Black-hole evaporation in the Klein-Sauter-Heisenberg-Euler formalism

Thibaut Damour* and Remo Ruffini†
Joseph Henry Physical Laboratories, Princeton, New Jersey 08540
and Physics Department, University of Western Australia, Nedlands, Australia
(Received 3 November 1975)

A generalization of the classical approach of barrier penetration introduced by Klein, Sauter, Heisenberg, and Euler to curved spaces endowed with future horizons is given. This technique allows one to recover most directly results obtained by Hawking recently. The treatment here presented encompasses, as special cases, the works of Deruelle and Ruffini, of Damour and Ruffini, and of Nakamura and Sato.

One of the most important results obtained in recent years in black-hole physics has been the realization that the total mass-energy of a black hole can be separated into three components[1]: the irreducible mass, the Coulomb energy, and the rotational energy. That both rotational and Coulomb energy could be in principle extractable by a set of classical gedanken experiments has been known for some time.[2] It has been only recently, however, that the quantum analog of these processes occurring in the "effective ergosphere" have been analyzed.[3] The use of the Klein-Sauter-Heisenberg-Euler formalism has led to a most direct understanding of these processes of vacuum polarization[4] and to detailed analyses of possible astrophysical interest.

Hawking[5] has suggested, however, that, also, by vacuum polarization processes the irreducible mass of a black hole could be radiated away. In the present paper we show how a generalization of our previous treatment of barrier penetration[4] leads to a clear understanding of this phenomenon.

We consider (a) a Kerr-Newman geometry endowed with a vacuum future horizon, (b) a massive charged scalar field Φ fulfilling the covariant Klein-Gordon equation in that background geometry, and (c) we assume analyticity properties of the wave function Φ in the complexified manifold.

The result can be obtained mathematically thanks to the existence of explicit asymptotic expressions for the field Φ near the horizon and at spatial infinity. Physically, it comes from the existence, inside the horizon, of a spacelike Killing vector ξ_t, which allows a classical particle as "seen" from infinity to reach a negative-energy state. In the quantum description, this phenomenon allows an antiparticle to reach positive-energy states. These states, classically confined in the black hole, can be tunneled out by a wave function "over" the horizon which gives rise to the creation of a pair: one particle (positive energy) going out and one antiparticle (negative energy) falling back toward the singularity. Note that this approach only requires the existence of a future horizon and is totally independent of any dynamical details of the process leading to the formation of this horizon.

As usual we consider the Kerr-Newman metric

$$\begin{aligned} ds^2 &= \Sigma(\Delta^{-1} dr^2 + d\theta^2) \\ &\quad + \Sigma^{-1}\sin^2\theta\,[(r^2+a^2)\,d\phi - a\,dt]^2 \\ &\quad - \Sigma^{-1}\Delta(dt - a\sin^2\theta\, d\phi)^2, \end{aligned} \tag{1}$$

with $\Delta = r^2 - 2Mr + a^2 + e^2 = (r-r_+)(r-r_-)$, where $r_\pm = M \pm (M^2 - a^2 - e^2)^{1/2}$ and $\Sigma = r^2 + a^2\cos^2\theta$, M being the mass, e the charge, and a the specific angular momentum of the black hole. (Here and in the following we choose $G = c = \hbar = 1$.) We also indicate by H_+ the future horizon. Introducing the coordinate r_*,

$$dr_*/dr = (r^2 + a^2)/\Delta\,, \tag{2a}$$

we have when $r \to r_+$ $(r > r_+)$

$$r_* \sim \frac{1}{2\kappa}\ln(r - r_+),$$

with (2b)

$$\kappa = \frac{1}{2}\,\frac{r_+ - r_-}{r_+{}^2 + a^2}\,.$$

We shall first treat the case of a Schwarzschild metric $(a = e = 0)$. The scalar function Φ fulfilling the covariant Klein-Gordon equation in this given metric can be separated as

$$\Phi_\omega = (2\pi|\omega|r^2)^{-1/2} E_\omega(r_*, t)\, Y_l^m(\theta, \phi)\,, \tag{3}$$

the Y_l^m being the usual spherical harmonics and E_ω being monochromatic in time. In the following we take $\omega > 0$, that is, a flux of *particles* at infinity, the flux of *antiparticles* being treated as usual by charge conjugation. It is easy to show that just outside the horizon H_+ $(r > 2M)$ two linearly independent solutions exist:

$$E_\omega^{\rm in} = e^{-i\omega(t + r_*)} = e^{-i\omega v} \tag{4a}$$

and

$$E_\omega^{\text{out}} = e^{-i\omega(t-r_*)} = e^{2i\omega r_*}e^{-i\omega v} = (r-2M)^{i4M\omega}e^{-i\omega v}, \tag{4b}$$

where we use the usual advanced Eddington-Finkelstein coordinates, $t+r_* = v, r, \theta, \phi$ in which the metric is well behaved and, in fact, analytic over the whole coordinate range $0<r<\infty$, $-\infty<v<\infty$ including H_+ $(r=2M, -\infty<v<\infty)$.

While Eq. (4a) corresponds to a wave purely ingoing on H_+ and can be extended inside $r<2M$, Eq. (4b) represents an outgoing wave and has an infinite number of oscillations as $r\to 2M$ and therefore cannot be straightforwardly extended to the region inside H_+.[6] We will in the following use and generalize to analytic curved spaces the well-known result of flat-space relativistic wave theories[7,8]: The wave function $\Phi(x)$ describing a particle state (positive frequencies) can be analytically continued to complex points of the form $z=x+iy$ if y lies in the past cone; similarly, for an antiparticle state (negative frequencies) y has to lie in the future cone.

Since in Finkelstein coordinates the vector $\partial/\partial r$ is everywhere null and past-directed, the prescription[9] $r\to r-i0$ will yield the unique continuation of Eq. (4a) describing an *antiparticle* state,

$$\overline{P}_\omega = \overline{N}_\omega \Phi_\omega^{\text{out}}(r-2M-i0), \tag{5a}$$

or introducing the Heaviside function Y,

$$\overline{P}_\omega = \overline{N}_\omega[Y(r-2M)\Phi_\omega^{\text{out}}(r-2M) + e^{4\pi M\omega}Y(2M-r)\Phi_\omega^{\text{out}}(2M-r)], \tag{5b}$$

where $\overline{N}_\omega$ is a normalization factor such that

$$\langle \overline{P}_{\omega_1}, \overline{P}_{\omega_2}\rangle = -\delta(\omega_1-\omega_2)\delta_{l_1 l_2}\delta_{m_1 m_2}. \tag{5c}$$

As Φ_ω was already normalized[10] it is very simple to obtain $\overline{N}_\omega$

$$|\overline{N}_\omega|^2 = (e^{8\pi M\omega}-1)^{-1}. \tag{6}$$

Now Eq. (5b) describes the splitting of $\overline{P}_\omega$ in a wave outgoing from the horizon and a wave falling on the singularity (see Fig. 1). The probability flux carried away by this outgoing wave is simply $|\overline{N}_\omega|^2/2\pi$ per unit of time [see Eq. (3)] and only a fraction Γ of this flux will be transmitted to infinity, where Γ is the transmission coefficient of the potential and centrifugal barrier (Fig. 1). Using Eq. (6) we get at infinity an outgoing flux of particles of

$$(\Gamma/2\pi)(e^{8\pi M\omega}-1)^{-1} \tag{7}$$

per unit of time and per unit range of frequency, which is Hawking's result.[5]

If we consider now a scalar field in a Kerr-Newman geometry, it can be shown,[11] using the analogs of the Eddington-Finkelstein coordinates[12] and a corresponding gauge transformation for the electromagnetic field, that the normalized ingoing wave Φ_ω^{in} is regular at H_+ but that Φ_ω^{out} contains a factor $(r-r_+)^{i(\omega-\omega_0)/\kappa}$, where κ is given by Eq. (2b) and where

$$\omega_0 = m\Omega + \epsilon V, \tag{8}$$

m being the usual azimuthal quantum number of the particle, ϵ its charge, and Ω and V being

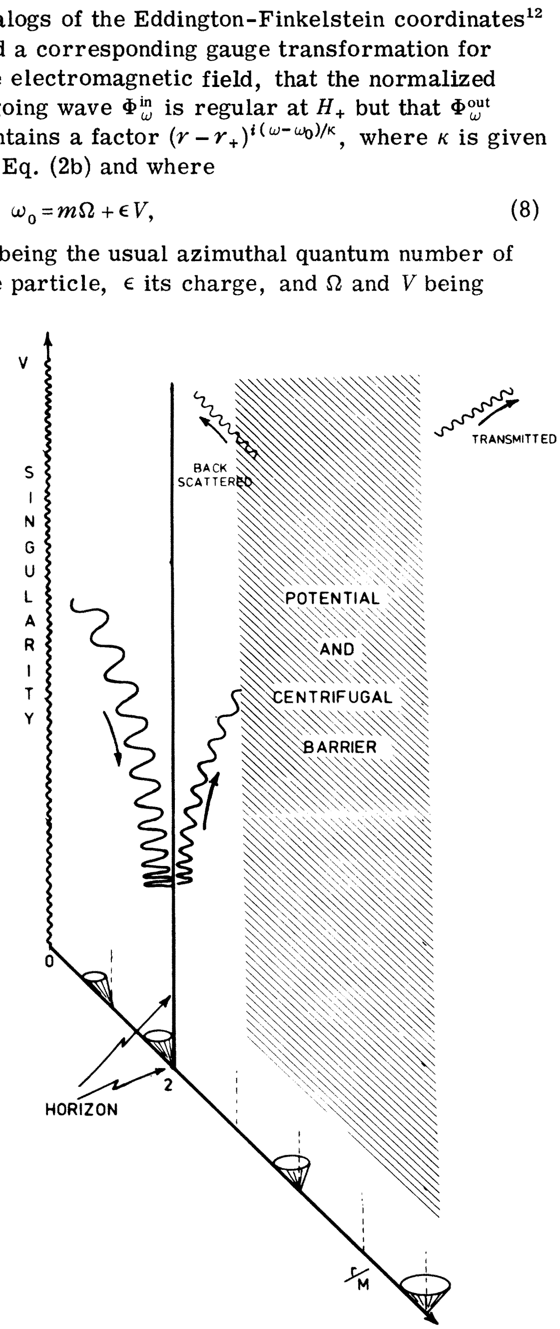

FIG. 1. In usual Eddington-Finkelstein coordinates, a qualitative representation is given of the splitting of the antiparticle state $\overline{P}_\omega$ [see Eq. (5b)] into two components: (a) a particle wave of strength $|\overline{N}_\omega|^2$ outgoing from the horizon and partially transmitted to infinity (if $\omega>\mu$) and partially backscattered into the hole, and (b) an antiparticle wave of strength $|\overline{N}_\omega|^2 e^{8\pi M\omega} = 1+|\overline{N}_\omega|^2$ with positive energy flux outgoing in the past from the singularity. This can always be interpreted as a negative-energy flux of antiparticles $|\overline{N}_\omega|^2$ ingoing in the future toward the singularity. $\overline{P}_\omega$ being normalized, $|\overline{N}_\omega|^2/2\pi$ yields the rate of pair creation per unit frequency from H_+.

respectively the angular velocity[13] and the electric potential[12] of the black hole. As in these coordinates the vector $\partial/\partial r$ is still null and past-directed we can use the same prescription as before, $r \to r - i0$, to describe an antiparticle, which yields the splitting analogous to Eq. (5b):

$$\overline{P}_\omega = \overline{N}_\omega [\, Y(r - r_+)\Phi_\omega^{\text{out}}(r - r_+) + e^{\pi(\omega-\omega_0)/\kappa} Y(r_+ - r)\Phi_\omega^{\text{out}}(r_+ - r)] . \qquad (9)$$

However, in the present situation of a Kerr-Newman geometry two drastically different situations occur for an energy of the wave $\omega > \omega_0$ or $\omega < \omega_0$.

For $\omega > \omega_0$ the norm of Φ_ω^{out} is positive and its flux is $+(2\pi)^{-1}$ so that one gets $|\overline{N}_\omega|^2 = (e^{2\pi(\omega-\omega_0)/\kappa} - 1)^{-1}$. And as usual only a positive fraction Γ of this flux is transmitted to infinity through the combined potential and centrifugal barriers.

For $\omega < \omega_0$ the norm of Φ_ω^{out} is negative as well as its flux $-(2\pi)^{-1}$ (antiparticles) so that one gets $|\overline{N}_\omega|^2 = (1 - e^{2\pi(\omega-\omega_0)/\kappa})^{-1}$. But we are precisely in the conditions of level crossing[4] *between* the horizon and spatial infinity so that a *negative* fraction Γ of this flux will be transmitted to infinity.

Therefore in both cases one observes a positive flux at infinity (particles) of

$$(\Gamma/2\pi)(e^{2\pi(\omega-\omega_0)/\kappa} - 1)^{-1} \qquad (10)$$

per unit of time and per unit range of frequency.

The drastic difference between those two regimes appears clearly if we consider the limit where the effective temperature $\kappa/2\pi$ of the black hole tends to zero. In the Schwarzschild case the rate of particle creation goes then to zero. In the Kerr-Newman solution the rate goes also to zero if $\omega > \omega_0$, but in the range $\mu < \omega < \omega_0$ (where μ is the mass of the particles) the rate of particle creation tends to $-\Gamma/2\pi$ which is the phenomenon studied in Ref. 4.

If we now take into account that the effective temperature is given by $\kappa/2\pi \simeq 10^{-6} \times (M_\odot/M)$ °K we can conlcude that the only place where thermal contribution to the vacuum polarization of a black hole can exist is in the early stages of cosmology (black holes of $M < 10^{15}$ g). For macroscopic black holes, outcoming from stellar collapse, only the process of vacuum polarization described in Ref. 4 can possibly be observed.

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*E. S. A. International fellow.

†Alfred P. Sloan fellow.

[1]D. Christodoulou, Phys. Rev. Lett. 25, 1596 (1970); D. Christodoulou and R. Ruffini, Phys. Rev. D 4, 3552 (1971). The result that the irreducible mass of a black hole can never decrease for classical transformations was independently shown by S. Hawking, Phys. Rev. Lett. 6, 1344 (1971).

[2]The extraction of rotational energy has been treated by R. Penrose and R. Floyd, Nature 229, 193 (1971). The extraction of Coulomb energy has been studied by G. Denardo and R. Ruffini, Phys. Lett. 45B, 259 (1973), and by G. Denardo, L. Hively, and R. Ruffini, Phys. Lett. 50B, 270 (1974). The region in which energy extraction processes can occur has been defined as the "effective ergosphere" in Ref. 1.

[3]The pioneering work in this field has been done by Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 62, 2076 (1972) [Sov. Phys.—JETP 35, 1085 (1972)], and A. A. Starobinsky, *ibid.* 64, 48 (1973) [*ibid.* 37, 28 (1973)], for the rotational energy and by G. W. Gibbons and S. Hawking, in *Gravitational Radiation and Gravitational Collapse*, edited by C. DeWitt (Reidel, New York, 1974), and W. T. Zaumen, Nature (London) 247, 530 (1974), for the Coulomb energy. Then the second-quantized formulations of these phenomena were given respectively by W. Unruh, Phys. Rev. D 10, 3194 (1974), and G. Gibbons, Commun. Math. Phys. 44, 245 (1975).

[4]The general outline of this approach is in the work by N. Deruelle and R. Ruffini, Phys. Lett. 52B, 437 (1974). The analysis of vacuum polarization processes were given in a Kerr-Newman geometry by T. Damour and R. Ruffini, Phys. Rev. Lett. 35, 463 (1975); in a Kerr geometry by N. Deruelle and R. Ruffini, Phys. Lett. 57B, 248 (1975); in a Reissner-Nordström geometry by T. Nakamura and H. Sato, Phys. Lett. 61B, 371 (1976).

[5]S. Hawking, Nature (London) 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).

[6]Some of the properties of this solution have been recently analyzed by D. G. Boulware, Phys. Rev. D 11, 1404 (1974). We disagree with the physical conclusions reached in that article and we explicitly show how our prescription of analytic continuation of the wave function leads to opposite physical conditions.

[7]See, e.g., R. Jost, in *The general theory of quantized fields* (American Mathematical Society, Providence, Rhode Island, 1965), p. 73.

[8]R. Penrose, Int. J. Theor. Phys. 1, 61 (1968), especially pp. 77 and 78.

[9]See, e.g., I. M. Guelfand and G. E. Chilov, *Les distributions* (Dunod, Paris, 1962), p. 43 ff.

[10]The norm is determined as usual by examining broad wave packets.

[11]The computation is similar to the one for the simpler case of a Reissner-Nordström geometry dealt with by G. Gibbons, Commun. Math. Phys. 44, 245 (1975); see p. 257.

[12]See, e.g., B. Carter, in *Black Holes*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973).

[13]D. Christodoulou and R. Ruffini, in *Black Holes* (Ref. 12).