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Applications of KdV

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1. Introduction

The discovery of the remarkable interaction properties of solitary wave solutions to KdV by Zabusky and Kruskal [62] and the invention by Gardner, Greene, Kruskal and Miura [9] of the Inverse Spectral Transform for the solution of the Cauchy problem for KdV stand as two of the most far-reaching breakthroughs in the development of modern nonlinear mathematical science. Of all the completely integrable systems discovered since 1967, KdV certainly remains the most fully understood and arguably the most important for applications to macroscopic phenomena and processes.

Here we shall survey some of these applications, with emphasis on several cases where there is extensive experimental confirmation of the predictions of KdV theory. Not surprisingly, many of these applications are in fluid mechanics – not surprisingly, because fluid mechanics continues to maintain its seminal role in physics as the discipline in which many key nonlinear structures were first discovered (shocks, bifurcations, solitons, deterministic chaos, hypercomplex systems, ...) and in which the underlying theories receive their most convincing experimental validation. As the prototypical integrable nonlinear system, KdV has also had enormous indirect impact on many parts of theoretical physics, pure mathematics, and the areas in between. Vast areas of mathematics, including ordinary differential equations, algebraic geometry, Lie group theory, differential geometry and asymptotics have been opened up 'on the back', as it were, of the solving of KdV, and brought to bear on issues in quantum field theory, string and conformal field theory, quantum gravity and classical general relatively, to say nothing of the myriad applications in concrete settings of other famous integrable systems including nonlinear Schrödinger (NLS) and sine-Gordon (SG). These latter applications range from condensed matter and semiconductor physics through nonlinear optics and laser physics, hydrodynamics, meteorology and plasma physics to protein systems and neurophysiology. Robin Bullough (private communication) has devised a 'map' of considerable complexity, showing the astonishingly rich range of interconnections between mathematical and physical structures in integrable nonlinear systems, while the applications receive many treatments, though not in recent survey form, in the very many monographs and collections of reviews on soliton systems that now exist, and in the large number of volumes of Proceedings of Advanced Research Workshops and Advanced Study Institutes organized under the NATO Special Programme *Chaos, Order and Patterns: Aspects of Nonlinearity*, to say nothing of the large number of articles that have appeared not only in the long-established scientific journals, but also in the dozen or so journals devoted to nonlinear science that have sprung up in the early 1990's.

It is tempting to include with KdV its one-dimensional modified form MKdV and its two-dimensional counterpart KP, but to do so would be to divert attention from the remarkable success of KdV theory, to which we turn in a moment. Amusingly, the first application of KdV theory was actually made some years or even decades before the publication of KdV in 1895, as pointed out by Bullough and Caudrey ([4, p. 374]). John Scott and Russell, in his book *The Wave of Translation in the Oceans of Water, Air and Ether*, published posthumously in 1882, applied the KdV formula for the propagation speed of a soliton, discovered experimentally by Russell himself, to two problems. First, given the velocity of sound, the depth of the Earth's atmosphere was calculated to be five miles on the basis that sound is evidently carried by waves free of distortion and therefore by Scott Russell's solitary waves. Remarkably, this prediction is correct numerically, or would be if the matter in the atmosphere were distributed with uniform density. The same argument was used, second, with the velocity of light as the propagation speed for a distortion-free signal, to infer the radius of the universe as 5×10^{17} miles. Bullough and Caudrey point out that this estimate is wrong by at least five orders of magnitude, and in any event is incorrect not least in using a value of g which Russell arbitrary reduced by a factor 10^{-5}. Largely free of contention, however, are Russell's own experiments on solitary waves in water channels in which, as is now well known, he not only correctly extracted the propagation velocity from measurements, and convincingly showed that solitary waves of depression are impossible and that an arbitrary initial elevation would break up into a finite number of solitary waves, but also that the interaction between solitary waves had the particle-like property which was not picked up for more than a further century.

It is appropriate to turn, for our first example, to modern careful experiments in the water-wave problem that started the whole subject off.

2. Surface Gravity Waves

Weakly nonlinear surface waves on uniform water of shallow depth provided the first recorded sighting of a soliton, the first formulation, by Boussinesq and by

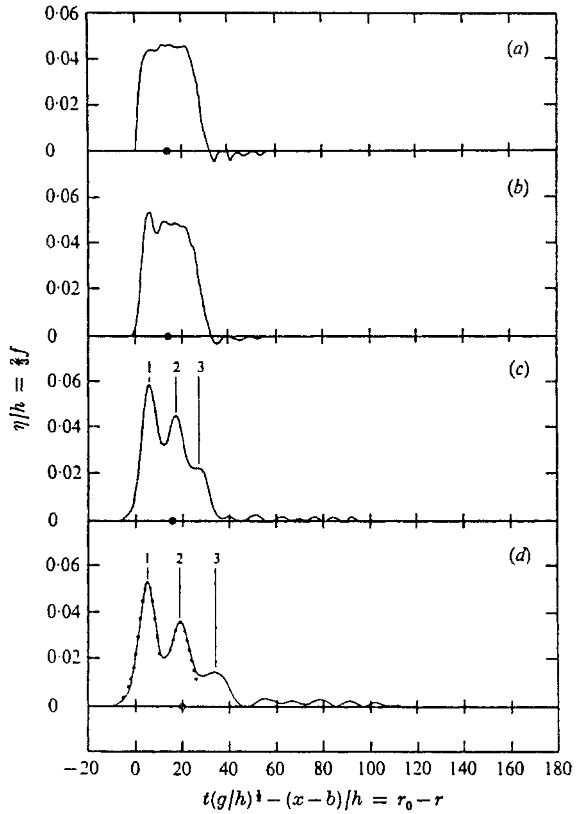

Fig. 1. Emergence of KdV solitons of elevation in water-channel experiments. The plots correspond to measuring stations (a) $x/h = 0$, (b) $x/h = 20$, (c) $x/h = 180$, (d) $x/h = 400$. Plotted on the abscissa is retarded time, so that a soliton should, in the absence of dissipation, shift to the left as time increases. (From Hammack and Segur [13], with permission.)

Korteweg and de Vries, of model equations to describe it, and the first example through KdV of an integrable system solvable by IST. These waves also find, in the work of Hammack and Segur [12–14], one of the most complete and convincing confirmations of IST theory for KdV. This work comprised experiments in a tank of length 31.6 m and breadth 39.4 cm, with waves of typical length-scale $10h$ and amplitude $0.05h$ generated in water of depth $h = 5$ cm or 10 cm by the controlled and recorded action of a rectangular wavemaker at one end. Measurements were taken for $h = 5$ cm at $x/h = 0, 20, 180, 400$ and for $h = 10$ cm at $x/h = 0, 50, 100, 150, 200$; in each case the first two stations correspond to linear propagation, the last two to ranges equivalent to the long time-scale for separation out of nonlinear KdV features. The initial wave profile was measured in each case and used to determine the number of solitons expected ($N = 0, 3, 4$ typically).

First, the prediction of N was very satisfactorily checked, and then confirmation that the emerging features (see Figure 1) were KdV solitons was made by measuring the amplitude of each feature and plotting points, as the dots in Figure 1, on the profile of an isolated KdV soliton with that amplitude. Second, an attempt was made to predict the amplitude of the leading soliton at the most remote measuring station. For this it was necessary to determine the largest Schrödinger eigenvalue of the initial waveform, and then to use results due to Keulegan [26] for the viscous decay, mainly in side-wall and bottom boundary layers, of a solitary wave of the corresponding amplitude. The viscous corrections are large enough to prevent any of the soliton velocities exceeding $(gh)^{1/2}$, which they should all do in lossless theory (the corresponding excess over the linear speed was detected, and found to be in good agreement with theory, in the propagation of pressure pulses in a tube of bubbly liquid: see Section 4), but if they are taken into account, the decaying amplitude and speed of the surface wave solitons can be reasonably well predicted.

But especially valuable in the work of Hammack and Segur is the study of the radiation corresponding to the continuous part of the spectral problem, particularly in the case of an initial wave of depression, which has no discrete eigenvalues and leads only to decaying oscillations. Although for realistic initial wave profiles there are no exact solutions of KdV involving radiation, the asymptotic evolution as $t \to +\infty$ is well understood in overall structure (see [1, Section 1.7c]).

(i) For $x \gg t^{1/3}$ the solution $\eta(x,t)$ is exponentially small.

(ii) For $|x| \lesssim t^{1/3}$ the solution is self-similar, and given by the solution $\eta = (x + x_0)/6t$ of the nondispersive simple wave equation $\eta_t + 6\eta\eta_x = 0$.

(iii) The transition away from (ii) takes place in a thin 'collisionless shock' at $x \sim -t^{1/3}(\ln t)^{p+2/3}$ for some p, $0 \leqslant p \ll 1$.

(iv) Beyond the collisionless shock, for $(-x) \gg t^{1/3}(\ln t)^{p+2/3}$, the solution breaks into a set of decaying wave packets, each packet having many oscillations at a fixed wavenumber and travelling at the (linearized) group velocity for that wavenumber. The nodes separating adjacent wave packets are defined by the zeros of the reflexion coefficient of the spectral problem for $\eta(x, 0)$.

Figure 2, from Hammack and Segur [13], shows that this structure is indeed borne out in experiment; and in the paper just quoted, the prediction of the leading wave (the simple wave ramp) is brought even more closely into agreement with experiment if account is taken of viscous effects. This leading wave is completely different, in form and strength, from that predicted by the solution of the linearized KdV with the same initial data (see Hammack and Segur [13]).

An interesting and important issue also settled by Hammack and Segur [13] is as to whether the presence of extraneous fine-scale waves, often present in practice and not correctly described by KdV, might invalidate the KdV predictions. That it would not was shown by driving the wavemaker with the same

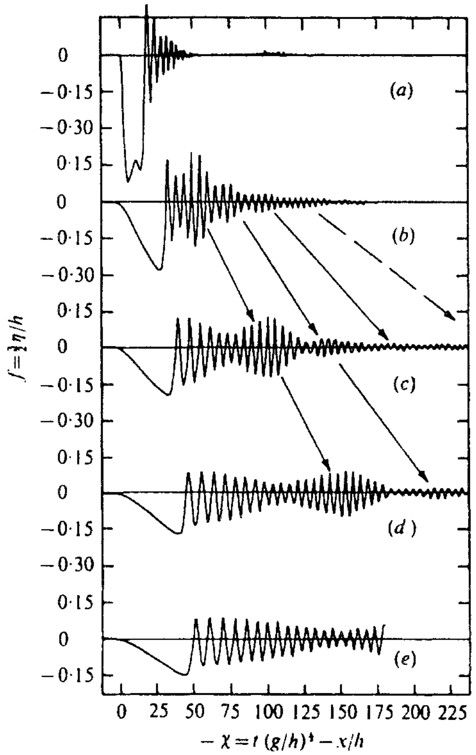

Fig. 2. Dispersive radiation produced by an initial depression of a water-channel surface. The plots correspond to measuring stations (a) $x/h = 0$, (b) $x/h = 50$, (c) $x/h = 100$, (d) $x/h = 150$, (e) $x/h = 200$. Plotted on the abscissa is retarded time. The arrows indicate wave packet trajectories based on the average wavenumber of each packet and the linear dispersion relation. Note the simple wave ramp at the left. (From Hammack and Segur [13], with permission.)

mean motion in each of three experiments, but with different types of high-frequency fluctuation superimposed. The fine-scale fluctuations were found to disperse rapidly, and at $x/h = 400$ precisely the same (four) solitons were found, identical to within experimental accuracy.

Segur and Hammack [55] used the same water channel to study internal waves at the interface between two layers of fluid of different densities. (The dispersion relation has two branches, in general distinct, one representing surface gravity waves, one internal waves localized at the density discontinuity.) If $kH \ll 1$ where H is the total depth, then in terms of a coordinate X translating with the nondispersive long-internal-wave speed and a long time variable τ proportional to the square root of the (small) density difference, the interface elevation η satisfies, after scaling, KdV in the form

$$\eta_\tau + 6(h_1 - h_2)\eta\eta_x + \eta_{xxx} = 0. \tag{2.1}$$

Solitons were generated by the operation of the wavemaker, and the interface displacement η measured. Close agreement was again found between measured wave profiles and those of individual KdV solitons, and agreement in other aspects was much improved by the inclusion of viscous effects at the interface, as well as in the side-wall and bottom boundary layers, through an extension of the Keulegan theory [27]. Very similar conclusions – resoundingly in favour of KdV – were reached by Koop and Butler [31], who also provided theoretical extensions of KdV to include higher-order nonlinearity.

If $h_1 = h_2$, the coefficient of the quadratic term in (2.1) vanishes, and then MKdV is the appropriate evolution equation, but for a still longer time-scale with τ now proportional to the density difference itself. The corresponding propagation length for the emergence of MKdV solitons may then be beyond the range of the Segur and Hammack facility.

Returning to KdV internal solitons, the theory for these has proved crucial in understanding some extraordinary observations of very large amplitude waves in the Andaman Sea, and we now turn to Osborne's work on this topic.

3. Internal Solitons in the Ocean

The internal solitons studied in the laboratory by Segur and Hammack [55] have been seen at full scale in the seas in the Far East, with amplitudes dangerously large in terms of human activities. In the Andaman Sea, between Sumatra and Thailand, severe sub-surface currents were experienced by oil drilling rigs, one of which, in a depth of 1100 m, was spun around through 90 degrees and translated more than 30 m as a large-amplitude internal soliton propagated along the thermocline beneath it. Satellite photographs confirmed, through surface wave activity of much lower amplitude, and generated by an internal-surface wave resonance mechanism, the presence of these solitons in very distinctive groupings, and Osborne and Burch [49] made extensive *in situ* temperature and velocity measurements in the Andaman Sea over a four-day period in 1976. The analysis of their work lends strong support to the KdV theory of internal solitons (rather than the competing weakly-nonlinear models, namely the intermediate long-wave equation for intermediate depth and the Benjamin–Ono equation for large depth). Similar observations of large-amplitude internal solitons were made in the Sulu Sea by Apel *et al.* [2]. The longevity of these waves is attested to by the fact that the research vessel followed an internal soliton, of wavelength 1700 m and amplitude 100 m, for more than two days as it propagated at about 8 km/h. Since then many studies have shown the ubiquitous presence of oceanic internal solitons where there are topographic features to excite them via tidal action, as, for example, in the Strait of Messina, the Strait of Gibraltar, and the Gulf of California, but not, for example, in the Gulf of Mexico.

The situation in the Andaman Sea is reviewed by Osborne [48]. The internal waves can be thought of, in first approximation, as localized on the thermocline at a depth of several hundred metres, and at such depths the vertical excursions during the passage of an internal soliton are large, of the order of 100 m, with much smaller surface waves of amplitude 1–2 m. Nevertheless, the surface waves are locked by the resonant coupling to the internal wave, are similarly one-dimensional, with coherent crests extending over 100 km or more, and for each soliton are confined to bands about 1 km wide normal to the crests. These surface wave 'rips' are very prominent in satellite photographs, in which each band in a rather regularly spaced sequence of six or more appears as a dark striation separating bright bands of wave-free activity between the rips. Successive sequences are separated by 12.4 h, the local tidal period. Topographical obstructions to the tidal flow in different locations generate such packets of solitons and rips travelling in different directions, but satellite photographs indicate that each packet is able to maintain its identity over hundreds of kilometres despite oblique interaction with other soliton packets.

All this, together with the shapes of the pulses measured in the displacements of isotherms (or isopycnals) and the fact that generally the pulses are rank-ordered by amplitude and correspond to a depression of the thermocline, suggests an underlying soliton mechanism, and Osborne [48] compares the merits of KdV, the intermediate depth equation of Kubota, Ko and Dobbs [32], and the Benjamin–Ono equation [3, 47]. The typical pulse widths are of order 1–3 km and the depth between 300 m and 1100 m, so the temptation is to assume the depth small compared with the pulse width and expect that (of these three models, all integrable) KdV will provide the best approximation. That is confirmed by the analysis of Osborne and his collaborators, with extensive measurements over the full water depth and analysis for a two-layer model and for continuous density profiles. The measurements show that throughout most of the depth, and certainly for a considerable range on either side of the thermocline, the wave-induced particle velocities are independent of depth, as required in the approximations leading to KdV.

For the analysis the streamfunction is written as $\psi(x, z, t) = \phi(z)\eta(x, t)$, where the vertical mode shape $\phi(z)$ is determined by a differential eigenvalue problem and $\eta(x, t)$, which to the same approximation represents the displacement of an isopycnal, satisfies KdV

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0. \tag{3.1}$$

With a two-layer model (upper and lower layer thicknesses and densities h_1, ρ_1, and h_2, ρ_2) the coefficients are simply

$$c_0 = \left[\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)\left(\frac{2gh_1h_2}{h_1 + h_2}\right)\right]^{1/2},$$

$$\alpha' = -\frac{3c_0(h_2 - h_1)}{2h_1 h_2}, \qquad \beta = \frac{c_0 h_1 h_2}{6}, \tag{3.2}$$

for $\rho_2 - \rho_1 \ll \rho_1, \rho_2$. Since $h_2 > h_1$ here, a soliton is, as observed, one in which the thermocline is depressed. In the detailed application of KdV theory, Osborne and his collaborators computed the coefficients in (3.1) by numerically solving the eigenvalue problem using measured mean density profiles and performing the necessary integrals involving the eigenfunction (only the lowest mode was considered, other measurements showing that more than 90% of the total energy was carried by this mode) and the mean state.

A time series of recorded data in the Andaman Sea was then used as the scattering potential for the Schrödinger problem, and the number and magnitudes of the discrete eigenvalues were calculated numerically, along with the reflexion coefficient as a function of scattering frequency (or wavenumber). In some cases, as many as 12 bound states were predicted, and the first seven of these gave soliton magnitudes in good agreement with observed maxima in the time series. Attention was also paid to the continuous spectrum, which was generally dominated by a single lobe in the frequency range for which KdV is applicable, corresponding to a single radiative wave packet in the time series (cf. Section 2).

Some anomalies remain, for example, in the observation that in some similar experiments involving internal waves in the ocean, the solitons were not always ordered according to amplitude. This may simply reflect the fact that a sufficiently large time had not elapsed to quality for the asymptotic limit for the particular complicated evolution in question, or it may have to do with the fact that the infinite-line spectral problem is not strictly appropriate to these problems where there is periodic tidal excitation. This would be settled if, as promised by Osborne, the periodic-initial-value problem for KdV were solved by numerical implementation of the periodic spectral transform. Whatever the outcome, it is clear that these large-amplitude waves (large in relation to human marine activities) are actually sufficiently weakly nonlinear and of sufficiently long wavelength that they should be described by KdV, and that KdV theory does indeed give an excellent description of all their essential features.

4. Nonlinear Acoustics of Bubbly Liquids

No account of applications of KdV should omit reference to theoretical and experimental work on the propagation of nonlinear acoustic waves in liquids with small volume concentrations of gas bubbles. Such a suspension has remarkable acoustic properties, even for very small volume concentrations of gas bubbles. First, for example, for air bubbles in water at standard pressure and temperature, the low-frequency sound speed c_0 is of the order of 40 m s^{-1} for a concentration $\alpha_0 = 0.1$, not simply well below the pure water sound speed $c_l = 1500$ m s^{-1} but also well below that of the pure gas phase (340 $\mathrm{m\,s^{-1}}$). This can be understood by noting that in the expression c_0^2 = (bulk modulus)/(density) the bubbles endow the suspension simultaneously with the low modulus of the gas phase and the high density of the liquid phase. These low values of c_0 imply that nonlinear effects are much more important for bubbly liquids, even with very small bubble concentrations, than for monophase liquids.

Second, the bubbly liquid exhibits strong dispersion. For a lossless suspension of identical-size bubbles, the acoustic phase speed $c_{\mathrm{p}}(\omega)$ is reduced, parabolically, below c_0 for low frequencies, and reaches zero at the bubble monopole resonance frequency ω_0. There is then a forbidden band $\omega_0 < \omega < \omega_1$ in which $c_{\mathrm{p}}(\omega)$ is purely imaginary and all motion is purely reactive, and then $c_{\mathrm{p}}(\omega)$ becomes real again, decreasing from $+\infty$ to essentially c_l as ω increases from ω_1 to ∞. If losses are included, there is a small real part of $c_{\mathrm{p}}(\omega)$ in the forbidden band, and a high value of $\mathrm{Im}\, c_{\mathrm{p}}(\omega)$ there, peaking at resonance. These features have been substantially confirmed by experiment. See, for example, the review by van Wijngaarden [58], in which, in his Figure 3, we see measured phase speeds as low as 500 $\mathrm{m\,s^{-1}}$ and as high as 2800 $\mathrm{m\,s^{-1}}$ in experiments at different frequencies on the *same* bubbly mixture with α_0 very small, $\alpha_0 = 2 \times 10^{-4}$.

The parabolic form of $c_{\mathrm{p}}(\omega)$ for $\omega/\omega_0 \ll 1$ and the presence of the usual convective derivatives in the mass and momentum conservation equations indicate that low-frequency pressure pulses of moderate amplitude should satisfy KdV, or (Burgers) BKdV if dissipative effects are significant. Two groups have done impressive analytical and experimental studies for such cases. That led by van Wijngaarden was the first to carry out experiments to verify qualitatively the predictions of KdV theory for soliton formation and of BKdV theory for the formation of shocks with either monotonic or oscillatory structure depending upon the strength of dissipation relative to dispersion. Later, van Wijngaarden and his student Roelofsen carried out experiments in which a triangular compression pulse was delivered by a piston at one end of a tube of bubbly mixture. For such initial data the Schrödinger spectral problem for KdV can be solved explicitly in terms of Airy functions, and the number of solitons expected can be predicted in closed form as a function of the duration and amplitude of the initial wave and the properties of the medium. Gratifying agreement was obtained between the experiment and theory, as is explained in more detail in van Wijngaarden's contribution to these Proceedings. This also amounts to an impressive confirmation of the theory of bubbly liquids, modelled as two co-existing continua. There is a spectacular difference between prediction or observation for a pure liquid or gas phase, where the triangular pulse simply lengthens as $t^{1/2}$, remaining triangular of constant area, and with a leading diffusion or relaxation-controlled shock with amplitude decreasing as $t^{-1/2}$; and the prediction or observation of the response of a liquid with a minute gas bubble concentration, where a finite number of rank-ordered solitons may be produced, with no shock-like features under appropriate circumstances.

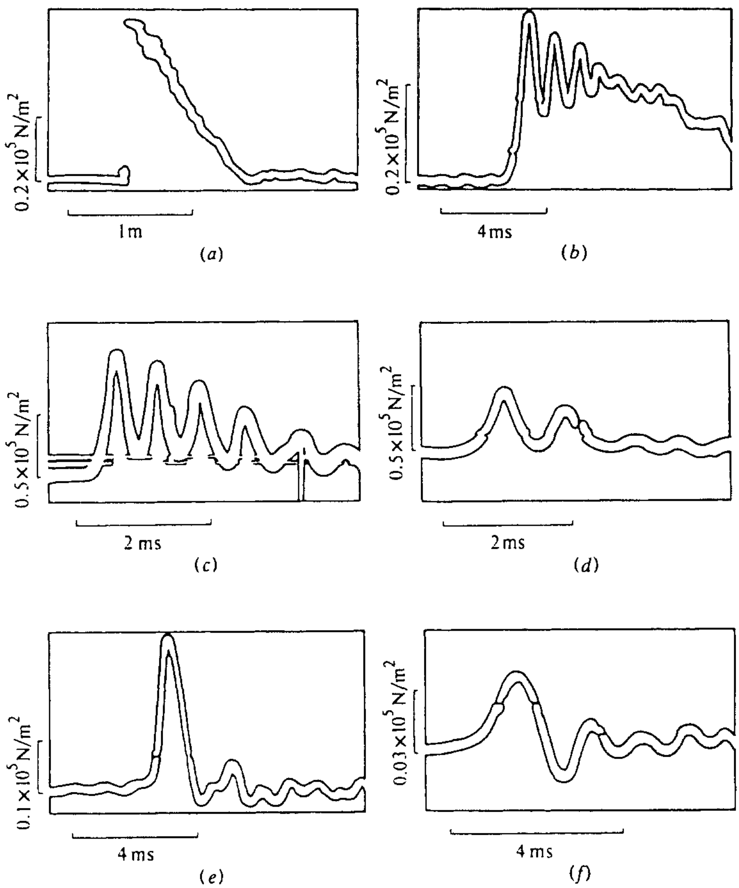

Fig. 3. Experimental observations of oscillatory shock and soliton features in propagation of pressure pulses in liquid with CO_2 bubbles. (From Kuznetsov *et al.* [33], with permission.)

Similar theoretical and experimental studies were also conducted in Novosibirsk (see Kuznetsov *et al.* [33] and Nakoryakov *et al.* [40]). A BKdV equation for the pressure $p(\xi, \tau)$,

$$\frac{\partial p}{\partial \tau} + p\frac{\partial p}{\partial \xi} - \frac{1}{\mathrm{Re}}\frac{\partial^2 p}{\partial \xi^2} + \frac{1}{\sigma^2}\frac{\partial^3 p}{\partial \xi^3} = 0 \tag{4.1}$$

was derived, in scaled variables, with Re a Reynolds number based on characteristic velocity and length-scales u_0, ℓ, respectively, of the initial signal and an effective viscosity coefficient incorporating viscothermal diffusion and acoustic radiation damping, and with σ a dispersion parameter,

$$\sigma = \ell_0 u_0^{1/2}\beta^{-1/2}, \qquad \beta = R_0^2 c_0/6\alpha_0(1-\alpha_0), \tag{4.2}$$

for a mixture of bubbles all of the same radius R_0, and with c_0 the low-frequency sound speed in the mixture. Low-amplitude wave packets were studied, and also larger-amplitude compression pulses produced by breaking a diaphragm separating a tube of bubbly liquid (CO_2 or He bubbles with $\alpha_0 \sim 0.01$) from a high-pressure chamber. Calculations were made for initial profiles of Gaussian form, and for triangular profiles corresponding to the diaphragm rupture.

For sufficiently small σ/Re, solitons were produced whose number agreed with spectral theory prediction. Reasonable agreement was obtained for the soliton propagation velocity (and, incidentally, for the propagation speed c_0 of low-amplitude wave packets) as a function of the medium parameters and soliton amplitude, for the dependence of soliton width on amplitude, and for the soliton shape, compared with KdV prediction. See Figure 3 above (Figure 10 of Kuznetsov *et al.* [33]) for oscillograms showing the emergence of 5, 2, 1 or 0 solitons in appropriate cases, together with dispersive radiation. The damping of solitons and wave packets was also studied, with agreement between theory and experiment which is astonishingly good given all the idealizations of the underlying model.

5. Voidage Slugs in Fluidized Beds

Multiphase media with dynamical microstructure will evidently display frequency dispersion, as is to be expected in a bubbly liquid where each micro-element has its own internal resonance, often lightly damped. It is less obvious that dispersion with only small dissipation will arise in the relaxation processes that take place in a gas or liquid phase with an included rigid particulate phase, such as one finds in dusty gas flows, in particulate matter sedimenting under gravity, or in fluidized beds as widely used in chemical processing. In a fluidized bed, gas or liquid is forced upwards under pressure in a vertical tube containing a fine bed of particles. When the fluidizing velocity exceeds a threshold, the bed expands – homogeneously one hopes – the particles being held at rest, on average, against gravity through their interactions with the fluid and with each other. But the homogeneously-expanded state is almost invariably instable, the instability leading in wide tubes to the production of large bubbles of particle-free fluid rising through the bed and creating a state visually similar to the boiling of a liquid. In sufficiently narrow tubes, a one-dimensional state persists, the instability here leading to 'slugging', the production of horizontal bands or slugs of particle-lean fluid propagating upward, with a region below each slug of increased particle concentration.

This instability is undesirable in technological applications, leading, for example, to highly uneven performance, excessive temperatures and fatigue damage. Much effort has been put into the modelling of fluidized beds (this is still a matter of considerable controversy) and into understanding the linear instability, although experimental study of the linearized stages is almost impossible. Weakly nonlinear analyses have been conducted by numerous authors, and Kluwick [28] was perhaps the first to show that in appropriate circumstances a fluidized bed or a sedimenting suspension would support KdV solitons. (In fact there is a change of sign of the nonlinear coefficient for a particular value $\phi_0 = n/(n+2)$ of the background voidage – the fraction of unit volume of mixture occupied by the gas phase – with n an empirically-determined constant in an assumed power law expression for the drag coefficient on a particle as a function of the local voidage ϕ, and Kluwick allowed for this, retaining quadratic and cubic nonlinearities to give an equation of Gardner type (KdV–MKdV).) However, weak nonlinearity is not necessarily stabilizing, and if the particle-phase interaction pressure is not sufficiently large, the essential linear instability will cause growth of the KdV solitons through what looks, for long waves, like negative diffusion. The question then is "what role does KdV have in the overall evolution, and what is the ultimate fate of any solitons that may be produced at the KdV stage of events?"

Harris and Crighton [15] sought to answer this question – and thereby in effect to show much more generally how the nongeneric integrable systems can be embedded, for limited ranges of time and space, in a much richer space-time evolution – by systematically analyzing one of several popular models for fluidized beds using asymptotic techniques model can, for a gas-fluidized bed, be reduced to the nondimensional system

$$-\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}[(1-\phi)v] = 0, \tag{5.1}$$

$$(1-\phi)\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = \frac{(1-\phi)}{F^2}\left(\frac{\phi_0}{\phi}\right)^{n+1}\left(1-\frac{v}{\phi_0}\right) - \frac{(1-\phi)}{F^2} - p_s'(\phi)\frac{\partial \phi}{\partial x} + \frac{1}{R}\frac{\partial^2 v}{\partial x^2}, \tag{5.2}$$

expressing conservation of the mass and momentum, respectively, of the particle phase which has voidage $\phi(x,t)$ and upward velocity $v(x,t)$. Terms on the right of (5.2) correspond, in order, to the drag force between particles and gas, to gravitational forces, to the interparticle pressure field, and to viscous effects in the particulate phase (this being treated as a Newtonian fluid with an isotropic interparticle pressure field and a deviatoric stress with a particulate phase viscosity). The parameters in (5.2) are the uniform background voidage ϕ_0, the empirical Richardson–Zaki index n, a Froude number F and a particle-phase Reynolds number R. Laboratory experiments in narrow tubes suggest $n = O(1)$, $R = O(1)$, $\phi_0 = O(1)$, and $F \ll 1$, so Harris and Crighton seek asymptotics for $F \to 0$ for fixed values of all other parameters.

Bypassing the linear instability and going straight to the weakly nonlinear stage via

$$\begin{aligned} v &= F^2 v_1 + F^3 v_2 + \cdots, \\ \phi &= \phi_0 + F^2\phi_1 + F^3\phi_2 + \cdots, \end{aligned} \tag{5.3}$$

with $X = x - (n+1)(1-\phi_0)t$, $\tau = F^2 t$, we get

$$\frac{\partial\phi_1}{\partial\tau} + \beta\phi_1\frac{\partial\phi_1}{\partial X} + \gamma\frac{\partial^3\phi_1}{\partial X^3} = -F\delta_0\frac{\partial^2\phi_1}{\partial X^2} + \mathrm{O}(F^2), \tag{5.4}$$

$$\beta = -(n+1)\left(n+2-\frac{n}{\phi_0}\right), \qquad \gamma = (n+1)\frac{\phi_0}{R}. \tag{5.5}$$

The left side contains cubic dispersion from what in the momentum equation for v was a diffusive term and predicts, for $\phi_0 < n/(n+2)$, the formation of KdV solitons of voidage greater than ϕ_0, rising vertically. The condition on ϕ_0 is usually satisfied, since typically $0.4 < \phi_0 < 0.5$ and $3 < n < 4$.

On the right of (5.4) the perturbing term is diffusive if $\delta_0 < 0$, which corresponds to a sufficiently strong particle-pressure $p_s'(0)$, and then perturbation theory [1, 24, 25, 29, 30] for nearly integrable systems shows that the KdV solitons simply decay adiabatically, with amplitude $\mathrm{O}(T^{-1/2})$, $T = F\tau$. If $\delta_0 > 0$ the linear elements of (5.4) represent the essential instability of the fluidized bed, and the solitons grow. (The problem is then ill-posed, but the high-wavenumber growth is actually controlled by a fourth derivative term (cf. Appendix A of Harris and Crighton [15], and Hayakawa *et al.* [16]).) Then we seek an expansion

$$\phi = \phi_0 + F^2\phi_1(X,\tau,T) + F^3\phi_2(X,\tau,T) + \cdots, \tag{5.6}$$

in which $\tau = F^2 t$, $T = F^3 t$, and ϕ_1 is an adiabatically changing soliton,

$$\phi_1 = \frac{12\gamma}{\beta}\kappa^2(T)\,\mathrm{sech}^2\big[\kappa(T)(X - X_0(T) - \xi)\big] \tag{5.7}$$

and where, as usual, it is important to uniformize the phase by writing

$$\xi = \frac{4\gamma}{F}\int \kappa^2(T)\,\mathrm{d}T. \tag{5.8}$$

Inspection of the problem for ϕ_2 gives the growth law

$$\kappa_T = \frac{8}{15}\delta_0\kappa^3, \qquad \kappa(T) = \left(\frac{T_0}{T_0 - T}\right)^{1/2}, \qquad T_0 = \frac{15}{16\delta_0}, \tag{5.9}$$

so that the soliton amplitude has a finite-time singularity. One can then complete the solution for ϕ_2 and by examining its asymptotics below the soliton we deduce the existence of the now-familiar 'shelf', in this context an extensive region of negative voidage. As in the first successful application of perturbation theory to KdV solitons, the amplitude change mandated by (5.9) is actually what one gets by substituting the slowly-varying soliton expression in the 'energy' balance equation obtained by multiplying (5.4) by ϕ_1 and integrating over all X. But the resulting soliton variation cannot simultaneously satisfy conservation of particles, expressed by integrating (5.4) itself over all X. In fact, particles must continually be lost from the soliton and they find their way into the shelf below, forming the particle-rich regions mentioned earlier. Although the amplitude of the shelf is formally small (as indeed is that of the soliton), it is proportional to $\kappa(T)$ and has the finite-time singularity too. This singularity suggests that the KdV solitons will amplify beyond the weakly nonlinear limit ($\phi - \phi_0 = \mathrm{O}(F^2)$), and possibly even to $\mathrm{O}(1)$.

Before that happens, however, other perturbations become comparable with, and then dominate over, the negative diffusion in (5.4), and the growth law changes. For $T_0 - T = F\widehat{T}$, $\widehat{T} = \mathrm{O}(1)$, various new quadratically nonlinear terms are comparable with diffusion, and the growth law – for the voidage fluctuation, which has now grown from $\mathrm{O}(F^2)$ to $\mathrm{O}(F)$, but is still given by a KdV soliton – is of the form

$$\kappa_{\widehat{T}} = b\kappa^3 + d\kappa^5, \tag{5.10}$$

and when the κ^5 term dominates

$$\kappa \sim \left[4d(\widehat{T}_0 - \widehat{T})\right]^{-1/4}, \tag{5.11}$$

so that the soliton continues to grow, but less rapidly. When $T_0 - T = \mathrm{O}(F^2)$ all perturbations are comparable and we must leave the KdV soliton in favour of a solution of the fully nonlinear system. However, we are permitted, through the matching backward in time, to seek a slowly-varying travelling wave solution of the full system, the slow variation being determined, once again, by examination of corrections to the leading order ($\mathrm{O}(1)$) voidage solitary wave.

Analysis of the travelling solitary wave is given by Harris and Crighton [15] along with a determination of the amplitude variation, and a proof that in the long-time limit the amplitude and propagation speed tend to *constant* values, dependent on the background voidage and related to each other in a certain functional way. Going backward in time, the expressions for the amplitude and velocity and for the solitary wave shape match asymptotically those for the explosively growing KdV soliton.

Computed solitary wave profiles are not much different from those of sech^2 solitons. Harris and Crighton [15] give estimates appropriate to typical small-scale laboratory experiments for the times and distances over which one may expect to see the full nonlinear development or merely the emergence of the weak solitons. For example, with glass beads of 50 μm diameter and a typical disturbance length-scale of 1 cm, the KdV soliton stage corresponds to upward propagation over about 0.5 m, and the fully developed with experiments has yet been made, but the theory provides a clear suggestion as to how the large-scale orderly structures initially form from general disturbances, and how they amplify to the large-amplitude state associated with voidage bands. It also provides the theory for the particle-rich shelf below each voidage band, and gives estimates for the length- and time-scales associated with solitons, solitary waves and shelves.

According to Harris and Crighton [15], other competing continuum models for fluidized beds lead to similar results; at some stage in the spatio-temporal development they all contain a KdV equation with unstable perturbation. This leads to the emergence of nascent voidage bands as solitons, though the details of their subsequent amplification and equilibration will differ with the model.

We have discussed this topic at some length because it is perhaps the first in which all stages of development in a complicated nonintegrable system can be at least partially analyzed and asymptotically related to each other. As part of this, the integrable KdV enters at one stage, and its solitons are subject to different types of perturbation as they grow. There is no need – indeed no possibility short of writing the whole system, very artificially, as a perturbed KdV system – to try to represent the system uniformly as perturbed KdV.

From this it is clear that although integrable, and indeed nearly-integrable, systems are a set of measure zero, a vast number of much more complicated systems generally far from integrable may contain an integrable element at some stage. Asymptotic methods tell us how to handle the complicated scenario, as above. Soliton large-scale order will emerge at the integrable stage, though whether it will persist, as here, and grow to large amplitude and a change of form, depends on the nonintegrable ingredients.

6. Magma Flow and Conduit Waves

KdV theory comes into play in several areas of geophysics. In conduit flows, buoyant fluid introduced below a layer of fluid of greater viscosity rises through a conduit which it creates, with buoyancy and viscous shear stress in balance for steady flow in a conduit of uniform area. If the supply rate varies, axisymmetric bulges propagate upward as conduit waves. Helfrich and Whitehead [17] offer simple elegant theory and laboratory experiments on large-amplitude conduit waves, and emphasize that they transport with them a substantial blob of recirculating fluid. It is thought that such essentially diffusion-free conduit-wave transport of magma from the Earth's interior to the surface could contribute to the formation of hot spots and of volcanic island chains.

If the conduit has area A and carries a volume flux Q, then mass conservation gives

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \tag{6.1}$$

while the momentum equation for a local Poiseuille flow and a small-slope approximation give

$$Q = A^2\left[1 + \frac{\partial}{\partial z}\left(\frac{1}{A}\frac{\partial Q}{\partial z}\right)\right], \tag{6.2}$$

where all quantities are dimensionless. In the weakly-nonlinear limit we replace Q by A^2 on the right of (6.2), substitute in (6.1) and drop certain derivatives on a long-waves basis. This immediately gives KdV for $A(z,t)$.

Olson and Christensen [46] studied solitary wave solutions of the full system (6.1), (6.2), showing, for instance, that the relation between propagation speed c and amplitude A_{m} is

$$c = (2A_{\mathrm{m}}^2 \ln A_{\mathrm{m}} - A_{\mathrm{m}}^2 + 1)/(A_{\mathrm{m}} - 1)^2, \tag{6.3}$$

which reduces to the KdV result for $(A_{\mathrm{m}} - 1)$ small, and that for $A_{\mathrm{m}} \gg 1$ the wave profile is Gaussian,

$$A(\xi) = A_{\mathrm{m}} \exp(-\xi^2/2c). \tag{6.4}$$

Helfrich and Whitehead tested these predictions, and those of KdV theory, against experiments in which controlled excitation of solitary waves was possible, and in which provision was made for marking – with dye – the fluid particles in a given solitary wave. They found that the predictions of the full nonlinear theory were generally well borne out in the experiments, which involved waves with A (nondimensionalized on the uniform area of the conduit with steady flow) as large as 20. The overtaking of one solitary wave by another was observed to take place essentially elastically, although for reasons as yet unexplained, the larger wave increased in amplitude by about 5% on average, and its speed decreased by 4%, while the smaller wave was unchanged in amplitude and speed to within experimental accuracy. Numerical solutions of (6.1) and (6.2) gave an over-prediction, by as much as 40%, of the phase shifts experienced in interaction. Clearly mass is not conserved by the two solitary waves in such interactions, and the excess must presumably be supplied unsteadily from the reservoir at the foot of the conduit.

As mentioned earlier, these conduit solitary waves transport matter with them, and for large amplitude waves the transported volume asymptotes to the whole volume excess

$$\int_{-\infty}^{+\infty} (A - A_0)(\xi)\,\mathrm{d}\xi$$

carried by a given solitary wave. This geologically significant transport property persists, though to a lesser degree, in the KdV limit. When a larger wave overtakes a smaller, matter from the smaller is retained in the larger afterward, while the smaller contains only fluid particles that were present in the original larger, as was shown very convincingly in the Helfrich and Whitehead [17] experiments.

Most of these studies, and indeed of the interest, refer to amplitudes far beyond the reach of the KdV limit, but KdV properties dominate the experiments, the way they are conducted, and the inferences drawn from them. Another geophysical process within the same general category is referred to as 'compaction-driven flow', where buoyant interstitial melt (corresponding to the intrusive conduit fluid) is forced through a deformable porous crystalline matrix representing the Earth's mantle, and corresponding to the viscous fluid exterior to a conduit. The functional relation corresponding to (6.2) is different for compaction-driven flows, but again KdV emerges in the weakly-nonlinear limit, with solitary wave solutions of the fully nonlinear system [51]. These solitary wave solutions have been called *magmons*, although numerical evidence is that the result of a collision between two magmons is not confined simply to phase shifts. The magmons are also quite different from the conduit waves in that they do not transport matter, the fluid particles simply experiencing a finite displacement in the passage of a magmon [54]. By contrast, conduit solitary waves are rather efficient in transporting matter right through the mantle with negligible diffusion; Whitehead and Helfrich [60] estimate that the occurrence of a single solitary wave (of the geologically plausible scales) every 500 million years would double the magma flux to the surface over what would be produced by steady flow in the same conduit!

7. Jupiter

Perhaps the boldest application of KdV to date has been to the understanding of the Great Red Spot (GRS) and other features in the Jovian atmosphere, seen in cloud patterns, such as the South Equatorial Disturbance, the Dark South Tropical Streak, the Hollow and the White Ovals and, in particular also, the South Tropical Disturbance (STD). The soliton theory is due to Maxworthy and Redekopp [38] and to Maxworthy, Redekopp and Weidman [39]. Although now seen to be inadequate quantitatively and even qualitatively, this work was highly influential ("a noticeable milestone in the history of contemporary ideas of the GRS nature" – Nezlin [43]) and contained many of the essential ingredients for current understanding in what has become, with the recent advent of high-quality spacecraft images, a topic of intense research interest. The reader is referred to Marcus [36], Nezlin [43], and a number of articles in the Focus Issue of *Chaos* (1994, Vol. 4, No. 2) coordinated by Nezlin, for accounts of the current position.

In rapidly rotating planetary atmospheres, the dominant linear wave modes are highly dispersive westward-propagating Rossby waves. The long waves have highest velocity and are weakly dispersive, with KdV-type cubic dispersion. Now in the 'shallow-water equations' for a thin atmosphere with a free upper surface, and with the β-plane approximation $f = f_0 + \beta y$ (y northward) for the local Coriolis parameter, a KdV nonlinearity $\beta h h_x$ arises (see [43, Equation (16)]) which has the effect of changing the phase velocity of long Rossby waves from $V_R = \beta r_R^2$ in the linear regime (r_R is the Rossby deformation radius, $r_R = (gH_0)^{1/2}/f_0$ in obvious symbols) to $V = V_R(1+h)$, $h = \delta H/H_0$, for a nonlinear solitary Rossby wave. Such a solitary wave can preserve itself over long times only if it is free of resonance (phase velocity matching) with any linear waves, so we must have $h > 0$, which means that the vortical structures producing the free surface elevation must be *anticyclonic*. By contrast, disturbances with cyclonic vorticity disperse by linear Rossby wave radiation. This anticyclonic circulation is completely in accord with almost all observations of large-scale organized structures in giant planets (Jupiter, Saturn, Neptune).

This further suggests that in a weakly nonlinear approximation, the vortical structure should be described by KdV, as first suggested in [38]. There the streamfunction for the quasi-geostrophic equations was expanded in the form of a mean horizontally-sheared east-west zonal shear flow plus a perturbation

$$\Psi = \int^{y} U(y)\,\mathrm{d}y + \varepsilon\psi(x, y, z, t) \tag{7.1}$$

and ψ taken as a single mode

$$\psi = A_n(\xi, \tau)\phi_n(y)\cos n\pi z + \text{integral harmonics}, \tag{7.2}$$

with scaled variables $\xi = \varepsilon(x - C_n^{(0)}t)$, $\tau = \varepsilon^3 t$ and with rigid top and bottom walls ($z = \pm 1$). Then A_n was found to satisfy MKdV for $n \neq 0$ (i.e. for baroclinic waves),

$$A_{n\tau} + \alpha_n A_n^2 A_{nx} + \beta_n A_{nxxx} = 0, \tag{7.3}$$

with coefficients defined through integrals of the basic state and the mode function $\phi_n(y)$ (itself the solution of an eigenvalue problem). The scalings $\xi = \varepsilon^{1/2}(x - C_0^{(0)}t)$, $\tau = \varepsilon^{3/2}t$ are needed for barotropic disturbances ($n = 0$), and A_0 satisfies standard KdV.

Maxworthy and Redekopp [38] calculated the coefficients for the simple shear flow $U(y) = S\tanh y$ ($S = \pm 1$ for anticyclonic and cyclonic basic shear) and displayed contours of constant ψ in a horizontal plane $z = \text{const}$ in coordinates travelling (westward) with a soliton solution of KdV or MKdV. They noted that such solitons have the forms, respectively,

$$A_0 = \operatorname{sgn}(\alpha_0\beta_0)\operatorname{sech}^2 X, \qquad A_n = \pm\operatorname{sech} X, \tag{7.4}$$

and so, at $z = 0$ say, one can have both an E-soliton of elevation and a D-soliton of depression for MKdV, corresponding to $\pm$ for A_n, but, for the same propagation speed only either an E-soliton or a D-soliton for KdV.

Of course, the features of a baroclinic D-soliton where $\cos n\pi z > 0$ become those of an E-soliton where $\cos n\pi z < 0$, and vice-versa. Typical streamline patterns for E- and D-solitons are shown in Figures 4a and 4b, together with, in Figure 4c, a suggested combination of D- and E-solitons on a jet profile modelling one of the east–west zones or belts on giant planets. This suggested structure was later confirmed by Maxworthy, Redekopp and Weidman ([39, Figure 16]), where analysis was carried out for a Bickley jet profile $U(y) = \operatorname{sech}^2 y$ and the MKdV equation. The morphology of Figure 4c is strongly reminiscent of

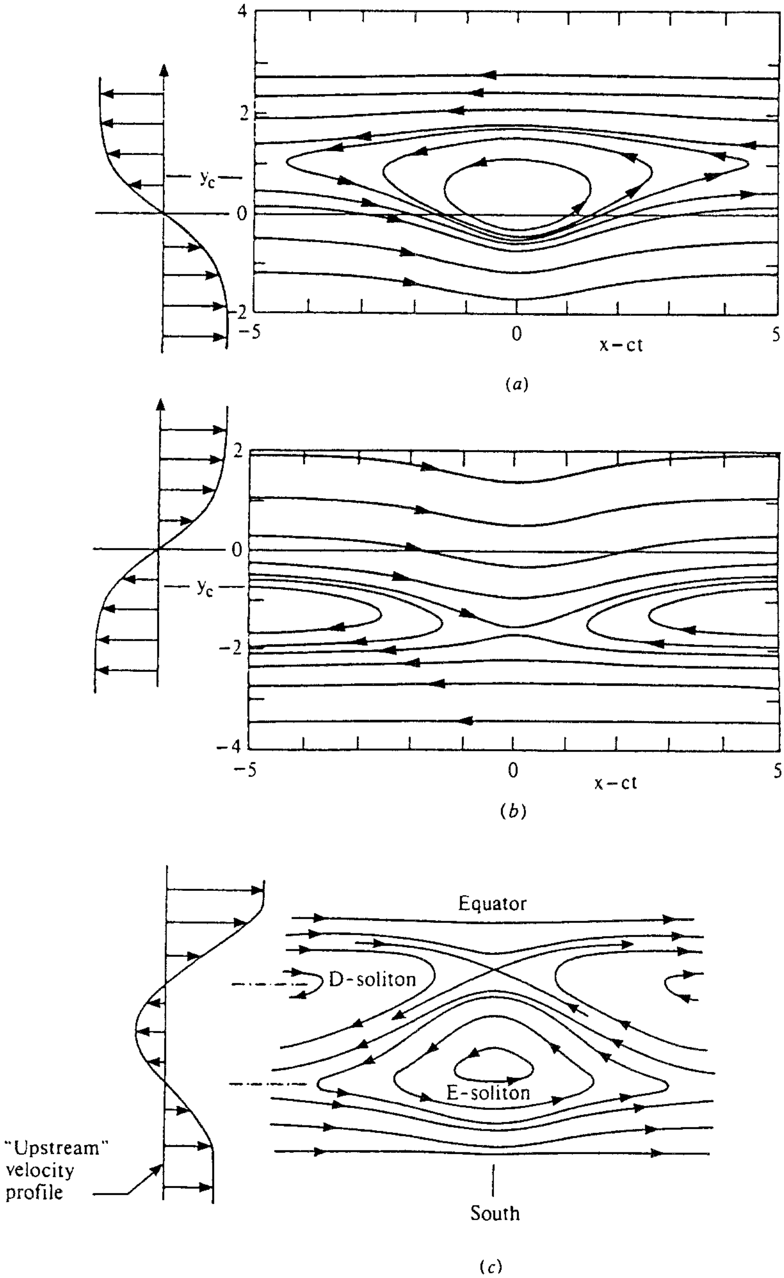

Fig. 4. (a) An E-soliton on an anticyclonic shear in the southern hemisphere; (b) A D-soliton on a cyclonic shear in the southern hemisphere; (c) A combined D–E soliton on an asymmetric jetlike profile. (Figure 4 from [38], with permission.)

the structure of the GRS and the Hollow immediately northward of it. Numerous other isolated Jovian features also have the typical structure of an isolated E- or D-soliton, the White Oval, for example, and the Dark South Tropical Streak, respectively.

A particularly striking aspect of these large-scale structures lies in their interaction properties, the most famous and well-documented of which [50] concerns that which occurred nine times between the GRS-plus-Hollow combination and the South Tropical Disturbance between 1901 and 1938. Peek's description of this has to be set alongside Scott Russell's vivid account of his first sighting of an individual soliton:

> "During the six weeks which would have been required for the p end of the Disturbance to pass from one end of the Hollow to the other... there was no sign whatever of any encroachment upon the region; instead, within a few days of its arrival at the f end of the Hollow, a facsimile of the p end of the Disturbance was seen... to be forming at the other end of the Hollow. ... The new development... proved to be a true p end of the Disturbance which drew away from the Red Spot at approximately the same rate (at which it had approached).... Thus its passage through... the Red Spot, which would have taken 3 months at its normal rate of progress, must have been accomplished in a matter of fourteen days."

Armed with KdV theory, it is impossible to refrain from speculating that Peek was describing the identity preservation of solitons and their generation of an abrupt phase shift on interaction. The phase shift in this case is equivalent to increasing the propagation speed of the STD from 7 $\mathrm{m\,s^{-1}}$ to 42 $\mathrm{m\,s^{-1}}$ as it traverses the GRS. Maxworthy, Redekopp and Weidman [39] analyzed the observations of this repeated interaction at length, modelling the GRS as an E-soliton and the STD as a double-D-soliton (while mentioning other possibilities for modelling the STD, namely a breather solution of MKdV, or a dispersive wave packet associated with the continuous spectrum). They calculated phase shifts not for their favoured MKdV system, but for Rossby waves in a homogeneous atmosphere. If the basic perturbation is the sum of two isolated solitary waves,

$$\psi = A_1(x,t)\phi_1(y) + A_2(x,t)\phi_2(y),$$

then the mode shapes $\phi_i(y)$ satisfy a barotropic Rayleigh equation, and the amplitudes satisfy coupled KdV equations,

$$A_{1t} + C_1 A_{1x} - 2r_1 A_1 A_{1x} - s_1 A_{1xxx} = \lambda_1 A_1 A_{2x} + \nu_1 A_2 A_{1x}, \tag{7.5}$$

$$A_{2t} + C_2 A_{2x} - 2r_2 A_2 A_{2x} - s_2 A_{2xxx} = \lambda_2 A_2 A_{1x} + \nu_2 A_1 A_{2x}. \tag{7.6}$$

The coefficients were evaluated for the periodic shear flow $U(y) = S\,\mathrm{sn}($[illegible]$\,m)$, and the interactions calculated by the asymptotic method of Oikawa and Yajima [45]. Direct comparison with the observations was not possible, however, because with the barotropic KdV solitons a D-soliton modelling the STD can approach an E-soliton modelling the GRS only from the east, and not from the west as on Jupiter. Nevertheless, the similarities between the various possible D–D, D–E and E–E interaction flow patterns and the observations are highly suggestive of the notion that although a realistic description cannot be attained at the level originally proposed by Maxworthy and Redekopp [38] (indeed, their rigid lid boundary condition precludes precisely the free surface variations that lead to the asymmetry between the anticyclonic and cyclonic solitary waves and thereby the exclusion, in agreement with observations, of the latter), there is nevertheless the crucial ingredient, within the more complete description, of an underlying integrable KdV model.

Finally, one might ask how large solitary Rossby waves are generated and maintained against dissipation. After all, the GRS has been observed now for more than 300 years, which exceeds by three orders of magnitude the lifetime of a Rossby wave packet of the same scale under linear dispersion mechanisms. Although it is possible that such waves are fed from some lower-atmosphere convective energy flux, it is more likely that they are formed from large-scale instability of the shear across the edges of the planetary belts and zones. Under the influence of linear amplification mechanisms, waves amplify to a weakly nonlinear stage, at which dispersion and nonlinearity enter in KdV or MKdV fashion. Integrability at this stage for the anticyclonic case implies the emergence of localized orderly soliton structures which may (cf. [38, p. 266]) be able to extract further energy from the shear and grow to larger amplitudes in which we have quasi-steady large-amplitude solitary Rossby waves with shear amplification and dissipation in balance. (Compare the corresponding evolution of large-amplitude solitary waves of voidage in fluidized beds, described in Section 5 above.) This would be consistent with the observations by Nezlin [43] of a laboratory model of Jovian flows where, in particular, he showed also how the number of such large features is a decreasing function of the velocity shear, only one feature, akin to GRS, being produced – and then maintained apparently indefinitely – for sufficiently strong shear.

Whatever the final theoretical framework might involve, it is clear that the astonishingly beautiful and well-ordered features of planetary atmospheres, and the attendant smaller-scale Lee waves, Kármán vortex streets, and hydraulic jumps – have been illuminated in an equally astonishingly beautiful way by KdV theory.

8. Plasma Physics

No catalogue of applications of KdV should fail to mention plasma physics. It was indeed in the context of hydromagnetic plasma waves [10] that the KdV equation was again encountered for the first time since 1895, and again in 1966 in the work of Washimi and Taniuti [59] on ion-acoustic waves in a cold plasma.

Ion-acoustic waves provided an early confirmation of the soliton aspects of KdV theory in the experiments of Hershkowitz, Romesser and Montgomery [18], the generation and interaction of ion-acoustic solitons having been demonstrated already by Ikezi, Taylor and Baker [23]; and Hershkowitz and Romesser [19] made observations of cylindrical ion-acoustic solitons before the cylindrical KdV has either been derived [37] or shown to be integrable [5].

Confining attention to one-dimensional ion-acoustic waves in a plasma with cold ions, negligible ion pressure and an isothermal perfect gas law for the electron field, one has, in the absence of any current, the set of four equations

$$\frac{\partial n_\mathrm{i}}{\partial t} + \frac{\partial}{\partial x}(n_\mathrm{i} v_\mathrm{i}) = 0, \tag{8.1}$$

$$\frac{\partial v_\mathrm{i}}{\partial t} + v_\mathrm{i}\frac{\partial v_\mathrm{i}}{\partial x} = E, \tag{8.2}$$

$$n_\mathrm{e} E + \frac{\partial n_\mathrm{e}}{\partial x} = 0, \tag{8.3}$$

$$\frac{\partial E}{\partial x} = n_\mathrm{i} - n_\mathrm{e}, \tag{8.4}$$

for the ion velocity v_i and number density n_i, the electric field E and the electron number density n_e. All quantities are nondimensional, the ambient state has $n_\mathrm{i} = n_\mathrm{e} = 1$, $v_\mathrm{i} = E = 0$, and ions and electrons are assumed to carry the same magnitude of charge. Equations (8.1)–(8.4) are, respectively, the expressions of mass and momentum conservation for the ions, a force balance equation for the electrons, whose inertia in a low-frequency wave is negligible, and Poisson's equation for the electric field given the charges.

The long-wave dispersion relation from (8.1)–(8.4) is

$$\omega^2/k^2 = 1 - k^2 + \mathrm{O}(k^4), \tag{8.5}$$

and therefore when this dispersion (valid for $\omega \ll \Omega_{\mathrm{p_i}}$ where $\Omega_{\mathrm{p_i}}$ is the ion-plasma frequency, and closely analogous to the dispersion in a bubbly liquid for $\omega \ll \omega_0$ (cf. Section 4)) is coupled with the convective derivatives in (8.1) and (8.2) in an appropriate weakly-nonlinear weakly-dispersive limit it is evident that fluctuations n_i' in the ion density must satisfy KdV (there being no symmetry under which the quadratic term would vanish).

Hershkowitz, Romesser and Montgomery [18] generated solitons from various square-wave input voltages in a narrow column of cold plasma, and were able to predict accurately the number of solitons produced in any given case from the well-known solutions to the Schrödinger equation for square-well potentials. The calculation also gave the soliton amplitudes in reasonable agreement with experiment, and the experiment further gave good agreement with theory on the speed-amplitude-width relation for KdV solitons. Ikezi [21] reported on similar experiments with one half-cycle of a cosine-squared wave parameters were reasonably predicted, the number of solitons observed was systematically greater than predicted, for reasons which seem not to have been explained. Ikezi [21] also showed the results of interaction between unequal solitons travelling in the same direction, and between equal solitons travelling in opposite directions. In the former case identity preservation was observed, though with very strong amplitude loss in both waves because of large damping experienced in the long interaction period; in the latter case, modelled by a Boussinesq equation, interaction was rapid and close to that expected theoretically. A further interesting point of these experiments was an observation of FPU recurrence. A sinusoidal input signal was transformed to one dominated by two solitons per period at 5 cm from input and back effectively to the original sinusoidal at 9 cm (excitation frequency 0.35 MHz, ion-plasma resonance frequency $\Omega_{p_i} = 4\pi$ MHz). Theory of the recurrence for the two-solitons-per-period case was developed by Ikezi, Taylor and Baker [23] and Tappert and Judice [57].

For the plane wave case, experimental observations of KdV solitons were also made in the context of the Trivelpiece–Gould plasma wave by Ikezi *et al.* [22] and Saeki [52]. Ion-acoustic plasma waves provided the first experimental evidence of cylindrical KdV solitons [19]. The plasma was contained within a circular cylinder, excitation being provided by the application of a bias voltage across a surrounding cylindrical plasma sheath. Break-up of the input signal was observed, with soliton-like pulses emerging in both the ingoing and outgoing (from the axis) waves for compression input, the soliton velocities exceeding the linear ion-acoustic low-frequency speed. The solitons had a pronounced asymmetry, trailing a wake generally familiar for two-dimensional linear waves, and a scaling law asserting constancy of the product of soliton width and square root of amplitude, and following from an adiabatically varying KdV soliton approximate solution to the cylindrical KdV, was found to be tolerably well satisfied. Application of rarefaction excitation produced no evidence of soliton-like structures.

Since the time of these early experiments, Cylindrical KdV has been proved integrable [5] and much of the apparatus of integrable systems has been at least partially established for it (e.g., Bäcklund transformations, together with a proof that BTs exist only for Cylindrical KdV out of the whole class of Generalized KdV equations [44]). Consequently it should be possible now to produce key relations for experimental test, though to the author's knowledge this has not been attempted. The experiments in plasma are difficult and costly at the scales that would allow detailed confirmation of CKdV theory. The two-wave nonlinear equation for weakly-nonlinear cylindrical waves (the counterpart to the Boussinesq equation for plane waves) is probably not integrable, so observation should be restricted to one-way waves in the period prior to any reflexion, and a preferable configuration maintaining the assumptions underlying CKdV for longer would involve excitation of a large annulus of plasma at some small inner radius with initially purely diverging waves. Experiments to test CKdV for cylindrically diverging waves on the surface of shallow water seem just as difficult to execute.

9. Electrical Transmission Lines

In the early days of soliton mathematics and physics, there were numerous simulations of physical systems through experiments on long transmission lines, and analytical and numerical solutions of discrete and continuum models for them. Such lines are relatively cheap and easy to construct, and versatile, in that they often lead to equations of the form

$$u_t + \alpha u^m u_x + \beta u_{xxx} = \delta u_{xx}, \tag{9.1}$$

where the relative importance of dissipation and dispersion is controllable, where the nonlinearity index m can be controlled by suitable choice of nonlinear elements, and where there is also the possibility of spatial variation of the coefficients. The nonlinear elements are typically variable-capacitance diodes, or saturating ferromagnetic inductances.

Lonngren [35] gives a review of numerous physical experiments on lines which may be as long as 20 m with as many as 1000 nominally identical sections. The typical nonlinear wave equation for the line voltage V, in the continuum limit and nondimensional variables, is

$$\frac{\partial^4 V}{\partial x^2 \partial t^2} + \frac{1}{LC}\frac{\partial^2 V}{\partial x^2} - \frac{1}{C}\frac{\partial^2 Q(V)}{\partial t^2} = 0, \tag{9.2}$$

where L, C are linear inductance capacitance, and the charge $Q(V)$ held by a nonlinear capacitance is assumed to have the form $Q(V) = C_0 V - C_N V^2$ as $V \to 0$. The linear dispersion relation is

$$\omega^2/k^2 = (LCk^2 + LC_0)^{-1} \tag{9.3}$$

and we readily find KdV,

$$\frac{\partial u}{\partial \tau} + \frac{C_N}{C_0} u \frac{\partial u}{\partial \xi} + \frac{C}{2C_0}\frac{\partial^3 u}{\partial \xi^3} = 0 \tag{9.4}$$

in appropriate variables for right-running waves. In principle it would be possible to go to higher nonlinearity and a higher leading-order dispersive term by appropriate choice of linear and nonlinear circuit elements.

Experiments on a line for which (9.4) should hold confirm quite reasonably the main predictions, namely the propagation of linear dispersive waves of either polarity in the linear limit; the emergence in strong waves of a definite number of compression solitons ($V > 0$) and a decaying oscillatory tail together conserving $\int_{-\infty}^{+\infty} V(t)\,dt$ along the line, with merely decaying radiation for rarefaction input signals; the amplitude–speed–width relation for KdV solitons; and identity preservation in both overtaking and head-on collisions. The generation of shocks with oscillations on the high-V side was also found for input signals corresponding to shock-type boundary conditions $u(\xi = -\infty) = u_1$, $u(\xi = +\infty) = u_2 > u_1$, also in accord with KdV theory. Fissioning of a KdV soliton into two solitons was demonstrated by Stewart and Corones [56] on a line comprising three segments, the first and third of uniform but different linear properties, with a spatial variation of dispersion coefficient in the middle section. FPU recurrence of an initial sinusoid, again for the two-solitons-per-period case as in Section 8, was observed by Hirota and Suzuki [20].

In all these cases dissipation is weak compared with dispersion. Gorshkov *et al.* [11] report similar experiments where MKdV is the basic equation and, in one limit, where the modified Burgers equation is appropriate. Among the distinctive features of solutions of that equation is the production of sonic shocks in finite time and their maintenance as sonic shocks thereafter, with an unusual shock profile; this is in agreement with asymptotic theory, for small diffusivity, for the (nonintegrable) modified Burgers equation.

Concluding his review, Lonngren [35] suggested that

> "With the rapid development of microcircuit technology, it is not inconceivable that 'pocket size' soliton (transmission line) experiments could be eventually built and sold to serious soliton students."

That this suggestion has not been taken up is perhaps simply a reflexion of the fact soliton theory has rapidly acquired an accepted place in so many areas of science and that numerical simulation of soliton processes has becomes so easy.

10. Epilogue

There are countless other contexts in which KdV arises, of which just a few can receive a brief note here.

First, although in the water wave context the first application was to the profile of long waves, KdV also turns up in modulation problems where one immediately thinks instead of NLS (see, e.g., [42, pp. 40–48]). For the modulation on long scales of a carrier wave we do have NLS,

$$ia_T + \tfrac{1}{2}\,\omega_0'' a_{\xi\xi} + \beta a|a|^2 = 0, \tag{10.1}$$

for the amplitude $a(\xi, T)$, where ω_0'' refers to the dispersion relation at the carrier wavenumber k_0 and β is defined in terms of k_0 and the medium properties. However, if $\beta\omega_0'' > 0$ the monochromatic wave given by $|a| = A = A_0$, $\phi = \arg a = \beta A_0^2 T$ is unstable to long-wave perturbations (the celebrated Benjamin–Feir instability), but if $\beta\omega_0'' < 0$ the perturbation $\tilde{A}$, where $|a| = A = A_0 + \tilde{A}$, evolves under KdV, with solitons and radiation. The solitons have $\tilde{A} < 0$ and represent a local reduction in intensity of the otherwise uniform intensity, and are called *dark solitons*.

Second, although the issue of energy transport over long ranges in the protein α-helix is dominated by Davydov solitons (see [6] and [34] for a recent review) on a discrete lattice, with the Zakharov equations of plasma physics and NLS in the continuum limit, KdV nevertheless arises in this field too. In the Yomosa [61] model, the emphasis is different, and is on the large-amplitude dynamics of the peptide groups in the nonlinear hydrogen-bonded spines of the α-helix, rather than on the interaction, as in Davydov theory, between a molecular mode and a low-frequency intermolecular acoustic mode. In the Yomosa lattice model we have a polypeptide chain with the nth peptide group undergoing a displacement u_n, with $r_n = u_{n+1} - u_n$ the extension of the nth peptide bond, and with an assumed potential $V_n(r_n) = Ar_n^2 - Br_n^3$ for the nth hydrogen bond. The equation of motion of the nth group is

$$m\ddot{r}_n = 2A(r_{n+1} + r_{n-1} - 2r_n) - 3B(r_{n+1}^2 + r_{n-1}^2 - 2r_n^2), \tag{10.2}$$

and in the continuum limit $n\ell \to x$, $r_n \to r(x,t)$ with right-propagating waves we get KdV,

$$u_\tau - 6uu_\xi + u_{\xi\xi\xi} = 0 \tag{10.3}$$

with

$$\xi = x/\ell - (2A/m)^{1/2}t, \quad \tau = (2A/m)^{1/2}t/24, \quad u = (6B/A)r. \tag{10.4}$$

The KdV solitons are supersonic, relative to the sound velocity $V_0 = (2A/m)^{1/2}\ell$. In his review Lomdahl [34] says that such supersonic lattice KdV solitons represent a 'reasonable alternative' to the Davydov and Takeno models for long-range coherent transport of biological energy.

In the field of solid mechanics, a number of authors, beginning with Nariboli and Sedov [41], have given analysis leading to KdV for the propagation of longitudinal waves in a nonlinearly elastic medium, but no successful confirmation of the KdV predictions has come from experiments. Indeed, Samsonov and Sokurinskaya [53] showed that for a thin circular rod of radius R, the weakly-nonlinear wave equation is not Boussinesq,

$$v_{tt} - c_0^2 v_{xx} = \frac{1}{2}\left[\frac{\beta}{\rho}v^2 + \nu^2 R^2 v_{tt}\right]_{xx}, \tag{10.5}$$

which leads to KdV under the one-wave restriction, but rather, through coupling of longitudinal and transverse waves, a double-dispersion equation

$$v_{tt} - c_0^2 v_{xx} = \frac{1}{2}\left[\frac{\beta}{\rho}v^2 + \nu^2 R^2 (v_{tt} - c_1^2 v_{xx})\right]_{xx}. \tag{10.6}$$

In these, v is the longitudinal strain, ρ and ν the density and Poisson's ratio, β a nonlinearity parameter and c_0, c_1 the speeds of linear P and S waves. Equation (10.6) has sech^2 solitary wave solutions, which coincide with those of KdV in the appropriate limit, though (10.6) appears generally nonintegrable. Experiments were carried out by Dreiden *et al.* [8] on the excitation of intense longitudinal waves in a polystyrene rod, for which all the numerous restrictions leading to (10.6) were reasonably well satisfied. These were the first in which such solitary waves had successfully been identified in solid matter, and confirmed the solitary wave features predicted by (10.6) to the exclusion of the earlier predictions of KdV.

In many of these applications, the initial stimulus came from the recognition that some complicated partial differential or difference system would reduce to KdV in an appropriate limit, suggesting an association between observed long-lived large-scale orderly structures and KdV solitons. Currently, the focus is much more on large-amplitude phenomena, well beyond the range of KdV theory and exemplified by the *magmons* of compaction-driven or conduit flows in geophysics (cf. Section 6), the *planetons* (large-amplitude Rossby waves (cf. Section 7)) introduced by Dewar [7], and perhaps, to coin a new term, the *voidons* (large-amplitude voidage slugs (cf. Section 5)) in fluidized beds. This shift of focus is reflected in the early dates, in the history of soliton systems, associated with the many discoveries of the application of KdV theory to diverse physical phenomena. It must be seen as a tribute to Korteweg and de Vries that their celebrated equation has had such profound effects, not simply on its own terms as a rich differential equation, nor simply for the great range of phenomena that it describes in some limited part of parameter space, but also for the impetus it has given to the analysis of more complex and general fully nonlinear systems.

The material for a review such as this is now to be found in literally thousands of papers in journals covering many disparate branches of science. I apologize to all those many researchers whose work has inevitably been overlooked, or omitted, or inadequately described in the above, and I hope that at least some feel has been given for the importance – impossible to exaggerate – of KdV in applications.

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