Selection of Homework Questions

Prev PDF Next

Topic 16: Cosmological Context

(1) Energy in the CMB

a. A spherical human (!) of radius 0.5m floats in the IGM, exposed only to the CMB. Their density and heat capacity are those of water (1 gm cm^{-3}; 4.2 J gm^{-1} K^{-1}). Starting at normal body temperature (36^oC), if they absorb all radiation and emit none, how long is it before they boil:
   (a) at the current epoch
   (b) at z = 1100, the time of the creation of the CMB.
   (make sure you get the right factors of π for the absorbed energy!)

b. For a thermal gas of number density n, and mean particle velocity v, how many particles strike unit area per second? Using the current baryon density Ω_b = 0.04 and Hubble Constant H_o = 72 km s^{-1} Mpc^{-1}, what was the proton and electron density at the time of the CMB (assume the gas is fully ionized hydrogen). How much heat energy from the gas enters our unlucky subject at the time of the CMB (when $T_{gas} = T_{photons}$) from (a) protons, and (b) electrons. How does this compare to the energy coming from the photons?

(2) Geometry in Curved Spaces :

a. **Derive** the metric for a spherical 2-D surface of radius R using the coordinates r, θ (For example using the Earth, r is from the pole **along** the surface, θ is a change in longitude)

   (a) What is ds along a "line of latitude" at constant r from the pole for a small sector dθ of a circle. Hence, what is the circumference of a circle radius r.

   (b) If you can measure lengths to 1 km accuracy (e.g. a car odometer over a long journey), how big must r be to detect the curvature of the Earth by driving around a line of constant latitude ($R_\oplus \approx$ 6000 km; assume you know r exactly and your uncertainty is in the circumference).

   (c) What's the relationship between the area of a spherical triangle and the sum of its interior angles (you do not need to derive this relation)? If you can measure angles to 1 arcmin, how big (side length) must an equilateral triangle be to detect the curvature of the Earth?

b. 3-D spatial curvature radius within a region of mass density ρ is roughly $R_c^2 \sim 3c^2 / (8\pi G\rho)$. Show that this can be re-expressed as $R_c = c \times P_{orb} / \sqrt{8\pi}$, where P_{orb} is the period of a circular orbit about the system. [Roughly: $R_c \approx c \times$ dynamical time].

   (a) Hence estimate the **local** spatial radii of curvature (i) near the Earth's surface, (ii) within the solar system (e.g. near the Earth's orbit), (iii) within the galaxy (e.g. near the sun's orbit), (iv) within the universe.

   (b) Does this curvature affect metrology with the following levels of accuracy: (a) two satellite GPS triangulation to 1 cm on earth (GPS altitude 20,000 km); (b) wide angle planetary separations to 1 arcsec; (c) wide angle globular cluster separations to 1 arcsec (GC radii out to 50 kpc); (d) angular separations within the local supercluster (50 Mpc) to within 1 arcmin.

   (Hint: use the triangle area relation from part c above).

(3) Equations of State

a. Write down the equation for the total (relativistic) energy, E, of a particle of rest mass m_o and momentum p. The de Broglie wavelength of this particle is given by $\lambda = h / p$, which increases with the scale factor, just like light: $\lambda \propto a$. Derive an expression for the equation of state parameter, w, for a gas of these particles, assuming they all have the same m_o and p, and that the total energy density is given by u = nE for n such particles per unit volume. Show that in the relativistic limit $w \rightarrow 1/3$ and in the non-relativistic limit $w \rightarrow 0$. (Recall: pressure $P = w u = w\rho c^2$ where $u = \rho$ is the total energy density). [Ryden: Q 4.5].

b. Imagine if, for some bizarre reason, all the matter in the Universe (both baryonic and dark matter) suddenly annihilated to become photons. A stupendously spectacular, but energy conserving, conversion. How would this change the future evolution of the scale factor, both in the short term, and the long term?

c. Compare the histories of the scale factor for two slight variants on our actual universe:

   (a) Perfect matter-antimatter symmetry led to full annihilation in the first second.
   (b) Something suppressed annihilation, so all CMB photons are instead proton/electron pairs.

(4) Observing De/Acceleration

a. Can we measure de/acceleration **directly** by watching for a gradual change in z over time for an object? Start with the fundamental definition of redshift: $1 + z = a(t_o)/a(t_e)$. Now, as time passes, **both** scale factors change, since obviously $a(t_o)$ is increasing, but so is $a(t_e)$ since the light we see sets out a little later. Now find dz/dt by differentiating (1 + z) w.r.t. time. What are $da(t_o)/dt$ and $da(t_e)/dt$? Substitute in for these, using E(z), to find an expression for dz/dt.

b. Currently, the highest accuracy with which spectral features can be measured is ~0.01 A at 8000A. For an Einstein de-Sitter universe, how long must we wait between observations to detect a change in redshift of a spectral feature at a redshift of (a) 1, (b) 4, (c) 8. Assume that in each case, a spectral feature is identified near 8000A. Compare your answer for z = 1 with the concordance model.

c. Of course, galaxies also experience "peculiar accelerations" -- i.e. their peculiar velocities build up/change over time. Consider a typical peculiar velocity of 800 km/s for a galaxy in a cluster ($P_{orb} \sim 1$ Gyr) and in the field ($t_{fall} \sim t_{Hubble}$). Will the peculiar accelerations of these two galaxies undermine the measurement of cosmic de/acceleration?

d. Discuss, briefly, the advantages of using Lyman-alpha forest absorption lines (rather than galaxies) in this experiment.

(5) Proper Distances

A galaxy is at redshift z = 1. What are the three proper distances: $r(t_o)$, $r(t_e)$, $c(t_o - t_e)$, to the galaxy in a single component flat universe filled with (a) matter, (b) radiation, (c) vacuum? Express your answers in units of r_H, the Hubble distance.

(6) Concordance Model

1. Use the concordance model parameters ($\Omega_m = 0.27$, $\Omega_v = 0.73$, $\Omega_{rel} = 8.4 \times 10^{-5}$, $H_o = 72$), to plot the following as a function of redshift. Use three separates graphs for a, b, c. Plot linear z ranges of 0 - 5 for a and b and log z from -1.0 to 5.0 for c. Mark on each plot the times of matter/vacuum equality (and for c, the time of relativistic/matter equality).

   a. The comoving (toady's) distance, $r(t_o)$; the emission distance, $r(t_e)$; the light-travel distance, $c(t_o - t_e)$.

   b. Angular diameter distance, D_A;
   Luminosity distance, D_L.

   c. Hubble parameter H(z);
   The velocity history, v(z), of a galaxy which is **currently** at 1 Mpc [ie a × H(z)].

   You will need to write a routine to evaluate E(z) and its integral. I suggest you make use of the integrator qromb (which also calls trapzd and polint) in Numerical Recipes.

2. You observe a magnitude 28.0 supernova at z = 3, located 1.2 arcsec from its host galaxy nucleus. Spectra show emission line widths implying an expansion speed of 10^4 km/s. What's the absolute magnitude of the supernova; its projected distance from the galaxy nucleus; and what angular velocity (in mas/yr) does the expansion velocity correspond to? (Ignore K corrections in your calculation of the absolute magnitude).

(7) The Age Problem

The observed age of the oldest globular clusters, 13 Gyr, and the Hubble Constant, 72 km/s/Mpc together place an interesting constraint on the density of a pure matter Universe.

a. What is the Hubble time: $t_{H,o}$?

b. What is t_{age} for $\Omega_m = 0$ (empty) and 1 (flat; Einstein-de Sitter).

c. Write an integral equation for t_{age} and solve it (numerically) to find $\Omega_{m,o}$ such that t_{age} = 13.0 Gyr -- the age constraint from GCs. Alternatively, you may solve the parametric equations for an open pure matter universe, finding η_1 such that $a(\eta_1) = 1$, and then finding $t_{age} = t(\eta_1)$.

d. Measurements of cluster dynamics suggest $\Omega_{m,o} \approx 0.3$. What age does this give?

Do you now see why there was an "age problem" with pure matter models.

(8) Vacuum Energy's Accelerating Expansion

The fact that a vacuum dominated universe accelerates in its expansion is often viewed as deeply puzzling. Phrases like "repulsive gravity", or "negative pressure" only confuse one's intuition. This question aims to provide an intuitive understanding. Although the analysis isn't rigorous, since it is essentially Newtonian, it nevertheless captures the essence of what's going on.

a. Consider a sphere of (non-relativistic) matter: radius R, mass M, and uniform density ρ_m. What energy resides in the gravitational field of this sphere, i.e., what is it's gravitational binding energy, U_{grav}? Why is the sign of U_{grav} negative?
b. Consider an expansion of this sphere by ΔR. Recalling that matter is conserved (i.e. M is constant), what is dU_{grav}/dR? What is its sign?
c. When things naturally "fall", the motion is always such that U_{grav} becomes MORE NEGATIVE. Since energy is conserved, POSITIVE energy is released, usually kinetic energy (objects accelerate "downwards"). With this in mind, answer the (obvious) question: when you release the sphere of matter, in which direction does it fall, inwards (to smaller R) or outwards (larger R).
d. Now replace the matter in our sphere by vacuum, with uniform density ρ_v. Recall that the strange thing about vacuum energy/density is that it is CONSTANT -- when you expand a sphere of it, more of it appears in the new shell. Repeat the evaluation of dU_{grav}/dR but this time subject to the condition that ρ_v is constant, not M. What is the sign of dU_{grav}/dR?

e. For a first guess, then, if we let go of this new sphere of vacuum, which way will it "fall" -- inwards (to smaller R) or outwards (to larger R)?
f. Not so fast, we've forgotten something important. It requires energy to "make" the new shell of vacuum: $4 \pi R^2 dR \rho_V c^2$. Find the expression for dU_{tot}/dR where $U_{tot} = U_{grav} + Mc^2$.
g. Notice that the condition for "infall" or "outfall" depends on the size of the region, R. What is this condition, expressed first in terms of the gravitational radius of the sphere, R_g (defined as $R_g = GM/c^2$), and then in terms of the vacuum density, ρ_V. For a vacuum density equal to that of water, how big must the sphere be before it continues to expand, making more and more "water" as it does so?
h. Assume our sphere represents an expanding cosmological volume with critical density $\rho_V = 3H^2/8\pi G$. Show that the critical radius for "runaway expansion" is roughly r_H, the Hubble radius.
i. For a related calculation, find the radius of a critical density sphere such that its negative gravitational energy is equal (in magnitude) to its positive rest-mass-energy. What is the TOTAL energy of such a sphere?
j. Although we've applied this discussion to present day dark energy, transfer these ideas to the epoch of inflation, and describe how at very early times, a small region of dense vacuum might launch the "big bang", creating a huge massive expanding universe from essentially nothing.

(9) The Flatness Problem

In the radiation-dominated phase of our Universe, the temperature and time were related by $T_K \approx 1.3 \times 10^{10}\, t_{sec}^{-1/2}$.

a. At what time was the temperature 3×10^{25}K ?
b. Suppose that at that time Ω_{tot} were, in fact, a bit less than one. Considering the Friedmann equation, with just radiation and curvature terms, show that curvature quickly dominates, and that the evolution of the scale factor quickly becomes $a(t) \propto t$.
c. How would the temperature-time equation be changed, and how old would the Universe be when its temperature fell to 3K?

   This is an example of how a Universe that is slightly curved (unbound) at early times evolves very differently from a perfectly flat Universe. In this case, the expansion is much faster and reaches the same size (scale factor) and temperature very quickly. [Question 13.3 from Liddle].

(10) The Monopole Problem

a. The underlying cause of the monopole problem stems from the fact that Magnetic monopoles are non-relativistic from very early times (because they are so massive). Suppose that at a temperature corresponding to the Grand Unified era, about 3×10^{28}K, magnetic monopoles were created with a density of $\Omega_{mon} = 10^{-10}$. Assuming that the Universe has a critical density and is radiation dominated, what was the temperature when the density of monopoles equalled that of the radiation of the radiation?
b. In the present Universe, $T \sim 3$K. Calculate the value of $\Omega_{mon}/\Omega_{rad}$ at the present time. Is this ratio compatible with observations?
c. If we have a period of inflation, the monopole density still dilutes as $\rho_{mon} \propto a^{-3}$, but the total density, dominated by vacuum, remains fixed. Since that density will be converted to radiation after inflation, we can imagine that the radiation density remains constant during inflation. How much inflationary expansion is necessary so that the present density of monopoles is equal to that of radiation? [Questions 13.5-6 from Liddle].

(11) Big Bang Nucleosynthesis

a. If neutrinos decouple when $T \sim 0.8$ MeV near 1 second, calculate the neutron/proton ratio at this time.
b. Since the deuteron binding energy is 2.2 MeV, one might naively expect deuterium to be stable at this time. Why do we need to wait about 200 seconds before deuterium can, in fact, survive?
c. If the neutron half-life were 100 seconds, instead of 614 seconds, estimate the helium mass fraction, Y, produced during the BBNS era.
d. Derive an expression for the number of electrons per baryon as a function of Y. What is this number for Y = 0.24?

(12) Origin of the CMB

This question explores the period of recombination, by calculating how the fractional ionization changes as the Universe cools. We also aim to show that the region we see (the cosmic photosphere, where $\tau = 1$) is located towards the end of the recombination period, when the gas is roughly 85% neutral. Throughout, we're assuming a pure hydrogen gas.

From statistical mechanics, a non-degenerate, non-relativistic gas of particles with mass m_x, chemical potential μ_x, and statistical weight g_x, has a number density n_x given by the Maxwell-Boltzmann equation:

$n_x = g_x (m_x k T / 2 \pi \hbar^2)^{3/2} \exp[(\mu_x - m_x c^2) / kT]$

Consider the "chemical" recombination reaction: $p + e = H + \gamma$. Another fundamental result from statistical mechanics is that when this reaction is in equilibrium, $\mu_p + \mu_e = \mu_H + \mu_\gamma$ where the general definition of chemical potential is $\mu/T = (\partial S/\partial N)_{U,T}$ (this relation arises from demanding that at a given temperature and energy, the total entropy, S, is at a maximum w.r.t. changing the particle numbers, N, of each species).

Note: Although this treatment seems to follow Ryden (pp 155-159), her MB relation is wrong -- by excluding μ she obscures the true logic behind the derivation of the Saha equation. The correct MB relation, and logic, is taken from Peebles (Principles of Physical

Cosmology, pp 165 - 167).

a. Recast the MB relation in the form $\mu_x = F(n_x,kT,m_x)$, and then use the above relation between the μ's, together with the special case of $\mu_\gamma = 0$ for photons, to derive the Saha equation:

$n_H / n_p n_e = (g_H / g_p g_e)\ (m_e kT / 2\pi\hbar^2)^{-3/2} \exp(Q / kT)$

where Q = 13.6 eV is the binding energy of the electron in hydrogen.

b. Here are some further simple relations/definitions: $g_p = g_e = ½\, g_H = 2$; the baryon number density $n_b = n_p + n_H$; the baryon-to-photon ratio is $\eta = n_b / n_\gamma$; the photon density for a black body radiation field is $n_\gamma = (2\ \zeta(3) / \pi^2)\ (kT/\hbar c)^3$, where ζ(3) = 1.202 is the Riemann zeta function; the fractional ionization is $X = n_p / n_b$; and charge neutrality demands $n_e = n_p$.

Re-express the LHS of the Saha equation in terms of X, η and n_γ, and after bringing η and n_γ to the RHS, recast n_γ explicitly in terms of kT and now simplify the RHS. You should have a quadratic relation for X of the form: $(1-X)/X^2 = f(\eta,kT)$. Hence, show that the ionization fraction, X, is given by:

$X = [\sqrt{(1 + 4f)} - 1\ ]/2f$ where $f(\eta,kT) = 3.84\ \eta\ (kT/m_e c^2)^{3/2} \exp(Q/kT)$

c. Calculate η for today's Universe, using the fact that $\Omega_b = 0.044$; $H_0 = 72$ km/s/Mpc; and $T_{CMB} = 2.725$ K. Recall that η doesn't change with expansion (number densities all scale with a^{-3}), so the value of η that you've just calculated is true at all times.

d. To simplify evaluation, work in eV energy units for kT, so that $m_e c^2 = 511{,}000$, and Q = 13.6, and graph X(kT) and, equivalently, X(z), where kT is in eV (and 1 eV = 11,600K). At what temperature and redshift is X = 0.5 (i.e. 50% ionized and 50% neutral)? Note that this is a significantly lower temperature than the simple estimate using kT ~ 13.6 eV (~160,000K).

e. Having established X(z) -- how the fractional ionization drops across the epoch of recombination -- let's estimate the redshift of the cosmic photosphere, namely, where τ(z) = 1, where τ is the optical depth to Thompson scattering.

In general, optical depth is given by: τ = n σ L, where n is the particle density, σ is the particle scattering cross section, and L is the path length. In our case, $\sigma = \sigma_T = 6.6 \times 10^{-25}$ cm^2 is the Thompson cross section; $n = n_e = X\ \eta\ n_\gamma$ is the electron density, and L = cΔt is proper path length corresponding to a difference in epoch of Δt. Thus the full expression for optical depth, as a function of redshift is:

$\tau(z) = \int X(z)\ \eta\ n_\gamma\ \sigma_T\ cdt/dz'\ dz'$ where the integral is from z' = 0 to z, and we've transformed our path length, cdt, into a redshift interval.

Now, although many of these quantities are functions of redshift (e.g. n_γ and cdt/dz') the recombination transition occurs over such a narrow window in redshift that we can effectively set these to constants. Thus we have:

$\tau(E) = \sigma_T\ \eta\ n_\gamma\ cdt/dz'\ dz'/dT\ dT/dE \int X(E)\ dE$, where E = kT in units of eV.

Pick a redshift, z, that is in the middle of recombination, and use relations: T = 2.725 (1+z) and T = 11,600 E to obtain values for dz'/dT and dT/dE; and the cosmological relation (see section 6ci): $cdt = r_{H,0}\ dz / (1+z)E(z)$ to obtain a value for cdt/dz' (you'll need to use the concordance parameters for E(z), and don't confuse Peeble's E(z) with our energy variable E). Combine all these to get the pre-factor A in the relation $\tau(E) = A \int X(E')\ dE'$, with limits of E' from 0 to E.

To find the redshift of the cosmic photosphere, where τ(E) = 1, find the value of E such that the integral = 1/A (you will need to do the integral numerically). You should find that the photosphere occurs where X ≈ 0.1, so the gas that we see in the CMB is in fact pretty neutral.

f. What are some of the shortcomings of this relatively simple equilibrium derivation, and what does a more realistic derivation give for X(E) and z(τ = 1)?

(13) Growth of Structure

This question explores, heuristically, several features of the way in which structure grows with cosmic expansion. You may find the class notes, the Schneider text, and the article by Peacock useful when answering this question.

a. In the linear regime, the uneven cosmic density field, δρ/<ρ>, can be described using its power spectrum, P(k), where k is a wavenumber, 2π/λ. What is the form of P(k) thought to be generated by inflation (i.e. the "Initial Power Spectrum")? Why is this often called a "scale invariant" spectrum?

b. Consider first large-scale variations, meaning they are super-horizon in scale. Why does δρ/<ρ> increase with cosmic expansion? In the interval between inflation (assumed to be at the GUT time) and radiation/matter equality, by what factor is δρ/<ρ> amplified? What is the corresponding boost to P(k) ?

c. Now consider small-scale variations, meaning they are sub-horizon in scale. Why can't such perturbations grow during the radiation era but they can during the matter era? In the interval between the CMB and today, by what factor has δρ/<ρ> been amplified (assuming δρ/<ρ> << 1 throughout)? Why can't such perturbations grow during the dark energy era?

d. Explain how the initial power spectrum changes over time to give today's power spectrum which has the following form: $P(k) \propto k$ for small k, passes through a broad peak at $k \sim 0.02$ (h Mpc^{-1}) and then drops as $P(k) \propto k^{-3}$ for large k.

e. Explain two quantitative aspects of today's P(k): (i) the physical length scale of the peak; (ii) the change by 4 in the index of P(k) on either side of the peak.

f. Why is the dimensionless version of the power spectrum, $\Delta^2(k) \propto k^3 P(k)$ more useful if we want to consider whether regions of a given size can break away from Hubble expansion and collapse to form objects?
