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On Quantum Resonances in Stationary Geometries.

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Detailed analyses have recently been made (1) of relativistic quantized fields in a classical background geometry described by Einstein field equations. Much emphasis has been directed toward the analysis of *a*) resonances of spin-0 and spin-$\frac{1}{2}$ fields in a Schwarzschild background geometry (2), *b*) scalar fields fulfilling the Klein-Gordon equation in a stationary geometry and their classical limits ($h \to 0$, Hamilton-Jacobi equation) (3), *c*) pair creation processes occurring in stationary geometries endowed, as well, with electromagnetic fields (4).

The aim of this letter is to use some of the results presented in ref. (3,4) and show by an explicit example how in a stationary geometry or in the field of a collapsed object endowed with electromagnetic structure resonance states with $\Gamma < 0$ (growing with time) can be found.

The existence of these states is most clear if the effective-potential approach (3,4) is used and has its physical justification in the interplay of the two processes of pair creation (4) and resonance states (2,3) in the field of a collapsed object.

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(1) See, *e.g.*, *Proceedings of the Marcel Grossman Meeting*, edited by R. RUFFINI (Amsterdam, 1976).

(2) See, *e.g.*, J. A. WHEELER: *Transcending the law of conservation of leptons*, in *Quaderno No.* 157, Accademia Nazionale dei Lincei (Roma, 1971), p. 133.

(3) N. DERUELLE and R. RUFFINI: *Phys. Lett.*, **52** B, 437 (1974); in this paper the positive- and negative-root solutions introduced by D. CHRISTODOULOU and R. RUFFINI: *Phys. Rev. D*, **4**, 3552 (1971), are identified with the classical limits of the positive- and negative-energy states of a relativistic quantized field. See also T. DAMOUR: *Lett. Nuovo Cimento*, **12**, 315 (1975), where this correspondence is made manifest by a suitable choice of the co-ordinates.

(4) See, *e.g.*, T. DAMOUR and R. RUFFINI: *Phys. Rev. Lett.*, **35**, 463 (1975); N. DERUELLE and R. RUFFINI: *Phys. Lett.*, **58** B, (1975) and references mentioned there.

Let us consider a spin-0 boson field Φ of mass μ and charge ε in a Kerr-Newman geometry

$$(\nabla^\alpha - i\varepsilon A^\alpha)(\nabla_\alpha - i\varepsilon A_\alpha)\,\Phi = \mu^2\,\Phi \tag{1}$$

and

$$\mathrm{d}s^2 = \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\,[(r^2+a^2)\,\mathrm{d}\varphi - a\,\mathrm{d}t]^2 - \frac{\Delta}{\Sigma}\,[\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi]^2\,, \tag{2}$$

$$A = -\frac{er}{\Sigma}\,(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)\,, \tag{3}$$

where $\Delta = r^2 - 2Mr + e^2 + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$, with M the mass, e the charge and a the specific angular momentum of the background geometry (here and in the following we choose $G = c = \hbar = 1$).

The function Φ in eq. (1) is separable [5], we then have

$$\Phi = \psi(r)\,\chi(\theta)\,\exp[i(m\varphi - \omega t)]\,, \tag{4}$$

where $\chi(\theta)$ is expressible as a function of spheroidal harmonics [6] and m is the usual azimuthal quantum number.

Introducing a new radial co-ordinate x such that [4]

$$\mathrm{d}x = [(r_+^2 + a^2)/r_+^2]\,r^2\,\mathrm{d}r/\Delta\,, \tag{5.1}$$

where

$$r_+ = M + (M^2 - a^2 - e^2)^{\frac{1}{2}}\,, \tag{5.2}$$

and such that $r = +\infty$ corresponds to $x = +\infty$ and $r = r_+$ to $x = -\infty$, we have for the radial dependence of the wave function

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = Wu \tag{6.1}$$

with $u = r\psi(r)$ and

$$W = \left(\frac{r_+^2}{r_+^2 + a^2}\right)^2 \left\{\frac{\Delta}{r^2}\left[\mu^2 + \frac{K}{r^2} + \frac{2M}{r^3} - \frac{2(a^2+e^2)}{r^4}\right] - \frac{1}{r^4}\,[(r^2+a^2)\,\omega - am - e\varepsilon r]^2\right\}, \tag{6.2}$$

where K is given in the first approximation by [6]

$$K = l(l+1) - 2ma\omega + a^2\omega^2 + \left[1 - \frac{(2m-1)(2m+1)}{(2l-1)(2l+3)}\right]\frac{\gamma^2}{2} + O(\gamma^4) \tag{6.3}$$

with $\gamma^2 = a^2(\mu^2 - \omega^2)$.

([5]) See, *e.g.*, B. CARTER: *Comm. Math. Phys.*, **10**, 280 (1968); D. BRILL, P. M. CHRZANOWSKI, C. M. PEREIRA, E. D. FACKERELL and J. R. IPSER: *Phys. Rev. D*, **5**, 1913 (1972).

([6]) J. MEIXNER and F. W. SCHÄRFKE: *Mathieusche Funktionen und Sphäroidfunktionen* (Berlin, 1954).

We are here interested in analyzing quantum states corresponding to classical circular and elliptical orbits. It is well known ([1-3]) that the corresponding quantum states are described by resonances of the quantum field of the kind first considered by GAMOW ([7]) in the classical problem of the α-decay from a nucleus ([8]).

In sharp contrast with the resonance states studied in ref. ([2]) we are here interested in resonances which present a level crossing between the positive- and the negative-energy states. We give an explicit example of this new kind of resonances in fig. 1.

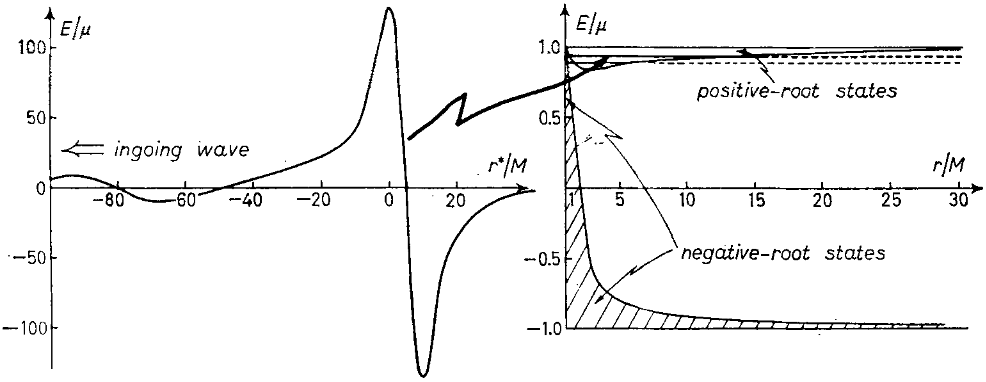

Fig. 1. – Example of resonance with level crossing between the positive- and the negative-energy states in the field of an extreme Kerr black hole with $a = M$. The scalar field is assumed to have $l = m = 2$ and a mass μ such that $\mu M = 1$. The energy of the resonance $E/\mu = 0.930$ corresponds to the first excited state $(1/\pi)\int_b^c (-W)^{\frac{1}{2}}\,\mathrm{d}x = 1.5$. The width of the resonance is $\Gamma/\mu = -4\cdot 10^{-5}$.

We approach the problem by the Gamow method using complex eigenvalues and the WKB approximation. Let us then first indicate by a, b and c the values of the x–co-ordinate corresponding to the zeros of the function $W(x, \omega)$, for a real value of ω, such that

$$-\infty < a < b < c < +\infty\,.$$

We then choose for $x > c$

$$u(x, \omega) = W^{-\frac{1}{4}} \exp\left[-\int_c^x W^{\frac{1}{2}}\,\mathrm{d}x\right] \tag{7}$$

for $x < a$, $u(x, \omega)$ is a mixture of equal-intensity ingoing and outgoing waves, the

(7) G. GAMOW: *Zeits. Phys.*, **51**, 204 (1928). See also G. GAMOW: *Structure of Atomic Nuclei and Nuclear Transformations* (Oxford, 1937).

(8) There is one important difference between the study of the resonances of a nucleus and the resonances of a quantized field around a collapsed object: the leakage in the case of a nucleus occurs always toward spatial infinity ($r \to \infty$) while in the case of a collapsed object occurs toward the horizon ($r \to r_+$ or $x \to -\infty$).

amplitude of which are complex conjugates:

$$(8.1)\qquad u(x,\omega)=\exp[-i\pi/4](-W)^{-\frac{1}{4}}\left[2\cos(I/2)\exp[\zeta/2]+\frac{i}{2}\sin(I/2)\exp[-\zeta/2]\right]\cdot \exp\left[-i\int_a^x(-W)^{\frac{1}{2}}\,\mathrm{d}x\right]+\exp[i\pi/4](-W)^{-\frac{1}{4}}\cdot \left[2\cos(I/2)\exp[\zeta/2]-\frac{i}{2}\sin(I/2)\exp[-\zeta/2]\right]\exp\left[+i\int_a^x(-W)^{\frac{1}{2}}\,\mathrm{d}x\right],$$

where

$$(8.2)\qquad I=2\int_b^c(-W)^{\frac{1}{2}}\,\mathrm{d}x$$

and

$$(8.3)\qquad \zeta=2\int_a^b(W)^{\frac{1}{2}}\,\mathrm{d}x\,.$$

The function $W(x)$ has the asymptotic behaviours

$$(9.1)\qquad \lim_{x\to+\infty}W(x)=\left(\frac{r_+^2}{r_+^2+a^2}\right)^2(\mu^2-\omega^2)$$

and

$$(9.2)\qquad \lim_{x\to-\infty}W(x)=-(\omega-m\Omega-\varepsilon V)^2\,,$$

where we have indicated by (9) $\Omega=a/(r_+^2+a^2)$ the angular velocity of the black hole and by (10) V its electric potential $V=er_+/(r_+^2+a^2)$.

Analytically continuing now the functions in eqs. (7) and (8) in the complex plane (11), we consider a complex eigenvalue

$$(10.1)\qquad \omega=\omega_0-i\Gamma/2$$

with the following additional conditions

$$(10.2)\qquad \omega_0\gg\Gamma\,,$$

$$(10.3)\qquad \omega_0<\mu\,,$$

$$(10.4)\qquad \omega_0<m\Omega+\varepsilon V\,.$$

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(9) D. Christodoulou and R. Ruffini: *On the electrodynamics of collapsed objects*, in *Black Holes*, edited by B. de Witt and C. de Witt (London, 1973).
(10) B. Carter: in *Black Holes*, edited by B. de Witt and C. de Witt (London, 1973).
(11) See, *e.g.*, G. Breit and F. L. Yost: *Phys. Rev.*, **48**, 203 (1935).

Equation (10.4) guarantees the condition of the level crossing between the positive- and the negative-energy solutions at resonance. This allows in the classical Gamow way [7] one to consider the entire spatial and time evolution of the resonance. However it is important to stress that the boundary conditions at the horizon are drastically influenced by the existence of the level crossing. We have to impose a *physically* ingoing wave at the horizon which corresponds here to a stream of antiparticles having a group velocity directed towards the hole. Therefore we have to keep in eq. (8.1) only the term in

$$\exp\left[+i\int_a^x(-W)^{\frac{1}{2}}\,\mathrm{d}x\right]\sim\exp\left[-i(\omega-m\Omega-\varepsilon V)x\right],$$

that is we must impose

$$2\cos(I/2)\exp[\zeta/2]+\frac{i}{2}\sin(I/2)\exp[-\zeta/2]=0\,. \tag{11}$$

But this stream of antiparticles carry a *negative flux* out of the potential well and therefore the conservation of flux implies that the function inside the potential well must be growing with time. This is easily checked by expanding eq. (11) which yields both the resonance condition

$$I(\omega_0)=(n+\tfrac{1}{2})\,2\pi\,, \tag{12.1}$$

where n is a positive or null integer, and

$$\Gamma=-\left[\exp[\zeta(\omega_0)]\,\frac{\mathrm{d}I(\omega_0)}{\mathrm{d}\omega_0}\right]^{-1}. \tag{12.2}$$

One can check straightforwardly that $\mathrm{d}I/\mathrm{d}\omega_0>0$.

In all the resonances having level crossing between the positive- and negative-energy states, we then have

$$\Gamma<0\,. \tag{13}$$

If following the Gamow approach we consider not only the time dependence of the resonance but also the space dependence implied by the presence of an imaginary component in the eigenvalue, we conclude that the eigenfunction is exponentially decreasing both for $x\to+\infty$ and $x\to-\infty$.

The radial dependence of the wave function at $x\to-\infty$ will, in fact, contain a factor $\exp[-i\omega x]=\exp[-i\omega_0 x-(\Gamma/2)x]$. The absence of the usual divergence in the spatial dependence of the eigenfunction [7] should be interpreted as a direct consequence of the fact that the amplitude of the resonance goes to zero when $t\to-\infty$.

The exponential decrease of the function as $x\to-\infty$ also implies that the norm of the function Φ, defined by the conserved current, is convergent and is identically zero. Since $\Phi\to 0$ as $t\to-\infty$, this implies that an equal number of particles and antiparticles have been created from the vacuum. Let us stress that, if eq. (10.4) is not fulfilled and if

$$\omega_0 > m\Omega + \varepsilon V \,, \tag{14.1}$$

then the resonant state will have its usual behaviour with a $\Gamma > 0$. If however

$$\omega_0 = m\Omega + \varepsilon V \,, \tag{14.2}$$

it is possible to have a resonance state with $\Gamma = 0$. This limiting case has as boundary condition at the horizon a wave function which tends to a constant value, which corresponds to a running wave.

It is possible, as usual ([11]), to build spatially bounded wave packets reproducing this wave till a cut-off in space which moves with the speed of light towards the horizon.