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Instabilities of Massive Scalar Perturbations of a Rotating Black Hole

THEODOROS J. M. ZOUROS* AND DOUGLAS M. EARDLEY†

Department of Physics, Yale University, New Haven, Connecticut 06520

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We study the stability of a massive scalar field in the exterior metric of a rotating Kerr black hole. An argument based on energy conservation shows, under some strong technical assumptions, that unstable normal modes exist. These unstable modes can be interpreted as wave packets in bound, superradiant orbits. A JWKB estimate of the fastest growth rate gives $10^{-7}M^{-1}\exp(-1.84M\mu)$ in the case $M\mu \gg 1$, where M is the mass of the hole and μ is the mass of the field. The existence of unstable normal modes has significant implications for quantum particle creation by rotating black holes, which we attempt to assess.

I. INTRODUCTION

Relativists believe that black holes, both nonrotating and rotating, are stable to small perturbations, including gravitational perturbations, by massless fields. This belief is supported by several results:

(a) The Schwarzschild metric has been proven to be stable to all massless fields [1, 2, 3, 4].

(b) The Kerr metric has been proven to be stable to axially symmetric ($m = 0$) perturbations by massless scalar and gravitational fields [5, 6].

(c) For the physically most worrisome case, that of nonaxisymmetric perturbations by massless fields of "superradiant" frequency ($0 < \omega/m < a/2Mr_+$; see §II for notation and equations) no proof has been given. But numerical solution of the scalar and Teukolsky wave equations for real frequency has shown no hint of instability [4, 7–11], and theoretically any instability must set in via a real-frequency mode [12]. In addition, direct evaluation of the scattering amplitude for complex frequency has shown no instability [5, 13].

In this paper we report on a situation that probably does give rise to a true instability as pointed out by Damour, Deruelle, and Ruffini [14]. We study solutions of the *massive* scalar wave equation $(\Box - \mu^2)\psi = 0$, the Klein-Gordon equation, in the

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Kerr background metric. Roughly the idea is this. Due to the nonzero rest mass it is possible to construct wave packets in stable bound orbits around the black hole. Such a packet may leak down the hole and decay forever. However, when the hole is rotating, superradiant amplification of the packet may occur [14, 15, 9], and in this case one expects it to grow forever. Hence instability. A similar idea was suggested by Press and Teukolsky in the "black hole bomb" [9].

We do not have a rigorous proof of instability; in §III we present a fairly convincing argument. A similar argument goes through for any integer-spin field theory with nonzero rest-mass normal modes, whose stress-energy tensor obeys the Dominant Energy Condition [16], and we expect instabilities to occur under these general circumstances for a rotating black hole; in this paper we confine ourselves to the free Klein–Gordon field as a simple case.

Instability ought to occur for all values of rest mass $\mu > 0$. In §IV and §V we estimate the actual growth rate for the case $\mu \gg M^{-1}$ (M is the mass of the hole) using the JWKB approximation, extending the results of Damour *et al.* [14]. The fastest growth rate we find in this approximation is $10^{-7}M^{-1}$ at $M\mu \sim 1$, but the approximation is breaking down badly. Meanwhile, Detweiler (to be published) has estimated growth rates in the opposite approximation $M\mu \ll 1$, using a method of Starobinsky [10]. Again his approximation breaks down for the fastest growth rate, at $M\mu \sim 1$. Direct numerical solution of the eigenvalue problem seems necessary to find the actual fastest growth rate of the instability; we hope to return to this question in a future paper.

The existence of unstable normal modes has significant, although probably not profound, consequences for quantum particle creation by rotating black holes [17]. In §VI we evaluate the probable consequences of our results.

II. Equations and Boundary Conditions

a. *Normal Modes*

The Klein–Gordon equation for a real, spinless, massive, free, classical field is

$$(\Box - \mu^2)\Psi = 0 \tag{1}$$

where μ is the inverse Compton wavelength associated with the rest mass M_ψ of the field, $\mu = M_\psi/\hbar$. We use geometrized units ($G = 1 = c$) but preserve $\hbar$, so that μ has units of 1/length. The boundary conditions on Ψ for the problem of instability are: (a) Ψ is regular at infinity and there is no incoming radiation, (b) Ψ is regular across the future event horizon. A normal mode is any solution Ψ of Eq. (1) under these boundary conditions that has harmonic time dependence, $\Psi \propto e^{-i\omega t}$, for some complex frequency ω. The normal modes may be found by solving Eq. (1) as an eigenvalue problem in ω.

The normal modes arise [4, 5] when an arbitrary solution $\Psi(t, r, \theta, \phi)$ is recovered from its Fourier transform $f(\omega, r, \theta, \phi)$,

$$\Psi(t, r, \theta, \phi) = (2\pi)^{-1} \int_C d\omega\, e^{-i\omega t} f(\omega, r, \theta, \phi) \tag{2a}$$

$$= \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} a(\omega) f_{\text{real}}(\omega, r, \theta, \phi) + \sum_j e^{-i\omega_j t} a_j f_j(r, \theta, \phi). \tag{2b}$$

Here C is a contour from $-\infty$ to $+\infty$ in the complex ω-plane that passes above all singularities of f. In the second formula, C has been deformed downward to just above the real axis, and the f_{real} belong to a continuous set of real frequency, normalized wave functions. There are two square root branch points at the "threshold points" $\omega = \pm\mu$ on the real axis, and the contour must always pass above them. The f_j are normalized wave functions of a discrete spectrum of unstable normal modes at frequencies ω_j in the upper half-plane, which appear as poles in the integrand as C is deformed. The $a(\omega)$ and the a_j are then expansion coefficients for an arbitrary solution Ψ.

If no modes f_j appear, then the field is stable, and one expects the f_{real} to form a complete basis of wave functions for either the classical or the quantum field Ψ. If the f_j appear, then the f_{real} are incomplete, and one must include the f_j to obtain a complete basis. In this case the fastest-growing normal mode will be that with the largest imaginary part ω_I of frequency ω_j, and this mode will generally dominate at late times,

$$\Psi \sim f_j \exp(\omega_I t) \qquad \text{as} \quad t \to +\infty \tag{3}$$

so that an arbitrary initial disturbance Ψ will grow without bound and the field is unstable.

If C is further deformed below the real axis, more poles will generally appear in the lower half-plane, which correspond to stable, dying normal modes.

b. *Boyer–Lindquist Coordinates*

These are convenient for the JWKB calculation. Here the contravariant Kerr metric is [18, 19]:

$$\begin{aligned}\left(\frac{\partial}{\partial s}\right)^2 &\equiv g^{\mu\nu}\, dx_\mu\, dx_\nu \\ &= \Sigma^{-1} \Big\{\Delta \left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial\theta}\right)^2 + ((\sin\theta)^{-2} - a^2\Delta^{-1}) \left(\frac{\partial}{\partial\phi}\right)^2 - 4Mra\Delta^{-1} \frac{\partial}{\partial\phi}\frac{\partial}{\partial t} \\ &\quad - [(r^2 + a^2)^2 \Delta^{-1} - a^2 \sin^2\theta] \left(\frac{\partial}{\partial t}\right)^2\Big\}\end{aligned} \tag{4}$$

with

$$\Delta \equiv r^2 - 2Mr + a^2, \qquad \Sigma \equiv r^2 + a^2\cos^2\theta \tag{5}$$

Here M is the mass of the hole, and $a \equiv J/M$ $(0 < a < M)$ is the specific angular momentum.

The Klein–Gordon equation separates in these coordinates [18, 10], and we can write its solution Ψ as:

$$\Psi = R(r)\,\Theta(\theta)\,e^{im\phi}e^{-i\omega t} \tag{6}$$

where m is an integer and ω is a complex number. Substituting solution (6) into Eq. (4), we obtain two separate ordinary equations,

$$\Delta \frac{d}{dr}\left(\Delta \frac{dR}{dr}\right) + [a^2m^2 - 4Mram\omega + (r^2 + a^2)^2\,\omega^2 - \mu^2r^2\Delta]R$$
$$= (Q + m^2 + \omega^2a^2)\,\Delta R \tag{7}$$

$$(\sin\theta)^{-1}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + [a^2(\omega^2 - \mu^2)\cos^2\theta - m^2(\sin\theta)^{-2}]\Theta$$
$$= -(Q + m^2)\Theta \tag{8}$$

where Q is the separation constant. See also [28].

We consider each equation separately:

i. *Angular Equation*

If we set

$$Q + m^2 \equiv \lambda_{ml}, \qquad c^2 \equiv a^2(\omega^2 - \mu^2), \qquad \Theta(\theta) \equiv S_{ml}(c, \cos\theta) \tag{9}$$

the angular equation becomes the well known [20] oblate spheroidal angular wave equation with m, l integers and $|m| \leqslant l$. The λ_{ml} are eigenvalues which in general cannot be analytically expressed in terms of l and m.

As a consequence of Eq. (8) the eigenfunctions satisfy orthogonality relations. If we define normalized spheroidal harmonics

$$Z_l^m(\theta, \phi) = \left[\frac{(2l+1)}{4\pi}\cdot\frac{(l-m)!}{(l+m)!}\right]^{1/2} S_{ml}(c, \cos\theta)\,e^{im\phi}, \tag{10}$$

these relations take the standard form

$$\int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi\; Z_l^{m*}Z_{l'}^{m'} = \delta_{ll'}\delta_{mm'}\,. \tag{11}$$

We note that for $c = 0$ we have $\lambda_{ml} = l(l+1)$, $S_{ml} = P_l^m(\cos\theta)$, i.e., Eq. (8) becomes the generalized Legendre equation. To find λ_{ml} and Z_l^m for small values of c

we can treat $c^2 \cos^2 \theta$ in Eq. (8) as a perturbation term on the generalized Legendre equation. To first order in c^2 we then get:

$$\lambda_{ml} \cong l(l+1) + \frac{2c^2[m^2 - l(l+1) + 1/2]}{(2l-1)(2l+3)} \tag{12}$$

$$Z_l^m(\theta, \phi) \cong Y_l^m + \frac{1}{2} c^2 \Big\{ (2l-1)^{-1} \left[\frac{(l^2 - m^2)((l-1)^2 - m^2)}{(2l+1)(2l-2)^2(2l+3)} \right]^{1/2} Y_{l-2}^m - (2l+3)^{-1} \left[\frac{((l+1)^2 - m^2)((l+2)^2 - m^2)}{(2l+1)(2l+3)^2(2l+5)} \right]^{1/2} Y_{l+2}^m \Big\}. \tag{13}$$

ii. *Radial Equation*

The radial equation Eq. (7) can be written as a one dimensional equation with an effective potential by defining a new radial function u and a new radial coordinate r^*:

$$u \equiv (r^2 + a^2)^{1/2} R, \qquad dr^* \equiv (r^2 + a^2) \Delta^{-1} dr \tag{14}$$

we then find

$$\frac{d^2 u}{dr^{*2}} + [\omega^2 - V(\omega)] u = 0 \tag{15}$$

with

$$V(\omega) = \Big\{ \frac{\Delta \mu^2}{r^2 + a^2} + \frac{4Mram\omega - a^2 m^2 + \Delta[\lambda_{ml} + (\omega^2 - \mu^2) a^2]}{(r^2 + a^2)^2} + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2 r^2}{(r^2 + a^2)^4} \Big\}. \tag{16}$$

Equation (15) is then readily amenable to the JWKB approximation.

The correct boundary conditions for Eq. (15) have been discussed elsewhere [4, 7] (see also below); they are:

$$R \sim e^{-ik_+ r^*} \qquad \text{for} \quad r \to r_+ \ (r^* \to -\infty) \tag{17a}$$

$$R \sim \frac{e^{i(\omega^2 - \mu^2)^{1/2} r^*}}{r} \qquad \text{for} \quad r \to \infty \ (r^* \to +\infty) \tag{17b}$$

where $k_+ \equiv \omega - m\omega_+$, $\omega_+ \equiv a/2Mr_+$ ("angular velocity of hole"), and $r_\pm \equiv M \pm (M^2 - a^2)^{1/2}$ ("location of horizons"). In Eq. (17b) we must assume that ω, for the bound case $\omega_R < \mu$, lies on the "first Riemann sheet" or "physical sheet," $0 \leqslant \arg(\omega^2 - \mu^2)^{1/2} < \pi$.

c. *Ingoing Timelike Kerr Coordinates* [19] $\{\tilde{t}, r, \theta, \tilde{\phi}\}$

This coordinate system is well suited to the imposition of boundary conditions and derivation of conservation laws because it extends smoothly across the future event horizon, unlike the Boyer–Lindquist system, to which it is related by:

$$\tilde{t} = t + \int 2Mr\Delta^{-1}\, dr, \qquad \tilde{\phi} = \phi + \int a\Delta^{-1}\, dr. \tag{18}$$

The contravariant metric in this case is:

$$\left(\frac{\partial}{\partial s}\right)^2 = \Sigma^{-1}\Big\{\Delta\left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial \theta}\right)^2 + (\sin\theta)^{-2}\left(\frac{\partial}{\partial \tilde{\phi}}\right)^2 + 2a\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial \tilde{\phi}}\right)$$
$$+ 4Mr\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial \tilde{t}}\right) - (\Sigma + 2Mr)\left(\frac{\partial}{\partial \tilde{t}}\right)^2\Big\} \tag{19}$$

where Δ and Σ are as previously defined.

The Klein–Gordon equation is likewise separable in these coordinates,

$$\Psi = \tilde{R}(r)\,\Theta(\theta)\,e^{im\tilde{\phi}}e^{-i\omega\tilde{t}} \tag{20}$$

where Θ is exactly as in Eq. (8), and R is related to $\tilde{R}$ by

$$R(r) = \tilde{R}(r)\exp\left[-i\int(2Mr\omega - am)\,\Delta^{-1}\,dr\right]. \tag{21}$$

The boundary condition that $\tilde{R}$ be regular around $r = r_+$ then leads to Eq. (17a). Condition (17b) at infinity holds for both R and $\tilde{R}$.

III. Existence of Unstable Normal Modes

A fairly convincing argument for instability follows from the integral law of energy conservation for a normal mode of the form (20) in ingoing timelike Kerr coordinates. To derive this law, multiply Eq. (1) by $i\omega^*(-g)^{1/2}\sin\theta\Psi^*$, integrate over $r_+ \leqslant r < \infty$ and $0 \leqslant \theta \leqslant \pi$, and take the real part, to obtain

$$0 = \tfrac{1}{2}B(\omega_R^2 + \omega_I^2 - m\omega_R\omega_+) + A\omega_I \tag{22}$$

where ω_R and ω_I are the real and imaginary parts of ω, and

$$B = 4Mr_+\left[\int_0^\pi d\theta\,\sin\theta\,|\,\Psi\,|^2\right]_{r=r_+} \tag{23}$$

$$A = \int_{r_+}^\infty dr\int_0^\pi d\theta\,\sin\theta\,\Big\{(\Sigma - 2Mr)\left|\frac{\partial\Psi}{\partial r}\right|^2 + \left|\frac{\partial\Psi}{\partial\theta}\right|^2$$
$$+ [(\Sigma + 2Mr)\,|\,\omega\,|^2 + \mu^2\Sigma]\,|\,\Psi\,|^2 + \left|a\sin\theta\frac{\partial\Psi}{\partial r} + im\csc\theta\Psi\right|^2\Big\}. \tag{24}$$

In Eq. (22), the first term is proportional to the energy flux down the hole, and the second term is proportional to the total energy content in a space slice $\tilde{t}$ = const outside the hole. The boundary integral B is positive-definite (unless $\Psi(r = r_+) \equiv 0$, in which case $\Psi \equiv 0$). In the slice integral A, the integrand is positive-definite outside the ergosphere, $\Sigma > 2Mr$; A will be positive if Ψ peaks outside the ergosphere, but may be negative for certain Ψ ("negative-energy wave packets") which peak inside the ergosphere. A is infinite for unbound "scattering states," $|\,\omega\,| > \mu$ along the real axis, but is finite elsewhere on the physical sheet.

Let us now *assume* (and thereby the argument is technically not rigorous as it stands) that normal modes exist, either stable or unstable, that are bound, superradiant, and peak far from the hole; specifically:

(1) The mode is bound, $0 < \omega_R < \mu$;

(2) Ψ peaks far outside the ergosphere, so $A > 0$, $A/B \gg 1$;

(3) ω is nearly real, $|\omega_I| \ll \omega_R$; and

(4) ω_R is "superradiant," $0 < \omega_R \leqslant m\omega_+$.

Now $A < \infty$ for any ω in the strip $0 < \omega_R < \mu$ of the physical sheet. Then Eq. (22) constrains ω to lie on a large circle of radius $\sim A/B$,

$$(\omega_R - \tfrac{1}{2}m\omega_+)^2 + (\omega_I + A/B)^2 = A^2/B^2 + \tfrac{1}{4}m^2\omega_+{}^2 \tag{25}$$

with center at $\omega = \frac{1}{2}m\omega_+ - iA/B$ far down in the lower half-plane, which passes through $\omega = 0$ and $\omega = m\omega_+$; see Fig. 1. Under assumptions (1), (3) and (4),

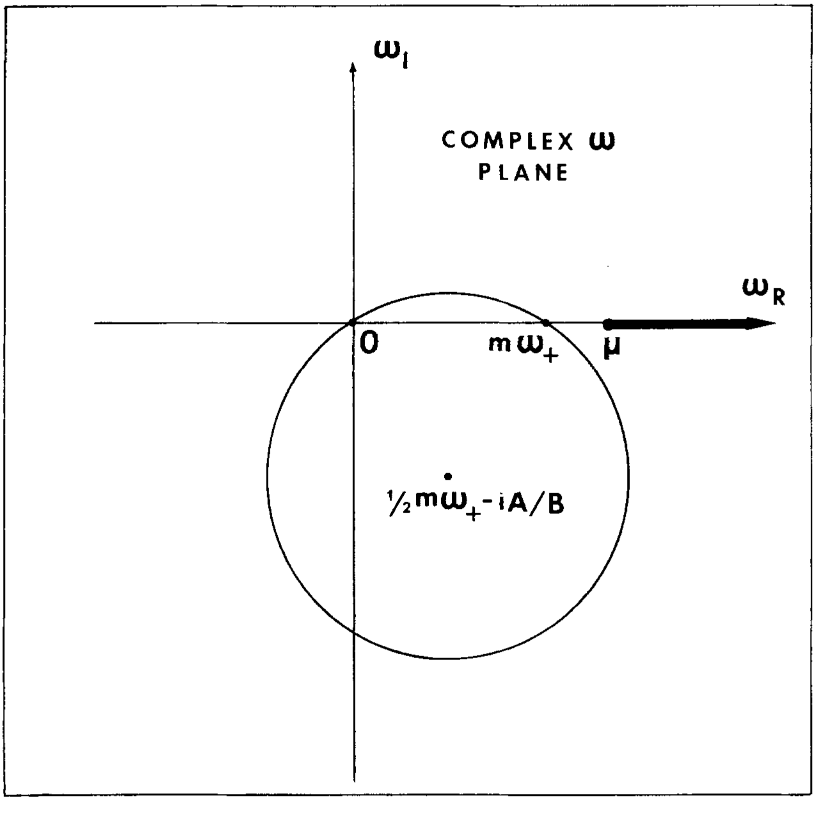

FIG. 1. The complex ω-plane. There are "threshold" branch points at $\omega = \pm\mu$, with cuts which may be taken out on the real axis to infinity. Poles belonging to normal modes must lie on the circle, under assumptions 1) – 4) of § III.

ω then must lie on the short arc in the upper half-plane between $\omega = 0$ and $\omega = m\omega_+$; therefore $\omega_I > 0$ and the normal mode must be unstable.

It would be most surprising if such modes satisfying assumptions (1)–(4) did not exist; one would need such modes to build wave packets in bound distant Keplerian orbits around the hole. The physical interpretation of this argument is that the mode represents such a packet. By the "superradiance" assumptions (2), which is always achieved by a sufficiently distant prograde orbit, and by (3), the energy flux down the hole is negative (there is a net energy flow out of the hole), i.e., the first term in Eq. (22) is negative; as usual this can be shown directly from the Area Theorem [16, Prop. 9.2.7]. Since the packet has positive total energy, $A > 0$, and since it cannot lose any energy to infinity because it is bound, it will grow indefinitely.

Why does this argument not go through for massless Boson fields, if we ignore (1)? Under the hypothesis $\omega_I > 0$ the argument above still holds. However, under the alternate hypothesis $\omega_I \leqslant 0$, we have $A = \infty$ because of the boundary condition $\Psi \sim e^{i\omega r}$ at infinity, unlike above, so that the argument is vacuous. Therefore either $\omega_I > 0$ or $\omega_I \leqslant 0$ is possible (but not both), and no conclusion regarding stability can be drawn. Physically one can say that there are no superradiant bound states of a massless field around a rotating black hole.

IV. JWKB Calculation of ω_I

How big or how small is ω_I for unstable modes ? We can estimate in the short-wavelength regime $M\mu \gg 1$ by the JWKB approximation [21].

First we approximate ω as real, $\omega \equiv \omega_R$, and construct a JWKB solution Ψ to Eq. (16) in some time slice $t = 0$ in Boyer–Lindquist coordinates. The potential $V(\omega)$ is now real and for the case of bound superradiant modes has the form seen in Fig. 2. The particle is bound in region III but can tunnel through the barrier region II. The boundary conditions are that Ψ die exponentially in region IV and that the phase velocity is *positive* in region I. Therefore there is a flux toward *increasing r* in the barrier region II.

The conserved current vector J_μ for Ψ is

$$J_\mu \equiv -i(\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*) \tag{26}$$

At $t = 0$ we normalize Ψ by imposing

$$E(0) \equiv 1 \tag{27}$$

where

$$\begin{aligned} E(t) &\equiv \int_{r_2}^{r_3} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, \Sigma \sin\theta J^0(t) \\ &\cong \int_{r_+}^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, \Sigma \sin\theta J^0(t). \end{aligned} \tag{28}$$

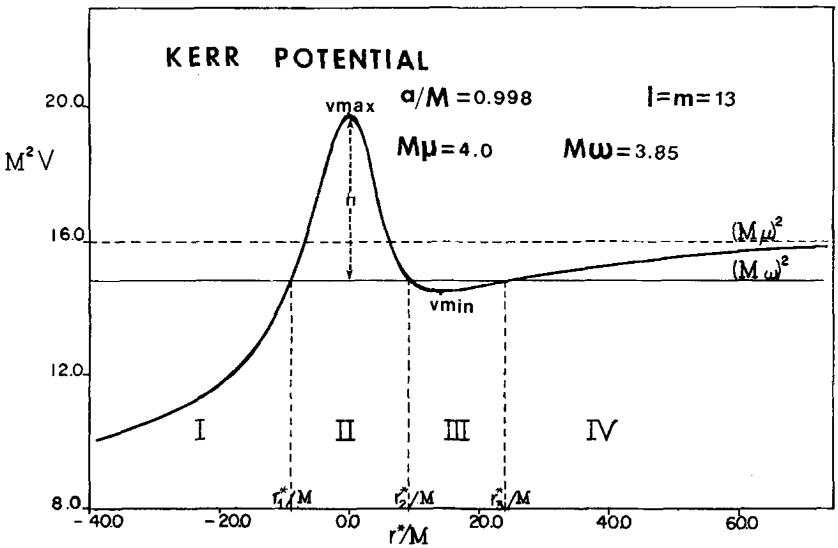

FIG. 2. Effective potential for motion in Kerr geometry as seen by a Bose field, expressed as a function of the Wheeler tortoise coordinate r^*, for a typical set of parameters a, μ, ω, l, m. The three turning points r_1^*, r_2^*, r_3^*, define the regions I, II, III, IV used in the JWKB approximation. $V_{\max}$ and $V_{\min}$ define the local maximum and minimum of the potential V. The barrier height h is defined as the distance from $V_{\max}$ to ω^2. The tortoise coordinate:

$$r^* = r + 2Mr_+(r_+ - r_-)^{-1}\log(r/r_+ - 1) - 2Mr_-(r_+ - r_-)^{-1}\log(r/r_- - 1)$$

is given in units of M, while the effective potential V and ω^2, μ^2 are given in units of $1/M^2$. For these parameters $r_+ = 1.0632M$, $M^2V_{\max} = 19.7246$, $M^2V_{\min} = 14.4488$, $M\omega_+ = 0.4693$, $M^2\omega^2 = 14.8225$.

The flux of J_μ downward into the hole across a 2-surface $r = r_0$, $r_+ < r_0 < r_1$, at $t = 0$ is

$$F = -\int_0^\pi d\theta \int_0^{2\pi} d\phi (-g)^{1/2} J^r \tag{29}$$

F can be computed using the JWKB solution Ψ to be

$$F = -\gamma \exp(-I) \tag{30}$$

with

$$I = 2\int_{r_1}^{r_2} \kappa(r^2 + a^2)\, \Delta^{-1}\, dr$$

$$= \text{“barrier integral”}; \tag{31}$$

$$\gamma = \gamma(\omega_N, a, M, l, m)$$

$$= \left\{\int_{r_2}^{r_3} \frac{4\cos^2\beta(r)}{k}\left[\omega_N\left(\frac{r^2 + a^2}{\Delta} - \frac{a^2\zeta}{r^2 + a^2}\right) - \frac{2Mmar}{\Delta(r^2 + a^2)}\right] dr\right\}^{-1}$$

$$= \text{“normalization integral”}; \tag{32}$$

$$\beta(r) = \int_r^{r_3} k(r^2 + a^2)\,\Delta^{-1}\,dr - \pi/4 \qquad (r_2 \leqslant r \leqslant r_3) \tag{33}$$

$$\zeta = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta Z_l^{*m} \sin^2\theta Z_l^{m} \tag{34}$$

$$k \equiv (\omega_N^2 - V(\omega_N))^{1/2} \tag{35}$$

$$\kappa \equiv (V(\omega_N) - \omega_N^2)^{1/2}. \tag{36}$$

Here ω_N are those frequencies for which

$$\int_{r_2}^{r_3} k(r^2 + a^2)\,\Delta^{-1}\,dr = (N + 1/2)\pi, \qquad N = 0, 1, 2,... \tag{37}$$

with N being the "energy" quantum number. Condition (37) is a result of matching the JWKB solution in IV so as not to give an exponential increasing mode.

The normalization integral γ is always positive by construction and neglecting terms of order $1/r^2$ and $1/r^3$, which is valid for sufficiently distant wave packets, we have

$$\gamma \cong \left[\int_{r_2}^{r_3} \frac{4\cos^2\beta(r)}{k}\,\omega_N(r^2 + a^2)\,\Delta^{-1}\,dr\right]^{-1}. \tag{38}$$

Finally we can estimate the value of ω_I as a correction to the initial approximation $\omega = \omega_R$. From $|\Psi| \propto \exp(\omega_I t)$ we have

$$E(t) = E(0)\exp(2\omega_I t) \tag{39}$$

from current conservation, $\nabla_\mu J^\mu = 0$, we have

$$\frac{dE(0)}{dt} = -F \tag{40}$$

so that

$$\omega_I = -\tfrac{1}{2}F = \tfrac{1}{2}\gamma\exp(-I). \tag{41}$$

V. Results of JWKB Calculations

Using the formulas we derived in §II and §IV we calculate ω_I and search for the most unstable modes. Noting that the Kerr potential $V(\omega) \equiv V(\omega, m) = V(-\omega, -m)$ and that if $\omega_I > 0$ and $\omega_R \gtrless 0$ then $m \gtrless 0$ respectively, we pick $\omega_R > 0$ and $m > 0$ with no loss of generality [5]. The results are best expressed in the dimensionless quantities a/M, $M\mu$ and $M\omega_I$.

Our JWKB approximation has to satisfy the condition $dk/dr^* \ll k^2$ (k as defined in Eq. 35), and since λ_{ml} is approximated to first order in c^2 (Eq. 12) we must keep $c^2 = a^2(\omega^2 - \mu^2) \ll 1$.

We first explore the condition for minimizing the barrier integral I of Eq. (31). It is known [cf. 22] that $l = m$ minimizes I. We therefore set $l = m$ and note that the barrier height h (see Fig. 2) decreases with increasing frequency ω. Therefore choose m and ω according to $\omega = m\omega_+ = \min\{\mu, (V_{\mathrm{MAX}})^{1/2}\}$. Then I reaches a minimum as $a \to M$.

For $M\mu \gg 1$, we then have $l = m \gg 1$, $\omega = \mu$; and under these conditions I can be expressed in terms of complete elliptical integrals [23]:

$$I = 2\sqrt{2}\, M\mu \left(\frac{M}{r^2}\right)^{1/2} \left\{ \frac{(r_1 + r_2 - 2M)}{M} F(\rho) + \frac{A_+}{r_2 - r_+} \Pi\left(\frac{r_2 - r_1}{r_2 - r_+}, \rho\right) + \frac{A_-}{r_2 - r_-} \Pi\left(\frac{r_2 - r_1}{r_2 - r_-}, \rho\right) - \frac{r_1}{M} \Pi(\rho^2, \rho) \right\} \tag{42}$$

with

$$\rho \equiv \left(\frac{r_2 - r_1}{r_2}\right)^{1/2} \tag{43}$$

$$r_{1\,2} = \frac{1 \pm H}{4M\Omega^2} \qquad (r_2 > r_1) \tag{44}$$

$$H \equiv [1 - 16M^2\Omega^2(a\Omega - 1)^2]^{1/2} \tag{45}$$

$$\Omega \equiv \omega/m \tag{46}$$

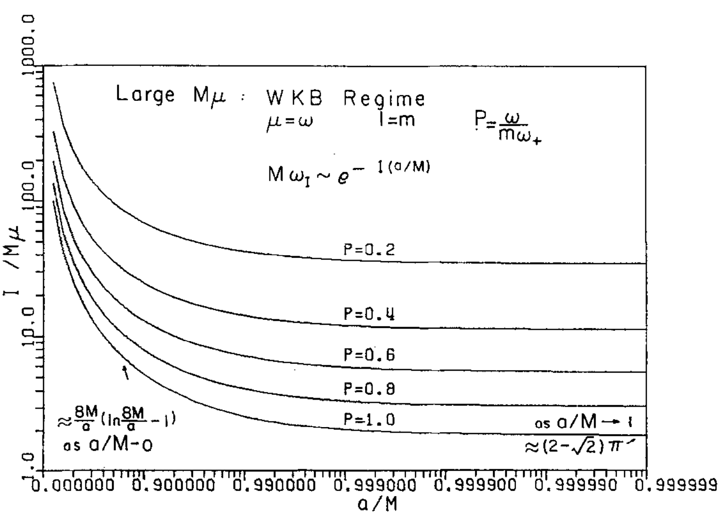

FIG. 3. Barrier integral I in units of $M\mu$, expressed as a function of a for the case of very large l ($l = m \gg 1$) and $\omega = \mu$, for different values Ω. For $\omega = m\omega_+$ the Barrier integral is smallest and has the limiting form $I \sim M\mu(8M/a)(\ln 8M/a - 1)$ as $a \to 0$ and $I \sim M\mu(2 - \sqrt{2})\pi$ as $a \to M$.

and

$$A_{\pm} \equiv \frac{r_{\pm}[a^2 + 2(r_1 + r_2)M - r_1 r_2 - 4M^2] - a^2(r_1 + r_2 - 2M)}{M(r_{\pm} - r_{\mp})} \tag{47}$$

$$F(x) \equiv \int_0^{\pi/2} (1 - x^2 \sin^2 \rho)^{-1/2}\, d\rho \tag{48}$$

$$\Pi(\xi, \eta) \equiv \int_0^{\pi/2} (1 + \xi \sin^2 \rho)^{-1}(1 - \eta^2 \sin^2 \rho)^{-1/2}\, d\rho. \tag{49}$$

In Fig. 3 we plot I as a function of a/M at different values of $P = \omega/(m\omega_+) = \Omega/\omega_+$ in this approximation. We conclude that I is minimized for $P \to 1$. For $P = 1$, I is of the form

$$I = M\mu \cdot f(a/M) \tag{50}$$

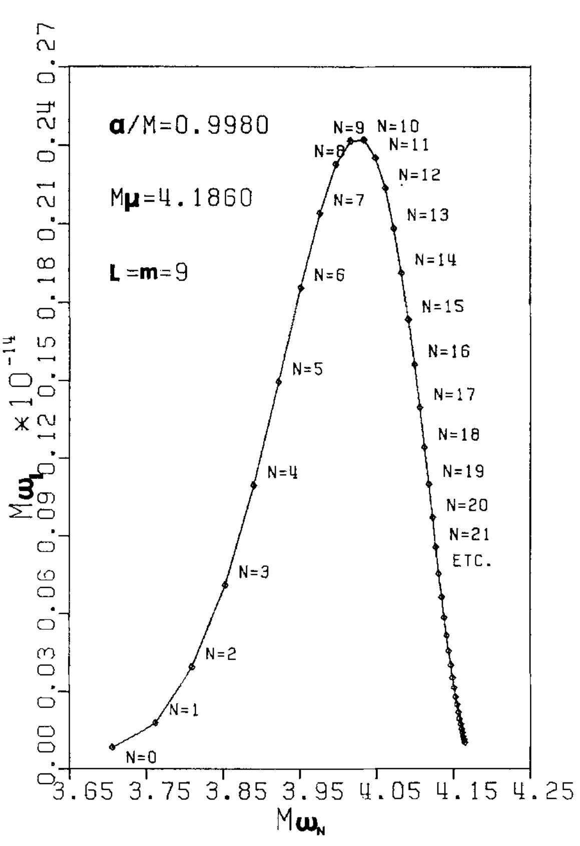

FIG. 4. Imaginary part of the frequency (ω_I) expressed as a function of the quantized real part of the frequency (ω_N). As the energy quantum number N increases, the barrier height h (see Fig. 1) decreases and both the barrier integral I and the normalization factor γ diminish. Since $\omega_I \sim \gamma \exp(-I)$ these effects act in opposite ways. For small N, I dominates, causing ω_I to increase even though γ decreases. After a certain point ($N = 10$), ω_I reaches a maximum and then γ takes over and dominates the behavior of ω_I causing it to decrease to zero as $N \to \infty$. Both ω_I and ω_N are given in units of $1/M$.

and in the limits $a/M = 0$ and $a/M = 1$ is

$$I \approx M\mu(8M/a)[\ln(8M/a) - 1] \quad \text{as} \quad a/M \to 0; \tag{51}$$

$$I \to M\mu(2 - \sqrt{2})\pi \approx 1.84M\mu \quad \text{as} \quad a/M \to 1. \tag{52}$$

We next investigate the conditions for maximizing γ (Eq. 38). For $\omega_N \to \mu$ (corresponding to marginally bound orbits), $r_3 \to \infty$ and $\gamma \to 0$. We note that there exist cases where $V_{MAX} < \mu^2$ and therefore "energy" quantum number N cannot always go to infinity.

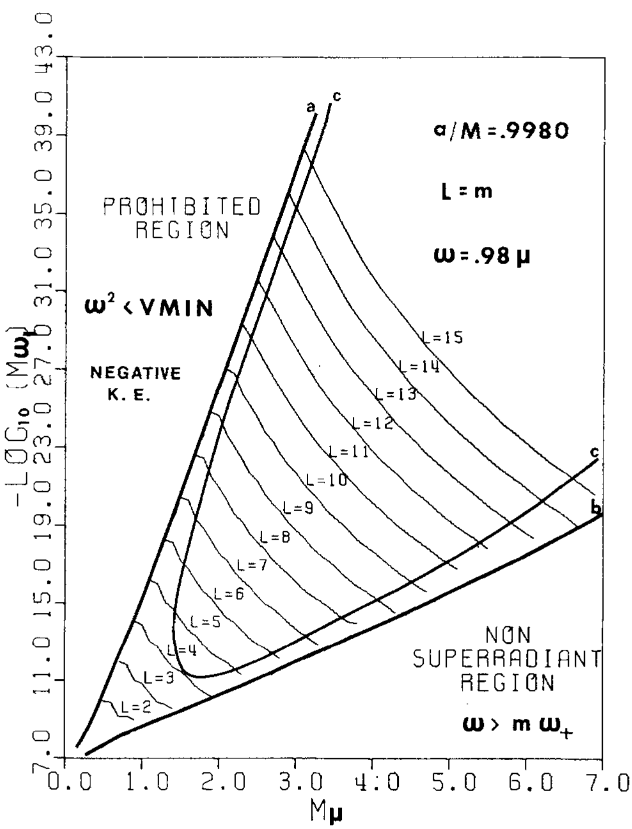

FIG. 5. Summary of *JWKB* results for fastest growing modes. Plotted is imaginary part of the frequency, ω_I, expressed as a function of the mass of the Bose field μ, for $a = 0.9980\ M$ and $\omega = 0.98\mu$ at different values of $l = m$. The region of bound superradiant orbits is separated by line *a* from the region of negative radial kinetic energy and by line *b* from the region of non-superradiant orbits. Close to lines *a* and *b* the JWKB approximation breaks down due to the closeness of the turning points. Line *c* marks off the region within the bound superradiant regime which satisfies the JWKB validity criterion for more than 60% of the distance between turning points. Line *c* is very qualitative and should only be used as a rough guideline.

The JWKB criterion used is $(dk/dr^*)/k^2 \ll 1$ with values of $(dk/dr^*)/k^2 \lesssim 0.2$ considered adequate.

In Fig. 4 we present a typical plot (for $M\mu = 4.186$, $a/M = 0.998$) of ω_I as a function of ω_N and observe a sharp maximum occuring roughly at $\omega \sim 0.95\mu - 0.98\mu$. This behavior is a result of I monotonically decreasing with increasing ω_N, but of $\gamma \to 0$ as $\omega_N \to \mu$ $(N \to \infty)$.

Finally we investigate the behavior of ω_I as a function of $M\mu$. As can be seen in Fig. 5, for a fixed $M\mu$ the most unstable modes occur at the smallest permissible value of l (closest orbit), with $\omega \simeq m\omega_+$.

In summary we conclude that for a certain mass $M\mu$, the most unstable (bound, superradiant) mode is the one with:

(a) Smallest permissible l.

(b) Largest permissible m $(m = l)$.

(c) Largest permissible a $(a/M = 1)$.

(d) Largest permissible real frequency $\omega_R \equiv \omega$ $(\omega \sim 0.98\mu \leqslant m\omega_+)$.

These optimal conditions lead to a growth rate that is approximately

$$M\omega_I \sim 10^{-7} \exp(-1.84 M\mu). \tag{53}$$

Our results suggest that the fastest growing modes occur for $M\mu \lesssim 1$ and have growth rates of at least $M\mu \sim 10^{-7}$. The small magnitude of this estimate is due mainly to the normalization factor $\gamma \sim 10^{-6}$, and not to the barrier factor $\exp(-\Gamma)$, which is only ~ 0.1 in favorable cases. These are all JWKB estimates.

In the opposite approximation $M\mu \ll 1$, Detweiler [to be published] finds that the bound states of the scalar field approximate those of the (spinless) nonrelativistic hydrogen atom; the fastest growth is in the $2P$ state with

$$M\omega_I \cong (M\mu)^9/12. \tag{54}$$

VI. Conclusions

If, as we have argued, unstable normal modes exist, then calculations of quantum particle creation by black holes are incomplete for the case of rotating holes and integer-spin, massive free particles with $M\mu \sim 1$. In fact, the main mechanism by which the hole gives up its angular momentum and free energy of rotation has been missed. Technically, these calculations are incomplete because they have implicitly assumed that a real-frequency basis of classical wave functions is complete; but this is not so if there exist unstable normal modes of the classical field, as we discussed in §II. Several technical difficulties crop up when one sets out to build a proper quantum field theory in these circumstances; e.g. the Hamiltonian is non-self-adjoint. However, we expect that quantum corrections will be small, and that the effect of the instability can be assessed correctly in the classical approximation. In contrast, the Hawking Process is essentially quantum in nature, and is of much greater fundamental interest [17].

For a free field, the instability will continue growing until the total energy and angular momentum content of the field begins to approach that of the hole. At this point, back-reaction becomes important, the hole begins to spin down, and gravitational radiation accompanied by some unbound radiation in the massive field goes off to infinity, carrying energy and angular momentum. Presumably the system asymptotically approaches a static final state consisting of a nonrotating black hole and some outgoing radiation at infinity. The horizon area of the final black hole is greater than that of the initial one; the Area Theorem [16, Prop. 9.2.7] applies because the process is classical.

Whether any of the above applies to a rapidly rotating black hole of some mass M in the real world is quite another question. Firstly, there must exist a massive Boson field of an appropriate mass μ so that $M\mu \sim 1$, in order that the instability can grow on a time-scale short enough to be interesting. Secondly, the other interactions, for instance decays, of this field must be sufficiently feeble that the instability is not quenched.

If $M \gtrsim 10^{15}$ g, then the evaporation of the black hole takes longer than the age of the universe $\sim H_0^{-1}$, where $H_0 \sim 10^{-18} s^{-1}$ is the Hubble constant, and the rotational instability will be interesting if the hole spins down in the age of the universe; from the results of §V, then,

$$(12MH_0)^{1/9} \lesssim M\mu \lesssim 0.54 \ln(1/MH_0) - 8.8 \tag{55}$$

for $M = M_\odot = 2 \times 10^{33}$ g,

$$7 \times 10^{-12}\,\text{eV} \lesssim M_\Psi \lesssim 4 \times 10^{-8}\,\text{eV} \tag{56}$$

$$5 \times 10^{2}\,\text{cm} \lesssim \lambda\!\!\!^{-}_\Psi \lesssim 3 \times 10^{6}\,\text{cm} \tag{57}$$

where M_Ψ is the mass of the field (in eV) and $\lambda\!\!\!^{-}_\Psi = \mu^{-1}$ is the reduced Compton wavelength. These limits scale nearly linearly in M. No such fields are known, but it has been speculated that such a field might contribute to macroscopic gravitation so that the inverse square law would break down for separations $r \sim \lambda\!\!\!^{-}_\psi$ [24, 25]; it seems that this wild possibility cannot be ruled out today [26, 27]. The definite observation of a rapidly rotating black hole could rule it out. Similar instabilities involving rotating neutron stars might similarly rule it out.

If $M \lesssim 10^{15}$ g, then the rotational instability will be interesting if it grows faster than the timescale [17] $\sim M^3/M_{\text{Pl}}^2$ for Hawking evaporation, where $M_{\text{Pl}} = 2 \times 10^{-5}$ g. From §V, then

$$(M_{\text{Pl}}/M)^{2/9} \lesssim M\mu \lesssim 1.1 \ln(M/M_{\text{Pl}}) - 8.8 \tag{58}$$

which for two representative values of M gives

$$10\,\text{keV} \lesssim M_\psi \lesssim 10\,\text{GeV} \quad (\text{for } M = 10^{15}\,\text{g}) \tag{59}$$

$$50\,\text{MeV} \lesssim M_\psi \lesssim 10^{4}\,\text{GeV} \quad (\text{for } M = 10^{12}\,\text{g}) \tag{60}$$

Again these limits scale nearly linearly in M. These ranges span the presently known or conjectured spectrum of elementary particle masses; for instance the π^0, the J/Ψ, and the Z^0 are possible candidates. (The latter two particles have spin 1; the estimates for spin 0 in §V should be conservative for spin 1.)

Now the question of other interactions and decays become crucial, and we can only speculate on possible processes. All know massive neutral Bosons decay with a half-life $t_{\frac{1}{2}} \sim 10^{-16}s$, and we expect that the instability will be quenched unless $\omega_I \gtrsim 1/t_{\frac{1}{2}}$, which will place more stringent limits on $M\mu$. Further, the π^0 and the J/Ψ are probably not elementary fields; according to the standard QCD theory of hadrons, these particles are quark-antiquark bound states, so that at sufficiently high occupation number they will stop behaving like Bosons. The rotational instability will spin down the hole only if the field grows to very high occupation numbers $\sim(M/M_\psi)^2$ (for this reason, incidentally, the corresponding process for Fermions can be ignored); therefore we rather doubt that there are instabilities for π^0 and J/Ψ. On the other hand there could be an instability in the purely gluonic sector of QCD, with the role of μ played by the characteristic mass scale $\sim$1 GeV of the strong interactions; this would be a "glueball" instability.

In contrast, the (as-yet-unobserved) neutral intermediate Boson Z^0 is an elementary field, at least according to the Weinberg–Salam and similar theories of the weak and electromagnetic interactions, and we expect the Z^0 field to be unstable if ω_I exceeds $(t_{\frac{1}{2}})^{-1} \sim 10^{-3}\mu$. The effect of the instability is a little different in this case, because the mass of the Z_0, $M_{Z^0} \sim 100$ GeV, is in theory due to spontaneous breakdown of a gauge symmetry. The free energy released from the hole will go into healing the symmetry breakdown in a small, growing region around the hole. In this phase transition region the intermediate Bosons $W^\pm$ and Z^0 will be massless along with the photon γ; but the instability probably will presumably continue to grow because of reflection from the boundary between the normal and symmetric phases. The region might grow to an eventual size $r \sim M(M/M_{\rm Pl})^{2/3}(M\mu)^{-4/3} \sim 10^{-6}$ cm; ultimately all the energy will be released in decay products such as lepton pairs, and the symmetry will break down again when the hole is spun down. In this way, a rotating black hole can cause a temporary phase change in the vacuum of an ambient gauge field theory.

In summary, the actual occurrence of the rotational instability for a mini black hole will depend very much on the detailed behavior of elementary particles. It will be necessary both to calculate numerically the actual rate of growth for the fastest growing classical field, and also to take real elementary particle physics into more careful account than we have done, in order to decide if the process actually can occur. The possible astrophysical interest of this process is that the sudden growth of the instability for a mini black hole might produce a sudden burst of energetic particles, γ's and $e^\pm$'s, rather like that from the explosion due to the Hawking process, but of greater magnitude if it occurs at greater mass M.

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