Radial charged particle trajectories in the extended Reissner–Nordstrom manifold

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It is shown that the trajectory of a charged particle on the extended Reissner–Nordström manifold can be such as to carry it into regions of the manifold where the definition of energy at infinity is different from the one at its point of origin. The various types of radial trajectories are classified. In the event one considers the manifold as having been produced by a collapsed star, there exist trajectories which go through both horizons, reach a minimum value of r, and go through two more horizons to a copy of the space in which it originated (flat at $r=+\infty$) without colliding with the matter of the collapsed star.

In a recent paper[1] it was shown that there are two distinct types of radial geodesics in the complete Kerr manifold, which can be classified by their place of origin on the manifold. This manifold contains infinitely many copies of two distinct spaces, both flat at $r=\pm\infty$. It is also shown that geodesics cannot cross over from one space to the other. However, this is possible if there is properly applied acceleration. It is the purpose of this paper to show in detail how this crossing over occurs for a very similar manifold: the complete Reissner—Nordström manifold. There has been renewed interest lately in this manifold. Ruffini has suggested that a magnetized rotating object should have a nonzero net charge in order to achieve a minimum energy configuration, and also that a very rapidly rotating, sufficiently small star would be able to maintain this charge in interstellar space.[2]

We will start with the Reissner—Nordström metric in Schwarzschild-like coordinates[3]

$$ds^2 = H^{-1}\,dr^2 + r^2\,d\Omega^2 - H\,dt^2, \tag{1}$$

where

$$H = H(r) = 1 - 2m/r + e^2/r^2,$$

and $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$ is the usual spherical surface element. Only the case $m^2 > e^2$ will be considered since otherwise the manifold is already complete. The complete extension was first determined by Graves and Brill[4] and given in a more convenient form by Carter.[5] Carter's extension is created by the repeated use of two null metrics. We define one coordinate system (r,u,θ,ϕ) with metric

$$ds^2 = 2dr\,du - H du^2 + r^2\,d\Omega^2 \tag{2a}$$

and another similar coordinate system (r,w,θ,ϕ) with metric

$$ds^2 = 2dr\,dw - H dw^2 + r^2\,d\Omega^2, \tag{2b}$$

where

$$u = \tfrac{1}{2}F(r) + t, \quad w = \tfrac{1}{2}F(r) - t, \quad \text{and} \quad \frac{dF}{dr} = \frac{2}{H}. \tag{2c}$$

This implies that

$$F(r) = 2r + mK_+^{-1}\log|r/r_+ - 1| + mK_-^{-1}\log|r/r_- - 1|, \tag{3a}$$

where $r_\pm$ are the roots of H with

$$K_\pm = (r_\pm - r_\mp)/(2r_\pm^2) \quad \text{and} \quad r_\pm = m \pm (m^2 - e^2)^{1/2}. \tag{3b}$$

Note that the function $F(r)$ is separately monotonic in each of the three regions

$$\begin{aligned} &\text{I: } r_+ < r,\\ &\text{II: } r_- < r < r_+,\\ &\text{III: } 0 < r < r_-. \end{aligned} \tag{4}$$

Each of these coordinate systems is analytic and extensible to a manifold larger than the one upon which the original coordinates were defined. Where these two manifolds overlap, one may introduce full null coordinates (u,w,θ,ϕ) with the metric

$$ds^2 = H\,du\,dw + r^2\,d\Omega^2. \tag{5}$$

This overlap region will be one of the three regions in Eq. (4); therefore, given u, w, and a region, one may uniquely determine r. We may then introduce, following Carter, a new coordinate system (ξ,ψ,θ,ϕ) by

$$\pm h(u) = \tan(\psi + \xi), \quad \pm h(w) = \tan(\psi - \xi), \tag{6}$$

where $h(z)$ must be a monotone increasing function such that $h(z) = O[\exp(-K_\pm z)]$ as $z \to \mp\infty$. The complete manifold will then consist of an infinite sequence of (r,u) patches labeled $(-,m)$, and superimposed on this, a similar sequence of (r,w) patches labeled $(n,-)$ running perpendicularly to the (r,u) sequence. By labeling each intersection by (n,m) the manifold consists of those intersections where $|n-m| \leqslant 1$. If $n=m$ is odd (even), then it is a II ($\overline{\text{II}}$) region; if n is even (odd) and $<$ ($>$) m, then it is a I (I′) region; if n is even (odd) and $>$ ($<$) m, then it is a III (III′) region. The choice in sign in the definition of ξ and ψ is determined by which of the regions I, I′, II, etc. is under consideration. Given an (n,m), the sign is $+h(u)$ $[-h(u)]$ for m odd [even], and equivalently for n with $\pm h(w)$.[6]

By denoting by E the constant of the motion associated with the timelike Killing vector, in the original coordinates of Eq. (1), and using a prime to denote the total derivative with respect to proper time τ, the equations of motion for a particle in radial motion with charge to mass ratio X are

$$(r')^2 = D^2 - H, \tag{7a}$$

$$t' = D/H, \tag{7b}$$

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where $D = D(r) = E - eX/r$. (7c)

Solving Eq. (7a) for the constant E (which has the interpretation of the energy per unit mass in unprimed regions and the negative of the energy per unit mass in primed regions[7]), we have

$$E = eX/r \pm [H + (r')^2]^{1/2}$$
$$= eX/r \pm [1 + (r')^2 - 2m/r + e^2/r^2]^{1/2}$$
$$\approx eX/r \pm [1 + \tfrac{1}{2}(r')^2 - m/r + e^2/2r^2], \quad (8)$$

since, for large r, both $(r')^2$ and $H-1$ are small compared to 1. Since eX/r is just the classical potential energy per unit mass of the electromagnetic interaction between the black hole and the test particle, this equation has a reasonable appearance for an energy equation, where the term $e^2/2r^2$ is an additional gravitational term due to the energy of the electric field associated with the charge e, and the $\pm$ sign is reminiscent of problems with the Klein—Gordon equation in particle physics.[8] Here, however, both signs are needed, since the sign of E must be negative at $r = \infty$ in primed regions.

The solutions to the equations can be written in the following form when $E^2 < 1$ (bound test particle), in terms of a parameter η which is adjusted to be 0 at maximum r,

$$r = m(\alpha + \beta\cos\eta), \quad (9a)$$

$$\tau - \tau_0 = m(\alpha\eta + \beta\sin\eta)/(1 - E^2)^{1/2}, \quad (9b)$$

$$t - t_0 = (\tau - \tau_0)E + (2mE - eX)\eta/(1 - E^2)^{1/2}$$
$$+ \tfrac{1}{2}K_+^{-1}[\mathrm{sgn}D(r_+)] \log\left|\frac{\tan(\eta/2) + \tan(\eta_+/2)}{\tan(\eta/2) - \tan(\eta_+/2)}\right|$$
$$+ \tfrac{1}{2}K_-^{-1}[\mathrm{sgn}D(r_-)] \log\left|\frac{\tan(\eta/2) + \tan(\eta_-/2)}{\tan(\eta/2) - \tan(\eta_-/2)}\right|, \quad (9c)$$

where $\eta_\pm$ are the values of η at which $r = r_\pm$, while $\alpha \pm \beta$ are the roots of $r'(r) = 0$:

$$\alpha = (m - EXe)/(1 - E^2), \quad \beta = [m^2 - 2EXem + e^2 - 1)]^{1/2}/|1 - E^2|. \quad (9d)$$

Solutions for $E^2 > 1$ are similar and may be obtained from Eqs. (9a)—(9c) by the following substitutions. Change everywhere $(1-E^2)^{1/2}$ to $(E^2-1)^{1/2}$. Then there are two cases: If β is real, replace $\cos\eta$ by $[\mathrm{sgn}(r - \alpha - \beta)]\cosh\psi$, $\sin\eta$ by $[\mathrm{sgn}(r - \alpha - \beta)]\sinh\psi$, and $\tan(\eta/2)$ by $\tanh(\psi/2)$, where ψ increases from $-\infty$ if $r > \alpha + \beta$ and from 0 if $r < \alpha - \beta$. If β is complex, define $\gamma^2 = -\beta^2$ and replace $\beta\cos\eta$ by $\gamma\sinh\psi$, $\beta\sin\eta$ by $\gamma\cosh\eta$ and $\tan(\eta/2)$ by $\tanh(\psi/2)$, where ψ increases from $-\infty$. In particular instances $\psi_\pm$ may both be complex, which means the particular trajectory never crosses the horizons, $r = r_\pm$. From Eq. (9a) we see that these radial trajectories are oscillatory in the coordinate r, although we shall see that they do not actually come back to their starting point on the extended manifold (unless, of course, one identifies various different regions of the same type, which leads to serious causal problems); $\alpha \pm \beta$ are just the turning points of this r motion. It is, however, possible for $\alpha - \beta$ to be negative in which case the particle strikes $r = 0$ first, which is a singularity. It is also clear that t becomes infinite at $r = r_\pm$, which merely indicates that it is no longer a good coordinate; however, either u or w is finite at $r = r_\pm$. From Eqs. (3) and (7) we find that

$$u' = [D + (\mathrm{sgn}r')(D^2 - H)^{1/2}]/H, \quad (10a)$$

$$w' = [-D + (\mathrm{sgn}r')(D^2 - H)^{1/2}]/H. \quad (10b)$$

It is then easily seen that at $r = r_\pm$ (roots of $H = 0$) u' (w') is finite if $\mathrm{sgn}(r'D) = -1$ (+1).

We now proceed to discuss the possible trajectories in more detail. In particular we divide all trajectories originating in a given region I into classes, as a function of E, X, and e, which have a given future history. From Eq. (7a) one sees that D may vanish along a trajectory only when H is negative, which happens in regions II and $\overline{\text{II}}$. There may then exist trajectories for which D changes sign while the particle is passing through such a region. This would then change which of u or w is finite as the boundary of the region is crossed, and therefore change which boundary is crossed. For sufficiently large r, D and E must have the same sign [Eq. (7c)], so that D is positive in region I and negative in I′. We now restrict consideration to particles originating in region I, while in this region the energy is given by Eq. (8) with a plus sign and is, of course, a fixed number for a given trajectory thereafter. Defining

$$V_\pm = eX/r \pm [H(r)]^{1/2}, \quad (11)$$

we see that for $r > r_+$, $E \geq V_+$, but for $r < r_-$, we have either $E \geq V_+$ (if $D > 0$) or $E \leq V_-$ (if $D < 0$). There are then five possible types of trajectories. In Fig. 1 is exhibited an E, X plane, for a specific choice of $e = 0.8\,m$, which is divided into regions according to the future history of a trajectory with those initial conditions. If $X \leq -1$ then the trajectory ends at the singularity $r = 0$ in region III [type (a)]. If $-1 < X \leq 0$, then the trajectory enters region $\underline{\text{III}}$, reaches a minimum value of r and rebounds through $\overline{\text{II}}$ back into another I region [type (b)]. However, when $X > 0$, there are more possibilities since D now may change sign. For $0 < X < 1$, if $E > eX/r_-$ the minimum r lies in region III as above. But for $E < eX/r_-$ an infalling particle starting in region I enters region II and, at some point in region II, D becomes negative. The particle must then continue into region III′, reach a minimum value of r there and rebound back into $\overline{\text{II}}$, where D becomes positive again, allowing it to exit into

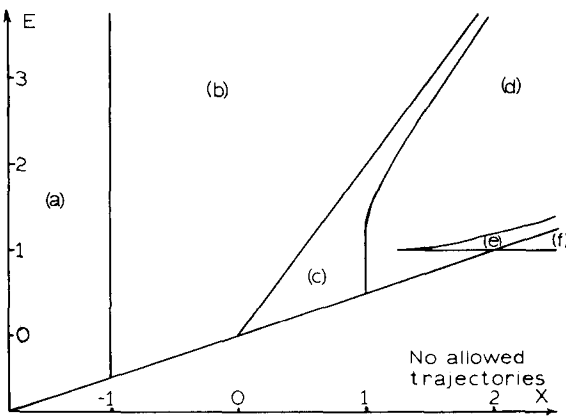

FIG. 1. Determination of the future history of a trajectory which originated in region I with given values of the energy per unit mass, E, and the charge per unit mass, X.

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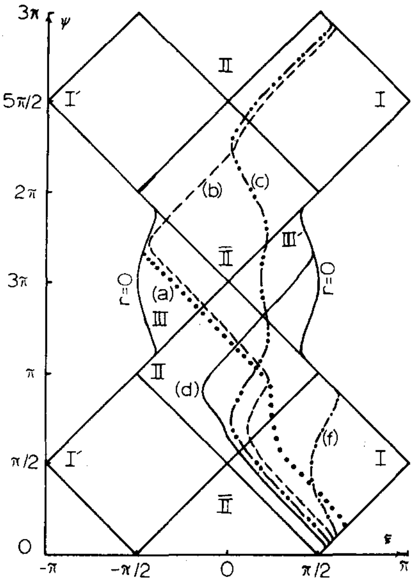

FIG. 2. Typical examples of different types of radial trajectories on the extended Reissner—Nordström manifold. The particular values chosen for these curves were: (a), $E=05$, $X=-2$, $t_0=-13.57$ m; (b), $E=0.3$, $X=-0.7$, $t_0=-4.88$ m; (c), $E=0.7$, $X=0.5$, $t_0=-8.74$ m; (d), $E=0.96$, $X=1.2$, $t_0=-23.09$ m; (e), $E=2.0$, $X=4.2$, $t_0=-2.71$ m.

another region I [type (c)].

If $1 \leq X \leq m/e$, then there are three possibilities. If $E < \Delta_+$, the trajectory enters into region III′ ($D<0$) and hits the singularity at $r=0$ [type (d)], where $\Delta_\pm = [mX \pm (X^2-1)^{1/2}(m^2-e^2)^{1/2}]/e$. If $m/e < X$, there are four possibilities. If $E<1$, the trajectory is of type (c), ending at $r=0$ in III′. If $1 \leq E < eX/r_+$ the trajectory will stay in region I, eventually going toward $r=+\infty$ [type (f)]. If $\Delta_- < E$, then one has trajectories of types (d), (c), and (b), as shown in Fig. 1. However, for $1 < E < \Delta_-$, the situation is more complicated because $V_+(r)$ has a maximum at $r=s$,

$$s = e^2\{m - X\,|\,(m^2-e^2)/(X^2-1)\,|^{1/2}\}^{-1} \geq r_+. \tag{12}$$

So if the initial value of r is greater than s, the trajectory will stay always in region I, eventually going toward $r=+\infty$. If the initial value of r is less than s, the trajectory will end at $r=0$ in region III′ [type (e), a choice between motions of types (d) and (f).]. For larger values of E there are trajectories of types (d), (c), and (b), as is shown in Fig. 1. In Fig. 2 typical examples of these various possible trajectories are shown on the extended manifold for a fixed θ and ϕ.

We note that for $X<0$ there exist trajectories for which the energy is negative; i.e., states in region I for which $E<0$ even though $D>0$. These trajectories are an indication that the energy of electrical attraction can be so negative as to overwhelm the energy associated with the rest mass.[9] In the case $E<0$ the maximum value of r for the orbit, d, must satisfy

$$m + (m^2-e^2)^{1/2} = r_+ \leq d \leq m + (m^2 - e^2 + e^2X^2)^{1/2}. \tag{13}$$

For any particular fixed value of E, with $D>0$, there is a maximum value of X for which that E can be realized by a particle on a radial orbit—$X_{max} = Er_+/e$. For $X=X_{max}$, $d=r_+$ and the gap between states with positive D is zero. Therefore, increasing X so that $X > X_{max}$ causes D to become negative, and the value of d now increases with increasing X, but the starting point of the motion is in region I′. Also the energy is now positive since the energy has been seen to be $-E$ in region I′.

We consider in detail a sequence of particles all released at the same starting point, $d > r_+$, but such that the members of the sequence have increasing charge to mass ratio, X. Since all the particles are momentarily at rest at $r=d$, the energy depends on d, and is given by

$$E_d = eX/d + [H(d)]^{1/2}, \tag{14a}$$

with the turn around point at minimum r given as

$$d_- = e^2(1-X^2)/[d(1-A^2)]. \tag{14b}$$

Starting with $X=0$ and looking at particles with larger and larger values of X, one obtains trajectories of type (b), above, similar to geodesic trajectories. But as X approaches

$$X_0 = \frac{r_-}{e}\left|\frac{d-r_+}{d-r_-}\right|^{1/2} < 1,$$

d_- approaches r_- and u_-, the value of u at $r=r_-$, approaches $+\infty$. For $X > X_0$, $D(r_-)<0$ and w_- is finite rather than u_-, while d_- is again less than r_- but in region III′. So the trajectory now exits from region II into III′ [type (c)]. Increasing X further to $X=1$, we find that the particle hits the singularity at $r=0$ in region III′ [type (d)]. However, there is a point at which the charge to mass ratio gets so large that there is no longer an attractive force at $r=d$. For X greater than

$$X_1 = [m - (e^2/d)]/[e^2H(d)]^{1/2} > 1,$$

a particle released at $r=d$, momentarily at rest there, will be repelled toward $r=\infty$, all in region I [type (f)].

It is seen that a full set of (radial) trajectories on the extended manifold requires use of both the plus and the minus sign for the energy in Eq. (8). On those trajectories for which D changes sign, one must use both signs in Eq. (8) for a single trajectory. Also note that even in the case where the collapsing matter which caused the horizon is not ignored, the trajectories of types (c) and (d), as well as (f) are perfectly feasible since the matter lies only in unprimed regions[10] and no collision with it occurs for these orbits.

[1]R.H. St. John and J.D. Finley III, J. Math. Phys. 15, 147 (1974).
[2]R. Ruffini and A. Treves, Astrophys. Lett. 13, 109 (1973) and R. Ruffini, in *Black Holes*, edited by C. deWitt and B.S. deWitt (Gordon and Breach, New York, 1973), p. 525.
[3]We use units in which $c=1=G$. When observed from very far away, the central region has mass m and electric charge e,

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which we assume is positive. For purposes of comparison, in Gaussian units, with $c=1=G$, the charge of a single proton is 1.381×10^{-39} km $= 9.353 \times 10^{-40}$ solar masses.

[4]J. C. Graves and D. R. Brill, Phys. Rev. **120**, 1507 (1960).

[5]B. Carter, Phys. Lett. **21**, 423 (1966). Note, however, that his figure (Fig. 1b) relevant to the case $0 < e^2 < m^2$ does not correctly indicate the locations of the singularities $r=0$. [A similar figure is also in Misner, Thorne, and Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Fig. 34.4.] There is no allowable choice of the function $h(z)$ [Eq. (6)] which makes the singularity a vertical line in the ξ, ψ plane, since $h(z)$ may not be solely even or odd, except when $e^2 = m^2$. It is always a curve with two symmetrical bulges toward the ψ axis, as indicated in Fig. 2, where the choice $h(z) = e^{-K_- z} - e^{-K_+ z}$ has been made.

[6]A more complete description of the manifold will be found in B. Carter, Phys. Rev. **141**, 1242 (1966).

[7]See Ref. 1 for a complete discussion of this.

[8]See also R. Ruffini and D. Chrsitodoulou, Phys. Rev. D **4**, 3552 (1971), for some reference to this problem.

[9]These trajectories are discussed in more detail, in region I, by R. Ruffini, in *Black Holes*, p. 503 (see Ref. 2).

[10]For example, see Ya. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics* (Univ. of Chicago Press, Chicago, 1971), p. 147.

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