Computing the Universe, Dynamics I: Newton's equations of motion, conservation laws, degrees of freedom, phase space

Martin Weinberg

June 21, 1999

1 Getting from here to there…

The main goal of these four dynamics lectures is provide a framework for understanding the evolution of matter under the influence of its own collective gravity. In general, these problems are impossible to solve exactly. One might say, "We all know Newton's Laws, let's just code 'em up and let 'er rip!" The problem is that this, too, is impossible and requires a myriad of approximations as you will see in the next few weeks. The only way to make progress with confidence is to develop a deeper, more general understanding of the underlying structure implied by Newton's Laws. An example of such a use of smarts rather than brawn you are familiar width is exploiting symmetries that lead to conservation laws. I will begin with these today but this sort of thinking leads the even deeper and more general approaches of Hamilton and Jacobi, and to ideas of modern dynamics pioneered by Kolmorgov, Moser, Sinai, Arnold and others.

2 Newton's Laws: a local description of evolution

Let's begin with a brief, but unusual review of Newton's Laws. There are three basic concepts, nearly common sense for physicists, that lead axiomatically to Newton's Laws:

1. Space & Time: space is Euclidean (3-d) and time is one dimensional ($R^3 \times R$).
2. Galilean relativity
   - Laws of physics are the same at all times in inertial reference frames;
   - All coordinate systems in uniform rectilinear motion are themselves inertial.
3. Newton's Principle: The initial state of a system (positions and velocities of its points) uniquely determine its future.

Now, let's define what we mean by motion:

$$\begin{aligned} \text{motion:} &\quad \mathbf{x}(t_o) \to \mathbf{x}(t_o + dt) \\ \text{velocity:} &\quad \frac{d\mathbf{x}}{dt} \\ \text{acceleration:} &\quad \frac{d\mathbf{x}^2}{dt^2} \\ \text{etc.} & \end{aligned} \tag{1}$$

We can visualize this as a system of points moving in configuration space with time:

Finally, *derivation* of Newton's equations. According to determinacy, the motion of a system is determined uniquely by its initial positions and velocities. *BUT NOT* its acceleration. Therefore, if

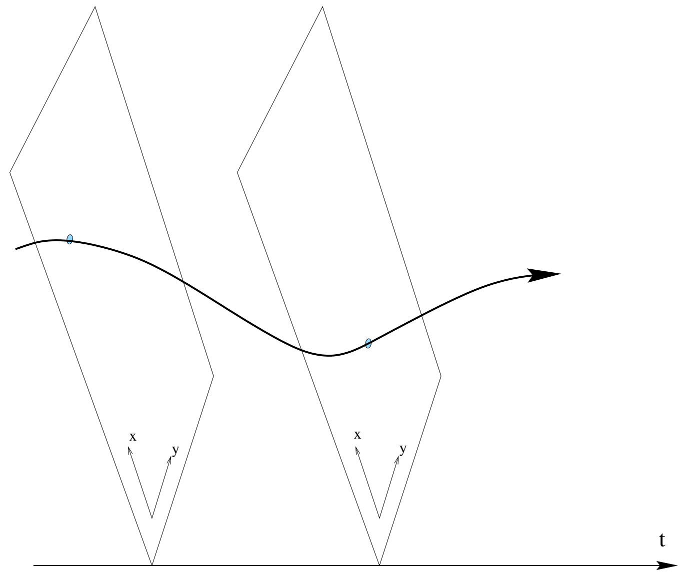

Figure 1: Motion of the system

each point in the system is not at inertial rest, then the position and velocities must determine the acceleration:

$$\ddot{\mathbf{x}} = \frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t).$$

By looking at symmetries of this equation and using the 3 principles, one can straightway, derive the all of the classic conservation laws such as the independence to an absolute origin, isotropy of space, and others. For the remainder of today, I will describe some simple but important examples: 1-d, 2-d and 3-d motion.

3 1-d motion

Physicists commonly use 1-d systems to understand basic principle. However, the 1-d dynamical system is *very* special and quite different than general reality. Nonetheless, it has the advantage of being fully solvable (remember, in general, Newton's Laws can not be solved).

Newton's equation for a particle in 1-d motion is:

$$\ddot{x} = f(x).$$

We usually define the kinetic energy for such as system as:

$$T = \frac{1}{2}\dot{x}^2 = \frac{1}{2}mv^2.$$

Similarly the work done is the force expended along the path. Since the path is one-dimensional, we have:

$$U(x) = -\int_{x_o}^{x} f(q)dq$$

where $x_o = x(t_o)$ and the minus sign accounts for the work done *on* the system. Note that

- $U(x)$ determines $f(x)$ uniquely[1]: $f(x) = -dU/dx$;
- $U(x) + U_o$ leaves the equations of motions unchanged.

The total energy is $E = T + U$. Alternatively, we know what to do with equations like this. Multiplying by $\dot{x}$, we have a perfect differential:

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = \dot{x}\ddot{x} = \frac{dU}{dt} = \dot{x}(\ddot{x} - f(x)) = 0.$$

Ok, this should have been mostly review. Now let's take a different point of view. The equation of motion, $\ddot{x} = f(x)$ is a second order, non-linear differential equation. One may always write an n-th order ODE as n first-order ODEs. In this case, the equations of

[1] This only works in 1-d in general.

motion are equivalent to

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= f(x).\end{aligned} \tag{2}$$

Rather than considering this to be a formal scheme for finding the explicit solution $x = p(t)$, consider the solution in coordinates x, y:

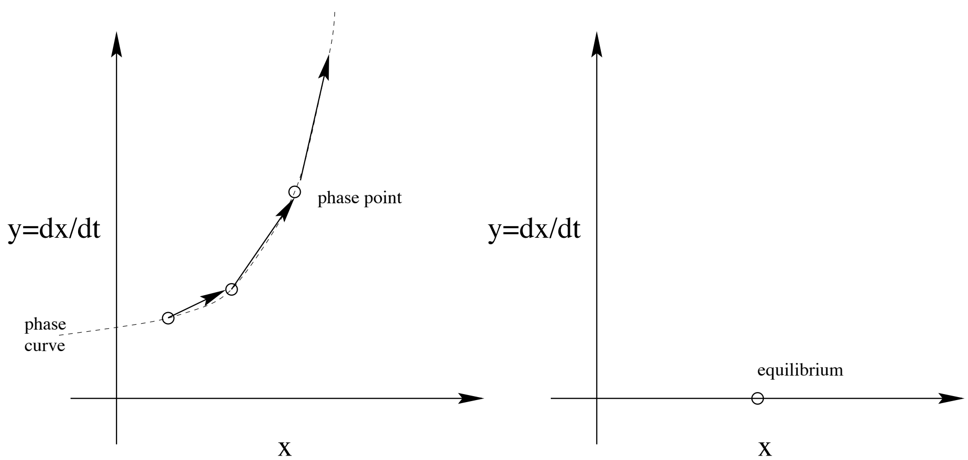

Figure 2: The phase curves for a one-dimensional system of motion

Note that this construction is true to the spirit of Newton's Principle.

3.1 Examples

3.1.1 Simple harmonic motion

3.1.2 General potential

3.1.3 Solution for phase curves

4 Systems with many degrees of freedom

Nearly all systems of interest will have many degrees of freedom. However, one can not solve the general case of an arbitrary potential with even two degrees of freedom! We have to rely on examples and attempt to identify fundamental models (e.g. simple harmonic motion).

4.1 2-d motion

The equation of motion becomes:

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{f} = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, etc. This is easily generalized to higher dimensions.

Following the one-dimensional case, we call the system *conservative* if $\mathbf{f} = -\frac{\partial U}{\partial \mathbf{x}} = -\nabla U$. Continuing, we have:

$$\begin{aligned} \ddot{\mathbf{x}} &= -\nabla U(\mathbf{x}) \\ \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} &= -\dot{\mathbf{x}} \cdot \nabla U(\mathbf{x}) \\ \frac{d}{dt}\frac{\dot{x}^2}{2} &= -\frac{d}{dt}U(\mathbf{x}) \end{aligned}$$

which immediately leads to $dE/dt = 0$. As before, if we know $\mathbf{x}, \dot{\mathbf{x}}$ at $t = t_o$, then the trajectories like within the domain $\mathbf{x}$ s.t. $U(\mathbf{x}) < E$. In standard lingo, *the point is trapped in the potential well.*

As in the one-dimensional case, we may look at the solution geometrically. The second-order ODE in two variables becomes four first-order couple ODEs. Three-dimensional phase curves describing the motion on the now four-dimensional *space*[2]. Projecting the phase curves into (x_1, x_2) give the *trajectory* or *orbit* of the motion. Note:

- The orbits may self intersect but not the phase curves (Why?).

[2] Can't call it a plane any more...

- We may say that the conservation of energy, E, restricts the phase curves to a three-dimensional hypersurface of phase space. Often, other conservation laws are able to further restrict the dimensionality of the phase curves.

For every degree of freedom, the phase space has two dimensions. In particular:

For 1 degree of freedom,	we have 2-d space.
For 2 degrees of freedom,	we have 4-d space.
For 3 degree of freedom,	we have 6-d space.
And for 3N degrees of freedom,	we have a 6N-d space.

However, in one-dimension, we have one conserved quantity which means that motion is one-dimensional. Similarly, in two-dimensions, conservation of energy leads to motion in not more than a three-dimensional space. In a two-degree of freedom system, a additional conserved quantities may be found. For example, the potential $V(x_1,x_2) = (x_1^2 + qx_2^2)/2$ has admits two energy like conserved quantities because the system is separable. Each new conservation law (or *integral of the motion*) reduces the dimensionality of the hyperspace by one. And each one corresponds to some symmetry. (To what symmetry does the conservation of energy correspond?) This is a simpler statement of a deep theorem in Lagrangian dynamics called Noether's Theorem which states that conservation laws correspond to symmetries and vice versa.

5 Central fields

So far, everything we have discussed has been for a general force. Gravity has some very special properties. Consider the work done along a path in space for a conservative force. The work done is:

$$W = -\int_{s_1}^{s_2} d\mathbf{s} \cdot \nabla U = -\int_1^2 dU = U(2) - U(1).$$

In other words, the work done in going from Point (1) to Point (2) does not depend on the path taken. This implies that the work done along a round trip vanishes. Using Stokes' theorem:

$$\oint_C d\mathbf{s} \cdot \mathbf{f} = 0 = \int_S da\, (\nabla \times \mathbf{f}) \cdot \mathbf{n},$$

where S is the area inclosed by the path C and $\mathbf{n}$ is the outward normal to path. This implies that $\nabla \times \mathbf{f} = 0$ and therefore $\mathbf{f} = -\nabla U$. So we have proven: *A force field is conservative if and only if the work along any path depends only on the end points of the path.*

A potential is called *central* (with center at 0) if it depends only on the radius from the center. Clearly, every central field is conservative. In addition, since we may express the central force as $\mathbf{f} = f(r)\mathbf{e}_r$, we have:

$$W = \int_{s_1}^{s_2} d\mathbf{s} \cdot \mathbf{f} = \int_{r_1}^{r_2} dr f(r)$$

and it follows that

$$U(r) = -\int^r dr f(r),$$

just like a one-dimensional system! For example, Newton's law of Gravity is

$$\mathbf{f}(r) = -K\frac{\mathbf{x}}{r^3}$$

with potential

$$U(r) = -K\frac{1}{r}.$$

We can exploit this further. For a central force, angular momentum, $\mathbf{L} = \mathbf{x} \times \dot{\mathbf{x}}$, relative to the origin of the central force is also conserved. Explicitly,

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{x}} \times \dot{\mathbf{x}} + \mathbf{x} \times \ddot{\mathbf{x}} = -U'(r)\mathbf{x} \times \mathbf{e}_r = 0.$$

Because $\mathbf{L}$ is a vector quantity, this is really *three* conserved quantities. Referring to the table above, we see that we have $6 - 1 - 3 = 2$ dimensions in phase space and, yes, the central force problem is reducible to one-dimensional motion and therefore completely solvable!

Kepler's laws, explicit solution of orbits, etc.

5.1 Equations of motion for systems of points: 3N degrees of freedom

This is a simple consequence and generalization of everything we have discussed so far ...

We will look at the complexities of this system in a few days.