Fundamental energy cost for quantum measurement

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Measurements and feedback are essential in the control of any device operating at the quantum scale and exploiting the features of quantum physics. As the number of quantum components grows, it becomes imperative to consider the energetic expense of such elementary operations. Here we determine fundamental energy requirements for any general quantum measurement. In particular, we compute the exact costs for projective measurements which causes much stronger constraints on error correction and control protocols than previously known. On the other hand, certain noisy measurement processes allow to extract energy from the device. Our results constitute fundamental physical limitations against which to benchmark implementations of future quantum devices as they grow in complexity.

The ability to manipulate and measure individual quantum systems[1] enables ever more powerful devices that fully exploit the laws of the quantum world. This has facilitated the development of high-precision clocks[2] and quantum simulators[3] as well as the observation of fundamental decoherence processes[4]. Quantum measurements are crucial to the operation of scalable quantum computers[5]—for their final readout but importantly also for continual protection against external noise via error correction[6,7]—and quantum computation wholly based on measurements has been proposed[8,9]. The combination of quantum measurement with feedback is an essential primitive for all these applications, as it allows future actions to depend on past measurement outcomes. Thus, with quantum devices becoming increasingly complex, more measurements have to be performed and physical requirements such as the energy supply for implementing these elementary operations must be accounted for (Fig. 1). This is in parallel to the primitive of information erasure[10], whose energetic expense will become a limiting technological factor within a few decades[11,12] as the miniaturization of computers progresses[13]. The expense is needed for initializing a computer register and therefore accumulates when using a device repeatedly[14], as is typical of measurement apparatuses. The physical ramifications of information erasure are summarized by Landauer's Principle[10], which demands $k_B T \ln 2$ of energy to be dissipated into a heat bath of temperature T for the erasure of each bit of information.

So how much energy must be expended for measurements during quantum computing and error correction, or in central control protocols such as quantum Zeno stabilization[15,16]? We derive the fundamental physical energy cost for projective measurements, which places significant energetic constraints on real-world implementations of quantum devices. In particular, much more energy than previously known must be expended to operate quantum devices robustly via active error correction schemes[6,17]. This will ultimately become a fundamental physical limitation to quantum computers, in a way similar to the Landauer limit for classical computers[11]. More drastically, the energy cost of quantum Zeno control diverges in the limit of perfect stabilization.

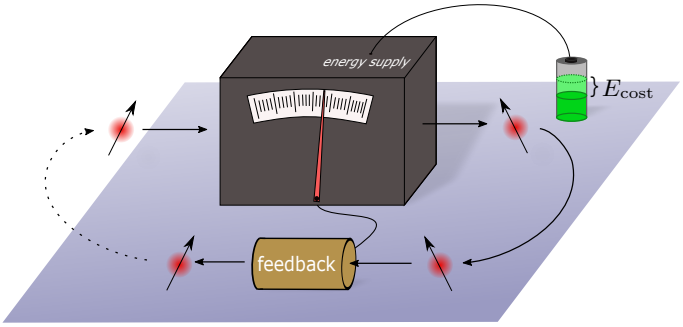

FIG. 1: **Role of quantum measurement.** Most quantum engineering protocols involve frequent measurements to maintain their stability or to control future actions. Whereas measurements are often considered as abstract primitives, we investigate their actual physical implementation and quantify the arising fundamental energy requirements E_{cost}.

We obtain these strong constraints because our purely quantum-mechanical framework goes, like refs.[18,19], beyond the usual state-transformation ideas in thermodynamics[10,14,20]. On the other hand, when the post-measurement state is irrelevant, our framework produces a simple explicit protocol that can even extract useful energy from the measurement device, all without assuming thermality or the Second Law[21–23].

Setup. Our framework allows to treat the most general quantum measurement[24], described by a collection of measurement operators $\{M_{ki}\}$, on a quantum system S. A device performing the measurement on any state ρ_S should thus obtain outcome k with probability $p_k = \sum_i \mathrm{tr}[M_{ki}\rho_S M_{ki}^\dagger]$, leaving S in the post-

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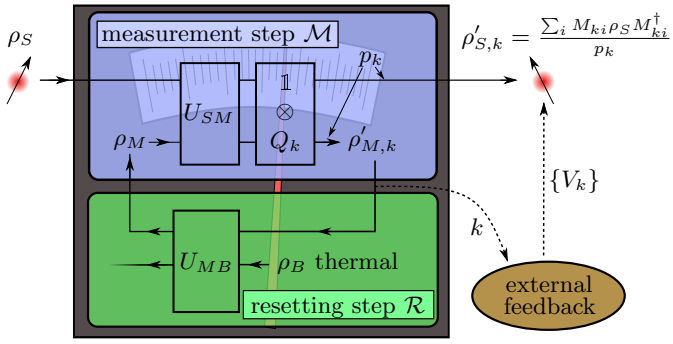

FIG. 2: **Physical implementation of a measurement.** To perform a general quantum measurement $\{M_{ki}\}$ on state ρ_S, system S is input to a measurement device (dark box) that subjects it to a measurement step $\mathcal{M}$ and leaves it in the final state $\rho'_{S,k}$, conditional on the outcome k. Subsequent feedback $\{V_k\}$ on S is possible as different outcomes belong to orthogonal projections Q_k on the memory M (Methods section). Before the device can be used again, step $\mathcal{R}$ must reset M to its proper initial state ρ_M, using a thermal resource ρ_B. Hamiltonians H_S, H_M, H_B determine the total energy cost $E_{\rm cost} = \Delta E_{\mathcal{M}} + \Delta E_{\mathcal{R}}$ required to operate the device physically. The fundamental bounds in (1) and (2) express this cost in terms of system quantities ρ_S, H_S and the measurement specification $\{M_{ki}\}$, and thus do *not* depend on microscopic details of the device (blue and green parts).

measurement state $\rho'_{S,k} = \sum_i M_{ki}\rho_S M_{ki}^\dagger/p_k$. This setup includes inefficient measurements[25], where an outcome can contain several jump operators $M_{k1}, M_{k2}, \ldots$; counterintuitively and unlike efficient measurements[21], this will allow to harvest energy from the measurement.

One may naively think that the energy cost for this measurement on state ρ_S equals the energy difference $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$ on S, where $\rho'_S = \sum_k p_k \rho'_{S,k}$ is the average post-measurement state and H_S the Hamiltonian. This however neglects the measurement device needed in a concrete physical *implementation* of the measurement $\{M_{ki}\}$, which consists of two steps (see Fig. 2): First, the *measurement step* $\mathcal{M}$ necessary to store the outcome k in a memory M for readout and feedback[26]. Secondly, in typical applications the same measurement device will be used repeatedly, so its state ρ'_M after step $\mathcal{M}$ has to be restored to the initial state ρ_M in a *resetting step* $\mathcal{R}$. Because the post-measurement state $\rho'_{S,k}$ has usually been altered by feedback following $\mathcal{M}$, step $\mathcal{R}$ cannot make use of it but must solely rely on a thermal bath B at temperature T as for usual Landauer erasure[10,27]. Importantly, such a physical implementation must yield the correct post-measurement states for *any* input state ρ_S (see Methods section for more details on the setup).

A quantum measurement $\{M_{ki}\}$ has many implementations, so we will look for the least expensive in terms of energy. This is done by expressing the energy cost only in terms of system quantities ρ_S, H_S and the measurement $\{M_{ki}\}$. We thus obtain fundamental results that are independent of the concrete physical measurement implementation[14,28].

Energy cost results. The total energy costs $E_{\rm cost} = \Delta E_{\mathcal{M}} + \Delta E_{\mathcal{R}}$ to operate the measurement device consist of those in the measurement step, $\Delta E_{\mathcal{M}}$, and resetting step, $\Delta E_{\mathcal{R}}$. Step $\mathcal{M}$ incurs an expense of $\Delta E_{\mathcal{M}} = \Delta E_S + \mathrm{tr}[H_M(\rho'_M - \rho_M)]$, where ΔE_S is already given in terms of system quantities. Step $\mathcal{R}$ requires energy $\Delta E_{\mathcal{R}} = \mathrm{tr}[(H_M + H_B)(\rho''_{MB} - \rho'_{MB})]$, where ρ'_{MB}, ρ''_{MB} denote the states of MB before and after this step, respectively.

We first state a fundamental constraint on the energy cost $E_{\rm cost}$ of any physical implementation of the measurement $\{M_{ki}\}$ on a state ρ_S (see Methods section):

$$E_{\rm cost} \;\geq\; \Delta E_S + k_B T\big[S(\rho_S) - \sum\nolimits_k p_k S(\rho'_{S,k})\big] \;, \quad (1)$$

where $S(\rho) = -\mathrm{tr}[\rho \ln \rho]$ denotes the von Neumann entropy. Thus, beyond the system energy change, the physical implementation incurs an additional expense of at least the average entropy decrease[29]. This result can also be obtained from ref.[22] through an argument based on the Second Law, and agrees with ref.[21] for efficient measurements; in this case $E_{\rm ext} = \Delta E_S - E_{\rm cost} \leq 0$, meaning that no useful energy can be extracted from the device[21]. Our framework, however, derives Eq. (1) by purely microscopic reasoning. It further shows that inefficient measurements can actually yield energy: In Fig. 3 we exhibit an implementation that extracts useful energy $E_{\rm ext} > 0$ from the device. This is surprising in light of refs.[20,21].

Our main result addresses the specific case of projective measurements[24], which constitute the textbook examples of quantum measurements and are of principal importance for applications, as exemplified below. These are efficient measurements $\{M_k\}$ with projection operators $M_k = M_k^\dagger = M_k^2$. Due to their rigid structure (Methods section), the energy costs for any implementation are determined by the exact equality

$$E_{\rm proj} \;=\; \Delta E_S + k_B T \sum\nolimits_k p_k \ln \frac{1}{p_k} \;, \quad (2)$$

where $p_k = \mathrm{tr}[\rho_S M_k]$ are the outcome probabilities and the last sum is simply their Shannon entropy $H(\{p_k\})$. This result is a significant improvement over Eq. (1) unless ρ_S was already classical, in which case both results agree. Quantum coherences thus require much more energy for measurement than previously known. Eq. (2) is stronger since, similar to single-shot treatments of general processes[18], it goes beyond the traditional state-transformation ideas in thermodynamics[19]. Interestingly, however, our result disagrees with the single-shot results in the asymptotic limit[18], because the latter does not require a faithful implementation on the *full* state space and therefore demands less energy (Methods section).

We now employ our main result Eq. (2) to determine the energetic cost of important quantum protocols, illustrating its strength in applications.

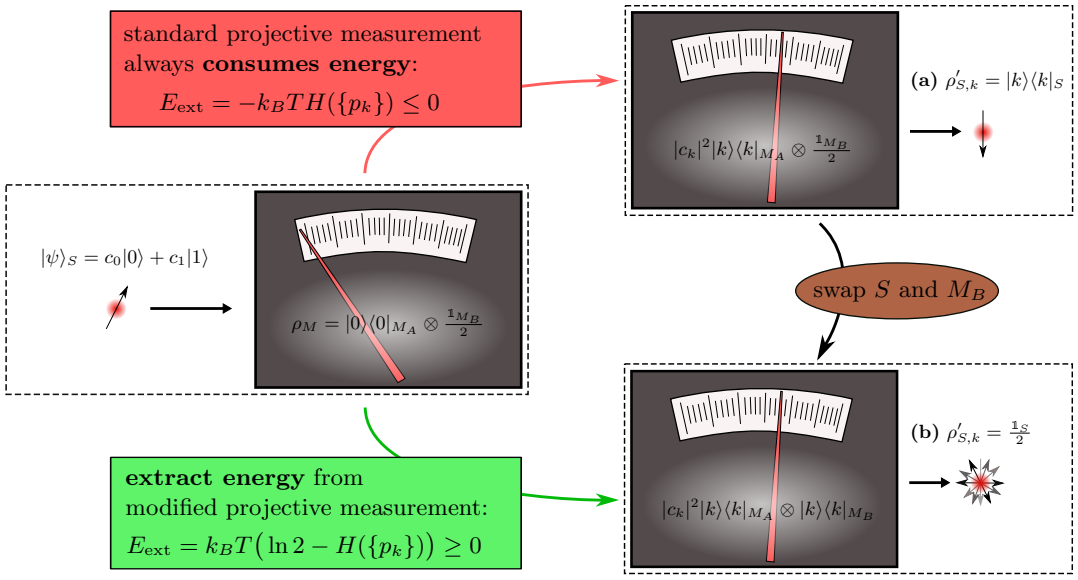

FIG. 3: **Extracting energy from measurement.** A pure qubit $\rho_S = |\psi\rangle\langle\psi|_S$ is measured in the spin-z basis by a device with bipartite memory $M = M_A M_B$ and initial state $\rho_M = |0\rangle\langle 0|_{M_A} \otimes \mathbb{1}_{M_B}/2$, in one of two different ways. Either measurement implementation yields the same outcome distribution $\{p_k\}$ and enables feedback via $Q_k = |k\rangle\langle k|_{M_A} \otimes \mathbb{1}_{M_B}$ but counterintuitively, the inefficient one allows to extract useful energy E_{ext} from the device, in contrast to previous results[20,21]. **(a)** The measurement is projective with $M_k = |k\rangle\langle k|$. Operating such a device can never yield energy, $E_{\text{ext}} \leq 0$ in accordance with Eq. (2). **(b)** The measurement is given by $M_{ki} = |i\rangle\langle k|/\sqrt{2}$ and thus is inefficient. Intuitively, the device implements the same unitary interaction $U_{SM_AM_B}$ as in (a), but with an additional swap of the systems S and M_B before the projections $\{Q_k\}$. This modified projective measurement always outputs a fully mixed state $\rho'_{S,k} = \mathbb{1}_S/2$ and yields energy $E_{\text{ext}} \geq 0$ (supplementary information).

Applications. As a first application, we study quantum Zeno stabilization[15], a paradigmatic quantum control protocol[16]. The task is to stabilize a qubit in a pure state $|0\rangle$ against its free Hamiltonian time evolution when $|0\rangle$ is not an eigenstate, such as for $H_S = E\sigma_X$ with the Pauli operator σ_X and energies $\pm E$. The protocol applies the projective measurement $\{M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|\}$, with M_0 the projector onto the desired state, at N regular time intervals $\delta t = t/N$ over the time span t whilst the disturbing Hamiltonian H_S is acting. The measurement will restore the system at each time step to either the state $|0\rangle$ or the undesired $|1\rangle$, hence $\Delta E_S = 0$. The n-th step returns the wrong state $|1\rangle$ with probability $\varepsilon_n \simeq n(E\delta t/\hbar)^2 = (Et/\hbar)^2 n/N^2$, so that the final state fidelity $F = 1 - \varepsilon_N$ becomes arbitrarily good when choosing the total number N of steps big enough. The energy expense of the whole protocol consists of the costs $E^{(n)}_{\text{proj}} \simeq k_B T \varepsilon_n \ln 1/\varepsilon_n$ necessary for all Zeno measurements $n = 1, \ldots, N$, where we have evaluated our sharp main result Eq. (2) to leading order. The total energy required to achieve high target fidelity F is then (supplementary information)

$$E_{\text{Zeno}} \simeq k_B T \frac{(Et/\hbar)^2}{2} \ln \frac{1}{1-F}\,. \tag{3}$$

This expense diverges as F approaches 1, so any restriction on the energy available for the stabilization scheme directly limits the achievable accuracy. Such drastic energy demands apply to Zeno schemes for dragging or holonomic computation[16,30] as well.

Measurement and feedback are essential also for stabilizer quantum error correction (QEC) schemes, which allow quantum computations to reach capabilities beyond classical computers even in the presence of noise[5,6]. After encoding the *logical qubits* $\mathcal{L}$ redundantly into a QEC code $\mathcal{C}$, which is simply a physical system subject to noise, the heart of QEC consists in performing repeated measurements of the *error syndrome* s on $\mathcal{C}$ followed by suitable feedback operations V_s. When these control operations are performed frequently enough, reliable computation is possible on noisy hardware by the threshold theorem[6]. Energetic considerations are paramount in this context since syndrome measurements with feedback must be performed many times and on many qubits for a scalable setup. As a paradigmatic example we examine the *5-qubit code*[24,31] $\mathcal{C}_5$, which encodes a single logical qubit $|\psi\rangle \in \mathbb{C}^2_{\mathcal{L}} \subset \mathcal{C}_5$ and whose syndrome measurement $\{P_s\}_{s=0}^{15}$ consists of two-dimensional projectors. We take each of the five physical qubits to be subject to excitation loss by amplitude damping noise[24] $\mathcal{N}_\gamma$ at strength $\gamma \in [0, 1]$.

The costs for each QEC step come from measuring $\{P_s\}$ on the noisy state $\rho_{S,\gamma} = \mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle\langle\psi|)$. According to our main result Eq. (2), these costs can depend

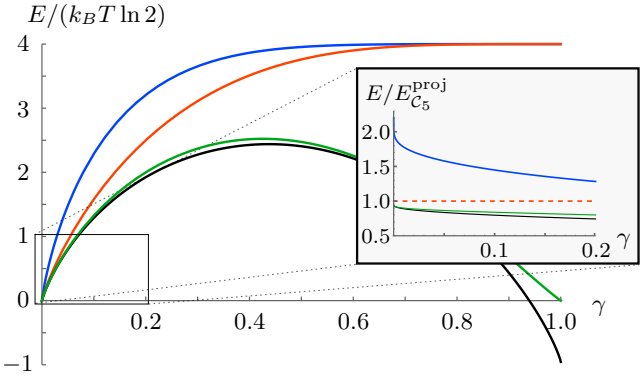

FIG. 4: **Energy cost of quantum error correction.** For the 5-qubit code under amplitude damping noise, the costs $E^{\mathrm{proj}}_{\mathcal{C}_5}$ from our exact result Eq. (2) for the syndrome measurement increase with the noise level γ *(red curve)*. These costs significantly surpass the best previous predictions, which are the lower bounds from Eq. (1) due to refs.[21,22] *(green)* and the naive Landauer bound[10,17] $E^{\mathrm{Lan}}_{\mathcal{C}_5}$ *(black)*. For an experimentally easier syndrome measurement by four separate stabilizers, Eq. (2) demands even larger costs $E^{\mathrm{sep}}_{\mathcal{C}_5}$ *(blue)*.

strongly on the system Hamiltonian H_S through the term ΔE_S, which is common to all bounds discussed in the following. For the sake of comparison we therefore neglect this contribution henceforth (see Fig. 4), effectively contrasting the additional energy costs caused by the physical implementation. The exact additional energy expense $E^{\mathrm{proj}}_{\mathcal{C}_5} = k_BTH(\{p_s\}_{s=0}^{15})$ from our Eq. (2) is shown in Fig. 4 (red curve), increasing from 0 in the noiseless case to $4k_BT\ln 2$ as $\gamma \to 1$. Much more energy must therefore be expended than the best previously known bound Eq. (1) predicts[21,22] (green curve), which for example vanishes at $\gamma = 1$. A naive application of Landauer's bound[10,17] to the code system $S = \mathcal{C}_5$ itself would yield an even weaker lower bound $E^{\mathrm{Lan}}_{\mathcal{C}_5} = k_BT[S(\rho_{S,\gamma}) - S(\rho'_{S,\gamma})] \leq E^{\mathrm{proj}}_{\mathcal{C}_5}$, negative at large noise γ (black curve). The comparison illustrates the strength of our main result Eq. (2) obtained by an exact treatment of the measurement device. Even in the sub-threshold regime $\gamma \lesssim 0.05$ where effectively noiseless quantum computing is possible[6] the improvement amounts to 15%. Practical QEC will exploit the stabilizer structure[24] to obtain syndrome $s \equiv (s^1, s^2, s^3, s^4)$ by four commuting measurements, each with two outcomes $s^j = \pm 1$ only. When these four devices operate independently the costs total $E^{\mathrm{sep}}_{\mathcal{C}_5} = k_BT\sum_j H(\{p^{(j)}_{s^j}\}_{s^j}) \geq E^{\mathrm{proj}}_{\mathcal{C}_5}$ by Eq. (2) (blue curve), showing that additional expenses can arise in simpler measurement realizations that disregard correlations (supplementary information).

Discussion. Our general result Eq. (2) determines the exact energetic costs for projective quantum measurements, imposing significant limitations on the performance of quantum error correction and control. Because it surpasses existing lower bounds in the truly quantum regime, our result also establishes a remarkable link between the viability of quantum technologies and the existence of uncertainty relations: Whenever incompatible measurements have to be performed, as in quantum state tomography[32] or quantum Monte Carlo sampling[33], entropic uncertainty relations[34] yield strictly positive bounds on the entropic term from Eq. (2), independent of the input state. Physical energy constraints together with uncertainty relations therefore place fundamental limitations on tomographic accuracy or sample quality.

Our energy results are in fact statements about thermodynamic work[35] as we accounted for all energetic contributions while employing unitary actions. It is then surprising that, in contrast to previous findings[21] the implementation in Fig. 3 can extract useful *work* from the measurement device, while still respecting the Second Law of Thermodynamics (supplementary information). Our study paves the way for investigations into the energy costs of further elementary operations[19] in the quantum sciences or engineering. It would be particularly interesting to extend the strength of our implementation requirement to the single-shot approach[18] and quantify the arising energy fluctuations[23].

Methods

Setup. To function as a memory, the Hilbert space $\mathcal{H}_M = \bigoplus_k \mathcal{H}_k$ of M must be composed of subspaces $\mathcal{H}_k$ with orthogonal projections Q_k corresponding to the classical measurement outcomes[36]. The measurement step $\mathcal{M}$ is then fully microscopically described via its physical implementation $(\rho_M, U_{SM}, \{Q_k\})$, where U_{SM} is a unitary interaction between S and M (Fig. 2) satisfying the following *implementation requirement*:

$$\mathrm{tr}_M[(\mathbb{1}\otimes Q_k)U_{SM}(\rho_S\otimes\rho_M)U^\dagger_{SM}] = \sum\nolimits_i M_{ki}\rho_S M^\dagger_{ki} \quad \forall\rho_S\forall k.$$

Such an implementation gives, for *any* initial ρ_S, the correct outcome probabilites p_k and post-measurement states $\rho'_{S,k}$ as required by the measurement specification $\{M_{ki}\}$. Unitary feedback operations $\{V_k\}$ on S can then be performed without disturbing M by acting via the overall unitary interaction $\sum_k V_k\otimes Q_k$ on the full state after measurement $\rho'_{SM} = \sum_k(\mathbb{1}\otimes Q_k)U_{SM}(\rho_S\otimes\rho_M)U^\dagger_{SM}(\mathbb{1}\otimes Q_k)$.

The projections $\{Q_k\}$ effect a dephasing operation which, unlike the measurement itself, does not return an outcome k. As shown in supplemental material, this dephasing operation can be implemented by a physical unitary operation at zero energy cost but not less, hence does not contribute to $\Delta E_{\mathcal{M}}$.

The resetting step $\mathcal{R}$ leaves S untouched, but should restore the memory to its initial state ρ_M. This is impossible by unitary evolutions on M alone because it is in general necessary to change the eigenvalues. Hence, we supply the thermal state $\rho_B = e^{-H_B/k_BT}/\mathrm{tr}[e^{-H_B/k_BT}]$ of a bath B at some temperature T. Step $\mathcal{R}$ proceeds then by a unitary U_{MB}[10,27],

$$\mathrm{tr}_B[U_{MB}(\rho'_M\otimes\rho_B)U^\dagger_{MB}] = \rho_M\,,$$

where $\rho'_M = \mathrm{tr}_S[\rho'_{SM}]$ is the marginal memory state after $\mathcal{M}$ (Fig. 2). Thermal states like ρ_B are free resources at ambient temperature T, but we must keep track of the energy expended during step $\mathcal{R}$, just as for step $\mathcal{M}$. By Landauer's Principle[10,27] the cost of the resetting step $\mathcal{R}$ satisfies $\Delta E_{\mathcal{R}} \geq -\Delta F_M$, where $\Delta F_M = F_M(\rho'_M) - F_M(\rho_M)$ is the change in free energy $F_M(\rho) = \mathrm{tr}[\rho H_M] - k_BTS(\rho)$. Moreover,

one can actually engineer the bath Hamiltonian H_B and interaction U_{MB} such that equality holds[27], i.e. $\Delta E_{\mathcal{R}} = -\Delta F_M$ (supplementary information); we therefore assume this in the main text, noting that even otherwise all our results remain valid lower bounds.

Proof of Eq. (1). We first derive the bound $\Delta E_{\mathcal{M}} \geq \Delta E_S + k_B T\big[S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\big] + \Delta F_M$ on the energy for the measurement step (supplementary information). Unlike prior work[21,22], our statement holds also for inefficient measurements and without assuming thermality of ρ_S. Taking the sum $E_{\text{cost}} = \Delta E_{\mathcal{M}} + \Delta E_{\mathcal{R}}$ yields the bound (1). Our derivation shows E_{cost} to be independent of the memory Hamiltonian H_M, as the terms $\pm\Delta F_M$ cancel in the sum; a similar observation for the special case of efficient measurements has been made in refs.[21,36].

Proof of Eq. (2). Using dilation theory[24], the requirement to perform a projective measurement $\{M_k\}$ correctly on *any* input ρ_S, fixes its implementations so rigidly that the memory's entropy increase is exactly given by $S(\rho'_M) - S(\rho_M) = H(\{p_k\})$. We show this in Supplementary Information along with the general statement[14,28] $\Delta E_{\mathcal{M}} + \Delta E_{\mathcal{R}} = \Delta E_S + k_B T[S(\rho'_M) - S(\rho_M)]$, which gives Eq. (2). This surpasses the bound in (1) by $k_B T[S(\sum_k M_k \rho_S M_k) - S(\rho_S)] \geq 0$.

To see that this implementation requirement is essential, consider as an example a measurement of the maximally coherent state $|\psi_S\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ in the computational basis, requiring energy $E_{\text{proj}} = \Delta E_S + k_B T \ln 2$ by Eq. (2). In contrast, less energy is needed if one demands the device to work only on the support subspace of $\rho_S = |\psi_S\rangle\langle\psi_S|$, in which case the minimal cost is $\Delta E_S + k_B T[S(E'|S'M') + S(M')] = \Delta E_S$ for both the single-shot and asymptotic regimes, as shown in ref.[18] (supplementary information). Note here that we always require the resetting step $\mathcal{R}$ not to make use of the system state ρ'_S, which is not available in feedback protocols such as quantum error correction (see main text); otherwise, the erasure term in the last result[18] would be $k_B T S(M'|S')$ instead of $k_B T S(M')$ and the measurement cost accordingly lower.

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Supplementary Material

Contents

Appendix A: The measurement model

A quantum measurement on a system S with Hilbert space $\mathcal{H}_S$ is mathematically described by a *quantum instrument*, i.e. a set of completely positive maps $\{T_k\}_{k=1,\dots,K}$ on $\mathcal{B}(\mathcal{H}_S)$ satisfying $\sum_k T_k^*(\mathbb{1}_S) = \mathbb{1}_S$, where k corresponds to the measurement outcome and T_k^* denotes the adjoint of T_k. The action of the map T_k on a state $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ can always be written in terms of Kraus operators M_{ki} via $T_k(\rho_S) = \sum_{i=1}^{I(k)} M_{ki}\rho_S M_{ki}^\dagger$, where $I(k)$ is the Kraus rank of T_k. A measurement is called *efficient* or *pure* if $I(k) = 1$ for all k. A measurement that is not efficient is called *inefficient*[22,S1].

A quantum instrument characterises both the probability $p_k = \mathrm{tr}[T_k(\rho_S)]$ to obtain outcome k and the corresponding post-measurement state $\rho'_{S,k} = T_k(\rho_S)/p_k$. In contrast, if the post-measurement state can be disregarded and only the outcome probabilities are of interest, it is sufficient to consider a POVM (positive operator valued measure) defined by positive operators $\{E_k\}_{k=1,\dots,K}$ satisfying $\sum_k E_k = \mathbb{1}_S$. The probability to obtain outcome k is then given by $p_k = \mathrm{tr}[E_k\rho_S]$. Any quantum instrument $\{T_k\}_k$ determines a POVM by $E_k = T_k^\dagger(\mathbb{1}_S) = \sum_i M_{ki}^\dagger M_{ki}$. In the following we will always consider quantum instruments in order to be able to describe applications such as quantum error correction, where the post-measurement state cannot be disregarded. Only in the energy extraction example (Appendix F) we employ the idea of POVMs to investigate the implications of disregarding the post-measurement state.

While a quantum instrument accurately describes the measurement as an abstract process on the measured system, we are considering physical *implementations* of an instrument that incorporate all relevant systems that are involved in the measurement process. In particular, the measurement outcome k has to be stored in degrees of freedom of a physical system M. We model this register by a quantum system with Hilbert space $\mathcal{H}_M = \bigoplus_{k=1}^K \mathcal{H}_k$ and Hamiltonian $H_M = \bigoplus_{k=1}^K H_k$, which naturally captures all the important properties one generally demands from a classical memory[36]. We consider a state $\rho'_{M,k} \in \mathcal{H}_M$ to store the measurement outcome k if it has support only on the subspace $\mathcal{H}_k$ corresponding to k. In this case, projection operators $\{Q_k\}_{k=1,\dots,K}$ which project onto the respective

subspaces $\mathcal{H}_k$, i.e. satisfying $\sum_k Q_k = \mathbb{1}_M$ with $Q_k^2 = Q_k = Q_k^\dagger$ for all k and $Q_k\psi_k = \psi_k$ for all $\psi_k \in \mathcal{H}_k$, can be applied to read out the measurement outcome from the register.

More formally, an implementation of a quantum measurement is a tuple $(\rho_M, U_{SM}, \{Q_k\})$ determining the initial state ρ_M of the memory register, the unitary dynamics U_{SM} that describes the interaction between measured system and register, and the projections $\{Q_k\}$ on M with which the outcome k can be read out from the register after measurement. To any such tuple $(\rho_M, U_{SM}, \{Q_k\})$ we associate a *measurement step* $\mathcal{M}$, i.e. the channel that takes as input an arbitrary initial state ρ_S of S and outputs the post-measurement state

$$\rho'_{SM,k} = (\mathbb{1} \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M) U_{SM}^\dagger (\mathbb{1} \otimes Q_k)/p_k$$

on S and M with probability

$$p_k = \mathrm{tr}[(\mathbb{1} \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M) U_{SM}^\dagger] \ ,$$

for each $k = 1, ..., K$. We say that a tuple $(\rho_M, U_{SM}, \{Q_k\})$ is *an implementation of a given measurement* $\{T_k\}_k$ if the associated measurement step outputs the correct post-measurement states on the measured system, $\mathrm{tr}_M[\rho'_{SM,k}] = \rho'_{S,k} = T_k(\rho_S)/p_k$, with correct probability $p_k = \mathrm{tr}[T_k(\rho_S)]$ *for all* possible input states ρ_S. In the main text we refer to this property as the *implementation requirement*. The measurement step $\mathcal{M}$ therefore outputs the state

$$\rho'_{SM} = \sum_k p_k \rho'_{SM,k} = \sum_k (\mathbb{1} \otimes Q_k) U_{SM}(\rho_S \otimes \rho_M) U_{SM}^\dagger (\mathbb{1} \otimes Q_k) \tag{A1}$$

on S and M, which correctly stores the outcome k on M, since $\rho'_{M,k} \equiv \mathrm{tr}_S[\rho'_{SM,k}]$ has by construction only support on $\mathcal{H}_k$. Note that the projections Q_k in (A1) effect a dephasing operation on the memory M; this is different from a full projective measurement on M since the values k are not stored in another register, but rather summed over. We investigate this dephasing operation in more detail in Appendix K.

Note that for any instrument $\{T_k\}_k$ there exists an implementation $(\rho_M, U_{SM}, \{Q_k\})$. Conversely, any $(\rho_M, U_{SM}, \{Q_k\})$ is an implementation of some instrument $\{T_k\}_k$. In this sense the above operational measurement model does not place any restrictions on the set of measurements described.

After the measurement, the final state $\rho'_M = \sum_k p_k \rho'_{M,k}$ of the register stores the information of the measurement outcome k. This information has to be erased by resetting the register to its initial state ρ_M before the same implementation of the measurement can be used another time. This process is called the *resetting step* $\mathcal{R}$ and employs an additional quantum system, called thermal bath B, with Hamiltonian H_B initially in a thermal state

$$\rho'_B = \exp(-\beta H_B)/Z_B$$

at inverse temperature $\beta = \frac{1}{k_B T}$, where $Z_B = \mathrm{tr}[\exp(-\beta H_B)]$ is the partition function. To achieve erasure, the register unitarily interacts with the thermal bath such that its state ρ'_M after $\mathcal{M}$ evolves back to the initial state

$$\mathrm{tr}_B[U_{MB}(\rho'_M \otimes \rho'_B) U_{MB}^\dagger] = \rho_M \ . \tag{A2}$$

This process is typically known as Landauer erasure[27]. Note that this process demands additional resources: Thermal states are needed since unitary dynamics on M alone cannot alter the rank or spectrum of the state. In this framework we consider thermal states a free resource as they can easily be obtained by weakly coupling quantum systems to thermal baths at the desired ambient temperature T. Still, the energy cost of the resetting step, specifically to implement the unitary U_{MB}, needs to be accounted for. The overall energy expense needed to run the measurement device is therefore the sum of the cost of the measurement step $\mathcal{M}$ and the cost of the resetting step $\mathcal{R}$.

Appendix B: Relating the cost $\Delta E_{\mathcal{M}}$ to operational quantities

Here we prove that the energy cost of implementing the measurement step $\mathcal{M}$ (see Eq. (A1)),

$$\Delta E_{\mathcal{M}} = \Delta E_S + \mathrm{tr}[H_M(\rho'_M - \rho_M)]$$

with $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$, splits into a sum of operational quantities. More concretely, we denote by $\Delta S = S(\rho_S) - \sum_k p_k S(\rho'_{S,k})$ the average change in state information about the system where $S(\rho) = -\mathrm{tr}[\rho \ln \rho]$ is the von Neumann entropy. Moreover, $\Delta F_M = F(\rho'_M) - F(\rho_M)$ denotes the difference between free energies $F(\rho) = \mathrm{tr}[\rho H] - S(\rho)/\beta$ of the memory before and after the measurement and $\mathcal{I} = \sum_k p_k I(S:M|k)$ is the average amount of correlations built

up between S and M as measured by the mutual information $I(S:M|k) := S(\rho'_{S,k}) + S(\rho'_{M,k}) - S(\rho'_{SM,k})$. Finally, we denote by $\Delta_Q = S(\rho'_{SM}) - S(\rho_{SM})$, where $\rho_{SM} = \rho_S \otimes \rho_M$, the total entropy increase during the measurement step induced by the projections $\{Q_k\}$.

Using this notation we show the following theorem:

Theorem 1. *Let $\mathcal{H}_S$ and $\mathcal{H}_M$ be finite-dimensional Hilbert spaces. Let $\rho_{SM} = \rho_S \otimes \rho_M$ be a quantum state on $\mathcal{H}_S \otimes \mathcal{H}_M$ and let H_S and H_M be a Hamiltonians on $\mathcal{H}_S$ and $\mathcal{H}_M$, respectively. For $\rho'_M = \mathrm{tr}_S[\rho'_{SM}]$, where ρ'_{SM} is the state after a measurement step $\mathcal{M}$ as in Eq.* (A1)*, the energy cost $\Delta E_{\mathcal{M}} = \Delta E_S + \mathrm{tr}[H_M(\rho'_M - \rho_M)]$ satisfies*

$$\beta\Delta E_{\mathcal{M}} = \beta\Delta E_S + \Delta S + \beta\Delta F_M + \mathcal{I} + \Delta_Q \ . \tag{B1}$$

Proof. Note that the post-measurement states $\rho'_{M,k}$ (and hence, also the states $\rho'_{SM,k}$) are mutually orthogonal due to the projection operators $\{Q_k\}$. Denoting the Shannon entropy of the probability distribution p_k by $H(\{p_k\}) = -\sum_k p_k \ln p_k$, we then find that the total entropy increase is

$$\begin{aligned}
\Delta_Q &= S(\rho'_{SM}) - S(\rho_{SM}) \\
&= H(\{p_k\}) + \sum_k p_k S(\rho'_{SM,k}) - \big(S(\rho_S) + S(\rho_M)\big) \\
&= H(\{p_k\}) + \sum_k p_k \left(S(\rho'_{S,k}) + S(\rho'_{M,k}) - I(S:M|k)\right) - \big(S(\rho_S) + S(\rho_M)\big) \\
&= S(\rho'_M) - S(\rho_M) + \sum_k p_k S(\rho'_{S,k}) - S(\rho_S) - \sum_k p_k I(S:M|k) \\
&= S(\rho'_M) - S(\rho_M) - \Delta S - \mathcal{I} \ ,
\end{aligned} \tag{B2}$$

where we used the additivity of the von Neumann entropy under tensor products, i.e. $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$ for all states ρ and σ.

The energy cost of the measurement step is therefore given by, using Eq. (B2) in the last step (note, the dephasing operation in Eq. (A1) can be performed at zero energy cost, but not less; see Appendix K):

$$\begin{aligned}
\beta\Delta E_{\mathcal{M}} &= \beta\Delta E_S + \beta\mathrm{tr}[H_M(\rho'_M - \rho_M)] \\
&= \beta\Delta E_S + \beta F(\rho'_M) + S(\rho'_M) - (\beta F(\rho_M) + S(\rho_M)) \\
&= \beta\Delta E_S + \beta\Delta F_M + \Delta S + \mathcal{I} + \Delta_Q \ .
\end{aligned}$$

□

Note that Eq. (B1) is an exact equality, but contains quantities such as $\mathcal{I}$ and Δ_Q that are often hard to control as they require precise knowledge over the internal state ρ_M of the measurement device. However, both $\mathcal{I}$ and Δ_Q are non-negative: $\mathcal{I}$ inherits this property from the non-negativity of the mutual information $I(S:M|k)$, whereas Δ_Q is non-negative because the measurement step $\mathcal{M}$ corresponds to a unital measurement channel[24] (see Eq. (A1)). Hence Eq. (B1) immediately implies the following inequality from the main text,

$$\Delta E_{\mathcal{M}} \geq \Delta E_S + k_B T\Big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big) + \Delta F_M \ .$$

Appendix C: Computing the cost $\Delta E_{\mathcal{R}}$ of the resetting step

Similar to the previous section, the cost $\Delta E_{\mathcal{R}}$ of the resetting step $\mathcal{R}$, Eq. (A2), can be expressed as a sum of operational quantities.

We employ the same notation as above. Additionally, we denote by $\Delta F_B = F(\rho''_B) - F(\rho'_B)$ the free energy increase in the thermal bath B with $\rho''_B = \mathrm{tr}_M[\rho''_{MB}]$ where $\rho''_{MB} = U_{MB}(\rho'_M \otimes \rho'_B)U^\dagger_{MB}$ is the joint state of MB after the resetting step. We also introduce the mutual information term $\mathcal{I}_{MB} = I(M:B)_{\rho''_{MB}}$ which measures the amount of correlations built up between M and B during step $\mathcal{R}$.

Theorem 2. *Let $\mathcal{H}_M$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Let $\rho'_{MB} = \rho'_M \otimes \rho'_B$ be a quantum state on $\mathcal{H}_M \otimes \mathcal{H}_B$ with $\rho'_B = \mathrm{e}^{-\beta H_B}/\mathrm{tr}[\mathrm{e}^{-\beta H_B}]$ thermal and H_M and H_B be Hamiltonians on $\mathcal{H}_M$ and $\mathcal{H}_B$, respectively. Consider the resetting step $\mathcal{R}$ as in Eq.* (A2) *and denote by ρ''_{MB} the final state of the process. Then the energy cost $\Delta E_{\mathcal{R}} = \mathrm{tr}[H_{MB}(\rho''_{MB} - \rho'_{MB})]$ of the resetting step satisfies*

$$\beta\Delta E_{\mathcal{R}} = -\beta\Delta F_M + \beta\Delta F_B + \mathcal{I}_{MB} \ . \tag{C1}$$

Moreover, ΔF_B and $\mathcal{I}_{MB}$ are both non-negative, such that

$$\Delta E_{\mathcal{R}} \geq -\Delta F_M \ . \tag{C2}$$

Proof. Note that the differing signs of the free energy terms appearing in Eq. (C1) are just due to our notation, where ΔF_M is defined as for the measurement step by $\Delta F_M = F(\rho'_M) - F(\rho_M) = -(F(\rho''_M) - F(\rho'_M))$, where $\rho''_M = \mathrm{tr}_B[\rho''_{MB}] = \rho_M$, whereas $\Delta F_B = F(\rho''_B) - F(\rho'_B)$. To show Eq. (C1) we compute

$$\begin{aligned}
\Delta E_{\mathcal{R}} &= \mathrm{tr}[H_{MB}(\rho''_{MB} - \rho'_{MB})] \\
&= \mathrm{tr}[H_{MB}(\rho''_{MB} - \rho'_{MB})] - \frac{1}{\beta}S(\rho''_{MB}) + \frac{1}{\beta}S(\rho'_{MB}) \\
&= \mathrm{tr}[H_{MB}(\rho''_{MB} - \rho'_{MB})] - \frac{1}{\beta}[S(\rho_M) + S(\rho''_B) - I(M:B)_{\rho''_{MB}} - S(\rho'_M) - S(\rho'_B)] \\
&= F(\rho_M) + F(\rho''_B) - F(\rho'_M) - F(\rho'_B) + \frac{1}{\beta}\mathcal{I}_{MB} \\
&= -\Delta F_M + \Delta F_B + \frac{1}{\beta}\mathcal{I}_{MB} \ ,
\end{aligned} \tag{C3}$$

where we used in Eq. (C3) that the unitary resetting step, Eq. (A2), does not change the entropies, $S(\rho''_{MB}) = S(\rho'_{MB})$.

To show the non-negativity of ΔF_B we first express the free energy $F(\rho)$ of any quantum state ρ w.r.t. a Hamiltonian H in terms of the relative entropy $D(\rho||\rho_{\mathrm{can}})$, where $\rho_{\mathrm{can}} = \mathrm{e}^{-\beta H}/Z$ with $Z = \mathrm{tr}[\mathrm{e}^{-\beta H}]$ is a thermal state w.r.t. the same Hamiltonian H. We have

$$\begin{aligned}
F(\rho) &= \mathrm{tr}[H\rho] - \frac{1}{\beta}S(\rho) \\
&= \mathrm{tr}[H\rho] + \frac{1}{\beta}D(\rho||\rho_{\mathrm{can}}) + \frac{1}{\beta}\mathrm{tr}[\rho \ln \rho_{\mathrm{can}}] \\
&= \mathrm{tr}[H\rho] + \frac{1}{\beta}D(\rho||\rho_{\mathrm{can}}) + \frac{1}{\beta}\mathrm{tr}[\rho(-\beta H - \ln Z)] \\
&= -\frac{1}{\beta}\ln Z + \frac{1}{\beta}D(\rho||\rho_{\mathrm{can}}) \ .
\end{aligned}$$

Hence, we find that the difference of free energies in the thermal bath,

$$\begin{aligned}
\Delta F_B &= F(\rho''_B) - F(\rho'_B) \\
&= -\frac{1}{\beta}\ln Z + \frac{1}{\beta}D(\rho''_B||\rho'_B) + \frac{1}{\beta}\ln Z - \frac{1}{\beta}D(\rho'_B||\rho'_B) \\
&= \frac{1}{\beta}D(\rho''_B||\rho'_B) \ ,
\end{aligned}$$

is, by Klein's inequality[24], indeed non-negative. □

It has been shown that an optimal process, in the sense that exact equality in the inequality (C2) holds, does in general not exist[27]. However, if the dimension of the thermal bath B is not restricted, one can approach the lower bound $-\Delta F_M$ arbitrarily closely, e.g. by conducting a process that consists of multiple intermediate steps in which the memory gets temporarily thermalised[27,S2]. Since we place no restriction on the Hilbert space dimension of B in our framework, we assume that the resetting step $\mathcal{R}$ is conducted in such a way that inequality (C2) is saturated as closely as desired, i.e.

$$\Delta E_{\mathcal{R}} = -\Delta F_M.$$

We emphasise that this assumption merely simplifies our results, but does not restrict their validity in a more general setting. Dropping this assumption will only increase the lower bounds by additional non-negative quantities.

Appendix D: Overall energy cost of a general quantum measurement

As described in the main text, the overall energy cost needed to implement a quantum measurement is the sum of the cost of step $\mathcal{M}$ and step $\mathcal{R}$,

$$E_{\mathrm{cost}} = \Delta E_{\mathcal{M}} + \Delta E_{\mathcal{R}} \ .$$

Combining Theorem 1 and the optimal implementation of the resetting step $\mathcal{R}$, $\Delta E_{\mathcal{R}} = -\Delta F_M$ (see Appendix C), the overall energy cost is given by

$$\beta E_{\text{cost}} = \beta \Delta E_S + \Delta S + \mathcal{I} + \Delta_Q \ . \tag{D1}$$

Note that the two terms $\pm\Delta F_M$ from (B1) and (C) have cancelled. Thus, although energy costs can be shifted between $\Delta E_{\mathcal{M}}$ and $\Delta E_{\mathcal{R}}$ by changing the memory Hamiltonian H_M arbitrarily, this does not affect the total energy costs E_{cost}[14,21,36]. To continue, as shown by the computation in Eq. (B2), we have $\Delta S + \mathcal{I} + \Delta_Q = S(\rho'_M) - S(\rho_M)$, so we find that the overall energy cost of a quantum measurement crucially depends on the entropy change in the memory

$$\beta E_{\text{cost}} = \beta \Delta E_S + S(\rho'_M) - S(\rho_M). \tag{D2}$$

This is consistent with applying Landauer's principle[10,27] to the measurement process (see main text), as has been first done by Bennett[14] and was generalized to feedback protocols by ref.[28]. The equality (D2) will turn out to be useful to derive the equality results in the following sections, but *requires knowledge about the internal state of the measurement device.* In contrast, inequality (1) in the main text

$$E_{\text{cost}} \geq \Delta E_S + k_B T \big[S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) \big] \tag{D3}$$

is independent of the specific measurement implementation $(U_{SM}, \rho_M, \{Q_k\})$ and immediately follows from (D1) by the non-negativity of $\mathcal{I}$ and Δ_Q. Our derivation of Eq. (D3) extends previous work by refs.[21,22], which was restricted to efficient measurements or assumed thermality of ρ_S and the Second Law (see Appendix H).

Appendix E: Lower bound in terms of inefficiency

Here we prove a lower bound on the energy cost in terms of the inefficiency of the measurement. A measurement is said to be *efficient* if each measurement outcome k has just one corresponding measurement operator M_k, i.e. the (unnormalized) post-measurement states on the measured system S all take the form $T_k(\rho_S) = M_k \rho_S M_k^\dagger$ or, in other words, the Kraus rank of all maps T_k is one. Efficient measurements however do not describe all possible quantum measurements. Instead, as described in Appendix A, the most general form of measurement is given through *inefficient* measurements described as $\rho_S \mapsto \rho'_S = \sum_{k,i} M_{ki} \rho_S M_{ki}^\dagger$. The index i ranges from 1 to the Kraus rank $I(k)$ of the channel T_k. We henceforth call the maximal Kraus rank of all elements T_k of a given quantum instrument the *inefficiency* I of the quantum instrument $\{T_k\}$. Clearly, if $I = 1$ we recover the case of efficient measurements.

For general measurements we prove the following theorem:

Theorem 3. *Let $(U_{SM}, \rho_M, \{Q_k\})$ be an implementation of a quantum measurement with inefficiency I. The energy cost of operating this device is then lower bounded as*

$$\beta E_{\text{cost}} \geq \beta \Delta E_S - \ln I \ . \tag{E1}$$

We highlight two consequences of this theorem: First, if we can construct a measurement device for an inefficient measurement with $I > 1$ that saturates the inequality (E1), then useful energy $E_{\text{ext}} = \Delta E_S - E_{\text{cost}}$ can be extracted from the device during this operation. Remarkably, as shown in Figure 3 in the main text, such devices exist. Further details on this example and how to construct such measurement devices will be presented in Appendix F. Second, such extraction of energy in a measurement is only possible for inefficient measurements. Efficient measurements can never yield energy[21,22], $E_{\text{ext}} \leq 0$.

Proof. In order to prove inequality (E1) let us denote by $p_k = \text{tr}\left[\sum_i M_{ki} \rho_S M_{ki}^\dagger\right]$ the probability of receiving outcome k and by $\rho'_{S,k} = \sum_i M_{ki} \rho_S M_{ki}^\dagger / p_k$ the corresponding post-measurement state on S. Furthermore, define $r_{ki} :=$

$\mathrm{tr}\left[M_{ki}\rho_S M_{ki}^\dagger\right]$. Then we have

$$\begin{aligned}
\sum_k p_k S(\rho_{S,k}) &= \sum_k p_k S\Big(\sum_i \frac{M_{ki}\rho_S M_{ki}^\dagger}{p_k}\Big) \\
&= \sum_k p_k S\Big(\sum_i \frac{r_{ki}}{p_k}\frac{M_{ki}\rho_S M_{ki}^\dagger}{r_{ki}}\Big) \\
&\leq \sum_k p_k \left[H\Big(\Big\{\frac{r_{ki}}{p_k}\Big\}_i\Big) + \sum_i \frac{r_{ki}}{p_k} S\Big(\frac{M_{ki}\rho_S M_{ki}^\dagger}{r_{ki}}\Big)\right] \qquad \text{(E2)} \\
&\leq \sum_k p_k \left[\ln I + \sum_i \frac{r_{ki}}{p_k} S\Big(\frac{\sqrt{\rho_S}M_{ki}^\dagger M_{ki}\sqrt{\rho_S}}{r_{ki}}\Big)\right] \qquad \text{(E3)} \\
&= \ln I + \sum_{k,i} r_{ki} S\Big(\frac{\sqrt{\rho_S}M_{ki}^\dagger M_{ki}\sqrt{\rho_S}}{r_{ki}}\Big) \\
&\leq \ln I + S\Big(\sum_{k,i} \sqrt{\rho_S}M_{ki}^\dagger M_{ki}\sqrt{\rho_S}\Big) \qquad \text{(E4)} \\
&= \ln I + S(\rho_S)\ ,
\end{aligned}$$

which by inequality (D3) proves the desired statement (E1). Inequalities (E2) and (E4) are obtained using the property of the von Neumann entropy that for any convex combination of quantum states, $\sum_j p_j\sigma_j$, we have that[24]

$$\sum_j p_j S(\sigma_j) \leq S\Big(\sum_j p_j\sigma_j\Big) \leq H(\{p_j\}_j) + \sum_j p_j S(\sigma_j)\ . \qquad \text{(E5)}$$

Inequality (E3) is obtained using the following two statements: First, the Shannon entropy of any probability distribution with I elements is upper bounded by $\ln I$ and, second, $S(LL^\dagger) = S(L^\dagger L)$ for any linear operator L since $LL^\dagger$ and $L^\dagger L$ have the same non-vanishing eigenvalues. □

Appendix F: How to extract energy through measurement

In this section we present and discuss in detail two examples of a measurement on a qubit S and compute how much energy is extracted during each process (see Figure 3 in the main text for a brief summary). The first example will be an efficient measurement, for which we already know that no energy can be extracted (see Theorem 3). The second example will be a slight variation of the first: Although quite similar to the first measurement, the second is inefficient and allows to extract $k_B T \ln 2$ of energy from the device for specific initial states of the measured system S (as depicted in Figure 3).

Our first example is a rank-1 projective measurement on a qubit system S with projection operators $\{|k\rangle\langle k|\}_{k=0,1}$ and we denote by ρ_S and ρ_M the initial state of the measured system S and memory M, respectively. The final state of S and M is of the form

$$\rho'_{SM} = \sum_{k=0,1} \langle k|\rho_S|k\rangle \cdot |k\rangle_S\langle k| \otimes \rho'_{M,k}\ , \qquad \text{(F1)}$$

where the states $\rho'_{M,k}$ have support on orthogonal subspaces such that the outcome value k is reliably stored on M. The outcome probabilities are then given by $p_k = \langle k|\rho_S|k\rangle$. A specific measurement device $(\rho_M, U_{SM}, \{Q_k\})$ that implements this projective measurement is characterised as follows: We take a memory M consisting of two qubits M_A and M_B with Hilbert spaces $\mathcal{H}_{M_A}$ and $\mathcal{H}_{M_B}$, respectively, that starts in the state

$$\rho_M = |0\rangle_{M_A}\langle 0| \otimes \frac{\mathbb{1}_{M_B}}{2}\ .$$

Additionally, we take projections $Q_k = |k\rangle_{M_A}\langle k| \otimes \mathbb{1}_{M_B}$ and consider the following unitary interaction between system and memory

$$U_{SM} = \Big(|0\rangle_S\langle 0| \otimes |0\rangle_{M_A}\langle 0| + |1\rangle_S\langle 1| \otimes |1\rangle_{M_A}\langle 0| + ...\Big) \otimes \mathbb{1}_{M_B}\ ,$$

where the dots indicate that we are free to choose any unitary extension. Indeed, evaluating Eq. (A1) for this implementation $(U_{SM}, \rho_M, \{Q_k\})$ we find that this measurement device outputs the desired final state in Eq. (F1) with $\rho'_{M,k} = |k\rangle_{M_A}\langle k| \otimes \frac{\mathbb{1}_{M_B}}{2}$.

In this paper, we provide various ways to calculate the energy cost of conducting this particular measurement: In the main text we claimed that the energy cost of any projective measurements is exactly given by $E_{\text{proj}} = \Delta E_S + k_B T H(\{p_k\})$, where $H(\{p_k\})$ denotes the Shannon entropy of the outcome probability distribution (see Eq. (2) in the main text). While this claim is proven in Appendix G, we can also verify this result for the example at hand using Eq. (D2). Indeed, since we specified the microscopic details of our showcase measurement device, we are able to compute the energy cost of this projective measurement

$$\begin{aligned}\beta E_{\text{proj}} &= \beta\Delta E_{\mathcal{M}} + \beta\Delta E_{\mathcal{R}} \\ &= \beta\Delta E_S + S(\rho'_M) - S(\rho_M) \\ &= \beta\Delta E_S + \big(H(\{p_k\}) + \ln 2\big) - \ln 2 \\ &= \beta\Delta E_S + H(\{p_k\}) \ .\end{aligned}$$

Hence, indeed we find that for all initial states of S no energy can be extracted, $E_{\text{ext}} = \Delta E_S - E_{\text{proj}} \leq 0$.

The situation changes if we consider the following slight variation of the above setup, which is our second example. Assume a situation where we are only interested in the outcome probabilities p_k of our measurement and not in the final state of S, i.e. we fix the POVM but not the quantum instrument. We can then construct a measurement device that in addition to the previous device performs, after U_{SM} but before the projections $\{Q_k\}$, a swap operation $U_{S\leftrightarrow M_B}$ between S and M_B (see Figure 3). The unitary interaction between measured system and memory in this device is therefore simply given by $U_{S\leftrightarrow M_B} \circ U_{SM}$. The post-measurement state then reads

$$\rho'_{SM} = \frac{\mathbb{1}_S}{2} \otimes \sum_{k=0,1} \langle k|\rho_S|k\rangle \cdot |k\rangle_{M_A}\langle k| \otimes |k\rangle_{M_B}\langle k| \ .$$

Note that the measurement device correctly outputs the outcome probabilities, i.e. the measurement outcomes can be read off from the memory via the projections Q_k with the correct probabilities $p_k = \langle k|\rho_S|k\rangle$, and therefore still allows for conditioning on the outcome. However, in contrast to the device in the first example, it always leaves the measured system in the completely mixed state. Remarkably, operating this device allows for extracting energy since by (D2), we have for this modified measurement that

$$\beta E_{\text{ext}} = \beta\Delta E_S - \beta E_{\text{cost}} = \ln 2 - H(\{p_k\}) \geq 0 \ .$$

Hence, if the measured system starts in any of the states $\{|k\rangle_S\langle k|\}$, this measurement device outputs $E_{\text{ext}} = k_B T \ln 2$ of useful energy. The reason why this slight modification of the setup allows for extracting energy is that the additional swap process introduces inefficiency into the measurement: A measurement that always outputs states of the form $\rho'_{SM} = \frac{\mathbb{1}_S}{2} \otimes \rho'_M$ cannot have a one-to-one correspondence between measurement operator M_k and outcome k. Indeed, our device implements the quantum instrument $\{T_k(\rho_S) = \sum_{i=1}^{2} \frac{1}{2}|i\rangle\langle k|\rho_S|k\rangle\langle i|\}$ with inefficiency $I = 2$ and hence saturates our inefficiency bound (E1).

Appendix G: Energy cost of projective measurements

Projective measurements are the textbook examples of "standard" quantum measurements. They are described by projective measurement operators $M_k = P_k$ with $P_k^2 = P_k^\dagger = P_k$ and map the initial state ρ_S of the measured system to the post-measurement state $\rho'_{S,k} = P_k\rho_S P_k/p_k$ with probability $p_k = \text{tr}[P_k\rho_S]$. In particular, projective measurements belong to the class of efficient measurements due to the one-to-one correspondence between measurement operator P_k and outcome k. A measurement device that implements such a projective measurement $\{P_k\}$ on S is described by a tuple $(U_{SM}, \rho_M, \{Q_k\})$ satisfying

$$\text{tr}_M\Big[(\mathbb{1} \otimes Q_k)U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(\mathbb{1} \otimes Q_k)\Big] = P_k\rho_S P_k \quad \forall\rho_S \forall k \ . \tag{G1}$$

Again, in accordance with our implementation requirement, we require this equality to hold for all states ρ_S on S, otherwise the device does not perform the projective measurement on all possible input states.

We now prove that the energy cost of implementing such a measurement is exactly given by Eq. (2) in the main text.

Theorem 4. *Let $(U_{SM}, \rho_M, \{Q_k\})$ be an implementation of a projective quantum measurement $\{P_k\}$, as prescribed by* (G1)*. Then the energy cost of operating this device on an initial state ρ_S is exactly given by*

$$\beta E_{\text{proj}} = \beta \Delta E_S + H(\{p_k\}) \ ,$$

where $H(\{p_k\}) = -\sum_k p_k \ln p_k$ is the Shannon entropy of the outcome probability distribution $p_k = \text{tr}[P_k \rho_S]$.

The proof of this theorem is based on the following lemma:

Lemma 5. *Let $\{P_k\}$ be a projective measurement on some quantum system S. Let a dilation of the "measurement channel" $T_S(\rho_S) := \sum_k P_k \rho_S P_k$ be given by*

$$\sum_k P_k \rho_S P_k = \text{tr}_E \left[U_{SE}(\rho_S \otimes \rho_E) U_{SE}^\dagger \right] \quad \forall \rho_S \ , \tag{G2}$$

where ρ_E is an initial state of a quantum system E and U_{SE} is a unitary on S and E.

Then there exist quantum states $\sigma_{E,k}$ with $S(\rho_E) = S(\sigma_{E,k})$ for all k such that, for all quantum states ρ_S, the post-measurement state of E, $\rho'_E = \text{tr}_S \left[U_{SE}(\rho_S \otimes \rho_E) U_{SE}^\dagger \right]$, can be written as

$$\rho'_E = \sum_k \text{tr}[P_k \rho_S] \sigma_{E,k} \quad \forall \rho_S \ .$$

If, additionally, U_{SE} and ρ_E together with projections $\{Q_k\}$ on E form an implementation $(U_{SE}, \rho_E, \{Q_k\})$ of the projective measurement $\{P_k\}$ on S, i.e. $\sum_k Q_k = \mathbb{1}$ and

$$P_k \rho_S P_k = \text{tr}_E \left[(\mathbb{1} \otimes Q_k) U_{SE}(\rho_S \otimes \rho_E) U_{SE}^\dagger (\mathbb{1} \otimes Q_k) \right] \quad \forall \rho_S \ \forall k, \tag{G3}$$

then $\sigma_{E,k} = Q_k \sigma_{E,k} Q_k$ for all k, i.e. the $\sigma_{E,k}$ are mutually orthogonal.

Proof. (Lemma 5) The proof is based on the Stinespring dilation theorem, according to which we can always write the channel $T_S(\rho_S)$ as a unitary U_{SA} acting on S and an ancilla A initially in a pure state $|0\rangle_A\langle 0|$,

$$T_S(\rho_S) = \text{tr}_A[U_{SA}(\rho_S \otimes |0\rangle_A\langle 0|) U_{SA}^\dagger] \ .$$

The minimal Stinespring dilation can be chosen to be any unitary extension U_{SA} of the operator $\sum_k P_k \otimes |k\rangle_A\langle 0|$, whose action is only defined on states of the form $|\psi\rangle_S \otimes |0\rangle_A$, where the ancilla Hilbert space A is spanned by the orthonormal basis $|k\rangle$[24]. The corresponding complementary channel takes the form

$$\begin{aligned} T_A(\rho_S) &= \text{tr}_S[U_{SA}(\rho_S \otimes |0\rangle_A\langle 0|) U_{SA}^\dagger] \\ &= \text{tr}_S \left[\sum_{k,k'} P_k \rho_S P_{k'} \otimes |k\rangle_A\langle k'| \right] \ , \\ &= \sum_k \text{tr}[P_k \rho_S] |k\rangle_A\langle k| \ . \end{aligned}$$

However, this channel is not the only possible complementary channel of T_S. Using (G2), we find another complementary channel,

$$T_{E\tilde{E}}(\rho_S) = \text{tr}_S[U_{SE\tilde{E}}(\rho_S \otimes \psi_{E\tilde{E}}) U_{SE\tilde{E}}^\dagger] \ ,$$

where $\tilde{E}$ is a purifying system of E such that the pure state $\psi_{E\tilde{E}}$ satisfies $\text{tr}_{\tilde{E}}[\psi_{E\tilde{E}}] = \rho_E$ and $U_{SE\tilde{E}} := U_{SE} \otimes \mathbb{1}_{\tilde{E}}$. The Stinespring theorem states that these complementary channels are related by an isometry $V : \mathcal{H}_A \to \mathcal{H}_E \otimes \mathcal{H}_{\tilde{E}}$, i.e.

$$T_{E\tilde{E}}(\rho_S) = V T_A(\rho_S) V^\dagger = \sum_k \text{tr}[P_k \rho_S] |\gamma_k\rangle_{E\tilde{E}}\langle \gamma_k|$$

with $|\gamma_k\rangle_{E\tilde{E}} := V|k\rangle_A$ again forming an orthonormal basis. Note that the complementary channel $T_{E\tilde{E}}$ and the final state ρ'_E of E are, by construction, linked via the partial trace,

$$\rho'_E = \text{tr}_S \left[U_{SE}(\rho_S \otimes \rho_E) U_{SE}^\dagger \right] = \text{tr}_{\tilde{E}} \left[T_{E\tilde{E}}(\rho_S) \right] \ .$$

Hence, for every ρ_S, the final state on E takes the form

$$\rho'_E = \sum_k \mathrm{tr}[P_k \rho_S] \sigma_{E,k} \qquad \forall \rho_S \ , \tag{G4}$$

where we define the states $\sigma_{E,k} = \mathrm{tr}_{\tilde{E}}[V|k\rangle_A\langle k|V^\dagger]$, which are independent of ρ_S.

To show $S(\rho_E) = S(\sigma_{E,k})$, let now $\rho_S = \psi_k$ be a pure state supported on the subspace characterised by one P_k, i.e. $P_k\psi_k = \psi_k$. Then by (G4)

$$\rho'_E = \mathrm{tr}_S[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger] = \sigma_{E,k} \tag{G5}$$

and the final state on S is pure,

$$\rho'_S = \mathrm{tr}_E[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger] = \sum_{k'} P_{k'}\psi_k P_{k'} = \psi_k \ .$$

Hence there are no correlations between the marginals of the final SE state, i.e. $U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger = \psi_k \otimes \sigma_{E,k}$. Since unitaries do not change the spectrum, we have $S(\rho_E) = S(\sigma_{E,k})$, which concludes the first part of the proof.

For the second part of the proof, we assume that we additionally have an implementation of the projective measurement on S, i.e. (G3) is satisfied. Note that we can obtain (G2) by summing (G3) over k; hence, all statements within the first part of the proof remain valid for this second part of the proof. We can thus take ψ_k to be a state in the support of P_k as above to find by (G5) that

$$\sigma_{E,k} = \mathrm{tr}_S\left[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger\right] \ .$$

Our aim is to show that by requiring (G3) we have, for all k, that

$$\sigma_{E,k} = Q_k \sigma_{E,k} Q_k \ . \tag{G6}$$

To this end observe that the quantity $Q_k\sigma_{E,k}Q_k$ is a positive operator with unit trace for all k since by (G3)

$$\begin{aligned}\mathrm{tr}_E\left[Q_k\,\mathrm{tr}_S[U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger]Q_k\right] &= \mathrm{tr}_{SE}\left[(\mathbb{1}\otimes Q_k)U_{SE}(\psi_k \otimes \rho_E)U_{SE}^\dagger(\mathbb{1}\otimes Q_k)\right] \\ &= \mathrm{tr}_S\left[P_k\psi_k P_k\right] = \mathrm{tr}\,\psi_k = 1 \ .\end{aligned}$$

But then we can compute

$$\begin{aligned}1 = \mathrm{tr}[\sigma_{E,k}] &= \mathrm{tr}[(Q_k + (\mathbb{1}-Q_k))\sigma_{E,k}(Q_k + (\mathbb{1}-Q_k))] \\ &= \mathrm{tr}[Q_k\sigma_{E,k}Q_k] + \mathrm{tr}[(\mathbb{1}-Q_k)\sigma_{E,k}(\mathbb{1}-Q_k)] \\ &= 1 + \mathrm{tr}[(\mathbb{1}-Q_k)\sigma_{E,k}(\mathbb{1}-Q_k)]\end{aligned}$$

to find that $(\mathbb{1}-Q_k)\sigma_{E,k} = \sigma_{E,k}(\mathbb{1}-Q_k) = 0$ which implies (G6). □

Let us finally prove Theorem 4:

Proof. (Theorem 4) To compute the energy cost $E_{\mathrm{proj}} = \Delta E_{\mathcal{M}} + \Delta E_{\mathcal{R}}$ of a projective measurement, we use (D2) to simplify the problem to computing the entropy difference in the memory, $\beta E_{\mathrm{proj}} = \beta\Delta E_S + S(\rho'_M) - S(\rho_M)$. While the entropy difference is hard to control for general measurements, Lemma 5 gives us enough information to compute it exactly in the case of projective measurements.

Recall that the state of the memory after the measurement is given by

$$\rho'_M = \mathrm{tr}_S\left[\sum_k (\mathbb{1}\otimes Q_k)U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(\mathbb{1}\otimes Q_k)\right] = \sum_k Q_k\tilde{\rho}_M Q_k \ ,$$

where we introduced the quantum state $\tilde{\rho}_M := \mathrm{tr}_S\left[U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger\right]$. Also note that $(U_{SM}, \rho_M, \{Q_k\})$ is, by assumption, an implementation of the projective measurement $\{P_k\}$, i.e. (G1) (resp. (G3) of Lemma 5) is satisfied. Hence, by Lemma 5, we know that the state $\tilde{\rho}_M$ takes the form

$$\tilde{\rho}_M = \sum_k \mathrm{tr}[P_k\rho_S]\sigma_{M,k} \ ,$$

where the $\sigma_{M,k} = Q_k \sigma_{M,k} Q_k$ are mutually orthogonal and have entropy $S(\sigma_{M,k}) = S(\rho_M)$ for all k. The post-measurement state of the memory is therefore given by

$$\rho'_M = \sum_k Q_k \tilde{\rho}_M Q_k = \sum_k Q_k \Big(\sum_{k'} \mathrm{tr}[P_{k'} \rho_S] \sigma_{M,k'} \Big) Q_k = \sum_k \mathrm{tr}[P_k \rho_S] \sigma_{M,k} = \sum_k p_k \sigma_{M,k}$$

From this it follows that

$$\begin{aligned} \beta E_{\mathrm{proj}} &= \beta \Delta E_S + S(\rho'_M) - S(\rho_M) \\ &= \beta \Delta E_S + \Big(H(\{p_k\}) + \sum_k p_k S(\sigma_{M,k}) \Big) - S(\rho_M) \\ &= \beta \Delta E_S + H(\{p_k\}) \ . \end{aligned}$$

□

Appendix H: Comparison with previous literature

After we have now derived the theoretical bounds Eqs. (1) and (2) from the main text, we compare them in this section with other results on the energy cost of quantum measurements that are given in, or can be extracted from, the literature. In particular, we will compare here, in greater detail than in the main text, our results and framework with those of Sagawa/Ueda[21,26], Jacobs[22], Bennett[10,14], and Faist *et al.*[18].

The framework most closely related to ours is the one from Sagawa and Ueda's work[21,26]. In this work the following lower bound on the energy cost for an efficient quantum measurement $\{M_k\}$ is found:

$$E_{\mathrm{cost}} \geq E_{SU} \equiv \Delta E_S + k_B T \mathcal{I} \ , \tag{H1}$$

where $\mathcal{I} = S(\rho_S) + H(\{p_k\}) + \sum_k \mathrm{tr}[M_k \rho_S M_k \ln M_k \rho_S M_k]$. As clarified in an Erratum[21], the given derivation relies crucially on the fact that the measurement is *efficient*[25], i.e. each measurement outcome k corresponds to a single measurement operator M_k. Our Eq. (1) in the main text and its derivation (given in Appendices B–D) give the correct generalization of this bound to *in*efficient measurements, where each outcome k can correspond to multiple associated jump operators $M_{k1}, M_{k2}, \ldots$. Indeed, the general bound Eq. (1) just reduces to the statement (H1) from ref.[21] in the special case of efficient measurements, since in this case $S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) = \mathcal{I}$ as is easily verified. The form of the bound given in ref.[21] does not give a correct generalization to general inefficient measurements.

The correct generalisation of the lower bound (H1) to general inefficient measurements can already be inferred from Jacob's work[22], which was directly following up on ref.[26]. Instead of directly calculating the energy expense of conducting the quantum measurement, ref.[22] answered the converse question: *After* some "black box" has performed the required measurement $\{M_{ki}\}$ on the state ρ_S, how much energy can be extracted by means of a feedback protocol that makes use of the measurement result k and the post-measurement state $\rho'_{S,k}$ on the measured system S? This feedback process is required to map each post-measurement state $\rho'_{S,k}$ to the initial state ρ_S such that the overall process (measurement and feedback) is cyclic. It is found that the amount of energy that can be extracted in an optimal feedback process is given by the average free energy difference

$$E_{ext,Jacobs} = \sum_k p_k (F(\rho_S) - F(\rho'_{S,k})) = \Delta E_S + k_B T \Big[S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) \Big] \ .$$

The derivation in ref.[22] assumes that the "measurement black-box" acts on a thermal state (also in[22] (Eq. (7)) a non-thermal state ρ_S is first reversibly transformed into a thermal state).

If one now assumes the Second Law of Thermodynamics, which states that no net amount of energy can be extracted in a cyclic process that involves a single thermal bath, one can use the optimal work extraction result (H) to argue that *at least* this much energy had to be invested by the measurement device ("black box"), i.e.

$$E_{cost} \geq \Delta E_S + k_B T \Big[S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) \Big] \ ,$$

which is identical to our Eq. (1) in the main text. Due to the above thermality restriction, the argumentation via (H) is valid only if ρ_S was thermal, whereas our general derivation from Appendices B–D holds without that thermality

requirement. Our result is therefore even applicable to systems far from thermodynamic equilibrium, which is the typical situation in quantum information processing tasks such as quantum computing. Also, our derivation does *not* assume the validity of Second Law, but instead provides a detailed microscopic picture of the implementation of the measurement device, whereas that device is treated like a black box in ref.[22]. Our explicit microscopic modeling is useful e.g. in order to display the work-extracting implementation of an inefficient measurement in our Fig. 3, which is surprising in light of ref.[21] and answers a theoretical question left open by refs.[21,26] whether such an implementation exists. The black-box treatment in ref.[22] can never display such an implementation capable of work extraction.

We also briefly compare our results with Bennetts's[14] and other implementation-dependent bounds, such as those in ref.[28] for feedback protocols. According to ref.[14], the total energy cost for measurement and erasure is $E_{\rm cost} = \Delta E_S + k_B T(S(\rho'_M) - S(\rho_M))$, where ρ_M and ρ'_M denote the *states of the measurement device* before and after the measurement step $\mathcal{M}$. We indeed rederived this statement within our framework in Eq. (D2), but argue that this result by itself is not very useful in our context, since it is *not* stated in terms of the measurement specification $\{M_{ki}\}$ (and ρ_S, H_S). Rather, for any fixed measurement $\{M_{ki}\}$ one can find *many* implementations $(\rho_M, U_{SM}, \{Q_k\})$ yielding different values of $S(\rho'_M) - S(\rho_M)$, see e.g. the red vs. blue curves in Fig. 4. It is the virtue of Eq. (1) and our main result Eq. (2) to give a bound on the energy cost *solely in terms of the measurement specification* $\{M_{ki}\}$ *and the system quantities* ρ_S, H_S. Only this feature allows us to compute fundamental energy costs for theoretical protocols (as in our applications, Appendices I–J).

General results to compute the thermodynamic energy costs for the implementation of any quantum operation were given in Faist *et al.*[18]. This work differs from all the previous more traditional thermodynamic treatments in that it goes beyond the usual "state-transformation ideas" and instead requires the implementation to act correctly on a larger subspace of states. This is similar in spirit to our implementation requirement (Appendix A), and we will comment on commonalities and differences further below. In ref.[18], the energy costs on the system S are not considered, effectively setting $H_S = 0$. All results in ref.[18] are derived within the so-called single-shot setting and later translated to the asymptotic i.i.d. setting; we will make a comparison below.

According to ref.[18] the single-shot energy cost to perform the measurement step is given by $k_B T \cdot H^\epsilon_{\max}(E|MS)$, where the *smoothed conditional max entropy* $H^\epsilon_{\max}(E|MS)$ is evaluated on the state

$$|\phi\rangle_{EMSR} = \sum_{k,i} |k,i\rangle_E \otimes |k\rangle_M \otimes (M_{ki}|\psi\rangle_{SR}) \ ,$$

where M is the memory that stores the outcome k, R is a purifying system of the measured system S such that $\mathrm{tr}_R\, |\psi\rangle_{SR}\langle\psi| = \rho_S$ is the initial state of S, and E is an ancilla system needed to calculate the Stinespring dilation of the measurement process $\mathcal{E}(\rho_S) = \sum_k |k\rangle\langle k|_M \otimes \sum_i M_{ki}\rho_S M_{ki}^\dagger$. Incorporating the single-shot cost $k_B T \ln 2 \cdot H^\epsilon_{\max}(M)$ for erasure, the total minimal cost for measurement in the single-shot setting amounts to (see ref.[18] (Supplementary Note 4))

$$E^\epsilon_{Faist} = k_B T \cdot (H^\epsilon_{\max}(E|MS) + H^\epsilon_{\max}(M)) \tag{H2}$$

We emphasize that here we require the resetting step *not to make use of the system state* ρ'_S, because in relevant situations (see e.g. Appendix J) this state is actually *not* available for the resetting step.

Below we will compare (H2) with our results Eqs. (1) and (2) in two different ways. For the first way of comparison, note that a non-vanishing error $\epsilon > 0$ implies that the process corresponds to a measurement only on a subset of the state space. In our framework however we demand that the process acts *correctly for all possible input states* according to our implementation requirement (still we evaluate the energy cost for a specific initial state ρ_S). For the comparison of the two frameworks below, we therefore first set $\epsilon = 0$ in the single-shot result (H2) which then reads[18]

$$E^0_{Faist} = k_B T \cdot (\ln ||\mathcal{E}(\Pi_S)||_\infty + \ln \mathrm{rank}[\rho'_M]) \ , \tag{H3}$$

where Π_S is the projector onto the support of the input state ρ_S. Intuitively, this result gives the worst-case estimate for the energy costs in a single shot of the measurement process.

Secondly, we will compare our results with the *i.i.d. limit* of (H2), where one evaluates the single shot quantities for fixed error $\epsilon \in (0,1)$ on product states $\rho^{\otimes n}$ in the limit of large n, see ref.[18] (Methods section). Taking this limit turns the term $H^\epsilon_{\max}(E|MS)$ from (H2) into

$$S(E|MS) = S(EMS) - S(MS) = S(\rho_S) - \Big[H(\{p_k\}) + \sum_k p_k S(\rho'_{S,k})\Big] \ ,$$

as one can easily compute, and the term $H_{\max}^{\epsilon}(M)$ from (H2) into

$$S(M) = H(\{p_k\}) \ .$$

The i.i.d. limit of (H2) therefore evaluates to

$$E_{Faist}^{iid} = S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) \ , \tag{H4}$$

which surprisingly equals exactly the lower bound presented in Eq. (1) for general measurements (again, $H_S = 0$ in ref.[18]).

The fact that the i.i.d. limit in (H4) agrees exactly with the bound presented in Eq. (1), implies that the i.i.d. costs $E_{Faist}^{iid,\mathrm{proj}}$ of *projective* measurements are in general strictly lower than the energy cost E_{proj} predicted by our main result Eq. (2) for projective measurements, $E_{Faist}^{iid,\mathrm{proj}} \leq E_{\mathrm{proj}}$ (the two values agree if and only if ρ_S was already diagonal with respect to the measurement basis). We find this a highly interesting disagreement which we explain as follows: The device constructed according to ref.[18] implementing the measurement step optimally, is required to *work correctly only on the support of the input state* ρ_S, rather than on the whole state space (see ref.[18] (Supplementary Note 4A) and ref.[S3]). As a consequence, in the macroscopic limit the constructed device is guaranteed to perform correctly *only on the typical subspace of* $\rho_S^{\otimes n}$, which is much smaller than the whole state space unless ρ_S was completely mixed. For our result Eq. (2) we require something *stronger*: The measurement device should perform correctly *on the full state space* (see the implementation requirement in the Methods Section). Hence, our energy cost result exceeds the one of ref.[18], as we will see explicitly in the examples below.

Whether to use the i.i.d. limit[18] in (H4) or our main result Eq. (2) depends on the physical situation. Here, we argue that Eq. (2) is more appropriate than Eq. (H4) to compute the energy cost of quantum error correction (which we describe in the main text and Appendix J). In the QEC scenario, a large number n of measurement operations $\mathcal{M}$ are applied independently to the n (logical) qubits of a quantum computer. However, a usual quantum computer is *not* a tensor product state $\rho^{\otimes n}$ but rather an n-qubit *correlated* (or even entangled) state $\rho_{\mathrm{ent}}^{(n)}$, making the macroscopic limit in ref.[18] inappropriate for that case.

To illustrate the disagreement between our main result Eq. (2), the single-shot result Eq. (H3) and its i.i.d. limit Eq. (H4), let us consider a projective measurement in the computational basis $\{|0\rangle, |1\rangle\}$ on a qubit, initially in the pure "superposition state" $\rho_S = (|0\rangle + |1\rangle)(\langle 0| + \langle 1|)/2$, which has also been considered in ref.[18] (Supplementary Note 4F, Example (III)):

(i) Our result Eq. (2) yields an energy cost for measurement and resetting of $E_{\mathrm{proj}} = k_B T \log 2$ (plus whatever energy change ΔE_S is incurred on the system).

(ii) Evaluating the entropic terms for Eqs. (H3) and (H4) yields

$$\begin{aligned} \ln ||\mathcal{E}(\Pi_S)||_\infty &= -\ln 2, \\ \ln \mathrm{rank}[\rho'_M] &= +\ln 2, \\ S(\rho_S) = S(\rho'_{S,k}) &= 0, \end{aligned}$$

so that the single-shot energy cost as well as the energy costs in the i.i.d. limit are predicted to be

$$E_{Faist}^{0} = E_{Faist}^{iid} = 0 \ .$$

This is strictly less than our prediction, for reasons explained above.

As a second (essentially classical) example, consider the same measurement on the state $\rho_S = p_0|0\rangle_S\langle 0| + p_1|1\rangle_S\langle 1|$ which is diagonal in the measurement basis. In this case one can easily compute that our result Eq. (2) agrees with the i.i.d. limit in Eq. (H4), but is generally less than the single-shot result Eq. (H3),

$$E_{\mathrm{proj}} = E_{Faist}^{iid} = k_B T H(\{p_0, p_1\}) \leq k_B T \log \mathrm{rank}[\{p_0, p_1\}] = E_{Faist}^{0} \ .$$

Equality holds in the chain if and only if ρ_S was completely mixed or one of the computational basis states.

Appendix I: Energy costs of quantum Zeno measurements

Here we compute the energy cost of conducting a stabilisation scheme via Zeno measurements – a process typical for the field of quantum control. As in the main text we consider a quantum system S, initially in the pure state $\rho_S =$

$|0\rangle\langle 0|$, with Hamiltonian $H_S = E\sigma_X$, where σ_X is the Pauli operator and $\pm E$ are the two energy eigenvalues of H_S. Our goal is to study the energy cost of conducting a quantum Zeno stabilisation protocol that stabilises S against the free Hamiltonian time evolution over the time span t by applying projective measurements $\{M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|\}$ at N regular time intervals $\delta t = t/N$. Since these measurements are projective, we can, by Theorem 4, compute the energy cost of each measurement exactly.

Note that the protocol employs multiple iterations of the *same* projective measurement. In our framework this may be equivalently described either by considering a single measurement device that is used repeatedly or by considering multiple devices, each possibly a different implementation of that measurement. The energy costs of both approaches are the same as the cost is by Theorem 4 independent of the specific implementation.

We find the following theorem for the total energy cost of Zeno stabilisation:

Theorem 6. *Consider a quantum Zeno stabilisation scheme as above. To achieve high target fidelity F, the energy cost of operating the devices that implement the projective measurements $\{M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|\}$ is given by*

$$E_{\text{Zeno}} \simeq \frac{1}{2} k_B T \left(\frac{Et}{\hbar}\right)^2 \ln\left[\frac{4.5}{1-F}\right] .$$

Hence, we find that the total energy required for stabilisation grows logarithmically in $1/(1-F)$ for increasing target fidelity F. In the asymptotic limit as $F \to 1$ the energy cost is given by Eq. (3) in the main text. Limited energy supply thus constrains our ability to stabilise a quantum system via Zeno control.

Proof. Let us denote the state on S after the n-th measurement by $\rho_S^{(n)} = (1-\epsilon_n)|0\rangle\langle 0| + \epsilon_n |1\rangle\langle 1|$. The probability that the process returns the wrong state $|1\rangle$ after n steps is then given by ϵ_n; the fidelity[S4] $F \equiv F(\rho_S^{(N)}, \rho_S) = \langle 0|\rho_S^{(N)}|0\rangle$ of the final state at the end of all N steps is $F = 1 - \epsilon_N$. Between the measurements, the system undergoes free time evolution according to the unitary $U = \exp(-i\delta t H_S/\hbar)$ such that the probabilities after the $(n+1)$-th measurement change to $\epsilon_{n+1} = \epsilon_n \cos(E\delta t/\hbar)^2 + (1-\epsilon_n)\sin(E\delta t/\hbar)^2$. Since $\epsilon_0 = 0$ by assumption, this recursion formula has the solution

$$\epsilon_n = \frac{1}{2}(1 - \cos(2E\delta t/\hbar)^n) = n\left(\frac{E\delta t}{\hbar}\right)^2 + \mathcal{O}(\delta t^4) .$$

According to our result for projective measurements (Theorem 4), the n-th measurement consumes energy $\beta E_{\text{proj}}^{(n)} = H(\{\epsilon_n, 1-\epsilon_n\})$, where we used that $\Delta E_S = 0$ for each step. The total energy required is then given by

$$\beta E_{\text{Zeno}} = \sum_{n=1}^{N} H(\{\epsilon_n, 1-\epsilon_n\}) .$$

We are interested in stabilisation schemes that yield high target fidelity F, which can be achieved by applying the measurements in shorter and shorter time scales, $\delta t = t/N \to 0$, or in other words by applying more measurements $N \to \infty$ in constant time span t. In this limit the higher order terms $\mathcal{O}(\delta t^4)$ of ϵ_n will not contribute to the energy cost of the measurements, so we set $\epsilon_n \simeq n\left(\frac{E\delta t}{\hbar}\right)^2$. We then have $F \simeq 1 - \frac{1}{N}\left(\frac{Et}{\hbar}\right)^2$ and

$$\begin{aligned}
\beta E_{\text{Zeno}} &= -\sum_{n=1}^{N} \epsilon_n \ln \epsilon_n - \sum_{n=1}^{N} (1-\epsilon_n)\ln(1-\epsilon_n) \\
&\simeq -\sum_{n=1}^{N} n\left(\frac{Et}{\hbar N}\right)^2 \ln\left[n\left(\frac{Et}{\hbar N}\right)^2\right] - \sum_{n=1}^{N}\left(1 - n\left(\frac{Et}{\hbar N}\right)^2\right)\ln\left[1 - n\left(\frac{Et}{\hbar N}\right)^2\right] \\
&= -\left(\frac{Et}{\hbar}\right)^2 \left(\sum_{n=1}^{N} \frac{n}{N^2}\ln\frac{n}{N} + \sum_{n=1}^{N}\frac{n}{N^2}\ln\left[\frac{1}{N}\left(\frac{Et}{\hbar}\right)^2\right]\right) \\
&\quad - \sum_{n=1}^{N}\left(1 - n\left(\frac{Et}{\hbar N}\right)^2\right)\ln\left[1 - n\left(\frac{Et}{\hbar N}\right)^2\right] \\
&\simeq -\left(\frac{Et}{\hbar}\right)^2 \left(\sum_{n=1}^{N} \frac{1}{N}\frac{n}{N}\ln\frac{n}{N} + \frac{N(N+1)}{2N^2}\ln\left[1-F\right]\right) \\
&\quad - \sum_{n=1}^{N}\left(1 - n\left(\frac{Et}{\hbar N}\right)^2\right)\ln\left[1 - n\left(\frac{Et}{\hbar N}\right)^2\right] .
\end{aligned}$$

In the limit $N \to \infty$ we have $\sum_{n=1}^{N} \frac{1}{N} \frac{n}{N} \ln \frac{n}{N} \simeq \int_0^1 x \ln x dx = -1/4$ and $\ln[1 - n\left(\frac{Et}{\hbar N}\right)^2] \simeq -n(\frac{Et}{\hbar N})^2$ (again higher orders of the expansion do not contribute). Hence, we have

$$\begin{aligned}\beta E_{\text{Zeno}} &\simeq \frac{1}{4}\left(\frac{Et}{\hbar}\right)^2 - \frac{1}{2}\left(\frac{Et}{\hbar}\right)^2 \ln\left[1-F\right] + \frac{1}{2}\left(\frac{Et}{\hbar}\right)^2 \\ &= \frac{1}{2}\left(\frac{Et}{\hbar}\right)^2 \left(\frac{3}{2} - \ln[1-F]\right) \\ &\simeq \frac{1}{2}\left(\frac{Et}{\hbar}\right)^2 \ln\left[\frac{4.5}{1-F}\right] \ .\end{aligned}$$

□

Appendix J: Energy cost of quantum error correction

In this section we investigate another application of our result on the energy cost of projective measurements (Theorem 4), namely computing the energy cost of conducting quantum error correcting protocols. In particular, we consider the 5-qubit code[31] in which the state of a single logical qubit $|\psi\rangle = \alpha_0|0_L\rangle + \alpha_1|1_L\rangle$ in the code space $\mathbb{C}^2_{\mathcal{L}}$, with $\alpha_0, \alpha_1 \in \mathbb{C}$ and $|\alpha_0|^2 + |\alpha_1|^2 = 1$, is encoded into the space $\mathcal{C}_5 \equiv (\mathbb{C}^2)^{\otimes 5}$ of five physical qubits by using the codewords[24]

$$\begin{aligned}|0_L\rangle =& \frac{1}{4}\left[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle\right. \\ &+ |01010\rangle + |00101\rangle - |11011\rangle - |00110\rangle \\ &- |11000\rangle - |11101\rangle - |00011\rangle - |11110\rangle \\ &\left.-|01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle\right] \ , \\ |1_L\rangle =& \frac{1}{4}\left[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle\right. \\ &+ |10101\rangle + |11010\rangle - |00100\rangle - |11001\rangle \\ &- |00111\rangle - |00010\rangle - |11100\rangle - |00001\rangle \\ &\left.-|10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle\right] \ .\end{aligned}$$

As in the main text we assume that each physical qubit is affected by the amplitude damping channel

$$\mathcal{N}_\gamma(\rho) = J_1 \rho J_1^\dagger + J_2 \rho J_2^\dagger$$

with Kraus operators $J_1 = \sqrt{\gamma}|0\rangle\langle 1|$ and $J_2 = \sqrt{\mathbb{1} - J_1^\dagger J_1}$, where $\gamma \in [0,1]$ determines the noise strength ($\gamma = 0$ corresponding to the noiseless case). Note however that our formalism applies to arbitrary noise models.

The error-correcting protocol is then a feedback scheme that allows to approximately recover the logical state $|\psi\rangle$ from the noisy state $\rho_{S,\gamma} = \mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle\langle\psi|)$ by applying so-called syndrome measurements

$$\begin{aligned}\mathcal{S}^1 = X \otimes Z \otimes Z \otimes X \otimes I \quad &, \quad \mathcal{S}^2 = I \otimes X \otimes Z \otimes Z \otimes X \ , \\ \mathcal{S}^3 = X \otimes I \otimes X \otimes Z \otimes Z \quad &, \quad \mathcal{S}^4 = Z \otimes X \otimes I \otimes X \otimes Z \ ,\end{aligned}$$

where X, Y, Z denote the Pauli operators and I is the identity matrix. Each syndrome measurement $\mathcal{S}^j$ has outcomes $s^j \in \{-1, 1\}$, occuring with probability $p_{s^j}^{(j)} = \text{tr}[P_{s^j}^{(j)} \rho_{S,\gamma}]$, where $P_{s^j}^{(j)}$ denotes the projector on the subspace corresponding to eigenvalue s^j. Furthermore all syndrome measurements commute and are hence jointly measurable. The measurement operators of the joint measurement $\mathcal{S}$ are given by projections $\{P_s\}_{s=0}^{15}$ with $P_s = P_{s^1}^{(1)} P_{s^2}^{(2)} P_{s^3}^{(3)} P_{s^4}^{(4)}$ and outcomes ("syndromes") $s \equiv (s^1, s^2, s^3, s^4)$ that occur with probability $p_s = \text{tr}[P_s \rho_{S,\gamma}]$.

We call a specific realisation of the 5-qubit code a *separate measurement scheme* if it employs four devices, each implementing one of the syndrome measurements $\mathcal{S}^i$. If only a single device is employed that implements a joint measurement $\mathcal{S}$ we call it the *joint measurement scheme*. Note that, for both separate and joint measurement scheme, any single-qubit Pauli error is uniquely identified by one of the 16 possible syndromes $s \equiv (s^1, s^2, s^3, s^4)$. Hence, such errors can be corrected with certainty by applying the conditional unitary

$$V_{SM} = \sum_s V_s \otimes P_s \ ,$$

which applies the unitary V_s on the measured system S if the syndrome (which is stored in the memory after the measurement and read out by projections P_s) is s. More concretely, since all single qubit errors (X, Y or Z) square to identity, we simply apply, say, $V_s = X \otimes I \otimes I \otimes I \otimes I$ if the syndrome s identified an X-error on the first physical qubit. In the following we denote the final state after applying the joint measurement $\mathcal{S}$ and corresponding feedback by

$$\tilde{\rho}_{S,\gamma} = \sum_s V_s P_s \rho_{S,\gamma} P_s V_s^\dagger \ .$$

Both the separate and the joint measurement schemes can be used for quantum error correction, but they come at different energy costs as may be verified by employing Theorem 4: Whereas the average energy change on S, ΔE_S, is the same for both measurement schemes, the joint measurement scheme demands an additional energy cost from the physical implementation

$$E_{\mathcal{C}_5}^{\text{proj}} = k_B T H(\{p_s\}_{s=0}^{15}) \ , \tag{J1}$$

which is always less or equal to the additional energy cost

$$E_{\mathcal{C}^5}^{\text{sep}} = k_B T \sum_{j=1}^{4} H(\{p_{s^j}^{(j)}\}_{s^j=\pm 1})$$

required to implement the four separate syndrome measurements. The reason for this is that a joint measurement scheme can exploit correlations in the measurement outcomes to reduce the cost of the resetting step $\mathcal{R}$. More concretely, for all joint probability distributions $p(s \equiv (s^1, s^2, s^3, s^4))$ of random variables S^1, S^2, S^3, S^4 the following relation holds:

$$\begin{aligned} H(S^1S^2S^3S^4) &= H(S^1) + H(S^2S^3S^4|S^1) \\ &= H(S^1) + H(S^2|S^1) + H(S^3|S^1S^2) + H(S^4|S^1S^2S^3) \\ &= \left(\sum_{j=1}^{4} H(S^j)\right) - I(S^1 : S^2) - I(S^1S^2 : S^3) - I(S^1S^2S^3 : S^4) \ , \end{aligned} \tag{J2}$$

where $H(S^1S^2S^3S^4) := H(\{p(s \equiv (s^1, s^2, s^3, s^4))\}_s)$ is the Shannon entropy of the joint probability distribution, $H(S^2|S^1) := H(S^1S^2) - H(S^1)$ is the conditional entropy and $I(S^1 : S^2) := H(S^1) - H(S^1|S^2)$ is the mutual information, which quantifies the correlations between the respective random variables and is always non-negative. Hence, only if the measurement outcomes are uncorrelated (i.e. all mutual information terms in (J2) vanish), we have that $H(\{p_s\}_{s=0}^{15}) = \sum_{j=1}^{4} H(\{p_{s^j}^{(j)}\}_{s^j=\pm 1})$ and the energy cost of the joint measurement equals the cost of all four separate measurements. In Fig. 4, the difference between the blue and red curve is due to the mutual information terms of the syndrome bits on the specific noisy states,

$$E_{\mathcal{C}^5}^{\text{sep}} - E_{\mathcal{C}_5}^{\text{proj}} = k_B T \left(I(S^1 : S^2) + I(S^1S^2 : S^3) + I(S^1S^2S^3 : S^4) \right) \ .$$

Note that (J1) *exactly* determines the minimum additional energy requirements for quantum error correction and is hence a drastic improvement to all previous results, which only provide lower bounds: Evaluating the result (D3) obtained in refs.[21,22] yields the best lower bound previously known

$$E_{\mathcal{C}_5}^{\text{SU}} := k_B T \Big[S(\rho_{S,\gamma}) - \sum_s p_s S(P_s \rho_{S,\gamma} P_s / p_s) \Big] \leq E_{\mathcal{C}_5}^{\text{proj}} \ ,$$

which itself is a slight improvement (due to concavity of the von Neumann entropy) upon the simple application of Landauer's principle as in ref.[10,17] directly to the system S,

$$E_{\mathcal{C}_5}^{\text{Lan}} := k_B T \big[S(\rho_{S,\gamma}) - S(\tilde{\rho}_{S,\gamma}) \big] \leq E_{\mathcal{C}_5}^{\text{SU}} \leq E_{\mathcal{C}_5}^{\text{proj}}.$$

Both lower bounds, $E_{\mathcal{C}_5}^{\text{SU}}$ and $E_{\mathcal{C}_5}^{\text{Lan}}$, predict much weaker energy requirements than actually needed for quantum error correction (see Figure 4 for a comparison).

Appendix K: Energy cost of the projections $\{Q_k\}$ during measurement step $\mathcal{M}$

In the main text we claimed that the projections $\{Q_k\}$ onto the different subspaces $\mathcal{H}_k$ of the memory M employed during the measurement step $\mathcal{M}$ can be implemented unitarily without any energetic costs using an environmental system E. Here we prove this statement. This result on the energy cost of dephasing operations may be of independent interest[19]. The following Theorem 7 also improves the main result of ref.[20]: The lower bound from ref.[20] on the energy cost of a dephasing operation is negative whenever the dephasing changes the state, whereas our optimal lower bound is always exactly zero, Eq. (7).

Recall from Appendix A that the Hamiltonian of the memory is given by $H_M = \bigoplus_{k=1}^{K} H_k$, where H_k is a Hamiltonian on the respective subspace $\mathcal{H}_k$ with corresponding projection Q_k.

Theorem 7. *Let $T_M(\sigma_M) = \sum_k Q_k \sigma_M Q_k$ be the dephasing operation on the memory M. Then for any dilation of T_M,*

$$T_M(\sigma_M) = \mathrm{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] \qquad \forall \sigma_M \ , \tag{K1}$$

where σ_E is a thermal state on an environment E with Hamiltonian H_E, the corresponding energetic cost

$$E_{\mathrm{deph}} := \mathrm{tr}[H_{ME}(U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger - \sigma_M \otimes \sigma_E)]$$

is non-negative,

$$E_{\mathrm{deph}} \geq 0 \ .$$

Conversely, there exist σ_E and U_{ME} such that the energetic cost E_{deph} of the corresponding dilation of $T_M(\sigma_M)$ is precisely zero.

Proof. Our goal is to quantify the energetic cost $E_{\mathrm{deph}} = \mathrm{tr}[H_{ME}(\sigma'_{ME} - \sigma_{ME})]$ of implementing T_M unitarily as in (K1), where $\sigma_{ME} := \sigma_M \otimes \sigma_E$ and $\sigma'_{ME} := U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger$ denote the initial and final state of M and E, respectively. Due to the direct sum structure of $H_M = \bigoplus_k H_k$ (see Appendix A) we know that $[Q_k, H_k] = 0$, which implies that the dephasing operation T_M does not change the average energy on M. Hence, all energy expenses of this implementation are due to energy changes in the environment E,

$$E_{\mathrm{deph}} = \mathrm{tr}[H_E(\sigma'_E - \sigma_E)] \ . \tag{K2}$$

The initial state of the environment is, by assumption, thermal, i.e. $\sigma_E = \exp(-\beta H_E)/Z_E$ with Z_E the partition function. The final state of E on the other hand can be characterised by applying Lemma 5 from which we know that there exist states $\sigma_{E,k}$ with $S(\sigma_{E,k}) = S(\sigma_E)$ for all k such that

$$\sigma'_E = \sum_k \mathrm{tr}[Q_k \sigma_M]\sigma_{E,k} \qquad \forall \sigma_M \ . \tag{K3}$$

But then

$$E_{\mathrm{deph}} = \mathrm{tr}[H_E(\sigma'_E - \sigma_E)] = \sum_k \mathrm{tr}[Q_k \sigma_M]\,\mathrm{tr}[H_E(\sigma_{E,k} - \sigma_E)] \geq 0 \ ,$$

where the last inequality is due to the non-negativity of the relative entropy D which implies that the thermal state minimizes the average energy on an entropic orbit, $0 \leq \beta\,\mathrm{tr}[H_E(\sigma_{E,k} - \sigma_E)] = D(\sigma_{E,k}, \sigma_E)$ for all k.

We now show the second part of Theorem 7, that is the existence of a thermal state σ_E and a unitary U_{ME} that implement the dephasing channel T_M, i.e.

$$\mathrm{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] = \sum_k Q_k \sigma_M Q_k \quad \forall \sigma_M \ , \tag{K4}$$

at vanishing energy cost, $E_{\mathrm{deph}} = 0$. Instead of just naming possible σ_E and U_{ME} to achieve the goal, we provide a characterisation of all unitaries U_{ME} which, given any fixed full rank state σ_E, satisfy (K4). In the end, we will describe a simple explicit construction.

Consider a pure state $\sigma_M = \psi = |\psi\rangle\langle\psi|$ on M in the support of a fixed projector Q_k, i.e. $Q_k\psi = \psi$. Then from (K4) we know that the marginal on M,

$$\mathrm{tr}_E[U_{ME}(\psi \otimes \sigma_E)U_{ME}^\dagger] = \psi \ ,$$

is pure, while by (K3) the marginal on E is

$$\mathrm{tr}_S[U_{ME}(\psi \otimes \sigma_E)U_{ME}^\dagger] = \sigma_{E,k} \ ,$$

where $\sigma_{E,k}$ is a state which is independent of ψ. This implies

$$U_{ME}(\psi \otimes \sigma_E)U_{ME}^\dagger = \psi \otimes \sigma_{E,k} \ . \tag{K5}$$

Let us now denote the spectral decomposition of σ_E by $\sigma_E = \sum_j \lambda_j |\phi_j\rangle_E\langle\phi_j|$. We note that all eigenvalues λ_j are strictly positive since σ_E is assumed to have full rank. Substituting the decomposition of σ_E into (K5) we obtain

$$\sum_j \lambda_j U_{ME}(\psi \otimes |\phi_j\rangle_E\langle\phi_j|)U_{ME}^\dagger = \psi \otimes \sigma_{E,k} \ . \tag{K6}$$

We therefore know that

$$\mathrm{tr}_E\left[U_{ME}(\psi \otimes |\phi_j\rangle_E\langle\phi_j|)U_{ME}^\dagger\right] = \psi \qquad \forall j \ ,$$

i.e. the marginal on M of each of the pure states $U_{ME}(\psi\otimes|\phi_j\rangle_E\langle\phi_j|)U_{ME}^\dagger$ must be pure itself and identical regardless j – otherwise the marginal of the sum could not be pure as required by (K6). A unitary U_{ME} that satisfies (K4) with full rank states σ_E is therefore always of the form

$$U_{ME}\left(|\psi\rangle_M \otimes |\phi_j\rangle_E\right) = |\psi\rangle_M \otimes V_\psi|\phi_j\rangle_E \ , \tag{K7}$$

with a unitary V_ψ depending on $\psi \in Q_k$. Note however that the right-hand side of (K7) must be linear in ψ due to the linearity of the left-hand side. Hence, V_ψ can only depend on the label k of the subspace corresponding to Q_k, so we write $V_k = V_\psi$.

Up to now we have evaluated (K4) for pure states $\sigma_M = \psi$ in the support of some Q_k only. To characterise the unitaries V_k we evaluate (K4) on all mixed initial states σ_M on M to find on one hand that

$$\mathrm{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] = \sum_k Q_k\sigma_M Q_k \ . \tag{K8}$$

On the other hand we can employ (K7) to compute

$$U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger = U_{ME}\Big(\sum_{ij} Q_i\sigma_M Q_j \otimes \sigma_E\Big)U_{ME}^\dagger = \sum_{ij} Q_i\sigma_M Q_j \otimes V_i\sigma_E V_j^\dagger \ .$$

which together with (K8) yields

$$\sum_{ij} \mathrm{tr}_E\left[V_i\sigma_E V_j^\dagger\right]Q_i\sigma_M Q_j = \sum_k Q_k\sigma_M Q_k \qquad \forall\sigma_M \ .$$

This holds if and only if the unitaries V_k satisfy

$$\mathrm{tr}[V_i\sigma_E V_j^\dagger] = \delta_{ij} \quad \forall i,j \ , \tag{K9}$$

implying that the unitaries V_k must form an orthonormal unitary operator basis with respect to the modified scalar product (K9). Such orthonormal unitary operator bases only exist if the Hilbert space dimension d_E of the environmental system E is sufficiently large compared to the number K of possible outcomes k, i.e. $d_E \geq \sqrt{K}$. For further properties of unitary operator bases we refer to ref.[S5].

Hence, given a full rank state σ_E on E, any unitary U_{ME} satisfying (K4) is of the form (K7) with unitaries V_k that meet the condition (K9).

To show that there exists an implementation of the dephasing channel (K4) with vanishing energy cost E_{deph}, we may therefore choose the Hamiltonian of the environment E to be trivial, $H_E = 0$, implying that the initially thermal state of E is maximally mixed for all β, $\sigma_E = \mathbb{1}_E/d_E$, and that the energy cost is $E_{\text{deph}} = 0$ by (K2). The corresponding unitary U_{ME} that implements the dephasing channel is given by $U_{ME} = \sum_k Q_k \otimes V_k$ with unitary operators V_k satisfying $\text{tr}[V_i V_j^\dagger] = d_E \delta_{ij}$ for all i, j, which can be easily checked by evaluating (K4)

$$\begin{aligned}\text{tr}_E[U_{ME}(\sigma_M \otimes \sigma_E)U_{ME}^\dagger] &= \text{tr}_E\left[\Big(\sum_j Q_j \otimes V_j\Big)\Big(\sigma_M \otimes \mathbb{1}_E/d_E\Big)\Big(\sum_k Q_k \otimes V_k^\dagger\Big)\right] \\ &= \frac{1}{d_E}\sum_{j,k} Q_j \sigma_M Q_k \,\text{tr}[V_j V_k^\dagger] \\ &= \sum_k Q_k \sigma_M Q_k \ .\end{aligned}$$

The unitaries V_k can for example be chosen as distinct elements from the set of unitaries

$$V_{l,m} = \sum_{r=0}^{d_E-1} \mathrm{e}^{\frac{2\pi i}{d_E} rm} |l+r\rangle\langle r| \ , \qquad l, m = 0, 1, ..., d_E - 1 \ ,$$

where the addition in $|l + r\rangle$ is taken modulo d_E. These d_E^2 operators can be understood as a discrete version of the Heisenberg-Weyl operators and indeed satisfy, as one can easily compute,

$$\text{tr}[V_{l,m} V_{s,t}^\dagger] = d_E \cdot \delta_{l,s}\delta_{m,t} \qquad \forall l, m, s, t \in \{0, ..., d_E - 1\} \ .$$

□

Appendix L: Work cost of quantum measurements and the Second Law of Thermodynamics

In this section we argue why our results on energy costs developed in this paper are, in fact, also statements about thermodynamic work.

More concretely, we argue that all energy costs of an implementation of a quantum measurement stem from unitary dynamics U only, so that thermodynamic work cost is given as the average energy change[35,S6,S2]

$$W = \text{tr}[H(U\rho U^\dagger - \rho)]$$

of a system with Hamiltonian H initially in the state ρ. Indeed, the energy expenses of the measurement step $\mathcal{M}$ are due to the unitary U_{SM} and not the projections $\{Q_k\}$ as shown in Appendix K and the resetting step is unitary by construction. Hence, the overall work cost W_{cost} of conducting a general quantum measurement is exactly equal to the energy cost E_{cost} in our results.

As a consequence of this identification of energy and work, the energy extraction example in Appendix F illustrates a means to *extract useful thermodynamic work* from the measurement device. This may seem intriguing in the context of the Second Law of Thermodynamics: Discussions on the net work gain in a whole cycle of a Szilard engine typically assume that no work can be extracted in the measurement itself and argue that all work gained in the extraction phase of the cycle is completely cancelled by the cost imposed by Landauer's principle[10] for resetting the memory that stores the measurement outcome[S7].

We show however that our findings do not contradict the Second Law of Thermodynamics because of the following reasoning: When considering the overall work gain of a Szilard engine that employs a measurement device as described in our work extraction example (Appendix F), one needs to incorporate the cost of completing the thermodynamic cycle by restoring the initial pure state on S. This restoring step consumes all work gained during the measurement. Indeed, one finds by Eq. (D3)

$$\begin{aligned}W_{\text{cost}} &\geq \Delta E_S + k_B T \Delta S \\ &= \Delta F_S + k_B T(S(\rho'_S) - S(\rho_S)) + k_B T \Delta S \\ &= \Delta F_S + k_B T\Big(S(\rho'_S) - \sum_k p_k S(\rho'_{S,k})\Big) \\ &\geq \Delta F_S \ ,\end{aligned} \tag{L1}$$

where $\Delta F_S = F_S(\rho'_S) - F_S(\rho_S)$ is the free energy change in the system during the measurement, which corresponds to the work cost of the aforementioned restoring step of the measured system S. The inequality (L1) follows from the concavity of the von Neumann entropy (see (E5)).

This shows that the overall work expense in a full thermodynamic cycle that includes measurements is always non-negative and proves the validity of the Second Law in our general setting.

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