QUANTUM AND CLASSICAL RELATIVISTIC ENERGY STATES IN STATIONARY GEOMETRIES

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The positive and negative root states (E^+, E^-) for a particle moving along a geodesic in a stationary background, introduced by Christodoulou and Ruffini, are here interpreted in the framework of a relativistic quantum field theory. It is shown how E^+ and E^- have to be considered as the classical correspondent of the positive and negative energy states of a quantized field. It is explicitly shown that crossing between the states E^+ and E^- can occur and consequently the necessary condition for particle creation as given by Klein, Sauter, Heisenberg and Euler can be encountered.

The equation of motion of a particle of mass μ in a given background geometry $g_{\alpha\beta}$ can be derived from the Hamilton Jacobi equation

$$g^{\alpha\beta}\partial_\alpha S\partial_\beta S = \mu^2 \tag{1}$$

In a Schwarzschild or Kerr metric we have‡

$$ds^2 = \left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \left[(r^2+a^2)\sin^2\theta + \frac{2Mra^2}{\rho^2}\sin^4\theta\right]d\phi^2 + \frac{4Mra}{\rho^2}\sin^2\theta\, d\phi\, dt \tag{2}$$

where M is the mass of the black hole, a its angular momentum per unit mass ($a = 0$ is the Schwarzschild case) and $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Eq. (1) can then be separated [1] and leads to the following radial Hamilton-Jacobi equation:

$$(ds/dr)^2 = \mu^2[E^2 - (1-2M/r)(1+L^2/r)]/(1-2M/r)^2, \qquad \text{for } a = 0, \tag{3.1}$$

and for $a \neq 0$

$$(ds/dr)^2 = [(r^2+a^2)^2E^2\mu^2 + a^2\Phi^2 - 4MraE\mu\Phi - \Delta(\mu^2r^2 + Q + \phi^2 + E^2\mu^2a^2)]/\Delta^2 \tag{3.2}$$

Here E, L, Φ/μ are the energy, the angular momentum and its projection along the rotation axis per unit mass and Q is Carter's [1] constant of motion.

We then obtain for the equation of motion of the particle

$$(dr/d\tau)^2 = \mu^2(E-E_0^+)(E-E_0^-)[(r^2+a^2)/(r^2+a^2\cos^2\theta)]^2 \tag{4}$$

where r and τ are respectively the radial coordinate and the proper time of the particle and E_0^+ and E_0^- are the effective potentials for the positive and negative energy solutions defined by Christodoulou and Ruffini [2]. We have for a Schwarzschild geometry ($a = 0$)

$$E_0^\pm = \pm(1-2M/r)^{1/2}(1+L^2/r^2)^{1/2} \tag{5.1}$$

and for a Kerr geometry ($a \neq 0$) in the equatorial plane ($Q = 0$)

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‡ Here and in the following we are going to use geometrical units $G = c = \hbar = 1$.

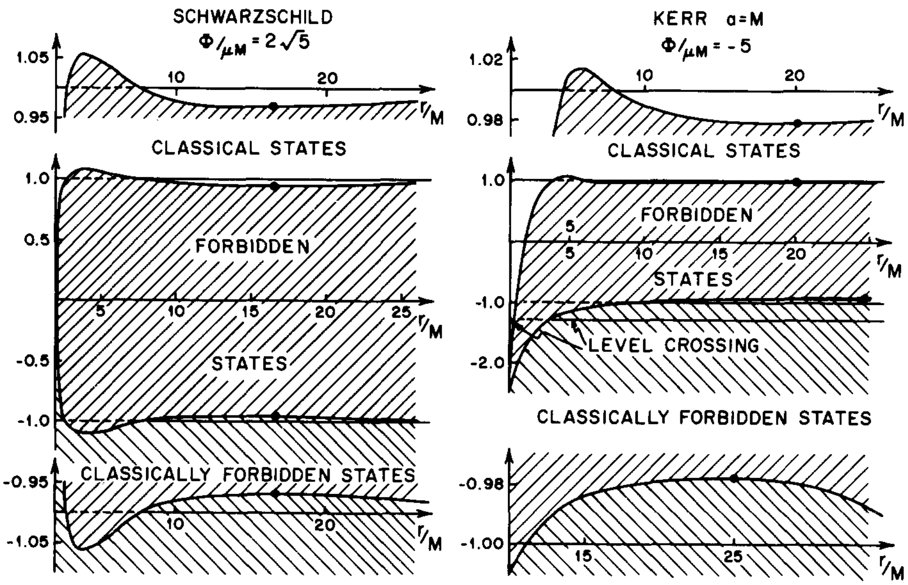

Fig. 1. The positive (E^+) and negative (E^-) root energy states are here plotted for the Schwarzschild and Kerr geometries. In both cases the effective potentials for circular (indicated by a solid dot) and elliptical orbits $E/M \leqslant 1$ are enlarged. The positive root states correspond to solutions which are the general relativistic generalization of the usual Newtonian trajectories [5]. The forbidden states have no physical interpretation, since they would correspond to trajectories with a complex value of their radial momentum [5]. Finally the negative root states have no classical correspondence since they correspond to particle states with negative mass or, equivalently, with oppositely directed velocity and momentum. These states, as shown here in fig. 2 and fig. 3, acquire a meaning in the framework of a relativistic field theory and they correspond to the negative energy solutions of a relativistic wave equation. Important is here to stress that in the case of a Kerr geometry (as well as in a Kerr-Newman [6] or in a Reissner-Nordstrøm geometry [7]) the positive root energy can cross the negative energy states, see e.g. solid line in this figure.

$$E_0^{\pm} = \{2\Phi aM/\mu \pm [4\Phi^2 a^2 M^2/\mu^2 + (r^3 + a^2 r + 2Ma^2)(r(r^2 - 2Mr + a^2) + (\phi^2/\mu^2)(r - 2M))]^{1/2}\}/(r^3 + a^2 r + 2Ma^2). \quad (5.2)$$

In the following we shall focus uniquely on the cases $a = 0$ and $a = M$. Diagrams of the effective potentials are given in fig. 1.

The purpose of this letter is to point out that the solutions E^-, meaningless in a classical theory, acquire significance as classical limits of a relativistic quantum field theory.

To prove this we first consider a scalar field satisfying the general relativistic Klein-Gordon equation

$$\nabla_\alpha \nabla^\alpha \phi + \mu^2 \phi = 0, \quad (6)$$

written in the metric (2). We can separate the variables [e.g. 3]

$$\phi = \exp(-\mathrm{i}\mu Et) y^l_m(\theta, \phi) R(r) \qquad \text{for } a = 0 \quad (7.1)$$

and

$$\phi = \exp(-\mathrm{i}\mu Et)\exp(\mathrm{i}m\phi) S_{ml}(\theta) R(r) \quad \text{for } a \neq 0 \quad (7.2)$$

where $S_{ml}(\theta)$ are spheriodal harmonics [e.g. 4], and obtain the radial equations

$$\mathrm{d}^2 u/\mathrm{d}r^{*2} = [(1 - 2M/r)(\mu^2 + l(l+1)/r^2 + 2M/r^3) - \mu^2 E^2] \qquad \text{for } a = 0 \quad (8.1)$$

and

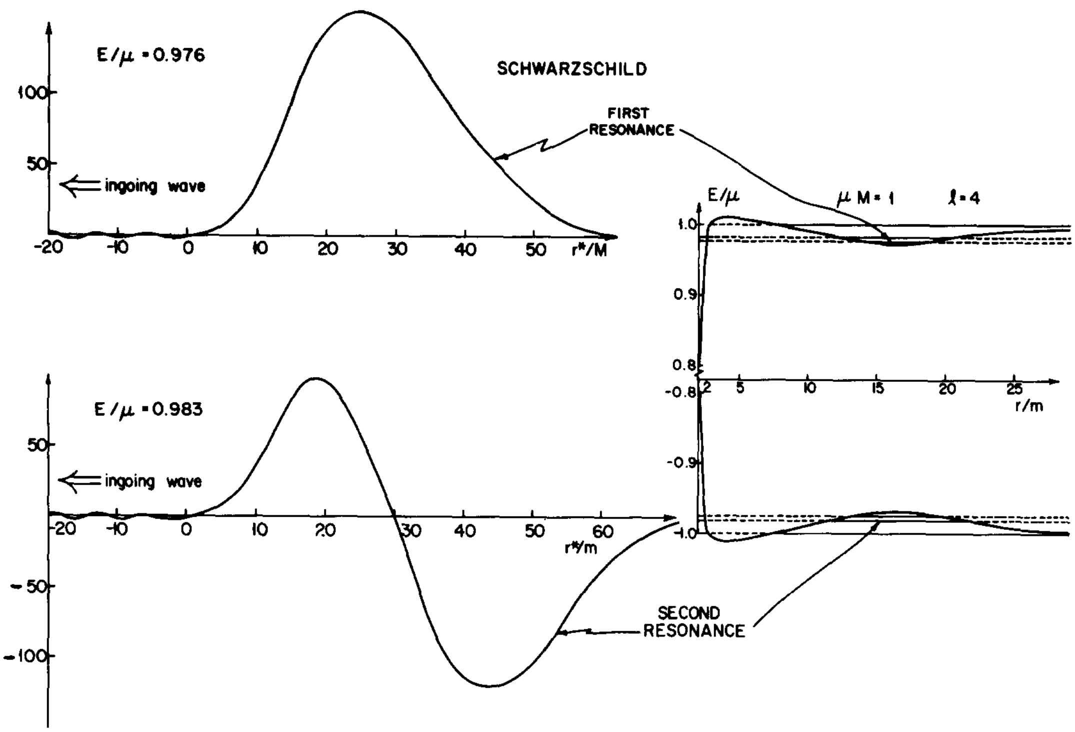

Fig. 2. The lowest two resonances for $\mu M = 1$ and $l = 4$ obtained by a direct numerical integration are here plotted as a function of the radial coordinate r^*. We have $r^* = r + 2M \ln(r/2M - 1)$. On the right side the energy levels of the resonances are represented on the classical effective potentials. In the Schwarzschild geometry the positive and negative root states are symmetric. The width of the resonances have been computed applying the standard W.K.B. approximation to eq. (8.1). We then have for the energy levels of the resonances $\int_b^c (-W)^{1/2} \mathrm{d}r^* = (n+1/2)\pi$, and for the width of the resonances $\Gamma = 2/|\mathrm{d}f/\mathrm{d}E|_{E=E_n}$ with $f(E) = \cos \int_b^c (-W)^{1/2} \mathrm{d}r^* \, [1 + \exp\{2\int_a^b (W)^{1/2} \mathrm{d}r^*\}]$, where a, b and c are the zeros of the function $W = (1-2M/r)(\mu^2 + l(l+1)/r^2 + 2M/r^3) - \mu^2 E^2$. We obtain for $\mu M = 1$, $l = 4$, $\Gamma_{E_1} \approx 2.7 \times 10^{-6}$ and $\Gamma_{E_2} \approx 4.1 \times 10^{-6}$ and for $l = 6$, $\Gamma_{E_1} \approx 1.7 \times 10^{-15}$ and $\Gamma_{E_2} \approx 2.5 \times 10^{-15}$. For $\mu M \to \infty$ the distance between successive energy levels goes to zero and $\Gamma \to 0$. Details in ref. [8].

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^{*2}} = \left\{ -E^2\mu^2 \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right) + \frac{4MamE\mu}{r^3} + \mu^2 \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right) - \lambda ml \frac{2M}{r^3} - \frac{1}{r^2} - \frac{a^2}{r^4} \right.$$
$$\left. - \frac{m^2 a^2}{r^4} + \frac{2}{r^6}\left[Mr^3 - r^2(a^2 + 2M^2) + 3Ma^2 r - a^4\right] \right\} u \qquad \text{for } a \neq 0 \tag{8.2}$$

where $u = R(r)r$ and $\mathrm{d}r/\mathrm{d}r^* = \Delta/r^2$. We also have $M + (M^2 - a^2)^{1/2} \leqslant r \leqslant +\infty$ and $-\infty \leqslant r^* \leqslant +\infty$.

Corresponding to the classical bound states (circular or elliptical orbits) we look for "resonances" states of the Klein-Gordon equation imposing as boundary conditions. (a) an exponential decay of the wave function for $r^* \to +\infty$, (b) a purely ingoing wave at $r^* \to -\infty$. The solutions of the problem have been found both by a numerical search of the eigenvalue and by the W.K.B. approximation. A few results from direct integration are shown in fig. 2 for the case $a = 0$ and in fig. 3 for $a = M$. Details of the techniques used the further results are given elsewhere [8].

Important here is to summarize the main conclusion:

(1) the continuous spectrum of the classical stable bound states is replaced by a discrete spectrum of resonances with the tunneling through the potential barrier giving the finite probability of the particle to be captured by the horizon (see fig. 2 and fig. 3).

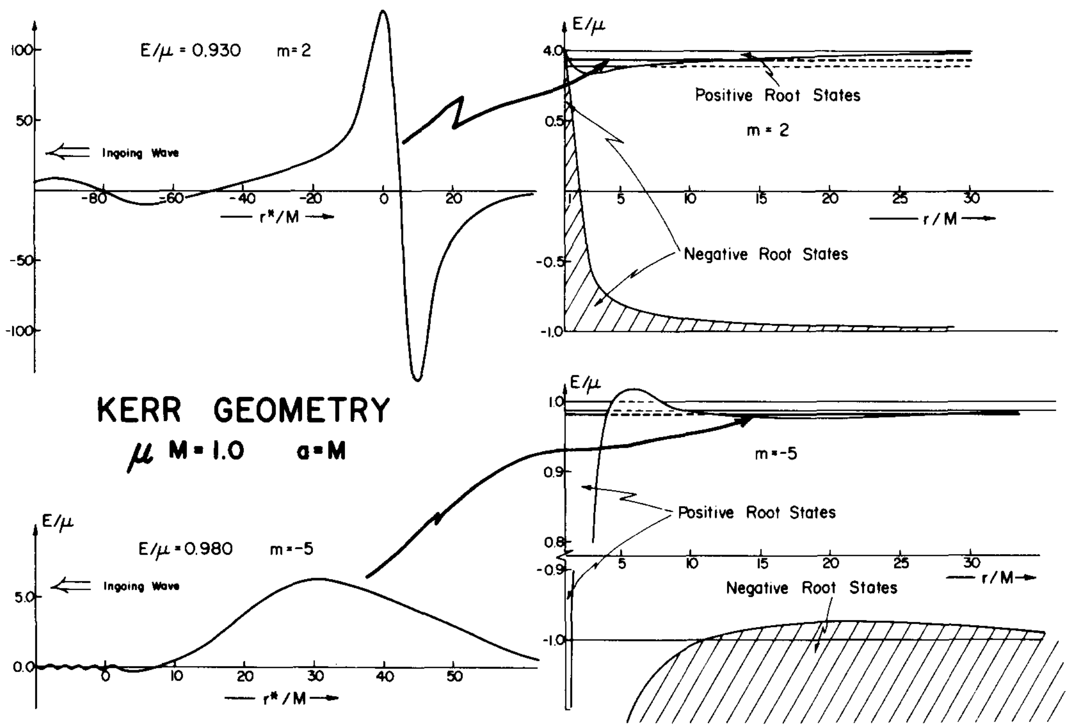

Fig. 3. Resonances in an extreme Kerr geometry with $\mu M = 1$ and $a = M$ corresponding to positive root states and positive energy states for $m = 2$ and $m = -5$. The case of a Kerr geometry differs strongly from the one of the Schwarzschild geometry in the asymmetry between the positive and negative root states. The equality $E(p_\phi) = -E(-p_\phi)$ is, however, fulfilled both in the classical and quantum level as evidenced by eqs. (3.2) and (8.2). The energy levels of the first two resonances both in the case of $m = 2$ and $m = -5$ are reproduced on the right side of the figure on the classical effective potentials. In both cases it is clear how tunnelling processes can occur between the positve and negative energy states (level crossing), an explicit example is given by the resonance for $m = 2$.

(2) These resonances states can be found for values of the angular momentum $l \geqslant 3\mu M$ in the case of a Schwarzschild geometry and for values of $l = m$ and $m \geqslant 2\mu M$ or $m \leqslant -5\mu M$ in the case of an extreme Kerr geometry

(3) When $\mu M \to \infty$ or $(GM/c^2)/(\hbar/\mu c) \to \infty$ the separation of the energy levels of the resonances tends to zero and the exponential in the forbidden region decreases more rapidly, the leakage toward the horizon also decreases and the width of the resonances $\Gamma \to 0$. (see fig. 2 and caption)

(4) From the results presented in figs. 2 and 3 follows that the negative root solutions do correspond to the classical limit ($\mu M \to \infty$) of the negative energy solutions of the Klein Gordon equations and consequently they can be thought of as antimatter solutions [10]. The negative root states can only be interpreted, then, in the framework of a fully relativistic quantum field theory.

(5) A fundamental difference exists between the positive and negative root solutions in the Schwarzschild and Kerr geometries. In the Schwarzschild case a positive root state never crosses a negative root state, in particular we always have $E^+ > 0$, $E^- < 0$. In the Kerr case we can have positive root states of negative energy in the ergosphere [2, 11] and we can also have for large enough value of the angular momentum of the particle states for which $E^+ < E^-$. See fig. 3. This corresponds to a classical example of level crossing as considered by Klein [12], Sauter [13], Euler and Kockel [14], Heisenberg and Euler [15], Pauli [16] and this leads directly in the quantum description to the possibility of particle creation [10].

Similar considerations can be made in the case in which the geometry is endowed with an electromagnetic field, either a Reissner-Nordstrφm or a Kerr-Newman solution. In these cases we have clearly to substitute to eq. (1) the generalized Hamilton-Jacobi equation

$$g^{\alpha\beta}(\partial_\alpha S + qA_\alpha)(\partial_\beta S + qA_\beta) = \mu^2 \tag{9}$$

and the Klein-Gordon equation

$$(\nabla_\alpha + iqA_\alpha)(\nabla^\alpha + iqA^\alpha)\phi + \mu^2\phi = 0$$

where q is the change of the test particle and A_α the four vector potential of the background geometry. The resonance states can be obtained much in the same way as in the case here considered, imposing purely ingoing waves at the horizon and exponentially decaying solutions at infinity. Once again we can have level crossing inside the effective ergosphere [6, 7] and therefore possible pair creation with consequent discharge of the black hole.

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