KLEIN PARADOX IN A KERR GEOMETRY

N. DERUELLE*
Joseph Henry Laboratories, Princeton University, Princeton 08540, N.J., USA

and

R. RUFFINI**
Institute for Advanced Studies, Princeton, N.J. 08540, USA

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The crossing of the classical positive and negative energy states E^+ and E^- introduced by Christodoulou-Ruffini and interpreted within the framework of a relativistic quantum field theory by Deruelle and Ruffini, leads to a Klein paradox. It has been shown by Euler and Heisenberg that when the transmission coefficient T^2 through the barrier between the E^+ and E^- states is small it is proportional to the probability of pair creation. Numerical computations show that, in the case of a small Kerr black hole ($GM/c^2 \lesssim \hbar/\mu c$), the probability of pair creation of particles of mass μ is maximum when $E \sim \hbar\Omega$, where E is the energy of the created particles and Ω and M the angular velocity and the mass of the back hole.

The Hamilton-Jacobi equation of a classical particle in a Kerr-Newman geometry leads to the following equation of motion:

$$(\mathrm{d}r/\mathrm{d}\tau)^2 = (E - E_0^+)(E - E_0^-)[(r^2 + a^2)/(r^2 + a^2 \cos^2\theta)]^2 \tag{1}$$

where

$$E_0^{\pm} = \frac{(a\phi + eQr) \pm [\Delta(\mu^2 r^2 + K)]^{1/2}}{r^2 + a^2} \tag{2}$$

with $\Delta = r^2 - 2Mr + a^2 + Q^2$. Here M, Q and a are the mass, the charge and the angular momentum per unit mass of the black hole, ϕ, e and K are the angular momentum, the charge and the Carter's constant of the motion of the particle of mass μ [1]:

$$K = p_\theta^2 + (\phi - Ea\sin^2\theta)^2/\sin^2\theta + a^2\mu^2\cos^2\theta. \tag{3}$$

In a previous letter [2], we have given an interpretation in the framework of a quantum field theory of the positive and negative root solutions $E^{\pm}$ introduced by Christodoulou and Ruffini [3] (details in ref. [4]). We showed that:

1) The classical $E^{\pm}$ bound states are the classical limit of the "resonances" of a quantum field satisfying the Klein-Gordon equation written in the Kerr-Newman metric.

2) The positive energy states ($E^+ > E_0^+, l, m$) correspond to a positive probability density J^0 (such that $\partial_\mu J^\mu = 0$) and therefore describe particles of energy E, quantum numbers l, m.

3) The negative energy states ($E^- < E_0^-, l, m$) correspond to a negative probability density J^0 and therefore describe, thanks to the properties of particle-antiparticle conjugation, antiparticles of energy $-E$, quantum numbers l, and $-m$.

4) When there is a crossing of the E^+ and the E^- states, the probability density J^0 has not a constant sign; we

* On leave of absence from Ecole Normale Superieure, Paris.
** Alfred P. Sloan fellow.

are then in the conditions of the Klein paradox as considered by Klein [5], Sauter [6], Euler and Kockel [7], Heisenberg and Euler [8], and Pauli [9].

In this letter we shall study the transmission coefficient through the barrier between the states E^+ and E^- of a scalar field satisfying the Klein-Gordon equation coupled to a classical gravitational background described by the Kerr-Newman metric. We write the Klein-Gordon equation

$$(\nabla_\alpha + ieA_\alpha)(\nabla^\alpha + ieA^\alpha)\,\phi + \mu^2\phi = 0 \tag{4}$$

in the Kerr-Newman metric:

$$ds^2 = [1 - (2Mr - Q^2)/\rho^2]\, dt^2 - (\rho^2/\Delta)\, dr^2 - \rho^2\, d\theta^2$$

$$- [r^2 + a^2 + a^2 \sin^2\theta\,(2Mr - Q^2)/\rho^2] \sin^2\theta\, d\varphi^2 + [2(2Mr - Q^2)\, a \sin^2\theta/\rho^2]\, d\varphi\, dt$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta; \qquad A = (Qr/\rho^2)(dt - a \sin^2\theta\, d\varphi).$$

The separation of the variables:

$$\phi = \exp i(m\varphi - Et) S_{ml}(\theta) R(r) \tag{6}$$

where S_{ml} are the spheroidal harmonics [10] of eigenvalues λ_{ml}, leads to the following radial equation:

$$r^4 d^2 u/dr^{*2} = W(r^*)u, \qquad W(r^*) = \Delta(\mu^2 r^2 + K) - [(r^2 + a^2)E - am - eQr]^2 + (2\Delta/r^2)(Mr - a^2 - Q^2) \tag{7a, b}$$

where

$$K = \lambda_{ml} + aE(aE - 2m)$$

$u = rR(r)$ and $dr/dr^* = \Delta/r^2$ (see fig. 1).

In the Schwarzschild case ($a = Q = 0$) there is no crossing of the E^+ and E^- solutions [2]. Whatever is the energy $E(E^2 > \mu^2)$ of the particle, the probability density has a constant sign (positive for particles, negative for antiparticles). Therefore in a Schwarzschild black hole there is no creation of particles due to the absence of level crossing. The problem is then a usual problem of diffusion in the potential (7) [11], the boundary condition being that nothing whatsoever can emerge from the horizon, i.e. that the wave is ingoing at the horizon of the black hole. Details in ref. [4].

On the other hand, when the black-hole is endowed with rotational energy or electromagnetic energy ($a \neq 0$, $Q \neq 0$), there is a level crossing of the E^+ and the E^- states inside the effective ergosphere [12, 13] (cf. fig. 1 and caption).

When such a crossing occurs, the probability density J^0 has not a constant sign. We are then in the conditions of the Klein paradox [5–9].

To clarify the discussion, we shall only consider the case $\mu < E < (am + eQr_+)/(r + a^2)$ where $r_+ = [M + (M^2 - a^2 - Q^2)^{1/2}]$ (incident *particle*; the case of an incident antiparticle $(am + eQr_+)/(r_+^2 + a^2) < E < -\mu$ is completely analogous).

The solution of the radial eq. (7) near the horizon is an *outgoing* wave corresponding to an infalling antiparticle:

$$u(r^*) \sim C \exp i|[(r_+^2 + a^2)E - am - eQr_+]/r_+^2|\, r^* \equiv C \exp i\,|k|\, r^*. \tag{8}$$

The solution at infinity is:

$$u(r^*) \sim A \exp[-i(E^2 - \mu^2)^{1/2} r^*] + B \exp[i(E^2 - \mu^2)^{1/2} r^*]. \tag{9}$$

The reflexion coefficient through the barrier (cf. fig. 1 and caption) is defined by $R^2 = (B/A)^2$ and the transmission coefficient by $T^2 = [|k|/(E^2 - \mu^2)^{1/2}]\,|C/A|^2$. The theorem of the Wronskian then implies:

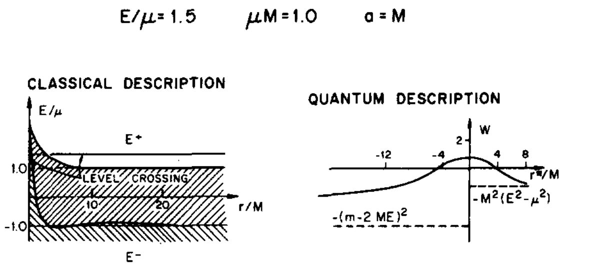

Fig. 1. The classical and quantum descriptions are here compared and constrasted in the extreme Kerr case. The effective ergosphere [3] in the region where the positive (negative) root solutions have a negative (positive energy. When there is a crossing of the E^+ and E^- states the corresponding quantum potential $W(r^*)$ (cf. eq. (7) has two zeros. There is the tunnelling from the E^+ to the E^- states through this Gamow barrier. The transmission coefficient is proportional to the probability for the incident particle to create a pair of particles [5–9]. In this figure and in the following we have chosen $h = c = G = 1$.

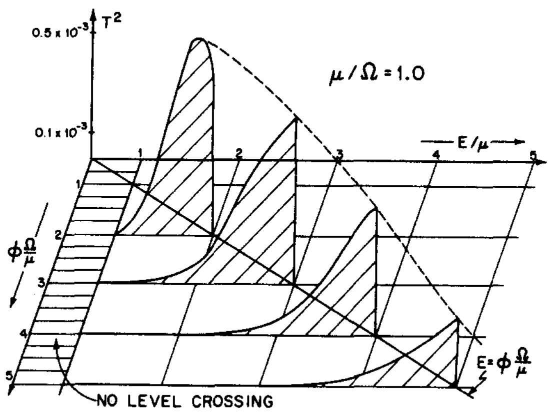

Fig. 2. The transmission coefficient through the potential barrier separating the E^+ from the E^- states is here represented as a function of E/μ the energy per unit mass of the particle and $\phi\Omega/\mu$ where $\Phi = m$ in the angular momentum of the particle and Ω the angular velocity of the black hole. In the extreme Kerr case here considered, $\Omega = (1/2)/M$; $\mu/\Omega = 1$ which means that the Compton wave-length of the particle is twice the radius of the black hole.

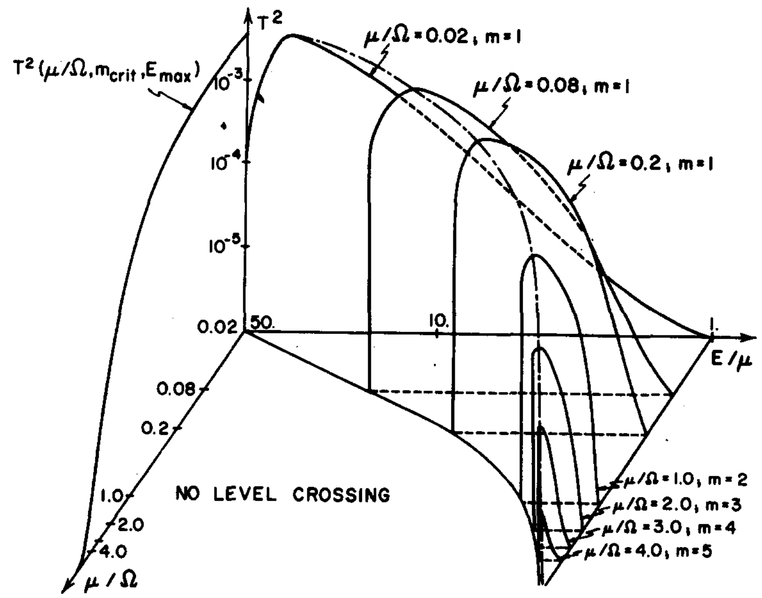

Fig. 3. The transmission coefficient T^2 is here given as a function of the energy E/μ of the particle and the parameter μ/Ω. It is then clear that the transmission coefficient goes to zero if $\mu/\Omega \gg 1$ and tends to a finite value ($T^2 \sim 0.35 \times 10^{-2}$) in the limit $\mu/\Omega \to 0$. In this last limit the transmission coefficient reaches a maximum value for an energy of the pair created $E \sim \Omega$ ($E \sim h\Omega$ in the conventional units).

$$T^2 + 1 = R^2. \tag{10}$$

Within the framework of a single particle theory, the transmission coefficient T^2 through the potential barrier separating the positive from the negative energy states is proportional to the probability for an incident particle to create a pair of particles of mass μ. This transmission coefficient was also studied in a different physical framework (superradiant scattering) and in the case of massless fields by Teukolsky and Press [14].

The transmission coefficient can be computed using the B.K.W. approximation:

$$T^2 = \exp\left(-2\int_a^b r^{-2}(W)^{1/2}\, \mathrm{d}r^*\right) \tag{11}$$

where a and b are the zeros of the potential $W(r^*)$ defined by eq. (7) (cf. fig. 1 and caption). T^2 can also be computed by a direct integration of eq. (7). A few results of a numerical computation of T^2 in the extreme Kerr case ($Q = 0, a = M$) are shown in fig. 2 and fig. 3. Further results in ref. [4,15,16].

Important here is to summarize the main results:

1) The probability of pair creation is negligible when the Compton wave-length of the incident particle is small compared to the radius of the black hole ($\hbar/\mu \ll GM/c$; classical limit).

2) When $\hbar/\mu \gtrsim GM/c$ the maximum of the probability of pair creation increases and tends asymptotically, for $\mu M/(\hbar C/G) \to 0$, to a finite value: $T^2 \sim 0.35 \times 10^{-2}$ independent of the particle mass μ (see fig. 3). This limiting value is the same as the one obtained for a massless field in ref. [14].

3) In the case of a small black hole ($GM/c < \hbar/\mu$) the transmission coefficient reached is maximum value for the energy of the particle.

$$E \sim \hbar\Omega \tag{12}$$

Here Ω is the angular velocity of a black hole [17] $\Omega = a/(r_+^2 + a^2)$. This result is in perfect accordance with the classical paper by Zel'dovich [18,19] concerning the amplification of a massless scalar wave scattered by a rotating cylinder. The result given in eq. (12) has a very satisfactory physical interpretation: the transmission coefficient reaches its maximum value for particles with a DeBroglie wavelength comparable to the characteristic length (in geometrical units) associated with the angular velocity of the black hole Ω.

4) From the law of conservation of the four momentum it follows that through the process of pair creation rotation energy can be extracted from a rotating black hole. It is important to stress that this process is always irreversible, in the sense defined in ref. [3] and increases the irreducible mass of the black hole. In this sense the process here considered differs drastically from the one studied by Hawking [20]. Physically the two processes are also drastically different. Here particle creation is produced by the shear in the space surrounding the horizon of the black hole due to the dragging of the inertial frames. There [20] particle creation is due to the time varying background geometry.

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