Exponential complexity and ontological theories of quantum mechanics

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Ontological theories of quantum mechanics describe a single system by means of well-defined classical variables and attribute the quantum uncertainties to our ignorance about the underlying reality represented by these variables. We consider the general class of ontological theories describing a quantum system by a set of variables with Markovian (either deterministic or stochastic) evolution. We provide the first proof that the number of continuous variables can not be smaller than $2N-2$, N being the Hilbert space dimension. Thus, any ontological Markovian theory of quantum mechanics requires a number of variables which grows exponentially with the physical size. This result is relevant also in the framework of quantum Monte Carlo methods.

I. INTRODUCTION

In classical mechanics, the number of variables describing the state of a particle ensemble scales as the number of particles, thus the calculation time and memory resources which are necessary in a numerical simulation are generally a polynomial function of the physical size. This friendly property of classical systems seems to be lost at the microscopic level, when quantum effects become relevant. It is well-known that the definition of a quantum state requires a number of resources growing exponentially with the number of particles. This characteristic is at the basis of the exponential complexity of quantum mechanics (QM) and forbids at present to efficiently simulate the dynamics of many-body systems, unless some approximation is involved or particular problems are considered [1, 2, 3, 4].

The origin of the exponential growth can be understood with the following example. A quantum scalar particle is associated with a complex wave-function $\psi(\vec{x})$ which lives in the three-dimensional space and evolves in accordance with the Schrödinger equation. Discretizing each coordinate with R points and evaluating the spatial derivatives by finite differences, the Schrödinger equation is then reduced to $2R^3$ real ordinary differential equations. One could interpret ψ as a physical field which pilots the particle and intuitively conclude that M interacting particles are trivially described by M complex wave-functions in the three-dimensional space. Thus, the number of variables would grow linearly with the number of particles. Instead, in the standard description of quantum phenomena, an ensemble of M particles is associated with a single complex wave-function $\psi(\vec{x}_1, ..., \vec{x}_M)$ which lives in the configuration space of the whole system. If each coordinate is discretized with R points, the number of real variables we have to integrate grows exponentially with M and is given by $2R^{3M}$.

In general terms, if two quantum systems A_1 and A_2 are associated with the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then the composite system $A_1 + A_2$ is associated with the tensorial product $\mathcal{H}_1 \otimes \mathcal{H}_2$. This characteristic was at the basis of the Born's argumentations against the realistic interpretation of the wave-function by Schrödinger, which would imply an exponential growth of physical variables. In the Born's interpretation the wave-function does not represent real quantities, but it is merely a mathematical tool of the theory enabling to evaluate the probabilities of physical events. It is similar to the classical concept of probability distribution, that has the form $\rho(\vec{x}_1, \vec{p}_1, ..., \vec{x}_M, \vec{p}_M)$ for an ensemble of M particles and lives in the phase space of the whole system. In spite of this similarity, the Born's interpretation constitutes a deep departure from classical statistical theories, where each system is described by well-defined (ontological) quantities and the probability distributions merely provide a statistical (epistemological) description of them (in the following, we use the terms "classical", "ontological" and "realistic" as synonyms).

This departure is not unavoidable, indeed ontological theories of quantum phenomena exist, such as Bohm mechanics [5, 6, 7], which describes single systems by means of well-defined variables (ontic variables) determining the outgoing values of measurements. However the Born's argumentation constitutes the main difficulty of these approaches, since they promote the abstract wave-function to the rank of a classical real field. Thus, it is natural to ask whether there exists an ontological theory of quantum phenomena which uses an alternative representation of a single system and is not subject to the exponential growth of the number of variables. This is not only a foundational problem, but it has also a concrete relevance, since such a theory would provide a new revolutionary quantum Monte Carlo method (QMC) for many-body problems [3, 4, 8]. Questions pertaining to the computational complexity of QMC are discussed in Ref. [9].

In this article, we consider this problem from a general point of view and give the first proof that the exponential growth of the number of variables is a common feature of any ontological Markovian (deterministic or probabilistic) theory of quantum systems. More precisely we prove that, for a system with a finite N-dimensional Hilbert space, the number of continuous ontic variables can not be smaller than $2N-2$. As a consequence, the Born's criticism does not concern only the pilot-wave theories, but every realistic Markovian theory. If we assume QM exact, then this result suggests two choices, either to reject ontological causal theories of quantum systems or to accept the exponential growth of the ontic variables as an intrinsic feature of quantum phenomena. After this work was completed, we became aware that similar questions were discussed by L. Hardy [10]. He proved that the number of ontic states can not be finite, but did not placed constraints on the dimensionality of the ontic state space. As far as we know, the question on the exponential growth of the ontological dimensionality was posed for the first time in Refs.[4, 8]. Very recently, similar studies has been tackled in Ref. [11]. However, until now no answer to this open question was given.

One could object that the use of the multi-particle wave-function as a variable field in ontological models is an obvious requirement, because of Bell's theorem on entangled states. Thus, our proof would not be necessary. However Bell's theorem imposes only a non-locality condition on realistic theories and says nothing about the dimensionality of the ontic state space. One can not reject *a priori* the possibility of a non-local realistic theory with a space of variables smaller than the Hilbert space without an explicit proof.

In section II we give the motivations for this study and introduce the general properties that an ontological theory of QM has to satisfy. In section III we provide some examples and in section IV discuss a particular case of dimensional reduction of the ontic state space. In section V A the proof of the theorem on the classical space dimension is presented and its consequences for quantum Monte Carlo methods are sketched. Finally, the conclusions are drawn in the last section.

II. CLASSICAL THEORIES OF QUANTUM MECHANICS

The state of a quantum system is described by a trace-one density operator $\hat\rho$ which is Hermitian and non-negative defined, i.e., it satisfies the properties

$$\begin{array}{l} Tr\hat\rho = 1 \\ \hat\rho^\dagger = \hat\rho \\ \langle\phi|\hat\rho|\phi\rangle \ge 0 \text{ for every } |\phi\rangle. \end{array} \tag{1}$$

When the maximal information on the system is obtained, the quantum state is described by a Hilbert space vector $|\psi\rangle$ and the density operator is the projector $|\psi\rangle\langle\psi|$. A von Neumann measurement is associated with a Hermitian operator $\hat M = \sum_k m_k |k\rangle\langle k|$, where m_k are the possible results and the vectors $|k\rangle$ are orthonormal. If $\hat M$ is not degenerate, when $\hat M$ is measured and the system is in the pure state $|\psi\rangle$, the probability of obtaining result m_k is $p_k = |\langle k|\psi\rangle|^2$. If m_k is obtained, the quantum state is projected into $|k\rangle$. This is the general framework of QM.

In spite of the evident success of QM in explaining secular experimental results, there are at least two reasons to ask for an alternative reformulation of the theory. The first one concerns the ambiguity in the state projection rule, which requires one to mark a boundary between the fuzzy microscopic quantum world and the macroscopic well-defined observations. This ambiguity has at present no practical consequence, since the quantum predictions are practically insensitive to the boundary, once the quantum domain is taken sufficiently large. This insensitivity is related to decoherence phenomena [12]. Our work is motivated by another more practical reason. In classical mechanics, the number of variables which specify the state scales linearly with the physical size, thus a numerical simulation of dynamics is generally a polynomial complexity problem. Conversely, the definition of a quantum state requires an exponentially growing number of resources, making the numerical integration of the Schrödinger equation impossible even for a small number of particles. Many approximate methods are used in order to circumvent this problem, such as quantum Monte Carlo and semi-classical methods, the Hartree-Fock approximation, the density-functional theory and so on. However, at present no general numerical method is known which is able to solve in polynomial time quantum many-body dynamics. It is interesting to observe that in the standard interpretation the wave-function does not represent a physical field, but it provides a complete statistical information on an infinite ensemble of realizations, thus the exponential growth of resources in the solution of the Schrödinger equation is not surprising. Indeed, this occurs also in classical mechanics with the definition of the multi-particle probability distribution $\rho(\vec x_1, \vec p_1, ..., \vec x_M, \vec p_M)$. For example, consider the problem of M mutually coupled classical Brownian particles, which are described by the stochastic equations

$$\frac{d\vec v_i}{dt} = \sum_{j\ne i} \vec F_{ij}(\vec x_i, \vec x_j) - \gamma \vec v_i + \vec\eta_i(t), \tag{2}$$

where $\vec x_i$, $\vec v_i$, $\vec F_{ij}$, γ and $\vec\eta_i \equiv (\eta_i^1, \eta_i^2, \eta_i^3)$ are the spatial coordinates, the velocity, the interaction force, the coefficient of viscosity and the noise term, respectively. The two-time correlation function of the noise term is $\langle \eta_i^k(t)\eta_j^l(t')\rangle = g\delta_{ij}\delta_{kl}\delta(t-t')$. The masses are set equal to 1. The solution of these equations is the trajectory of a single realization and its evaluation is a polynomial problem. Equation (2) is associated with the following Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} = \sum_i \left[\frac{\partial}{\partial \vec v_i} \cdot (\gamma \vec v_i - \sum_{j\ne i} \vec F_{ij}) + \frac{g}{2}\frac{\partial^2}{\partial \vec v_i^2} - \frac{\partial}{\partial \vec x_i} \cdot \vec v_i \right] \rho. \tag{3}$$

Its direct numerical integration is an exponential complexity problem. The difference of complexity between Eq. (2) and Eq. (3) is not amazing, since the first equation describes the trajectory of a single realization, conversely the second one provides a complete statistical description of an infinite number of realizations. Since this complete information is in general out of the experimental domain, a detailed evolution of the multi-particle probability distribution is not required and it is practically sufficient to evaluate the averages of some quantities over a finite number of trajectories by means of a Monte Carlo method. In QM, a similar approach is used for thermal equilibrium problems and, with some approximations, in dynamical problems (quantum Monte Carlo methods). The point of this discussion is the following. Suppose, in accordance with the Einstein's view, that there exists a more fundamental theory which does not provide a statistical description on ensembles and characterizes each quantum system by means of a set of well-defined physical variables. The dynamical laws of this ontological theory and the Schrödinger equation would be similar to Eqs. (2,3) of our example, respectively. Thus, it is natural to pose the following question: is the trajectory simulation in this fundamental theory a polynomial problem? more precisely, does the ontological space dimension grow polynomially with the physical size? In this case, the fundamental theory would provide in a natural way a revolutionary quantum Monte Carlo method. This is the non-obvious core question of this article and we will find that the answer is negative, i.e., the exponential complexity is not related to the ensemble description, but it is a general feature of any ontological Markovian theory of quantum phenomena. In the following subsections, we introduce the general properties of such theories.

A. Kinematics

We characterize a single system by means of a set of continuous and discrete ontological variables, say x_1,x_2,...,x_w and s_1,...,s_p, respectively. In the following we use synthetically the symbol X for this set, i.e., $X=\{x_1,...x_w,s_1,...s_p\}$. For a single system, it takes a well-defined value $X(t)$ at any time t. When a quantum system is prepared in a state $|\psi\rangle$, the ontological variable takes a value X with a probability $\rho(X)$ which depends on $|\psi\rangle$. $\rho(X)$ has to satisfy the following conditions

$$\int dX\rho(X)=1 \tag{4}$$

$$\rho(X)\ge 0. \tag{5}$$

In principle, it is possible that $|\psi\rangle$ does not fix unequivocally ρ; this one could then depend on the specific experimental setup used to prepare the pure state. Furthermore in the quantum formalism the pure state preparation deletes the memory of the previous history. This is not necessarily true in the ontological theory. In order not to lose in generality, we assume that each pure state can be associated with different probability distributions which can depend on the specific experimental setup and the previous history. This prevents us from writing a single-valued functional relation $|\psi\rangle\rightarrow\rho(X|\psi)$. Let $\mathcal{C}$ be a set which contains at least one element, we write the mapping

$$|\psi\rangle\rightarrow\{\rho(X|\psi,\eta),\eta\in\mathcal{C}\}. \tag{6}$$

This over-labeling of ρ can be found also in the positive P-functions [13], mainly used in quantum optics and degenerate boson gases, and in Ref. [14], where $\eta\in\mathcal{C}$ is identified as context for the quantum state preparation. Obviously, if $|\psi\rangle\langle\psi|\ne|\psi'\rangle\langle\psi'|$, then $\rho(X|\psi,\eta)\ne\rho(X|\psi',\eta')$ for any η and $\eta'\in\mathcal{C}$. Thus, Function (6) can be inverted and we have

$$|\psi\rangle\langle\psi|=\hat D(\rho), \tag{7}$$

where the operator $\hat D(\rho)$ is a function whose domain is the image of the functional $\rho(X|\psi,\eta)$.

Equation (6) and its equivalent form, Eq. (7), are our first hypothesis. As a practical example, consider a single mode of the electromagnetic field, whose annihilation and creation operators are $\hat a$ and $\hat a^\dagger$, respectively. The coherent state $|\alpha\rangle$ of the mode is

$$|\alpha\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha\hat a^\dagger)^n}{n!}|0\rangle, \tag{8}$$

where $|0\rangle$ is the vacuum state and α is a complex number. By means of the coherent state, it is possible to define some quasi-probability distributions associated with the quantum states. The Glauber distribution $P_G(\alpha)$ of the state $|\psi\rangle$ is defined as follows,

$$\int d\alpha P_G(\alpha)|\alpha\rangle\langle\alpha|\equiv|\psi\rangle\langle\psi|. \tag{9}$$

A Glauber distribution exists for any quantum state, but in general it is highly singular and non-positive and can not be interpreted as a probability distribution [Eq. (5)]. The positive-P distribution $P(\alpha,\beta)$ is a generalization of P_G and is such that

$$\int d\alpha d\beta P(\alpha,\beta)\hat B(\alpha,\beta)\equiv|\psi\rangle\langle\psi|, \tag{10}$$

where $\hat B(\alpha,\beta)\equiv e^{(|\alpha|^2+|\beta|^2-2\beta^*\alpha)/2}|\alpha\rangle\langle\beta|$. It is possible to prove that each quantum state $|\psi\rangle$ is associated with a positive-P distribution [13]. Equation (10) is a concrete example of Eq. (7), X being the variable set $\{\alpha,\beta\}$. Note that a single mode has an infinite dimensional Hilbert space, conversely the $\{\alpha,\beta\}$ space is four-dimensional, i.e, we can represent a quantum state as a statistical ensemble on a reduced space of variables. These variables and $P(\alpha,\beta)$ are analogous to the variables $\{\vec x_i,\vec v_i\}$ in Eq. (2) and the probability distribution ρ in Eq. (3), respectively. The dimensional reduction program would look, at this stage, feasible. However, we have still to define the dynamics of X and the connection between X and the measurements.

B. Dynamics

In quantum mechanics, the state evolution of a conservative system from time t_0 to t_1 is described by a unitary operator $\hat{U}(t_1;t_0)$, i.e., we have the Markovian and deterministic evolution

$$|\psi(t_1)\rangle = \hat{U}(t_1;t_0)|\psi(t_0)\rangle. \tag{11}$$

We retain in the ontological theory the Markovian property and define a conditional probability $P(X,t_1|\bar{X},t_0)$ such that $\rho_1(X) = \int d\bar{X} P(X,t_1|\bar{X},t_0)\rho_0(\bar{X})$, where $\rho_0(X)$ and $\rho_1(X)$ are two probability distributions associated with $|\psi(t_0)\rangle$ and $|\psi(t_1)\rangle$, respectively.

We assume that every unitary evolution is physically attainable. This hypothesis rests on the fact that, in quantum computers, every unitary evolution is in principle feasible by means of a finite number of quantum gates, that have a physical implementation (see for example Chapter 4 of Ref. [15]). As a consequence, every unitary operator has to be associated with a conditional probability. As for the probability distribution, in general each unitary operator can be mapped to many different conditional probabilities which can depend on the the physical implementation of the evolution. Thus, we introduce a set $\mathcal{E}$ which contains at least an element and define the mapping

$$\hat{U} \rightarrow \{P(X|\bar{X},\hat{U},\chi), \chi \in \mathcal{E}\}. \tag{12}$$

The label $\chi \in \mathcal{E}$ identifies the context for the unitary evolution [14].

The conditional probabilities have to satisfy the following conditions:

1. For any $\hat{U}$ and $\chi \in \mathcal{E}$
$$P(X|\bar{X},\hat{U},\chi) \geq 0, \tag{13}$$
$$\int d\bar{X} P(\bar{X}|X,\hat{U},\chi) = 1. \tag{14}$$

2. For any $\hat{U}$, $\chi \in \mathcal{E}$, $|\psi\rangle$ and $\eta \in \mathcal{C}$, there exists a $\eta_1 \in \mathcal{C}$ such that
$$\rho(X|\hat{U}\psi,\eta_1) = \int d\bar{X} P(X|\bar{X},\hat{U},\chi)\rho(\bar{X}|\psi,\eta); \tag{15}$$

3. For any $\hat{U}_{1,2}$ and $\chi_{1,2} \in \mathcal{E}$, there exists a $\chi_3 \in \mathcal{E}$ such that
$$\int dX_i P(\bar{X}|X_i,\hat{U}_2,\chi_2)P(X_i|X,\hat{U}_1,\chi_1) = P(\bar{X}|X,\hat{U}_2\hat{U}_1,\chi_3). \tag{16}$$

Note that the integral symbol $\int dX$ synthetically indicates the integration and sum over the continuous and discrete variables.

The set $\{P(\bar{X}|X,\hat{U},\chi), \chi \in \mathcal{E}\}$ is an equivalence class labeled by $\hat{U}$. The set of all the equivalence classes is a group, whose identity is $\{P(\bar{X}|X,\mathbb{1},\chi), \chi \in \mathcal{E}\}$.

These are the general hypotheses for the dynamics in the ontological theory. In the case of the positive-P distribution, it is possible to define conditional probabilities which satisfy these conditions, but we will not show it.

It is important for our purposes to deduce some properties of $P(X|\bar{X},\hat{U},\chi)$ and $\rho(X|\psi,\eta)$. We introduce the following

Definition 1. We denote by $I(\psi,\eta)$ the support of the probability distribution $\rho(X|\psi,\eta)$, i.e., the smallest closed set with probability 1 and define $I(\psi)$ as follows

$$I(\psi) = \{I(\psi,\eta), \eta \in C\} \tag{17}$$

In practice, if the set of values X is countable, then $X \in I(\psi,\eta) \Leftrightarrow \rho(X|\psi,\eta) \neq 0$. For continuous spaces this is still true apart from a zero probability set. In order not to be pedantic, we assume that these negligible sets are null, i.e., we assume that our probability distributions have the same properties of discrete distributions.

Property 1. If $\bar{X} \in I(\psi)$, then the support of $P(X|\bar{X},\hat{U},\chi)$ is a subset of $I(\hat{U}\psi)$. Equivalently, if $\bar{X} \in I(\psi)$ and $P(X|\bar{X},\hat{U},\chi) \neq 0$, then $X \in I(U\psi)$.

If the X space is discrete, this property is a consequence of Eqs (5,13,15). The proof is as follows: if $\bar{X} \in I(\psi)$ then there exists an η such that $\rho(\bar{X}|\psi,\eta) \neq 0$. Thus, if X is an element of the support of $P(X|\bar{X},\hat{U},\chi)$, then exists an η_1 (second enumerated conditions) such that $\rho(X|\hat{U}\psi,\eta_1) = \sum_{\tilde{X}} P(X|\tilde{X},\hat{U},\chi)\rho(\tilde{X}|\psi,\eta) \geq P(X|\bar{X},\hat{U},\chi)\rho(\bar{X}|\psi,\eta) \neq 0$, i.e., X is an element of $I(\hat{U}\psi)$. For continuous spaces, Property 1 is true apart from unimportant sets with zero probability. As previously said, we will assume them null.

Property 1 can formulated in these terms. If $X(t)$ is the deterministic/stochastic trajectory of the ontological variable and $X(t_0) \in I(\psi)$ at time t_0, then at a subsequent time $t_1 > t_0$ $X(t_1) \in I[\hat{U}(t_1;t_0)\psi]$. In general, this property is not invariant for time inversion and $X(t_0) \notin I(\psi)$ does not implies that $X(t_1) \notin I[\hat{U}(t_1;t_0)\psi]$. The non-invariance is due to the fact that a backward Markovian process is not in general a Markovian process. Thus, the set $I[\psi(t)]$ is as a black hole, the trajectories $X(t)$ can jump into it but can not escape from it. This is illustrated in Fig. 1a. The points (I), (II) and (III) in the figure are defined in the caption.

It is reasonable to assume that this confluence of the trajectories into $I[\hat{U}(t_1;t_0)\psi]$ corresponds to a transient behavior which becomes negligible after a suitable series of unitary evolutions. This is equivalent to say that for any $|\psi'\rangle$, there exists a probability distribution $\rho(X|\psi',\eta)$ whose support contains only points of type (I) and (III) (Fig. 1b).

In the following, we assume that, after a suitable transient evolution, the variable X evolves towards a classical subspace where the time symmetry is fulfilled and consider only this subspace as the space of ontic states. Thus, we enunciate

Property 2. For any $|\psi\rangle$, $\hat{U}$ and $\chi \in \mathcal{E}$, if $\bar{X} \notin I(\psi)$ and $P(X|\bar{X},\hat{U},\chi) \neq 0$, then $X \notin I(U\psi)$.

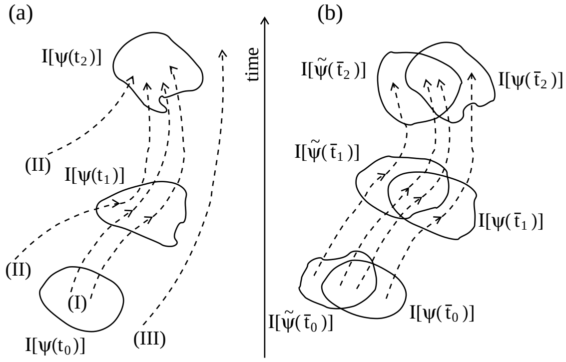

FIG. 1: Visual representation of the evolution of $I[\psi(t)]$ in the ontic state space. (a) Classical states in $I[\psi(t_0)]$ at time t_0 [labeled with (I)] can not escape from $I[\psi(t)]$ at subsequent time $t > t_0$. However, some states outside $I[\psi(t_0)]$ [labeled with (II)] can jump into $I[\psi(t)]$. Other states (III) may exist which remain always outside $I[\psi(t)]$. (b) The evolution of the support I for two different states $|\psi\rangle$ and $|\tilde{\psi}\rangle$. After a suitable transient evolution, no trajectory can escape from or to jump into $I[\psi(t)]$ and $I[\tilde{\psi}(t)]$.

Equivalently, if $X(t)$ is a deterministic/stochastic trajectory and $X(t_0) \notin I(\psi)$ at time t_0, then at a subsequent time $t_1 > t_0$ $X(t_1) \notin I[\hat{U}(t_1; t_0)\psi]$ (Fig. 1b).

It is interesting to note that the actual state of a system we seek to describe in the laboratory might not have undergone the suitable transient evolution and not be in a region with the time symmetry property. For example, this might occur for a system sufficiently isolated since the early universe. However, it important to realize that the ontic state space is fixed once and for all and it must be able to describe every potential future evolution of the system. For our purpose, it is sufficient to know that there exists a region with the time symmetry property established by Property 2. The constraint on the dimensionality that we will find for this sub-region will be valid for the whole classical space.

Properties 1-2 will be fundamental for our proof in Section V A.

C. Measurements

Let $\hat{M}$ be a Hermitian operator with eigenvectors $|k\rangle$ and eigenvalues m_k, k being an integer index. When a measurement of $\hat{M}$ is performed on the state $|\psi\rangle$, the result m_k is obtained with probability

$$p_k = |\langle k|\psi\rangle|^2 \tag{18}$$

if m_k is not degenerate.

In the ontological theory, we introduce a conditional probability $P_{\hat{M}}(k|X)$ such that [8]

$$\sum_k P_{\hat{M}}(k|X) = 1, \tag{19}$$

$$P_{\hat{M}}(k|X) \geq 0, \tag{20}$$

$$\int dX P_{\hat{M}}(k|X)\rho(X|\psi,\eta) = |\langle k|\psi\rangle|^2, \tag{21}$$

for any $\eta \in \mathcal{C}$ and $|\psi\rangle$. Equation (21) makes the classical measurement rule equivalent to the Born's rule.

When $P_{\hat{M}}(k|X)$ takes only the values 0 and 1, the ontological theory is called "dispersion-free" and the variable X determines exactly the measurement results. However, this property is not necessary for our scope and will not be assumed. It is useful to note that each Hermitian operator $\hat{M}$ may be associated with many conditional probability distributions, i.e., the measurement results may depend on the physical implementation of the measurement of $\hat{M}$. In order to account for this dependence, we have to introduce a set $\mathcal{D}$, akin to $\mathcal{C}$ and $\mathcal{E}$, and let the conditional probability depend on $\tau \in \mathcal{D}$, that is, denote the conditional probability by $P_{\hat{M}}(k|X,\tau)$ [14]. This function has to satisfy properties (19-21) for any $\tau \in \mathcal{D}$.

For our purpose, it is sufficient to consider only trace-one projectors and prove the following

Property 3. If $|\psi\rangle$ and $|\psi_\perp\rangle$ are two orthogonal states, then the supports of $\rho(X|\psi,\eta_1)$ and $\rho(X|\psi_\perp,\eta_2)$ do not contain common elements, for every η_1 and η_2, i.e., $I(\psi) \cap I(\psi_\perp) = \oslash$. Proof: Assume *ab absurdo* the opposite and let X_0 be a value such that $\rho(X_0|\psi,\eta_1)$ and $\rho(X_0|\psi_\perp,\eta_2)$ are not equal to zero. Let $P(1|X)$ be the conditional probability of obtaining 1 with the projective measurement $|\psi\rangle\langle\psi|$ and a fixed τ. By Eq. (21), we have

$$\int dX P(1|X)\rho(X|\psi,\eta_1) = 1. \tag{22}$$

Since $\rho(X|\psi,\eta_1)$ is positive and normalized to one [Eqs. (4,5)], and $\rho(X_0|\psi,\eta_1) \neq 0$, by Eq. (22) we have that $P(1|X_0) = 1$. This is obvious if the set of values X is countable. As said in Section II B, this is true also in the continuous case apart from events with zero probability which can be neglected. Similarly, we have

$$\int dX P(1|X)\rho(X|\psi_\perp,\eta_2) = 0. \tag{23}$$

Since $\rho(X_0|\psi_\perp,\eta_2) \neq 0$, we have that $P(1|X_0) = 0$, in contradiction with the previous deduction. Thus, X_0 can not be a common element of $I(\psi)$ and $I(\psi_\perp)$. In simple words, since $|\psi\rangle$ and $|\psi_\perp\rangle$ correspond to mutually exclusive events, they can not be associated with the same value of the ontological variable. This property has been used in Ref. [14] to derive a no-go theorem for noncontextual hidden variable models, and a similar property has been used in Ref. [10] to derive the "ontological excess baggage theorem".

It is interesting to note that every positive distribution introduced in quantum optics, such as the positive-P and the Husimi Q functions do not provide an ontological description of quantum mechanics, since in general their support is the whole phase space and two orthogonal quantum states can have overlapping probability distributions.

We have completed our characterization of an ontological theory of QM. In the following section, we introduce some ontological models of simple quantum systems and show that their space dimension is always equal to or larger than $2N-2$, where N is the Hilbert space dimension.

III. TWO EXAMPLES OF ONTOLOGICAL THEORIES

A. two-state quantum system

We consider the ontological model of a two-state system reported in Ref. [16]. The classical variable X is a three-dimensional unit vector, which we denote by $\vec{v}$. Let $|-1\rangle$ and $|1\rangle$ be two orthogonal states, we associate each quantum state $|\psi\rangle \equiv \psi_{-1}|-1\rangle + \psi_1|1\rangle$ with the following probability distribution in X,

$$\rho(\vec{v}|\psi) = \frac{1}{\pi}\vec{v}\cdot\vec{w}(\psi)\theta[\vec{v}\cdot\vec{w}(\psi)], \tag{24}$$

where θ is the Heaviside function and the three components of $\vec{w}(\psi)$ are

$$\begin{array}{l} w_1(\psi) \equiv \psi_{-1}^*\psi_1 + \psi_1^*\psi_{-1}, \\ w_2(\psi) \equiv -i\psi_{-1}^*\psi_1 + i\psi_1^*\psi_{-1}, \\ w_3(\psi) \equiv |\psi_{-1}|^2 - |\psi_1|^2, \end{array} \tag{25}$$

i.e., $\vec{w}(\psi)$ is the Bloch vector of $|\psi\rangle$. $\vec{w}(\psi)$ is the symmetry axis of the probability distribution, whose support is a hemisphere (the region where ρ is different from zero).

We write the Hamiltonian as

$$\hat{H} = \sum_{k=1}^{3} h_k(t)\hat{\sigma}_k, \tag{26}$$

$\hat{\sigma}_k$ and $h_k(t)$ being the Pauli matrices and three real time-dependent coefficients, respectively. The vectors $|\pm 1\rangle$ are the eigenstates of $\hat{\sigma}_3$ with eigenvalues ± 1. It is easy to prove that the equation of motion of $\vec{w}(\psi)$ is

$$\frac{dw_i}{dt} = 2\sum_{jk}\epsilon_{ijk}h_j(t)w_k, \tag{27}$$

where ϵ_{ijk} is the Levi-Civita symbol. Thus, the probability distribution evolves rigidly according to the Louville equation

$$\frac{\partial\rho}{\partial t} = -2\sum_{ijk}\epsilon_{ijk}h_j(t)v_k\frac{\partial\rho}{\partial v_i}, \tag{28}$$

which corresponds to the deterministic equation for the ontological variable $\vec{v}$

$$\frac{dv_i}{dt} = 2\sum_{jk}\epsilon_{ijk}h_j(t)v_k. \tag{29}$$

The measurement rule is as follows: the probability of an event associated with the vector $|\phi\rangle$ is

$$P(\phi|\vec{v}) \equiv \theta[\vec{w}(\phi)\cdot\vec{v}]. \tag{30}$$

It is possible to prove that

$$\int d\vec{v}P(\phi|\vec{v})\rho(\vec{v}|\psi) = |\langle\phi|\psi\rangle|^2, \tag{31}$$

the second side being the probability of the event $|\phi\rangle$ according to the Born rule. Thus, we can describe a two-state system as a classical one in a space with the dimension equal to $2N-2$, where $N=2$ is the Hilbert space dimension.

It is useful to remark that, in this model, each vector $\vec{v}$ is not associated with only one quantum state, i.e., probability distributions associated with different quantum states can overlap. In other words, if we define $\mathcal{S}(\vec{v})$ as the set of Hilbert space vectors $|\psi\rangle$ such that $\rho(\vec{v}|\psi)\neq 0$, then $\mathcal{S}(\vec{v})$ contains infinite elements. Note that this model satisfies property 3, i.e., if $|\psi\rangle$ and $|\psi_\perp\rangle$ are two orthogonal vectors and $|\psi\rangle\in\mathcal{S}(\vec{v})$, then $|\psi_\perp\rangle\notin\mathcal{S}(\vec{v})$ (This property will be further discussed in Section V A).

B. Higher dimension of the Hilbert space and Kochen-Specker theorem

Classical dispersion-free models are possible also for higher dimensional quantum systems. Here we discuss a simple example introduced in the first chapter of Ref. [6]. It is very artificial, but shows that ontological formulations of quantum mechanics are possible.

We consider a quantum system associated with an N-dimensional Hilbert space. Let $|1\rangle$, $|2\rangle$,...,$|N\rangle$ be an orthonormal basis. We associate the quantum state

$$|\psi\rangle = \sum_k \psi_k|k\rangle \tag{32}$$

with the probability distribution

$$\rho(\chi_1,....,\chi_N,\lambda|\psi) \equiv \prod_k \delta(\chi_k-\psi_k), \tag{33}$$

where the ontic state space is spanned by the N complex variables χ_k and λ, which takes values in the real interval $[0,1]$ with uniform probability. Obviously, the ontological variables χ_k evolves deterministically as ψ_k, i.e., by means of the Schrödinger equation

$$i\hbar\frac{\partial\chi_k}{\partial t} = \sum_l\langle k|\hat{H}|l\rangle\chi_l. \tag{34}$$

We can consider λ a constant of motion. At this point, we have to write a conditional probability for events. Let $|\phi(1)\rangle$,...$|\phi(N)\rangle$ be a set of orthonormal vectors associated with events. If the projective operator $\hat{P}^{(1)} =$

$|\phi(1)\rangle\langle\phi(1)|$ is measured, the probability of the event $\phi(1)$ is

$$P(1|\psi) \equiv |\langle\phi(1)|\psi\rangle|^2. \quad (35)$$

It is easy to prove that

$$P(1|\psi) = \int d^{2N}\chi \int_0^1 d\lambda P(1|\chi,\lambda)\rho(\chi,\lambda|\psi), \quad (36)$$

where

$$P(1|\chi,\lambda) \equiv \theta[|\langle\chi|\phi(1)\rangle|^2 - \lambda] \quad (37)$$

with $|\chi\rangle \equiv \sum_k \chi_k|k\rangle$. Thus, the function $P(1|\chi,\lambda)$ can be interpreted as the conditional probability of the event 1. If the projective operator $|\phi(2)\rangle\langle\phi(2)|$ is subsequently measured, the probability of the event $\phi(2)$ is

$$P(2|\psi) \equiv |\langle\phi(2)|\psi\rangle|^2. \quad (38)$$

We want to find a conditional probability $P(2|\chi,\lambda)$ such that

$$P(2|\psi) = \int d^{2N}\chi \int_0^1 d\lambda P(2|\chi,\lambda)\rho(\chi,\lambda|\psi). \quad (39)$$

Since the two events $\phi(1)$ and $\phi(2)$ are mutually exclusive, $P(1|\chi,\lambda)$ and $P(2|\chi,\lambda)$ cannot be different from zero for the same values of the conditional variables. Bearing this in mind, we put

$$P(2|\chi,\lambda) = \theta[|\langle\chi|\phi(1)\rangle|^2 + |\langle\chi|\phi(2)\rangle|^2 - \lambda] - P(1|\chi,\lambda), \quad (40)$$

i.e., $P(2|\chi,\lambda)$ is 1 for $|\langle\chi|\phi(1)\rangle|^2 < \lambda < |\langle\chi|\phi(1)\rangle|^2 + |\langle\chi|\phi(2)\rangle|^2$ and zero elsewhere. Analogous constructions can be made for the other projective measurements $|\phi(k)\rangle\langle\phi(k)|$.

It is interesting to note that the conditional probability of the event $\phi(2)$ depends by construction on $|\phi(1)\rangle$, i.e., a different choice of the first projective measurement modifies the outgoing result of the second one. This characteristic is called *contextuality* and is unavoidable when the Hilbert space dimension is higher than 2, as established by the Kochen-Specker theorem [16, 17].

We have shown that it is possible to construct an ontological theory which fulfills the three conditions established in Section II. The classical variables are $2N+1$ in number. However, since $\sum_k |\chi_k|^2 = 1$ and the global phase is unimportant, the manifold of ontic states can be reduced to $2N-1$. Another dimension can be eliminated if we give up the dispersion-free property. A very simple example of model which is not dispersion-free and satisfies our three general conditions is obtained with the following probability distribution and conditional probability for the state ψ and the event ϕ,

$$\rho(\chi_1,....,\chi_N|\psi) \equiv \prod_k \delta(\chi_k - \psi_k), \quad (41)$$

$$P(\phi|\chi) = |\langle\phi|\chi\rangle|^2. \quad (42)$$

The corresponding ontological manifold has $2N-2$ dimensions. Although this example sounds trivial, it shows that an ontological theory in a $2N-2$ dimensional space which satisfy our conditions is possible. As we will prove in Section V A, this dimensional value is also the lowest possible one.

IV. DIMENSIONAL REDUCTION OF THE ONTOLOGICAL SPACE IN A PARTICULAR CASE

In this section we discuss an example of dimensional reduction of the ontological space. This reduction is possible for particular quantum states and measurements. We will consider a bosonic mode and show that there exists a four-dimensional manifold in the Hilbert space whose elements can be represented in a two-dimensional classical space. Since this example is very well-known in literature, its discussion will be brief. For more details, see for example Ref. [4] and references cited there in.

We consider a one bosonic mode with a Hamiltonian quadratic in the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. The results can be extended to the case of a higher number of modes. The Wigner distribution of a quantum state $|\psi\rangle$ is by definition the function

$$W(\alpha) \equiv \frac{1}{\pi^2}\int \langle\psi|e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}|\psi\rangle e^{\lambda^*\alpha - \lambda\alpha^*} d^2\lambda, \quad (43)$$

where α is a complex number and the domain of integration is the complex plane.

Let $\hat{q}$ and $\hat{p}$ be two Hermitian operators such that $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$, they satisfy the canonical commutation relation $[\hat{q},\hat{p}] = i$. In the basis of the eigenvectors $|x\rangle$ of $\hat{q}$, Equation (43) becomes

$$W(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \psi(q + x/2)\psi^*(q - x/2)e^{-ixp}, \quad (44)$$

where $q + ip \equiv \alpha$ and $\psi(x) \equiv \langle x|\psi\rangle$.

The Wigner function satisfies the identity

$$\int dqdpW(q,p) = 1. \quad (45)$$

In general, it can take negative values, but for particular states it is positive in the whole phase space (q,p) and can be interpreted as a probability distribution. This is the case of the Gaussian states

$$\psi(x|q_0,p_0,a,b) \equiv \frac{1}{(\pi a)^{1/4}} e^{\frac{-(x-q_0)^2}{2a} + ip_0(x-q_0) + ib(x-q_0)^2}, \quad (46)$$

whose Wigner distribution is the two-dimensional Gaussian function

$$W(q,p|q_0,p_0,a,b) = \frac{1}{\pi} e^{-\frac{(q-q_0)^2}{a} - a[p - p_0 - 2b(q-q_0)]^2}. \quad (47)$$

q_0 and p_0 are the mean values of q and p, a and b set the squeezing and the symmetry axes of the distribution.

From Eq. (44) it easy to verify that

$$\int_{-\infty}^{\infty} dp W(q,p) = |\psi(q)|^2, \tag{48}$$

$$\int_{-\infty}^{\infty} dq W(q,p) = |\tilde{\psi}(p)|^2, \tag{49}$$

where $\tilde\psi$ is the Fourier transform of ψ. Thus, the marginal probability distributions of q and p are the probability distributions of the observables $\hat x$ and $\hat p$, respectively. In general, the probability distribution of the observables $\cos\theta\hat q + \sin\theta\hat p$, called in quantum optics *quadratures*, is the marginal probability distribution of $\cos\theta x + \sin\theta p$.

If only quadrature measurements of are considered, then the probability distribution of the outgoing values can be obtained by means of the classical probability rules of Section II C. For example, we have from Eq. (48)

$$|\psi(q)|^2 = \int d\bar q d\bar p P(q|\bar q,\bar p) W(\bar q,\bar p), \tag{50}$$

where $P(q|\bar q,\bar p) \equiv \delta(q-\bar q)$ is the conditional probability density associated with the measurement of $\hat q$. Furthermore, it is possible to prove that for any unitary evolution whose generator is quadratic in $\hat a$ and $\hat a^\dagger$, the Wigner function evolves according to the rules of Section II B. More precisely, the evolution equation of W is a Liouville equation [4].

The manifold of the Gaussian quantum states is four-dimensional, conversely the number of classical variables is two. The dimensional reduction becomes more drastic when M modes are considered. In this case, the number of classical variables grows linearly with M, conversely the dimension of the manifold of Gaussian quantum states grows quadratically. Multidimensional Wigner functions are used for example in the study of Bose-Einstein condensates [18, 19].

V. THEOREM ON THE DIMENSION OF THE ONTOLOGICAL SPACE

A. Definitions and properties

In Section II we have established three general conditions of ontological theories of quantum mechanics, let us enumerate them.

1. Let X, $\mathcal{H}$ and $\mathcal{C}$ be a set of discrete and/or continuous variables, a N-dimensional Hilbert space and a set which contains at least one element. There exists a functional $|\psi\rangle \rightarrow \{\rho(X|\psi,\eta), \eta\in C\}$ which associates each quantum state $|\psi\rangle\in\mathcal{H}$ with a set of probability distributions of the variables X.

2. Let $\mathcal{E}$ be a set with at least one element. There exists a functional $\hat U \rightarrow \{P(X|\bar X,\hat U,\chi), \chi\in\mathcal{E}\}$ which associates each unitary operator $\hat U$ of $\mathcal{H}$ with a set of conditional probabilities $P(X|\bar X,\hat U,\chi)$. These distributions satisfy the properties in Sect. II B.

3. Let $\hat P(\phi)$ be the trace-one projector $|\phi\rangle\langle\phi|$. There exists a conditional probability $P_M(\phi|X)$ such that

$$\int dX P_M(\phi|X)\rho(X|\psi,\eta) = \langle\psi|\hat P(\phi)|\psi\rangle = |\langle\phi|\psi\rangle|^2. \tag{51}$$

for every η and $|\psi\rangle$.

It is useful to introduce the following definitions:

Definition 2. We call $\mathcal{S}(X)$ the set of quantum states $|\psi\rangle$ such that $X\in I(\psi)$.

Definition 3. $\hat U\mathcal{S}(X)$ is the set of the vectors $\hat U|\psi\rangle$, with $|\psi\rangle\in\mathcal{S}(X)$. In simple words, $\hat U\mathcal{S}(X)$ is the unitary evolution of the set $\mathcal{S}(X)$.

By means of Properties 1-2, it is trivial to prove

Property 4. If $P(X|\bar X,\hat U,\chi)\neq 0$, then $\hat U\mathcal{S}(\bar X) = \mathcal{S}(X)$.

Proof. It is sufficient to prove that a state $|\psi\rangle$ is in $\mathcal{S}(\bar X)$ if and only if $\hat U|\psi\rangle\in\mathcal{S}(X)$. If $|\psi\rangle\in\mathcal{S}(\bar X)$, then $\bar X\in I(\psi)$. By Property 1, $X\in I(\hat U\psi)$, i.e., $\hat U|\psi\rangle\in\mathcal{S}(X)$. Similarly, it is proved by means of Property 2 that $\hat U|\psi\rangle\in\mathcal{S}(X)\Rightarrow|\psi\rangle\in\mathcal{S}(\bar X)$.

Property 4 can be formulated in these terms. If $X(t)$ is a deterministic/stochastic trajectory in the ontic state space and $\hat U(t_1;t_0)$ is the associated unitary evolution from time t_0 and to time t_1, then

$$\hat U(t_1;t_0)\mathcal{S}[X(t_0)] = \mathcal{S}[X(t_1)], \tag{52}$$

for any $t_{0,1}$. An illustration of this property is reported in Fig. 2, where the unitary evolution $\hat U(t_1;t_0)$ is synthetically denoted by $\hat U$. At left we have represented a trajectory in a one-dimensional classical space. At right the corresponding evolution of $\mathcal{S}$ is sketched. $\mathcal{S}[X(t_{0,1})]$ are drawn as cones with $|\Psi\rangle$ and $\hat U|\Psi\rangle$ as symmetry axis. The Hilbert space is represented as a three-dimensional Euclidean space.

Another fundamental property of $\mathcal{S}$ holds.

Property 5. The set $\mathcal{S}(X)$ can not contain every vector of the Hilbert space. This is a consequence of Property 3. If $|\psi\rangle$ is a vector of $\mathcal{S}(X)$, i.e. $X\in I(\psi)$, then any orthogonal vector of $|\psi\rangle$ is not an element of this set.

At this point, the proof of our theorem on the classical space dimension is very simple.

B. The theorem

By means of the outlined properties, we will prove the following

Theorem: The number of continuous variables in the set X can not be smaller than $2N-2$.

In the proof, we will use Properties 4 and 5, which are a synthesis of Properties 1-3.

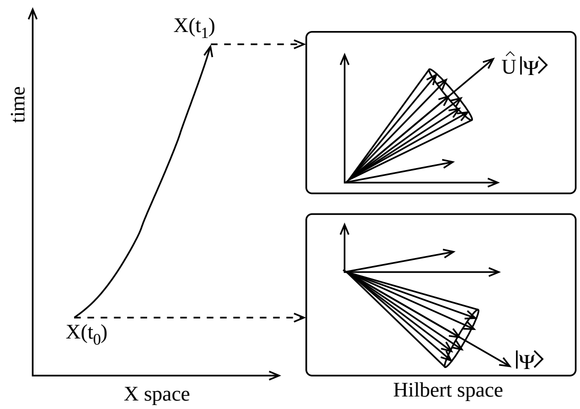

FIG. 2: Visual representation of a trajectory in a one-dimensional classical space and corresponding evolution of $\mathcal{S}$. At time t_0, the ontological variable takes the value $X(t_0)$. At the subsequent time t_1, the variable has evolved to a value $X(t_1)$ with probability $P[X(t_1)|X(t_0),\hat U,\chi]$. If this probability is finite, the set $\mathcal{S}[X(t_1)]$ of Hilbert space vectors (top right inset) is equal to the unitary evolution of $\mathcal{S}[X(t_0)]$ (bottom right inset), i.e., equal to $\hat U\mathcal{S}[X(t_0)]$. The sets $\mathcal{S}$ are represented as cones with $|\Psi\rangle$ and $\hat U|\Psi\rangle$ as symmetry axis. Note that if the Hilbert space has a dimension higher than 2, the points of the one-dimensional classical space can not map every possible orientation of $\mathcal{S}[X(t_1)]$.

Proof: Let $\bar X$ be a fixed value such that $\mathcal{S}(\bar X)$ contains at least one vector of the Hilbert space. Property 4 says that for every unitary evolution $\hat U$ there exists a value X such that

$$\hat U\mathcal{S}(\bar X)=\mathcal{S}(X), \tag{53}$$

This implies that the number of continuous ontological variables in X ($x_1,...x_w$; see Sec. II A) is at least equal to the number of parameters required to specify the orientation of any unitary evolution of $\mathcal{S}(\bar X)$ (see Fig. 2). If $\mathcal{S}(\bar X)$ would be invariant with respect to every $\hat U$, one would find 0 as lowest bound, but this is not our case, since $\mathcal{S}(\bar X)$ is not a null set and does not contains every vector (Property 5). As discussed in the following, the lowest number of required orientation parameters is $2N-2$. Thus, the theorem is proved. □

In an N-dimensional Hilbert space the orientation of $\hat U\mathcal{S}(X)$ can be specified by $N-1$ orthogonal vectors, which evolve according to $\hat U$, i.e., they are rigidly fixed in $\hat U\mathcal{S}(X)$. This representation is sufficient, but could be redundant when $\mathcal{S}$ has some symmetry. In order to clarify intuitively this point, one can consider some visual examples in the Euclidean three-dimensional space. The orientation of a sphere does not require any parameter, since it is invariant with respect to every rotation. A cylinder or a cone are invariant with respect to the rotation around their symmetry axis. Thus, in order to specify their orientation it is sufficient to give the direction of this axis, i.e., a vector. By contrast, a pyramid has no rotational symmetry and we have to rigidly fix two orthogonal axes and specify their directions. In general, for a set of elements in an N-dimensional Euclidean or Hilbert space the orientation is specified by $N-1$ orthonormal vectors when the set has no symmetry with respect to rotations, whereas no parameter is required if the set is completely symmetric. In our case, since $\mathcal{S}$ is not uniform, the number of required orientation vectors is at least one. This minimal labeling is possible when there exists one vector $|\Psi\rangle$ such that $\mathcal{S}$ is invariant with respect to every unitary evolution which leaves $|\Psi\rangle$ unchanged. In simple words, the orientation can be merely specified by one vector when there exists a symmetry "axis" and $|\Psi\rangle$ is the direction of this "axis" (see Fig. 2). Thus, the set $\hat U\mathcal{S}(\bar X)=\mathcal{S}(X)$ is identified by $\hat U|\Psi\rangle$, where $|\Psi\rangle$ is the symmetry axis of $\mathcal{S}(\bar X)$. This vector is defined by N complex numbers, but because of the normalization of the quantum states and the irrelevance of their global phase, it can be labeled by $2N-2$ real parameters, which are the minimal number of parameters required to specify the orientation of $\hat U\mathcal{S}$.

Let us consider the two-state model of Section III A as a practical illustration of the theorem. The space X is the set of three-dimensional unit vectors $\vec v$ and each quantum state is associated with the probability distribution in Eq. (24). In this case, the elements of $\mathcal{S}(\vec v)$ are the Hilbert space vectors $|\psi\rangle$ such that $|\langle\psi|\phi\rangle|^2>B\equiv 1/2$, $|\phi\rangle$ being the state whose bloch vector, defined by Eq. (25), is $\vec v$. It is clear that $|\phi\rangle$ is the symmetry axis of $\mathcal{S}(\vec v)$ and determines the orientation of the set. The dynamics of $\mathcal{S}(\vec v)$ is a mere rotation and the axis evolves according with Eq. (53). This model is dispersion-free. For more general models, it is possible to have different symmetric sets $\mathcal{S}(\vec v)$ with $0\le B<1/2$. However, B can not be larger than 1/2, because of Property 3. For $B=0$, set $\mathcal{S}(\vec v)$ contains only the vector $|\phi\rangle$ and the conditional probability defined by Eq. (30) is replaced by $P(\phi|\vec v)\equiv(1/2)[1+\vec w(\phi)\cdot\vec v]$. It is possible to have theories with an asymmetric set $\mathcal{S}$, but in this case the two dimensions of the Bloch sphere are not sufficient.

The hidden-variable model discussed by Aaronson in Ref. [20] is another practical illustration of our result. Let $|1\rangle$,...,$|N\rangle$ be a complete orthonormal basis of a N-dimensional Hilbert space. In Ref. [20], the quantum state $|\psi\rangle\equiv\alpha_1|1\rangle+...+\alpha_N|N\rangle$ is mapped to a probability distribution $\rho(n)\equiv\alpha_n$ whose domain is a space of N ontic states. The unitary evolution is mapped to a $N\times N$ stochastic matrix, which corresponds to the conditional probability P introduced in Section II B. As previous proven by Hardy [10], this mapping of quantum state does not work, since the number of ontic states has to be infinite. Furthermore, our theorem says that we need at least $2N-2$ continuous ontological variables. Aaronson makes up for such lack of a suitable number of ontic states by assuming that the stochastic matrix has to depend on the quantum state $|\psi\rangle$ (See page 4 in Ref. [20]). This corresponds to assume that $|\psi\rangle$ is an ontological variable, as in the Bohm theory. These models are typical examples of pilot-wave theories and the dimension of their ontic state space is consistent with our constraint.

Every known classical formulation of quantum mechanics satisfies our constraint on the dimension of the classical space. In section III and here, we have considered some examples, other examples are provided in Refs. [21, 22, 23, 24], where a N-dimensional Hilbert space is reduced to a classical phase space of $2N$ real variables. Our result is relevant also for quantum Monte Carlo methods. By means of them, one tries to map a quantum dynamics to a classical one with a reduced phase space dimension. We have proved that this mapping is not possible, unless some condition on the probability distributions and the conditional probability distributions is discarded. Quantum Monte Carlo methods in a reduced sampling space are introduced for example in Ref. [1]. In these methods, the kinematics and dynamics conditions in Sec. II are fulfilled, but they do not provide positive conditional probability distributions for measurements. As a consequence, they are subject to the celebrated "sign problem" and have a computational complexity which grows exponentially with the evolution time and the number of particles. This complexity is not due to the dimension of the sampling space, but to the necessity of an exponentially large number of realizations in order to reduce the statistical errors.

VI. CONCLUSIONS

The proved theorem clearly shows that, if quantum mechanics is formulated as a Markovian realistic theory, the corresponding phase space grows exponentially with the physical size of the system. This occurs for theories which are able to describe every attainable unitary evolution. The explosion of the variable number implies that an exact Monte Carlo approach, which simulates quantum processes by means of realistic Markovian chains, is in general subject to an exponential growth of numerical resources and integration time [25].

In some practical cases, polynomial algorithms could be feasible, for example when the trajectory of the quantum state is not dense in the Hilbert space or different states are indistinguishable experimentally. This is the case for example with dilute Bose-Einstein condensates in the mean field approximation. Decoherence could actually play an important role in the complexity reduction for concrete systems. Roughly speaking, decoherence is due to our inability to distinguish a pure quantum state from a mixture of other quantum states [26]. When a system is composed of many particles, such as a macroscopic gas, this inability is not merely technical, but fundamental. Thus, a large set of states could be described as statistical mixtures of a smaller set of pure states, enabling to simulate the dynamics by means of a Markovian theory in a reduced phase space, as expected in classical limit. However, it is generally surmised that systems as quantum computers require nearly unitary evolutions with negligible decoherence effects in order to execute efficiently quantum algorithms [15] and would not be suitably simulated by approximated stochastic methods which replace pure state by mixtures. These conclusions would support the conjecture that, in general, quantum algorithms cannot be efficiently simulated by classical computers in polynomial time [27]. Indeed, the actual speed-up of quantum computation requires further corroborations. For example, nobody has proven that factoring does not have a polynomial solution classically. Note that the subset of available quantum states can be parametrized with a number of variables which grows with the number of quantum gates, i.e., with the physical size. If this subset is dense in the Hilbert space or its parametrization is computational hard, then our theorem supports the conjecture on the polynomial non-computability, but these conditions cannot be assumed *a priori*. The subset parametrization is certainly possible for example in particular classes of quantum computers, as recently reported in Refs. [28, 29].

Finally, we conclude with a discussion on some possible implications of our theorem for the future study of the hidden variable theories. As said in the introduction, the main criticism against the known ontological models, such as Bohm mechanics, is indeed the exponential growth of the ontic space dimension with the physical size. We have shown rigorously that this feature is unavoidable in the framework of causal Markovian theories. Thus our result seems to put another nail in the hidden variable coffin and to imply that the realistic interpretations do not provide a practical advantage for the study of quantum systems. However, our intent is to give a constructive result and to suggest a novel direction for the investigations on the ontological theories. In order to avoid the exponential growth of the number of ontic variables, we have one possibility, to discard some hypotheses of the theorem. In our opinion, the Markovian property is the only one sacrifiable. More drastically, we could discard the causality hypothesis. It is interesting to observe that the Bell theorem and the Lorentz invariance seem to suggest the same conclusion. The Bell theorem establishes that an ontological theory of quantum mechanics can not be local and relativity implies that a non-local theory is also non-causal. We suspect that realism in quantum mechanics with non-pathological consequences for the ontic space dimension could be possible with non-causal rules for evaluating correlations among events.

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