X}}\cdot\hat{Y}\cdot\sqrt{\hat{X}}$, to define for the discrete sample space case a *collapse product* of two probability density generating operators or characteristic function generating operators as

$$\begin{aligned}
\dot{A}_u \blacktriangleright\!\!\circ \dot{B}_v = \delta(\hat{A}-u)\blacktriangleright\!\!\circ\delta(\hat{B}-v) &\doteq \sum_i \delta(u-\alpha_i)\hat{P}_i^{(A)}\delta(\hat{B}-v)\hat{P}_i^{(A)} = \sum_{i,j}\delta(u-\alpha_i)\delta(v-\beta_i)\hat{P}_i^{(A)}\circ\hat{P}_j^{(B)},\\
\mathring{A}_\lambda \blacktriangleright\!\!\circ \mathring{B}_\mu = \mathrm{e}^{\mathrm{j}\lambda\hat{A}}\blacktriangleright\!\!\circ\mathrm{e}^{\mathrm{j}\mu\hat{B}} &\doteq \sum_i \mathrm{e}^{\mathrm{j}\lambda\alpha_i}\hat{P}_i^{(A)}\mathrm{e}^{\mathrm{j}\mu\hat{B}}\hat{P}_i^{(A)} = \sum_{i,j}\mathrm{e}^{\mathrm{j}(\lambda\alpha_i+\mu\beta_j)}\hat{P}_i^{(A)}\circ\hat{P}_j^{(B)}
\end{aligned}$$

(where for projection operators we have $\sqrt{\hat{P}} = \hat{P}$), so we can write $p_{A,\text{collapse},B}(u,v) = \rho(\dot{A}_u\blacktriangleright\!\!\circ\dot{B}_v)$, and we can define the reverse case, $\dot{A}_u\circ\!\!\blacktriangleleft\dot{B}_v \doteq \dot{B}_v\blacktriangleright\!\!\circ\dot{A}_u$. We can think of this construction as applying a collapse operation to other measurements instead of to the state, following Bohr's preference for measurements affecting other measurements instead of collapse of the state[16]. We can confirm that $\dot{A}_u\blacktriangleright\!\!\circ\dot{B}_v$ is a positive semi-definite operator and that it is normalized appropriately to generate a joint probability density in any state, $\int\dot{A}_u\blacktriangleright\!\!\circ\dot{B}_v\mathrm{d}u\mathrm{d}v = \hat{1}$. If $[\hat{A},\hat{B}] = 0$, then $[\hat{P}_i^{(A)},\hat{B}] = 0$, so in that case $\dot{A}_u\blacktriangleright\!\!\circ\dot{B}_v = \dot{A}_u\cdot\dot{B}_v$.

We can generalize the collapse product to three or more characteristic function generating operators as, for example,

$$(\mathring{A}_{1,\lambda_1}\blacktriangleright\!\!\circ\mathring{A}_{2,\lambda_2})\blacktriangleright\!\!\circ\mathring{A}_{3,\lambda_3} = (\mathrm{e}^{\mathrm{j}\lambda_1\hat{A}_1}\blacktriangleright\!\!\circ\mathrm{e}^{\mathrm{j}\lambda_2\hat{A}_2})\blacktriangleright\!\!\circ\mathrm{e}^{\mathrm{j}\lambda_3\hat{A}_3} \doteq \sum_{i,j,k}\mathrm{e}^{\mathrm{j}(\lambda_1\alpha_i^{(1)}+\lambda_2\alpha_j^{(2)}+\lambda_3\alpha_k^{(3)})}(\hat{P}_i^{(A_1)}\circ\hat{P}_j^{(A_2)})\circ\hat{P}_k^{(A_3)},$$

where we have to include brackets because the collapse product and the sequential product are nonassociative as well as noncommutative. Furthermore, the collapse product is nonlinear in its first argument and the construction $(\hat{X}\circ\hat{Y})\circ\hat{Z} = \sqrt{\sqrt{\hat{X}}\cdot\hat{Y}\cdot\sqrt{\hat{X}}}\cdot\hat{Z}\cdot\sqrt{\sqrt{\hat{X}}\cdot\hat{Y}\cdot\sqrt{\hat{X}}}$ is significantly more complicated than $\hat{X}\circ(\hat{Y}\circ\hat{Z}) = \sqrt{\hat{X}}\cdot\hat{Y}\cdot\hat{Z}\cdot\sqrt{\hat{Y}}\cdot\hat{X}$. We can call the collapse product *power associative* insofar as $\mathring{A}_\lambda\blacktriangleright\!\!\circ\mathring{A}_\mu = \mathring{A}_\lambda\mathring{A}_\mu = \mathring{A}_{\lambda+\mu}$ and

$$\mathring{A}_{\lambda_1}\blacktriangleright\!\!\circ(\mathring{A}_{\lambda_2}\blacktriangleright\!\!\circ\mathring{B}_\mu) = \mathring{A}_{\lambda_2}\blacktriangleright\!\!\circ(\mathring{A}_{\lambda_1}\blacktriangleright\!\!\circ\mathring{B}_\mu) = \mathring{A}_{\lambda_1+\lambda_2}\blacktriangleright\!\!\circ\mathring{B}_\mu = (\mathring{A}_{\lambda_1}\blacktriangleright\!\!\circ\mathring{A}_{\lambda_2})\blacktriangleright\!\!\circ\mathring{B}_\mu.$$

For probability density generating functions, both $(\dot{A}_{1,u_1}\blacktriangleright\!\!\circ\dot{A}_{2,u_2})\blacktriangleright\!\!\circ\dot{A}_{3,u_3}$ and $\dot{A}_{1,u_1}\blacktriangleright\!\!\circ(\dot{A}_{2,u_2}\blacktriangleright\!\!\circ\dot{A}_{3,u_3})$ are positive semi-definite operators and are normalized appropriately to generate a probability density in any state. Note that $\sqrt{\hat{P}_i^{(A)}\hat{P}_j^{(B)}\hat{P}_i^{(A)}}$ is well-defined because $\hat{P}_i^{(A)}\hat{P}_j^{(B)}\hat{P}_i^{(A)}$ is a positive semi-definite operator, but neither operator is a projection unless it happens that $[\hat{P}_i^{(A)},\hat{P}_j^{(B)}] = 0$ for the particular eigenspaces.

Most commonly, the time ordering of measurements is taken to determine the ordering of collapse products, however, as a mathematical construction, the ordering of collapse products is independent of the time ordering of measurement operators, so that we could perfectly well use the collapse product out of time order if we were to find it useful to do so. Furthermore, time ordering is only a partial ordering for measurements that are associated with overlapping time intervals, in which case time ordering cannot by itself determine the ordering of collapse products, so that the collapse product may introduce some difficult decisions. Whereas time reversal is straightforward for no-collapse+commutativity models, time reversal for collapse+noncommutativity models in which time-order determines collapse order introduces significant complications.

We can *loosely* consider the collapse product to be a regularized form of the positive semi-definite but unnormalized construction

$$\begin{aligned}
\delta(\hat{A}-u)\delta(\hat{B}-v)\delta(\hat{A}-u) &= \sum_{i,j,k}\delta(u-\alpha_i)\delta(v-\beta_j)\delta(u-\alpha_k)\rho(\hat{P}^{(A)}_i\hat{P}^{(B)}_j\hat{P}^{(A)}_k)\\
&\overset{N}{=} \sum_{i,j,k}\delta(u-\alpha_i)\delta(v-\beta_j)\delta_{i,k}\rho(\hat{P}^{(A)}_i\hat{P}^{(B)}_j\hat{P}^{(A)}_k)\\
&= \sum_{i,j}\delta(u-\alpha_i)\delta(v-\beta_j)\rho(\hat{P}^{(A)}_i\hat{P}^{(B)}_j\hat{P}^{(A)}_i) \quad = \quad \dot{A}_u\ltimes\dot{B}_v,
\end{aligned}$$

where the regularization replaces the improper expression $\delta(u-\alpha_i)\delta(u-\alpha_k)$ by $\delta(u-\alpha_i)\delta_{i,k}$. We can understand both $\dot{A}_u\ltimes$ and $\mathring{A}_\lambda\ltimes$ acting on operators on their right to be a parameterized set of operations in Kraus form[13, §II.3.1].

2.1. Continuous sample spaces

If we decide to work in an idealized formalism instead of using the available instrumental discretization, we *can* introduce other additional structure, for example by choosing a finite set of probability densities $p^{(A)}_i(u)$, replacing the densities $\delta(u-\alpha_i)$, for each of which $\int p^{(A)}_i(u)\mathrm{d}u = 1$, each of which is followed by the application of a different state transformer to the initial state, as

$$\dot{A}_u = \sum_i p^{(A)}_i(u)\hat{\mathbf{A}}^\dagger_i\hat{\mathbf{A}}_i, \quad \sum_i \hat{\mathbf{A}}^\dagger_i\hat{\mathbf{A}}_i = 1, \quad p^{(A)}(u) = \rho(\dot{A}_u) = \sum_i p^{(A)}_i(u)\rho(\hat{\mathbf{A}}^\dagger_i\hat{\mathbf{A}}_i),$$

so that the probability density $p^{(A)}(u)$ is a convex sum of the probability densities $p^{(A)}_i(u)$, and, for the collapse product,

$$\dot{A}_u\ltimes\dot{B}_v = \sum_i p^{(A)}_i(u)\hat{\mathbf{A}}^\dagger_i\dot{B}_v\hat{\mathbf{A}}_i = \sum_{i,j} p^{(A)}_i(u)p^{(B)}_j(v)\hat{\mathbf{A}}^\dagger_i\hat{\mathbf{B}}^\dagger_j\hat{\mathbf{B}}_j\hat{\mathbf{A}}_i.$$

We can introduce ever more elaborate ways to use noncommuting operators to construct joint probability generating operators, with perhaps the simplest being a regularization of $\delta(\hat{A}-u)\delta(\hat{B}-v)\delta(\hat{A}-u)$ that can be used for operators that have either a continuous or discrete spectrum, for any $\epsilon > 0$,

$$\dot{A}_u \overset{\epsilon}{\ltimes} \dot{B}_v = \frac{1}{\sqrt{\pi\epsilon}}\mathrm{e}^{-(\hat{A}-u)^2/2\epsilon}\delta(\hat{B}-v)\mathrm{e}^{-(\hat{A}-u)^2/2\epsilon},$$

which is manifestly positive semi-definite and is normalized if we perform the v integration first, $\int\dot{A}_u \overset{\epsilon}{\ltimes}\dot{B}_v\mathrm{d}v\mathrm{d}u = \hat{1}$, but performing the u integral first may not be possible and the limit $\epsilon \to 0$ may well not exist. A given construction of a collapse product may well require some kind of regularization, and a sequence of collapse products may require increasingly elaborate care. The quantum measurement theory literature includes many such constructions of joint probabilities[17].

3. Equivalent noncommutative and commutative models

Because $p_{A,\mathrm{collapse},B}(u,v)$ *is* a joint probability density, with sample space $\{(\alpha_i,\beta_j)\}$, we can certainly introduce an operator $\hat{B}'$ that has the same sample space $\{\beta_j\}$ as $\hat{B}$ but that commutes with $\hat{A}$, $[\hat{A},\hat{B}'] = 0$, and a different state ρ', for which $\rho'(\dot{A}_u{\cdot}\dot{B}'_v) \doteq p_{A,\mathrm{collapse},B}(u,v)$ (which does not fix either $\hat{B}'$ or ρ' uniquely.) Although we have arrived at this mathematics by a somewhat different motivation, as a way to remodel collapse of the state to be classically natural, the operators $\hat{A}$ and $\hat{B}'$ are then *Quantum Non-Demolition* (QND) observables relative to each other[7, 8], for which there is a long history and experience in their use. Particularly when we model a stream of millions or billions of jointly recorded measurement events over time, as we do when we record signal levels on a signal line, it can be equally or more effective simply to use a commutative algebra of measurement operators to model a joint probability density instead of working with a mathematical formalism in which collapse of the state occurs millions or billions of times. Equally, however, where we have been accustomed to modeling joint probability densities in classical physics only using commutative algebras of measurement operators, it can be justifiable for a classical physicist or in signal analysis to use the collapse product to achieve a useful compression or to achieve other goals.

We can also use a single long sequence of noncommutative operators and collapse products in different ways.

$$(\cdots((\dot{A}_{1,u_1} \mathbin{\blacktriangleright\!\!\circ} \dot{A}_{2,u_2}) \mathbin{\blacktriangleright\!\!\circ} \dot{A}_{3,u_3}) \mathbin{\blacktriangleright\!\!\circ} \cdots) \mathbin{\blacktriangleright\!\!\circ} \dot{A}_{n,u_n},$$

which collapses the state after every measurement, gives a very different type of modeling than is given by

$$\dot{A}_{1,u_1} \mathbin{\blacktriangleright\!\!\circ} (\dot{A}_{2,u_2} \mathbin{\blacktriangleright\!\!\circ} (\cdots \mathbin{\blacktriangleright\!\!\circ} (\dot{A}_{n-1,u_{n-1}} \mathbin{\blacktriangleright\!\!\circ} \dot{A}_{n,u_n})\cdots)),$$

which can be thought of as applying a single, combined collapse only after the last-but-one measurement. We can consider the latter to be the mathematics behind a form of Many Worlds Interpretation, in the sense that there is no collapse except when the experiment is finally completed (which we could say happens when we begin analysis of the experimental raw data, or, more extravagantly, we could say that the final collapse will not happen until after the last human being dies or at some other cosmologically defined endtime.) Both constructions allow us to generate joint probability densities, because they are both positive semi-definite operators as functions of $u_1, ..., u_n$ and they are both normalized appropriately, but for the same statistics of experimental results we would have to use different states to achieve empirically equivalent models.

If we only allow unitary evolution, with no use whatsoever of collapse of the quantum state, as in the Many Worlds and some other interpretations of quantum mechanics, we can only model joint probabilities using commuting operators, giving an essentially classical perspective of QND measurements. With a no-collapse approach, we are free to adopt any interpretation of classical probability —Dutch Book, Propensity, Frequency, Many Worlds, or any other— but there must be a connection to experimental raw data, its statistics, and to the choice of subensembles of the data.

Yet another fairly natural construction can be thought of as applying a single, combined collapse in time-reverse order, immediately after the first measurement (taking $\hat{A}_i$ to be before $\hat{A}_{i+1}$),

$$(\cdots((\dot{A}_{1,u_1} \mathbin{\circ\!\!\blacktriangleleft} \dot{A}_{2,u_2}) \mathbin{\circ\!\!\blacktriangleleft} \dot{A}_{3,u_3}) \mathbin{\circ\!\!\blacktriangleleft} \cdots) \mathbin{\circ\!\!\blacktriangleleft} \dot{A}_{n,u_n},$$

again requiring a different state to generate the same statistics, but, as already noted, there are many other possibilities. Indeed, for a given ordering of n distinct operators, there are $\frac{(2n-2)!}{(n-1)!n!}$ distinct ways to introduce the required brackets, giving the sequence $1, 1, 2, 5, 14, 42, 132, ...$, with some choices of brackets being more mathematically natural than others but with none absolutely preferred.

For a given experiment, we can use whichever construction seems most helpful, but for each the state will be different and so will our understanding of its relationship to experimental raw data. The profusion of different states, depending on what approach we take to constructing joint probabilities, is straightforward on an epistemological or structural realist understanding of quantum states, however it is arguably incompatible with a naïve ontological understanding of physical states.

4. Discussion

For joint measurements at space-like separation, we do not need to introduce collapse of the state to model them, because operators at space-like separation commute and collapse of the state has no effect if measurement operators commute. For joint measurements at time-like separation, however, the collapse of the state after a measurement makes it *possible* to model those joint measurements using noncommutative operators: without collapse of the state or a similar construction it would not be possible to use noncommutative operators to model joint measurements for all prepared states, because of the elementary observation that in general $\rho(\hat{A}\hat{B})^* \neq \rho(\hat{A}\hat{B})$ unless $[\hat{A}, \hat{B}] = 0$. Any model that uses collapse to make it possible to use noncommutative operators to model joint measurements can be replaced by a model that uses only mutually commuting operators, but, conversely, it also may be useful when using Koopman classical mechanics to replace a commuting operator model by a more compact model that freely uses a noncommutative algebra of operators.

In the Schrödinger picture of phase space quantum mechanics, measurements are associated with regions of space and with a state that only models the statistics of measurement results at one time, evolving unitarily from time to time.

In this picture, it is in a sense straightforward to introduce a nonunitary collapse of the quantum state immediately after a measurement result, even though it introduces well-known interpretational concerns. In contrast, when modeling physical systems in a quantum field theoretic way or in the Heisenberg picture, measurements are associated with regions of space-time and with a state that models the statistics of measurement results at all times, so we cannot as straightforwardly introduce collapse of the quantum state at a particular time, but we can nonetheless use the collapse product acting on other measurements as a way to construct joint probabilities.

Although it has been stressed here that we *can* think about collapse of the quantum state in terms of the construction of joint probability densities straightforwardly and effectively, the much more elaborate mathematics of detailed models of the thermodynamic behavior and statistical mechanics of real experimental apparatus, of measurement as interaction, of decoherence, and even of the observer's brain or mind, is not thereby made unnecessary. The many discussions of such mathematics in the literature are of course just as valid and necessary as they ever were, but at the level of abstraction at which we consider probabilities and statistics of actually recorded experimental results, we can reasonably shut up and calculate joint probabilities using collapse of the state, knowing that we could also shut up and calculate using QND measurements with no collapse if we wished to do so.

Joint measurements, however, do not exhaust the use of noncommuting operators in quantum mechanics, because not all measurements are joint measurements. Algebraic formalisms for classical mechanics and signal analysis also can reasonably include the use of noncommuting operators[5]. For measurements that are not joint measurements, we can use noncommuting operators to represent the relationship between different analyses of recorded experimental raw data. For experiments in which Bell-CHSH-type inequalities are violated, for example, we *must* include noncommutative operators for an algebraic model to give us an effective model for the results[18][5, §7.2], because the use of arbitrary post-selection algorithms to create new datasets effectively creates distinct experimental contexts. To require classical or other operator formalisms not to use noncommutative operators is effectively to make them a straw man. As Pitowsky puts it (saying "commeasurable" for "jointly measurable"), quoted by Abramsky[19, p. 7],

> For certain families of events the theory stipulates that they are commeasurable. This means that, in every state, the relative frequencies of all these events can be measured on one single sample. For such families of events, the rules of classical probability —Boole's conditions in particular— are valid. Other families of events are not commeasurable, so their frequencies must be measured in more than one sample. The events in such families nevertheless exhibit logical relations (given, usually, in terms of algebraic relations among observables). But for some states, the probabilities assigned to the events violate one or more of Boole's conditions associated with those logical relations.

There are signal analysis algorithms that we can apply to jointly measured results that result in collections of post-selected data that cannot be commeasured and do not admit a joint probability density. There are other ways of discussing commeasurability: recent Generalized Probability Theory literature uses the word "incompatibility" when two probability densities do not admit a joint probability density, and there is a substantial literature on "contextuality"[20, 21].

A signal analysis approach to quantum mechanics takes as its starting point the recorded values of the jointly measured signal levels on the many signal lines out of an experimental apparatus. The subsequent signal analysis of those actually recorded signal levels, whether in real-time in hardware or software, or in post-analysis in software, may result in probability densities for the results of those computations that do not admit joint probability densities, but we take the actually recorded signal levels that are the source for that analysis to be essentially classical: we can share experimental raw data with other physicists with no concern that anything will change when they are read. Although we can continue to take notable features such as sudden transitions of the signal level on a signal line to have been caused by a "particle" or a "wave" or a "quantum particle" or a "quantum state" or a "quantum field state", because we have become skilled in the use of our existing quantum mechanics toolbox, we can also —because of the equivalence of collapse+noncommutativity models and no-collapse+commutativity models constructed above— take the finite collection of actual records of the signal levels over time to be a consequence of the way the whole experimental apparatus has been engineered.

For future experiments, we can ask a continuum of questions about how the actual records of signal levels would differ if small or large changes were made to the apparatus, with arbitrarily many additional devices and signal lines interpolated among those already there or with the signal levels recorded arbitrarily more accurately or more often. For established theories, such interpolation and extrapolation gives accurate enough results for a continuum of future experiments, in a wide range of conditions, that we can even use them as a good basis for engineering and we can justify thinking about them in a realist way. For more modestly established theories, however, realism is proportionately less justified. The overall perspective here, which may in time become "The signal analysis interpretation of classical and quantum mechanics", is the subject of further development.

Acknowledgements

I am grateful for comments from Martin Bohmann, Martin Plávala, and Spyridon Samios, and to Richard Gill for reminding me of Belavkin[8].

Appendix A. Conditions satisfied by a state

We take a complex-valued state ρ acting on a $*$-algebra $\mathcal{A}$, $\rho : \mathcal{A} \to \mathbb{C}; \hat{A} \mapsto \rho(\hat{A})$, to satisfy four conditions[1, §III.2.2][2, §3.2.1.3]:

- von Neumann complex-linearity: $\rho(\lambda\hat{A} + \mu\hat{B}) = \lambda\rho(\hat{A}) + \mu\rho(\hat{B})$, satisfied even if $[\hat{A}, \hat{B}] \neq 0$;
- positive semi-definiteness: $\rho(\hat{A}^\dagger\hat{A}) \geq 0$;
- compatibility with the adjoint: $\rho(\hat{A}^\dagger) = \rho(\hat{A})^*$;
- normalization: $\rho(\hat{1}) = 1$.

These conditions allow us to use a state to construct a Hilbert space $\mathcal{H}$ and a representation of $\mathcal{A}$ that acts upon it[5, §3]. Suitable different conditions, which we do not consider here, would allow the construction of Generalized Probability Theories (GPTs) that are not generated by a Hilbert space[22, Ch. 1][23, 24]. In an algebraic approach, a state should be distinguished from a "vector state", a normalized vector $|\psi\rangle \in \mathcal{H}$, $\langle\psi|\psi\rangle = 1$, which can be used to construct a pure state, $\rho_{|\psi\rangle}(\hat{A}) = \langle\psi|\hat{A}|\psi\rangle$. The Born rule expression for a probability density such as $|\langle x|\psi\rangle|^2 = \langle\psi|x\rangle\langle x|\psi\rangle = \rho_{|\psi\rangle}(\hat{X})$ can be understood as the expected measurement result for a projection operator $\hat{X} = |x\rangle\langle x|$ in the pure state $\rho_{|\psi\rangle}$.

References

[1] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, 2nd Edn., Springer, Berlin, 1996. https://doi.org/10.1007/978-3-642-61458-3
[2] F. David, *The Formalisms of Quantum Mechanics: An Introduction*, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-10539-0
[3] N. P. Landsman, "Algebraic Quantum Mechanics", in D. Greenberger, K. Hentschel, and F. Weinert (Eds.), *Compendium of quantum physics*, Springer, Berlin, 2009, pp. 6-10. https://doi.org/10.1007/978-3-540-70626-7_3
[4] S. Sekhon, "C*-algebras: The (Quantum) Path Less Traveled", https://arxiv.org/abs/2104.02038
[5] P. Morgan, "An algebraic approach to Koopman classical mechanics", Ann. Phys. **414** (2020) 168090. https://doi.org/10.1016/j.aop.2020.168090
[6] L. Cohen, "Time–Frequency Distributions — A Review", Proc. IEEE **77** (1989) 941. https://doi.org/10.1109/5.30749
[7] M. Tsang and C. Caves, "Evading Quantum Mechanics: Engineering a Classical Subsystem within a Quantum Environment", Phys. Rev. X **2** (2012) 031016. https://doi.org/10.1103/PhysRevX.2.031016
[8] V. P. Belavkin, "Nondemolition Principle of Quantum Measurement Theory", Found. Phys. **24** (1994) 685. https://doi.org/10.1007/BF02054669
[9] C. Anastopoulos, "Classical Versus Quantum Probability in Sequential Measurements", Found. Phys. **36** (2006) 1601. https://doi.org/10.1007/s10701-006-9077-5
[10] J. Kofler and Č. Brukner, "Classical World Arising out of Quantum Physics under the Restriction of Coarse-Grained Measurements", Phys. Rev. Lett. **99** (2007) 180403. http://dx.doi.org/10.1103/PhysRevLett.99.180403
[11] M. Schlosshauer, "Quantum decoherence", Phys. Rep. 831 (2019) 1. https://doi.org/10.1016/j.physrep.2019.10.001
[12] K. Landsman, *Foundations of Quantum Theory: From Classical Concepts to Operator Algebras*, Springer, Cham, Switzerland, 2017. https://doi.org/10.1007/978-3-319-51777-3
[13] P. Busch, M. Grabowski, P. J. Lahti, *Operational Quantum Physics*, Springer, Berlin, 1997. https://doi.org/10.1007/978-3-540-49239-9
[14] L. Cohen, "Rules of Probability in Quantum Mechanics", Found. Phys. **18** (1988) 983. https://doi.org/10.1007/BF01909934
[15] S. Gudder and R. Greechie, "Sequential Products on effect algebras", Rep. Math. Phys. **49** (2002) 87. https://doi.org/10.1016/S0034-4877(02)80007-6
[16] D. Howard, "Who Invented the "Copenhagen Interpretation"? A Study in Mythology", Philosophy of Science **71** (2004) 669. https://doi.org/10.1086/425941

[17] P. Busch, P. Lahti, J.-P. Pellonpää, K. Ylinen, *Quantum Measurement*, Springer, 2016. https://doi.org/10.1007/978-3-319-43389-9
[18] L. J. Landau, "On the violation of Bell's inequality in quantum theory", Phys. Lett. A **120** (1987) 54. https://doi.org/10.1016/0375-9601(87)90075-2
[19] S. Abramsky, "Classical Logic, Classical Probability, and Quantum Mechanics", in M. Hemmo, O. Shenker (eds.), *Quantum, Probability, Logic, Jerusalem Studies in Philosophy and History of Science*, Springer, Cham, 2020. https://doi.org/10.1007/978-3-030-34316-3_1
[20] F. Shahandeh, "Contextuality of General Probabilistic Theories", PRX Quantum **2** (2021) 010330. http://dx.doi.org/10.1103/PRXQuantum.2.010330
[21] D. Schmid, J. H. Selby, E. Wolfe, R. Kunjwal, R. W. Spekkens, "Characterization of Noncontextuality in the Framework of Generalized Probabilistic Theories", PRX Quantum **2** (2021) 010331. http://dx.doi.org/10.1103/PRXQuantum.2.010331
[22] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, Springer, Basel, 2011. https://doi.org/10.1007/978-88-7642-378-9
[23] P. Janotta, H. Hinrichsen, "Generalized probability theories: what determines the structure of quantum theory?", J. Phys. A **47** (2014) 323001. https://doi.org/10.1088/1751-8113/47/32/323001
[24] M. Plávala, "General probabilistic theories: An Introduction", https://arxiv.org/abs/2103.07469