Singularity Avoidance of Schwarzschild and Reissner-Nordström Black Holes in Loop Quantum Gravity

Master Thesis in Theoretical Physics

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Abstract

While describes large scale structures of the universe remarkably precise, Einstein's theory of general relativity (GR) fails to give a consistent picture at certain space-time points, called singularity, where physical observables of the theory, such as energy density, become divergent. It is widely believed that the true cure of such physically meaningless states of gravity, predicted by GR, lies out of the domain of validity of the theory; they must be resolved in a regime in which both gravitational and quantum mechanical effects are dominant.

In this thesis, I present the method of resolution of the intrinsic singularity of Schwarzschild and Reissner-Nordström black hole in loop quantum gravity (LQG). LQG is a non-perturbative, background independent approach toward constructing a theory of quantum gravity which presents a clear picture of the quantum structure of space-time at Planck scale. After outlining the main philosophy, methods and features of LQG, and defining classical black holes and singularities, based on symmetry reduced models of LQG, quantization of the spherically symmetric and homogeneous Kantowski-Sachs regions of space-times of charged and Schwarzschild black holes is illustrated. Furthermore, the classical phase space (Ashtekar variables) of different regions of Schwarzschild and Reissner-Nordström space-times are constructed. This calculations reveal that the classically divergent function, $\frac{1}{r}$, at $r = 0$ singularity, out of which all divergent components of scalar curvature are made can be written in terms of components of the classical momenta. Finally, quantum operator analogue to the classically divergent quantity is constructed and is shown to exhibit an upper bounded spectrum. This result together with non-singular evolution (quantum-Einstein) equation prove the avoidance of singularity in such a reduced model of quantum gravity, which can shed light on the generic issue of singularity avoidance in full quantum gravity.

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Notation and Conventions

Symbol	Description
$g_{\mu\nu}(x)$	metric field
∇_μ	covariant derivative on space-time
$\Gamma^\rho_{\mu\nu}$	Levi-Civia connection
$R^\mu_{\nu\rho\lambda}$	curvature of $\Gamma^\rho_{\mu\nu}$
$e^I(x)$	frame fields (tetrad)
ω^I_J	spin connection
$\tilde{R}^I_J$	curvature of ω^I_J
A_i	connection of gauge group
D_i	covariant derivative on gauge group
F^{ij}	crvature of A_i
$h(A,\ell)$	holonomy of A along path ℓ
K^i_a	extrinsic curvature
$\bar{\Gamma}$	spin connection of 3 teraids
ℓ_{Pl}	Planck length
γ	Immirzi parameter
GR	General Relativity
QM	Quantum Mechanics
QG	Quantum Gravity
LQG	Loop Quantum Gravity

- Throughout this work the constants c, $\hbar$, G are set equal to 1, unless otherwise stated.
- Indices $I, J, ...$ denote Minkowski, $\mu, \nu, ...$ space-time, $i, j, ...$ 3-space and $a, b, ...$ $\mathbb{R}^3$indices.

CHAPTER 1

Introduction

THE first half of twentieth century witnessed major scientific revolutions in theoretical physics: Einstein's curiosity to look at mechanics and electrodynamics in a consistent picture led to his special theory of relativity, and his philosophical ideas regarding the nature of space and time revealed that gravity is a manifestation of dynamics of space-time and formed the basis of general theory of relativity (GR). The modification they make to the pre-relativistic physics become significant at velocities comparable to the velocity of light, and in the vicinity of massive bodies. Meanwhile, the demand to find a coherent description of matter, made out of stable atoms, gave rise to a more profound conceptual revolution: quantum mechanics (QM), whose effects come into play in atomic scales. Naturally, then, the question is posed as what would happen to systems whose range of validity contain both quantum mechanics and special (or general) relativity? Investigation of behavior of special relativistic quantum mechanics, led to the introduction of the concept of quantum fields. The astonishing work of pioneering physicists in this area, during the second half of the last century, culminated in the standard model of particle physics and was tested experimentally to an incredible precision. On the other hand, novel predictions of general relativity, such as the evolution of universe, along with astronomical observations, formed the ground for what is known as the standard model of cosmology. However, the question of merging quantum mechanics and general relativity did not draw a great deal of attention till the last decades of the century. In fact, the striking progress of particle physics, being followed by experiment, left no time for physicists to be devoted to a physical regime in which no experimental evidence were known. On the other hand, physicists believed that the case of quantum mechanical behavior of gravity should follow the same general pattern as the quantum field theory (QFT) of other fields. Nevertheless, all attempts toward constructing a consistent quantum theory of gravity are pluged by a number of obstructions and appear to have failed. This failure lends support on the belief that a revolutionary new theory should arise able to solve all conceptual and technical obstacles concerning quantum gravity. Below, I state briefly such conceptual problems and discuss the role black holes can play in achieving such a tentative theory. Finally I conclude by outlining the most serious approaches toward resolving the problems.

1.1 The Problem with Quantum Gravity

A simple dimensional analysis shows that the typical scales on which quantum gravitational effects can be influential are of the order of Planck length,

$$\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35} cm, \tag{1.1}$$

or Planck energy

$$E_{Pl} = \sqrt{\frac{\hbar c^5}{G}} \approx 10^{28} eV. \tag{1.2}$$

These are far beyond our present day technological abilities to be probed, and hence leave us with an absence of any experimental evidence so far. However, we can still have speculations about how such a regime would look like, based on what we know from GR and QM.

General relativity proposes profound modifications in our perception of nature: space-time is not a fixed, absolute entity on which things move, instead, it is a dynamical network of relations between physical events. Physical laws governing nature must not depend on the choice of the coordinate system of a privileged observer. The causal structure of space-time and the geometry of space, dictated by the metric of space-time, are then subject to change and will be determined locally by the gravitational field; gravity is space-time. Quantum mechanics, on the other hand, reveals the laws governing short scale structures of the universe; the quantum observables come in quanta and the results of measurements highly depend on the existence of an external observer. There correspond some intrinsic quantum fluctuations to the conjugate pairs of observables which restrict one from measuring the pair simultaneously with arbitrary high precision. The framework of QFT, which arose as a consequence of special relativistic quantum mechanics, describes three other fundamental forces of nature (except gravity) in the language of gauge fields and proposes a picture to unify them. It associates with every field intermediate particles, carriers of the force, and presents a way to calculate the transition amplitude of the scattering processes between particles. However, such amplitudes suffer from UV divergences, where certain integrals diverge at the limit of short distances, which can be resolved by the renormalization process. The mentioned characteristics of gravitational and quantum mechanical systems bring up, naturally, a number of questions regarding a tentative theory which aims to describe a regime in which both theories are indispensable. Among many, the sharpest ones are:

1. If the metric of space-time is subject to quantum fluctuations, what does it mean to speak of causal structures? (the problem of time [1])

2. If the space-time, metric, is gravity and gravity is to be quantized, geometrical quantities, made out of metric, must come in quanta. How would it be realized?

3. If the quantum gravitational system happens to be the whole universe, what does it mean to speak of an external observer?

4. If the laws of physics must not depend on the particular choice of a metric, how QFT can be formulated without referring to the background metric?

5. Would quantization of space-time naturally lead to a UV divergence free theory?

A look at Einstein's equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.3}$$

reveals some aspects of the controversial issues concerning QG: (i) the left hand side, geometry, has a totally classical nature, while our complete description of matter, the right hand side, has a quantum nature. (ii) the equation has a covariant form which implies general covariance (or background independence), while the QFT, by means of which the right hand side (matter) is described, highly depends on the choice of a fixed metric of space-time. In fact, the questions 1, 2, 3 posed above fall into the category of the former issue which will be settled by introducing a consistence notion of "quantum space-time", whereas the former issue, containing question 4, will be dealt with by realizing a generally covariance QFT. Questions like the last one would then be addressed based on a background independence QFT on a quantum space-time.

Put it more technically, regardless of all conceptual difficulties mentioned above, one might try to quantize gravity, i.e. the metric field $g_{\mu\nu}(x)$, perturbatively according to standard methods of QFT [2]. To that end, one can define a generating functional integral for GR as

$$\int \mathcal{D}g_{\mu\nu} e^{-iS_{EH}(g)}, \tag{1.4}$$

with $S_{EH}[g] = \int d^4x \sqrt{-g}\frac{R(x)}{16\pi G} + \mathcal{L}_{matter}(x)$, being the Einstein-Hilbert action. $S_{EH}[g]$ is not a polynomial in $g_{\mu\nu}$. However, one can perform a perturbative expansion of metric field around a fixed background $g^{BG}_{\mu\nu}$

$$g_{\mu\nu}(x) = g^{BG}_{\mu\nu} + \sqrt{16\pi G} h_{\mu\nu}(x), \tag{1.5}$$

and study the quantum behavior of exitation fields $h_{\mu\nu}(x)$.

In performing expansion of 1.4 in terms of G, one encounters familiar divergences of QFT. However, for the case of gravity, apart from technical difficulties that finite parts of the counter terms can be freely chosen, physical problems arise as well which are combinations of two features: (i) perturbation expansion does not converge, and (ii) the expansion parameter becomes large if center-of-mass energies reach beyond the Planck value.

1.2 The Role of Black Holes

1.2.1 Black Hole Entropy

The closest theoretical physics has reached to a quantum gravitational effect, is the discovery of black hole radiation. In 1974, Stephen Hawking [3] showed that black holes, which are objects that light cannot escape from and hence classically are at absolute zero, do radiate, at temperature

$$T_H = \frac{\hbar c^3}{8\pi G M k_b}, \tag{1.6}$$

when quantum mechanical effects are taken into account. The presence of both gravitational and quantum mechanical constants reflects the fact that this result should somehow lie in the domain of quantum gravitational regime. In fact, this effect is predicted via studying quantum fields on the curved background of a black hole and the observation that the thermal spectrum of particle creation at infinity lies at temperature T_H. This, of course, is not a quantum theory of gravity since the gravitational field, manifest in curvature of space-time, is kept fixed and its dynamics is not intended to be studied. However, this is different from QFT on fixed Minkowski background and hence serves as a semi-classical calculation toward quantum gravitational effects. The seminal discovery of black hole radiation, together with laws of black hole mechanics, which is reminicent of the first and second laws of thermodynamics, lent support on the anticipation of Bekestein [4] that a black hole with horizon surface area A possesses an entropy

$$S_{BH} = \frac{k_b c^{3/2} A}{4G\hbar}, \tag{1.7}$$

where the subscript BH stands for either Black Hole for Bekenstein-Hawking. Statistical mechanics, on the other hand, presents a microscopic origin of entropy of thermodynamical systems in terms of ensemble averaging of quantum microstates of quantum systems. The semi-classical arguments of Hawking and Bekenstein, though, do not provide such a description. This is where the role of a more fundamental theory becomes crucial. It is widely believed that the statistical origin of S_{BH} must be sought for in a regime in which both quantum mechanical and gravitational effects are dominant. In the absence of any experimental evidence in such a regime, finding an accurate statistical origin of S_{BH} could then serve as a “test” for any theory aiming to unify quantum and gravity.

1.2.2 Black Hole Singularity

The birth of quantum mechanics stem from resolving the apparently unrealistic predictions of classical physics. In spite of the observation of stable atoms, classical physics has no explanation of why does the system of an electron orbiting the nucleus constitute a stable atom. Physically, this manifests in the fact that the energy of the system after a finite amount of time becomes *minus infinity*. Quantum mechanics cures the problem by demonstrating the fact that, following from its principles, the energy of the atom is *bounded below*. QFT, as well, illustrates a consistent way to prevent classical fields from having unbounded negative energies; the problem of unboundedness of Dirac field energy from below was cured by introducing the concept of quantum field obeying anti-commutation relations which leads to a positive definite Hamiltonian operator.

From the above point of view, quantization can be seen as a way of restricting unphysical configurations, manifest by the infinite value of an observable, to physical ones by putting a (lower or upper) bound on the values such observables can gain within quantum framework.

Classical GR, is a field theory for gravity. It gives rise to striking predictions many of them have been confirmed by experiment; from bending of light, and corrections to the prehelion precession in Mercury orbit, to gravitational waves, neutron stars and black holes. However, there are solutions to the Einstein’s equations which contain unphysical predictions: singularity. A singularity is, roughly speaking, a point in space-time where the curvature of space-time becomes infinite (for a more precise definition of singularity see section 3.2). A common situation in which a (irremovable) singularity can occur is at the center of a black hole. This means that, classically, black holes are unphysical configurations of gravitational fields.

It is strongly believed that, as were the case for QM and QFT, the true physical picture of black hole singularity must be given within a theory which takes the quantum effects into account as well. The imaginative picture is that in the same way that *quantum nature of matter* (the Fermi degenerate gas repulsive pressure) prevents a star from collapsing and leads to stable white dwarfs or neutron stars, the *quantum nature of space-time* should prevent the collapsing star (with $M > M_{crit}$) from forming a singularity by generating a stronger repulsive force than gravity.

1.3 Different Approaches

The various approaches toward quantum gravity [5, 6] can be categorized into 3 main paths:

- **covariant approach** Using Lagrangian formulation of GR and perturbative expansion, as discussed brifely above, this approach searchs for modifications to the Einstein-Hilbert action, and adding more degrees of freedom and new symmetries at Planck scale energies, such as extra dimensions and supersymmetry. String theory is the most serious attempt following this approach.

- **canonical approach** Based on Hamiltonian formulation of GR, this is a non-perturbative approach toward canonically quantizing gravity. Such a Hamiltonian formulation is traditionally considered as the Arnowit Deser and Misner [7] work which led to ill-defiend Wheeler-DeWitt equation. LQG is the modern well-defined attempt in this line of research.

- **sum over histories approach** Such an approach is an attempt to find an analogue of Feynman path integral quantization for quantizing gravitational field. Hawking's Euclidean quantum gravity [8], most of the discrete approaches, such as causal dynamical triangulation [9], and the spin foam models [10] falls into this category.

There are also other approaches such as: asymptotic safety [11], group field theory [12], 't Hooft deterministic quantum approach [13] ....

Beside technical obstacles in the way of constructing quantum theory of gravity, lack of any experimental evidence has generated a peculiar situation for theoretical physics which is in contrast to the conventional way it works. Conventionally theoretical, mathematical, models are linked to the physical data or fact via a conceptual framework that depends to some extent on the subject area concerned [14]

$$\text{theory} \longleftrightarrow \text{concepts} \longleftrightarrow \text{facts} .$$

In the absence of any experimental data, though, the above chain is shortened to include only the first two parts

$$\text{theory} \longleftrightarrow \text{concepts} .$$

Therefore, beside the mentioned technical categorization, the challenging issues which distinguish different lines of reaserch are that of conceptual and philosophical viewpoints. The two main active programs in this field, string theory [15] and LQG [16], then can be seen as consequences of two apparently distinguished viewpoints:

I **A particle physicist point of view** Take gravity as another field theory (than already known ones). Choose a fixed background space-time to start with. Try to quantize gravity perturbatively and, hopefully, seek background independence at the end of the day. In accordance with the tradition of particle physics, take unification with other 3 forces as a goal and add new symmetries and more degrees of freedom as the price of going to higher energies. Such a point of view is best realized in the framework of string theory.

II **A relativist point of view** Take gravity as a field theory plus a principle (general covariance). Seek for a most suitable formulation of the classical theory. Try to make sense of a background independent QFT. Try to quantize gravity directly, regardless of unification with other forces, in a way which respects general covariance. This is what LQG has realized to some extent and therefore is reffered to as “quantum geometry” as well.

In the present thesis, I try to look at the how of addressing resolution of singularity issue of Schwarzschild and charged black hole by taking the view point of a relativist. In such a quest, I will describe LQG, the way I have learned it throughout the year I am studying it, in chapter 2. To state the problem which is going to be addressed more transparent and mathematically more rigorous, I have devoted a chapter 3 on classical definitions of black hole and singularity and reviewing the main features of Schwarzschild and Reissner-Nordström black holes. Chapter 4 aims to sketch the basic ideas of a symmetry reduced model and illustrates it in the case of homogeneous Kantowski-Sachs and Spherically symmetric quantum geometry. In the last chapter 5, I will present the results: (i) classical phase space variables for both Schwarzschild and Reissner-Nordström black holes are calculated, and (ii) based on this calculations and the reduced models discussed in chapter 4, the divergence of classical singularities of such black holes are shown to be avoided in quantum gravitational regime. Finally, to be self-contained, I have included 2 appendices at the end, which serve as a brief introduction to the mathematical tools used throughout the thesis and the functional representation of QFT which is extensively used in LQG.

CHAPTER 2

Loop Quantum Gravity

Loop quantum gravity is a non-perturbative approach to canonically quantize gravity. This line of research is the most conservative approach toward QG in the sense that it just makes use of principles of QM and GR and does not postulate any new physical ingredient, such as supersymmetry or extra dimensions, to begin with. The program is based upon quantization of gravitational field expressed, instead of the metric field $g_{\mu\nu}(x)$, as an $SU(2)$ Yang Mills gauge theory. LQG is the only approach toward constructing a consistent theory of QG which directly adresses the questions concerning space-time structure at Planck scale. The salient feature of such an aprroach is its background independence; observables and states of such a quantum theory are all constraint to satisfy the diffeomorphism invariance constraint which gaurantees nothing depends on the choice of a background geometry for space-time. This is truely consistance with the spirit of Einstein's theory of general relativity which demand general covariance. In section 2.1, I will outline the suitable formulation of classical GR for a background independent quantization, and will sketch the basic ideas and achievments in LQG in section 2.2.

2.1 Classical Gravity as an $SU(2)$ Gauge Theory

Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore - Albert Einstein

Canonical quantization of classical fields rests on the Hamiltonian formulation of dynamics. The Hamiltonian function of a field theory, generating time evolution, is expressed in terms of dynamical variables of the field and their conjugate momenta. Such conjugate momenta are derived from the Lagrangian of the field as the functional derivatives of Lagrangian with respect to the field variables.

General relativity expresses gravity as a field theory of the basic field $g_{\mu\nu}(x)$, the metric of space-time. The dynamics of the theory, in vacuum and in the absence of cosmological constant, is governed by the vacuum Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0. \tag{2.1}$$

As is well known, this can be derived from the Einstein-Hilbert action

$$S[g_{\mu\nu}] = \int d^4x\mathcal{L} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R, \tag{2.2}$$

with R being the Ricci scalar and $g \equiv \det(g_{\mu\nu})$. To write the Hamiltonian formulation of the theory, it seems natural to take the metric (or 3-metric, see section 2.1.2) as the basic field variable and seek its conjugate momenta derived from the above Lagrangian. This constitutes the main idea of the ADM formalism [7] to cast the dynamics of GR in the Hamiltonian form. Canonical quantization, based on ADM variables, has been done which leads to highly non-linear evolution equations (Wheeler-Dewitt equation) and therefore did not gain much interests. However, classical mechanics is invariant under canonical transformation. These are the set of transformation which leaves the Poisson brackets invariant. Thus, the most natural variables might not necessarily be the most suitable ones for quantization. For instance, there is a formulation, the Palatini formulation which makes use of connections and tetrad fields as basic variable. Finally, Abhay Ashtekar in 1986 [17] presented a new set of variables, describing Hamiltonian GR using $SU(2)$ connections. Ashtekar variables drastically simplify canonical quantization equations and serve as the starting point of loop quantum gravity. I will discuss all the three mentioned approaches to GR in more details below.

2.1.1 Palatini Formalism

Gravity is described by evolution of the metric $g_{\mu\nu}(x)$, a symmetric second rank tensor field in four dimension. It has 10 independent components. The idea of Palatini formalism is to break this 10 into 4+6. To this end, it makes use of 4 frame fields $e^I(x)$ and 6 spin connection fields $\omega^{IJ} = -\omega^{IJ}$ compatible with e^I. At each point x of space-time M, consider four one-form fields $e^I(x)$, called tetrad or frame field,

$$e^I(x) = e^I_\mu(x)dx^\mu, \tag{2.3}$$

with values in the Minkowski space, and mutually orthonormal in the sense

$$g_{\mu\nu}(x) = \eta_{IJ}e^I_\mu(x)e^J_\nu(x), \tag{2.4}$$

where $I, J, \ldots$ denote Minkowski indices being raised and lowered by η_{IJ} the Minkowski metric. Working as the basic variables with e and ω, the action 2.6 takes the form:

$$S[e^I, \omega^{IJ}] = \frac{1}{16\pi G}\int d^4x\epsilon_{IJKL}e^I \wedge e^J \wedge \tilde{R}[\omega]^{KL}. \tag{2.5}$$

In the above expression $\tilde{R}^{IJ}$ is the curvature associated with the spin connection ω^{IJ}. They are analogues of the Riemann curvature tensor and the Levi-Civita connection (Christoffel symbols) notions respectively. A more rigorous definition of the concepts of connection and curvature is given in appendix A.1. In such a formulation of GR, field equations will be obtained by demanding the action to be stationary under variations of e^I and ω^{IJ} independently. It is easily seen that the variation with respect to e^I leads to:

$$\frac{\delta S}{\delta e^I} = 0 \Longrightarrow \epsilon_{IJKL}e^I \wedge \tilde{R}^{JK} = 0. \tag{2.6}$$

Defining $\tilde{R}^I_\mu = \tilde{R}^{IJ}_{\mu\nu}e^\nu_J$ and $\tilde{R} = \tilde{R}^I_\mu e^\mu_I$, it takes the form

$$\tilde{R}^I_\mu - \frac{1}{2}\tilde{R}e^I_\mu = 0. \tag{2.7}$$

Variation with respect to ω, on the other hand, shows that the spin connection coincides with the Levi-Civita connection. This implies that field equations 2.7 are equivalent to the familiar looking form 2.1.

2.1.2 Hamiltonian General Relativity: ADM Formalism

In classical mechanics, the Hamiltonian formulation expresses the n second order differential equations of a system with n degrees of freedom, as $2n$ first order differential equations for n pair of canonically conjugate variables constituting the phase space. These equations are generated by the Hamiltonian function $H = H(p, x)$ which is obtained from the Lagrangian $L = L(x, \dot{x})$ by a Legandre transformation

$$L(x, \dot{x}) \longrightarrow H(p, x) = px - L; \tag{2.8}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \longrightarrow \begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \tag{2.9}$$

In the above expressions, p, the momentum conjugate to x is defined via

$$p \equiv \frac{\partial L}{\partial \dot{x}}. \tag{2.10}$$

The Hamiltonian formulation, singles out the time parameter by defining the canonical momentum as above, and express the evolution of the phase space elements in time. While the Lagrangian formulation of GR describes the evolution of the metric in space-time, the idea of Hamiltonian GR is then, to see how the geometry of "space" evolves as "time" passes. This requires assuming a topology of space-time M of the form

$$M \cong \mathbb{R} \times \Sigma, \tag{2.11}$$

with coordinates $x^\mu = (t, x^a)$. In the above relation, Σ denotes a 3 dimensional manifold of arbitrary topology and space-like signature with coordinate x^a, and $\mathbb{R}$, the real line, is a one dimensional manifold with a coordinate t. $\Sigma = \Sigma_t$ can then be thought of as the one-parameter family of hypersurfaces $t = const$. This allows us to identify the coordinate t as a time parameter. However, this identification does not mean we are using an "absolute time". This is guaranteed by the diffeomorphism invariance of the theory (see appendix A.1 for the definition of a diffeomorphism); while working in a chosen Σ_t, we can always transfer to another family of hypersurfaces $\Sigma_{t'}$ by means of a diffeomorphism ϕ

$$t \longrightarrow t' = \phi t, \tag{2.12}$$

and the diffeomorphism invariance of GR ensures us that physical quantities are independent of this choice. Now, picking a particular time coordinate τ, one can decompose the corresponding time flow vector field $\tau^\mu(x) = (1, 0, 0, 0)$ into components tangent and normal to Σ

$$\tau^\mu(x) = N(x)n^\mu(x) + N^\mu(x), \tag{2.13}$$

where $n^\mu(x)$ is the unit vector field normal to Σ_t: $g_{\mu\nu}n^\mu n^\nu = -1$. (see figure 2.1)

N, the lapse function, measures the rate of follow of τ with respect to coordinate time t, as one moves normal to σ_t. On the other hand, N^μ, the shift vector field, measures the amount of shift tangent to Σ_t. The 3-metric on Σ_t is given by

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \tag{2.14}$$

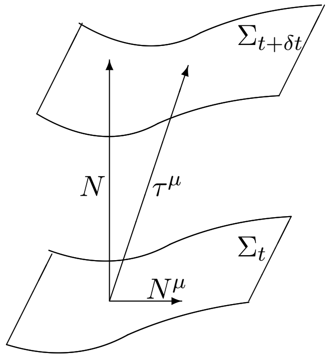

Figure 2.1: Evolution of Σ in time

and using 2.13 to write $n^\mu = (\tau^\mu - N^\mu)/N$, can be expressed as:

$$q_{\mu\nu} = g_{\mu\nu} - \frac{1}{N^2}(\tau_\mu - N_\mu)(\tau_\mu - N_\mu). \tag{2.15}$$

The line element of M can be written in terms of lapse and shift

$$ds^2 = -(N^2 N_a N^a)dt^2 + 2N_a dt dx^a + g_{ab}dx^a dx^b. \tag{2.16}$$

These formulas enable us to move in the direction of finding the Hamiltonian of GR. The first step is to identify our canonical variables. Since we are interested in time evolution of Σ_t, we take the 10 degrees of freedom, corresponding to 10 independent components of $g_{\mu\nu}$, to be 6 components of the 3-metric q_{ab}, 3 components of shift vector field N^a, and 1 lapse function N as the dynamical variables of the theory. To determine the conjugate momenta we need the Lagrangian, which is the integrand of the action 2.5. Expressed in terms of lapse and shift, the action takes the form

$$S[q_{ab}, N^a, N] = \frac{1}{16\pi G}\int d^4x\sqrt{q}N\left[\mathcal{R} - K^2 + Tr(KK)\right]. \tag{2.17}$$

In this expression, $\mathcal{R}$ is the Ricci scalar of Σ_t, $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_{\vec{n}}q_{\mu\nu}$ is the extrinsic curvature of Σ_t, and $q = \det q_{ab}$. From the above Lagrangian, the conjugate momenta to the q_{ab}, N^a, and N becomes:

$$\pi^{ab} \equiv \frac{\delta\mathcal{L}}{\delta\dot{q}_{ab}} = \sqrt{q}(K^{ab} - Kq^{ab}); \tag{2.18}$$

$$\pi^a_{\vec{N}} \equiv \frac{\delta\mathcal{L}}{\delta\dot{N}_a} = 0; \tag{2.19}$$

$$\pi_N \equiv \frac{\delta\mathcal{L}}{\delta\dot{N}} = 0. \tag{2.20}$$

Vanishing of the momenta conjugate to N^a and N, is similar to the situation in electrodynamics, where the momentum correspond to the zeroth component of the vector potential vanishes. It signals the first indication of pecularities in GR and suggests that the true dynamical variables of the GR are the 6 components of the 3-metric q_{ab}. In fact, N^a and N play the role of Lagrange multipliers in the Lagrangian. Now, everything is ready to express the Hamiltonian in terms of canonical variables. It turns out to be

$$H[q_{ab}, \pi^{ab}] = \frac{1}{16\pi G}\int dt \int d^3x\,[N^a H_a + NH]\,, \tag{2.21}$$

where

$$H_a = -2\sqrt{q}\nabla_b\left(\frac{\pi_a^b}{\sqrt{q}}\right), \tag{2.22}$$

$$H = \frac{1}{\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})\pi^{ab}\pi^{cd} - \sqrt{q}\mathcal{R}. \tag{2.23}$$

Since time derivatives of N and N^a does not appear in 2.21, the Hamilton's equations of motion are simply

$$H = 0, \tag{2.24}$$

$$H_a = 0. \tag{2.25}$$

This expresses the fact that the canonical variables of the phase space of gravity, are restricted to satisfy the above system of constraints which are called Hamiltonian and spatial diffeomorphism constraints respectively. Being proportional to the Lagrange multipliers, the Hamiltonian 2.21 is thus identically zero. This is the most obvious indication of the "problem of time" in the canonical formulation of GR; H is the generator of time translation, in the same sense that $\hat{H}$ is the generator of time translation in the Schrödinger equation $\hat{H} = i\partial_t$. Thus $H = 0$ implies that there is no dynamics with respect to t. However, Note that the interpretation of t in GR is different from that of t in Scrödinger equation. As discussed above, because of the diffeomorphism invariance of GR, t is just a parameter which has no physical content, therefore it is meaningless to speak of dynamics in time t and thus, $H = 0$ is expected. The parameter t in Schrödinger equation, on the other hand, is the absolute pre-relativistic notion of time in which the physical system evolve.

2.1.3 Ashtekar Variables

Whereas expresses GR in a Hamiltonian form, the ADM formalism results in a system of constraint equations which are difficult to solve. Ashtekar [17], then, looked for a new set of variables in terms of which, the constraint equations can take a simpler form. Finding the new variables is based on the observation that we can use as our 6 dynamical variables, coming from the 3-metric q_{ab} the 6 independent components of the spin connection used in

Palatini approach. However, instead of ω^{IJ} we use 3 complex one-form fields $A^i(x) = A^i_a dx^a$ containing all the information of 6 components of ω^{IJ}:

$$\omega^{IJ} = \begin{pmatrix} 0 & \omega^{01} & \omega^{02} & \omega^{03} \\ -\omega^{01} & 0 & \omega^{12} & \omega^{13} \\ -\omega^{02} & -\omega^{12} & 0 & \omega^{23} \\ -\omega^{03} & -\omega^{13} & -\omega^{23} & 0 \end{pmatrix} \longrightarrow \begin{cases} A^1 = \omega^{23} + i\omega^{01} \\ A^2 = \omega^{13} + i\omega^{02} \\ A^3 = \omega^{12} + i\omega^{03} \end{cases} .$$

There is, indeed, a mathematical motivation behind this choice of variables. The spin connections ω^{IJ} are elements of the Lorentz algebra, $\mathfrak{so}(3,1)$. The complex Lorentz algebra can be decomposed to two complex $\mathfrak{so}(3;\mathbb{C})$ algebra,

$$\mathfrak{so}(3,1;\mathbb{C}) = \mathfrak{so}(3;\mathbb{C}) \oplus \mathfrak{so}(3;\mathbb{C}), \tag{2.26}$$

which are called selfdual and anti-self dual components of complex Lorentz algebra. There is, then, two projector operators which project elements of $\mathfrak{so}(3,1,\mathbb{C})$ into its two components. The self-dual projector is defined by

$$P^i_{jk} = \epsilon^i_{jk}, P^i_{0j} = -P^i_{j0} = \frac{i}{2}\delta^i_j. \tag{2.27}$$

Then, Ashtekar variables A^i are simply the self-dual part of the spin connection ω^{IJ}:

$$A^i = \frac{1}{2}\epsilon^i_{jk}\omega^{jk} + i\omega^{0i}. \tag{2.28}$$

Now, the action 2.5 can be written in terms of A and e

$$S[e,A] = \frac{-i}{16\pi G}\int P_{iIJ} e^I \wedge e^J \wedge F^i(A), \tag{2.29}$$

where $F^i = dA^i + \epsilon^i_{jk}A^jA^k$ is now the curvature of the self dual connection A^i which coincides with the self dual component of the curvature $\tilde{R}$. The action 2.29 gives rise to the vacuum Einstein's equations in the simple form

$$F^i = 0. \tag{2.30}$$

The key observation here is that had we chosen to work with the anti-self dual part of the connection ω^{IJ}, we would have ended up with the equations of the form 2.30, with the anti-self dual part of the curvature equal to zero, which is equivalent to the same Einstein's equations in vacuum. Thus, in fact, using Palatini variables we are deriving Einstein's equations twice. This illustrates the origin of simplifications arising from using A instead of ω. It can be shown that a canonical transformation can be performed, by replacing the imaginary unit i with an arbitrary real parameter, γ, the Immirzi parameter, under which Einstein's equations remain unaffected. We therefore proceed by defining the new variable

$$A^i = \frac{1}{2}\epsilon^i_{jk}\omega^{jk} + \gamma\omega^{0i}. \tag{2.31}$$

Now, lets cast the variables A and Σ in the Hamiltonian formalism, and see what constraint equations will we get. Recall that 6 components of the q_{ab}, the spacial components of the metric of Σ, are the true dynamical variables. The 3 fields A^i, on the other hand, contains the 6 components of the spin connection field. They, together with their corresponding momenta, can serve as the true coordinates of the phase space of GR. They are, however, defined on the 4 dimensional space-time M. To be used in the Hamiltonian formulation, we need their induced components on the 3-space Σ:

$$A^i(x^\mu) \longrightarrow A^i(\vec{x}) = A^i_a(\vec{x})dx^a, \tag{2.32}$$

$$e^I(x^\mu) \longrightarrow e^I(\vec{x}) = e^I_a(\vec{x})dx^a. \tag{2.33}$$

From action 2.29, the conjugate momenta to the A^i_a turns out to be

$$E^a_i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i_a} = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}e^j_b e^k_c = (\det e)e^a_i, \tag{2.34}$$

which is called the gravitational electric field as is so in Yang-Mills theory. They are in fact densitised traids of 3 triad fields $e^i (i = 1, 2, 3)$ obtained by fixing the gauge through $e^0 = 0$.

The pair (A, E) form our basic conjugate fields, called the Ashtekar variables, and satisfy the canonical Poisson bracket

$$\left\{A^i_a(\vec{x}), E^b_j(\vec{y})\right\} = 8\pi\gamma G\delta^b_a\delta^i_j\delta(\vec{x}, \vec{y}). \tag{2.35}$$

The Hamiltonian in terms of (A, E) takes the form

$$H = \int d^3x \left(\lambda^i_0 G_i + N^b H_b + NC\right) = 0, \tag{2.36}$$

where,

$$G_i = D_a E^a_i; \tag{2.37}$$

$$H_a = E^b_i F^i_{ab}; \tag{2.38}$$

$$C = \epsilon_{ijk}F^i_{ab}E^a_i E^b_j - 2(1+\gamma^2)K^i_{[a}K^j_{b]}E^a_i E^b_j. \tag{2.39}$$

These are called Gauss, spatial diffeomorphism, and Hamiltoinan constraints respectively. The $K^i_a(x)$ field in the last expression is the extrinsic curvature which is related to A through

$$A^i_a = \bar{\Gamma}^i_a + \gamma K^i_a, \tag{2.40}$$

where $\bar{\Gamma}^i_a = \frac{1}{2}\epsilon^i_{jk}\bar{\Gamma}^{jk}_a$ is the spin connection of 3 teriad fields $e^i_a(\vec{x})$ 2.33, defiend by $de^i_a + \bar{\Gamma}^{ij} \wedge e^j_a = 0$.

Comparing with the constraints of the ADM formalism, we find the new Gauss constraint. But, what do these constraints mean? Remember that indices $i, j, \ldots$ in the fields A^i_a and E^a_i denote components of those fields in the algebra $\mathfrak{so}(3) = \mathfrak{su}(2)$. The constraint 2.37, similar to the Gauss law in electrodynamics, generates $SU(2)$ gauge transformations and hence $G = 0$ guarantees that the pair (A, E) are $SU(2)$ gauge invariant. The next constraint 2.38 produces spatial diffeomorphism, and consequently $H_a = 0$ ensures us that the theory is invariant under spatial diffeomorphism. $C = 0$, the Hamiltonian constraint, gurantees invariance under time translation.

Therefore, in this formulation, GR takes the form of a diffeomorphism invariant $SU(2)$ Yang-Mills gauge theoy.

2.2 Loop Quantum Gravity

Expressed an an $SU(2)$ gauge theory, general relativity is now in its most suitable formulation to be quantized canonically. The program of LQG is based on Dirac method for quantizing constraint systems [18] which consists of the following steps:

1. Find a representation of the classical phase space variables as operators in an auxiliary kinamatical Hilbert space $\mathcal{H}_{kin}$, by demanding them to satisfy

$$\{\star, \bullet\} \longrightarrow \frac{-i}{\hbar}\left[\hat{\star}, \hat{\bullet}\right]; \tag{2.41}$$

2. Implement classical constraints C_i on states in $\mathcal{H}_{kin}$ as operators $\hat{C}_i$;

3. Construct $\mathcal{H}_{phys}$ the Hilbert space of solutions of the constraints,

$$\hat{C}_i\Psi = 0, \forall\Psi \in \mathcal{H}_{phys}. \tag{2.42}$$

To quantize gravity, we are encountering three constraints. In the following sections we first introduce the suitable representation, the holonomy-flux algebra, then by implementing Gauss constraint the $SU(2)$ gauge invariant states, spin networks, will be selected which form the space $\mathcal{K}_{SU(2)}$. Applying diffeomorphism constraint, restrict the Hilbert space to those states invariant under spatial diffeomorphism, s-knots, and form the Hilbert space $\mathcal{K}_{diff}$, and finally by applying the Hamiltonian constraint, the physical Hilbert space of the theory will be known. Schematically this can be summarize as

$$\mathcal{H}_{kin} \xrightarrow{\hat{G}_i} \mathcal{H}_{SU(2)} \xrightarrow{\hat{H}_a} \mathcal{H}_{diff} \xrightarrow{\hat{H}} \mathcal{H}_{phys}$$

Dirac prescription, described above, makes use of functional representation of QFT. Since such a representation is not the conventional method for quantizing fields discussed in standard text books on QFT, I have briefly sketched its main features in appendix B.

2.2.1 Quantum Kinematics: Spin Networks

Kinematical Hilbert Space $\mathcal{H}_{kin}$

Our basic field describing gravity is A a connection in the Lie algebra $\mathfrak{su}(2)$

$$A(\vec{x}) = A^i_a(\vec{x})\tau_i dx^a, \tag{2.43}$$

where $\tau_i = -\frac{i}{2}\sigma_i$ is a basis of $\mathfrak{su}(2)$, with σ_i being the Pauli matrices. To represent states, as a functional of A, we make use of holonomy of A along a smooth oriented path γ in σ , which is an element of the group $SU(2)$, defined by:

$$h(A,\gamma) = \mathcal{P}\exp\int_\gamma A. \tag{2.44}$$

Holonomies are discussed in more details in appendix A.1. Now, consider a graph Γ which is a collection of paths γ_l for $l = 1, 2, \ldots, L$

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_L\}, \tag{2.45}$$

and a smooth function f, cylindrical function, of holonomies along the paths in graph Γ. We define a wave functional of the connection A via

$$\Psi_{\Gamma,f}[A] = f(h(A,\gamma_1), h(A,\gamma_2), \ldots, h(A,\gamma_L)). \tag{2.46}$$

The linear space of all such functionals S is not still a Hilbert space. To construct the Hilbert space $\mathcal{H}_{kin}$, we need a scalar product defined by

$$\langle\Psi_{\Gamma,f}|\Psi_{\Gamma,g}\rangle \equiv \int \prod_{n=1}^{L} dh_n f^*(h_1,\ldots,h_L)g(h_1,\ldots,h_L), \tag{2.47}$$

where $h_n \equiv h(A,\gamma_n)$ and dh is the Haar measure on $SU(2)$, which is the natural measure for the integral of functions on a group like $SU(2)$. The Hilbert space $\mathcal{H}_{kin}$ is then defined as the space in which for all Cauchy sequences Ψ_n the norm $||\Psi_m - \Psi_n||$, induced by the above inner product, converge to zero.

<u>Loop States</u>: As an example of function f and graph Γ consider the trace function tr on the group, and the loop α which is a single closed curve. For any state $|\alpha\rangle$, the wave functional $\Psi_{\alpha,tr}[A]$ becomes

$$\Psi_{\alpha,tr}[A] = \langle A|\alpha\rangle = trh(A,\alpha). \tag{2.48}$$

In fact the name of the theory, loop quantum gravity, springs from the interesting characteristics that such states posses.

<u>An Orthonormal Baisis</u>: Since our basic fields, holonomies, are elements of the Lie group $SU(2)$, an orthonormal basis can be obtained from the matrix elements of the irreducible representations j of $SU(2)$

$$U^{(j)\alpha}_{\beta}(h) = \langle h|j,\alpha,\beta\rangle\,. \tag{2.49}$$

The orthonormal basis for each graph γ is defined

$$|\Gamma, j_l, \alpha_l, \beta_l\rangle \equiv |\Gamma, j_1, \ldots, j_L, \alpha_1, \ldots, \alpha_L, \beta_1, \ldots, \beta_L\rangle\,, \tag{2.50}$$

which means that our wave functionals in this basis takes the form

$$\Psi[A] = \langle A|\Gamma, j_l, \alpha_l, \beta_l\rangle = U^{(j_1)\alpha_1}_{\beta_1}(h(A,\gamma_1))\ldots U^{(j_L)\alpha_L}_{\beta_L}(h(A,\gamma_L)). \tag{2.51}$$

<u>Operators</u>: The basic operators in $\mathcal{H}_{kin}$ are field operator and its conjugate momentum. However, the naive recipe $\hat{A}\Psi = A\Psi$ and $E\Psi = i(\partial/\partial A)\Psi$, turns out not to be suitable in this case; such operators map states out of $\mathcal{H}_{kin}$. The correct operators, are obtained by smearing A and E along curves γ and surfaces S respectively:

$$A^i_a(\vec{x}) \longrightarrow h_\gamma[A] = \mathcal{P}\exp\int_\gamma A(\vec{x}), \tag{2.52}$$

$$E^a_i(\vec{x}) \longrightarrow E_i[S] = \int d^2\sigma n_a E^a_i(\vec{x}), \tag{2.53}$$

with $\sigma = (\sigma^1, \sigma^2)$ and n_a being coordinates and normal vector on the surface S respectively. $h_\gamma[A]$ is the holonomy of the connection A along the path γ, and $E[S]$ is the flux of the gravitational electric field E through the surface S. Their corresponding operators, forming the holonomy-flux algebra, act by multiplication and differentiation

$$\hat{h}_\gamma[A]\Psi = h_\gamma[A]\Psi, \tag{2.54}$$

$$\hat{E}_i[S] = -i\hbar\int d^2\sigma n_a \frac{\delta}{\delta A^i_a}. \tag{2.55}$$

The action of flux operator on holonomies turns out to be

$$\hat{E}_i[S]h_\gamma[A] = h_{\gamma_1}[A](-i\hbar\tau_i)h_{\gamma_2}[A], \tag{2.56}$$

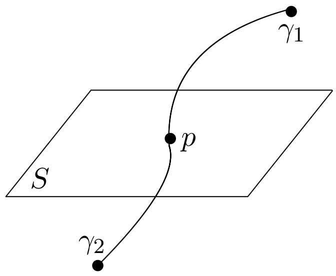

Figure 2.2: The flux field grasps the intersection point p of the surface S and the path γ.

where γ_1 and γ_2 are two segments of γ connected at the intersection point p of surface S and γ (see figure 2.2). Therefore, the action of flux operators through surface S on holonomies along path γ is only non-vanishing at p; E "graspes" γ.

Gauge invariant Hilbert Space $\mathcal{H}_{SU(2)}$

SPIN NETWORK STATES:

The states introduced above 2.50 form a basis for $\mathcal{H}_{kin}$ but are not gauge invariant; they manifestly depend on representation indices α_l and β_l. The basis which span $\mathcal{H}_{SU(2)}$, the Hilbert space whose elements are invariant under $SU(2)$ gauge transformation, must not depend on α_l and β_l. Thus, we should look for an object with the set of dual indices to contract with those of $|\Gamma, j_l, \alpha_l, \beta_l\rangle$. Such objects are called intertwiners and are introduced in appendix A.1. Recall that basic field operators of the theory are holonomies along paths, which generally intersect. Consider a graph Γ consisting of a set of intersecting paths. Call intersection points "nodes". The non-intersecting curves connecting two nodes are called "edges". Associate with each edge l an irreducible representation j_l of $SU(2)$, and with each node n an intertwiner i_n between representations associated to the edges adjacent to the node. We call such a triplet (Γ, j_l, i_n) a spin network (see figure 2.3). The desired gauge invariant states, the spin network states $|S\rangle$, are thus obtained

$$\begin{aligned} |S\rangle &= |\Gamma, j_l, i_n\rangle \\ &\equiv \sum_{\alpha_l, \beta_l} v_{i_1}^{\beta_1 \dots \beta_{n_1}}{}_{\alpha_1 \dots \alpha_{n_1}} v_{i_2}^{\beta_{n_1+1} \dots \beta_{n_2}}{}_{\alpha_{n_1+1} \dots \alpha_{n_2}} \dots v_{i_N}^{\beta_{n_{N-1}+1} \dots \beta_L}{}_{\alpha_{n_{N-1}+1} \dots \alpha_L} |\Gamma, j_l, \alpha_l, \beta_l\rangle \,, \end{aligned} \tag{2.57}$$

for a spin network with L edges and N nodes.

OPERATORS: One of the striking features of LQG is the prediction that space has a discrete nature. This is manifest in the discrete spectrum of the geometrical observables such as area and volume of a region of space-time. Below, I will illustrate the form of such operators and

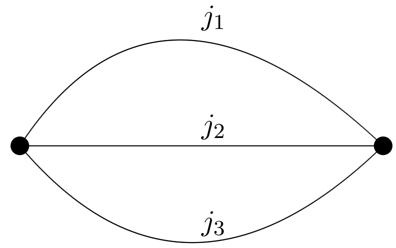

Figure 2.3: A spin network with 3 edges and 2 nodes

their action on spin network states. For a more detailed and technically subtle construction of such operators, the reader is referred to [16].

Area Operator: Classically, the area of a given surface S, coordinatized by σ^1 and σ^2, in 3-space is defined using determinant via:

$$A(S) = \int_S d\sigma_1 d\sigma_2 \sqrt{\det\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}\right)} \qquad \alpha, \beta = 1, 2, \tag{2.58}$$

which in terms of E_i^a, 4.47, turns out to be

$$A(S) = \int_S d\sigma_1 d\sigma_2 \sqrt{E_i^a n_a E_i^b n_b}. \tag{2.59}$$

The integral can be written as a Riemann sum

$$A(S) = \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{E^i(S_I)E_i(S_I)}. \tag{2.60}$$

Here we have divided S into N cells, and $E_i(S_I)$ is the flux of E_i through the Ith cell. At quantum level, fluxes of E in the above expression will be replaced by their corresponding flux operators. Therefore, to find out the action of $A(S)$ on a generic spin network state, we must investigate the action of operator $\hat{E}^i\hat{E}_i$ on such states. From 2.56 we have

$$\begin{aligned} \hat{E}_i\hat{E}^i h_\gamma[A] &= h_{\gamma_1}[A](-i\hbar)^2(\tau_i\tau^i)h_{\gamma_2}[A] \\ &= -\hbar^2 C^2 h_{\gamma_1}[A]h_{\gamma_2}[A] \\ &= -\hbar^2 C^2 h_\gamma[A], \end{aligned} \tag{2.61}$$

where $C^2 = \tau_i\tau^i$ is the Casimir operator of the algebra, and in the last step I have made use of the property of holonomies $h_\gamma[A] = h_{\gamma_1}[A]h_{\gamma_2}[A]$. This result can be extended to an arbitrary spin-j representation of $SU(2)$ where $C^2 = -j(j+1)$ and therefore, acting on an arbitrary spin network we have

$$\hat{E}_i\hat{E}^i \left|\Gamma, j_l, i_n\right\rangle = \hbar^2 j_l(j_l+1)\left|\Gamma, j_l, i_n\right\rangle. \tag{2.62}$$

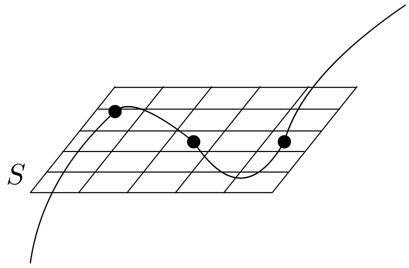

Figure 2.4: Intersection of a spin network and a surface

We can now easily see the spectrum of area operator. Puting back the gravitational constant G, and the speed of light c, we have

$$\hat{A}(S)\,|S\rangle = \frac{8\pi\hbar G}{c^3}\sum_i \sqrt{j_i(j_i+1)}\,|S\rangle\,. \tag{2.63}$$

Following descriptions below 2.56, this operator only has non-vanishing eigenvalues at the intersection points of edges of the spin network with the surface whose area is being measured (figure 2.4). Note that this is a diagonal operator on spin network states, with discrete eigenvalues. The lowest eigenvalue, corresponding to $j = 1/2$, suggest a fundamental minimum of area

$$A_0 = \frac{4\sqrt{3}\pi\hbar G}{c^3} \sim 10^{-66}cm^2, \tag{2.64}$$

however, this minimum of area depend on the choice of γ, the free parameter of the theory. It can be fixed by comparing LQG results with the black hole entropy (see section 2.3.1) and turns out to be $\gamma = 0.2375$, and thus does not affect the order of A_0. The crucial point about this is that the fundamental area of space is not put in theory *by hand*; its a direct and straightforward *result* of the basic principles of the theory, which are in fact the basic principles of GR and QM.

Volume Operator: The volume of a region $\mathcal{R}$ in space, can classically be expressed as:

$$V(\mathcal{R}) = \int_{\mathcal{R}} d^3x \sqrt{\frac{1}{3!}\left|\epsilon_{abc}\epsilon_{ijk}E^a_i E^b_j E^c_k\right|}, \tag{2.65}$$

which can be written as a Riemann sum

$$V(\mathcal{R}) = \lim_{\epsilon\to 0}\sum_{I_\epsilon}\epsilon^3\sqrt{|\det E(x_{I_\epsilon})|}, \tag{2.66}$$

where the region $\mathcal{R}$ is divided into cubes $\mathcal{R}_{I_\epsilon}$ of volume ϵ^3. Leaving technical details of taking the limit and regularizing the $\det E$ term, the corresponding operator turns out to be well-defined, self-adjoin and non-negative on $\mathcal{H}_{SU(2)}$, and acts on spin network states via

$$\hat{V}(\mathcal{R})\,|\Gamma, j_l, i_n\rangle = (16\pi\hbar G)^{3/2}\sum_n \mathcal{V}^{i'_n}_{i_n}\,|\Gamma, j_l, i_1\ldots i'_n\ldots i_N\rangle\,. \tag{2.67}$$

where $\mathcal{V}_{i_n}^{i'_n}$ are matrix elements of transition from an intertwiner i_n to i'_n of node n. Note that the operator just acts on nodes of the spin network and hence changes the intertwiner index of the corresponding node. Therefore, is not a diagonal operator on $|S\rangle$. The calculation of eigenvalues shows that the volume operator has a discrete spectra.

Diffeomorphism invariant Hilbert Space $\mathcal{H}_{\mathbf{diff}}$

Spin network states consist of a graph Γ with oriented edges which are labeled by irreducible representations with a certain ordering. Under a diffeomorphism, though, the spin network could generally change, either via changing the graph, or the orientations of edges, or the ordering of labels. The spin network states are thus not invariant under a spatial diffeomorphism U

$$|S\rangle \longrightarrow U\,|S\rangle \neq |S\rangle\,. \tag{2.68}$$

Spin networks can be partitioned into equivalence classes, K, of invariant graphs under spatial diffeomorphism. K is called a “knot”. The true spatial diffeomorphism invariant states which form an orthonormal basis for the space $\mathcal{H}_{\text{diff}}$ turns out to be *s-knot* states $|s\rangle$. These are, roughly speaking, states belonging to different knots. Therefore, they carry the same labellings of a spin network, as well as a label K characterizing the knot from which the state is chosen

$$|s\rangle \equiv |K, j_l, i_n\rangle\,. \tag{2.69}$$

Put it more intuitive, a knot is a spin network whose localization in space is not determined; they are not defined at a point of space. This is precisely the meaning of not depending on diffeomorphism, in the same way that in classical GR a valid solution to Einstein's equations does not depend on the particular choice of diffeomorphism i.e. on the particular coordinate system. Consequently, the area and volume operators act on s-knot states in the same way they do so on a spin network states.

Physical Picture of Quantum Geometry

The construction of area and volume operators provides us with a clear picture of how space structure looks like at Planck scale. The role they play in such a description can be investigated by considering their action on s-knot states and their spectrum.

1. *The volume operator* $\hat{V}(\mathcal{R})$

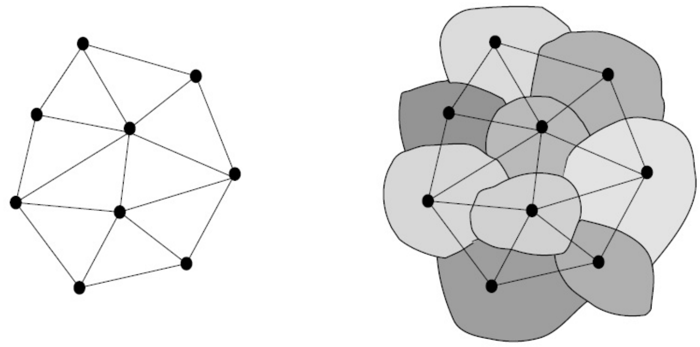

Figure 2.5: Abstract s-knots and quantum of volume

i measures the volume of a region $\mathcal{R}$ which contains a node;

ii has discrete eigenvalues.

2. *The Area operator* $\hat{A}(\mathcal{S})$

i measures the area of a surface S which contains an intersection with an edge;

ii has discrete eigenvalues.

The volume of space is divided into quanta of volume, V_0, each contains one node n. Every node is connected to a number of edges. The boundary surface, S, of V_0 intersects with edges coming from the node n. This partitions S into quanta of area A_0 containing one intersection. See figure 2.5

Remarks

- Being spatial diffeomorphism invariant, the positions of nodes is not determined with respect to a coordinate system; they are defined only relative to each other, and the spatial relation between nodes are only determine by means of edges connecting them to each other.

- This picture does not imply that the s-knots are a *fixed* network of nodes connected to each other with edges and physical fields lie over them, as is the picture which is used in lattice field theory. In fact, to imagine such a lattice is a classical picture; actual fabric of space is a quantum mechanical superposition of such states.

- The picture is presented at the kinematical level of LQG; the questions like how do the nodes of s-knots are distributed over the space, and how do they evolve in time, must be answered by the dynamics.

2.2.2 Quantum Dynamics

While the kinematics of LQG is well understood, the dynamics still needs to be developed further. The dynamical states of the theory are those satisfying Hamiltonian constraint. Such states build the physical Hilbert space of the theory $\mathcal{H}_{phys}$. Below, I sketch the Hamiltonian constraint operator, and its action on spin network states. There is another way of studying dynamics, as well, which is called spin foam models. This formulation of dynamics is the analogue of the Feynman path integral formulation of QFT. The reader is reffered to [10] for a review of the spin foam models.

Physical Hilbert Space $\mathcal{H}_{phys}$

The classical Hamiltonian constraint has the form

$$C[N] = \int d^3x N(\vec{x}) \left(\epsilon_{ijk} F^i_{ab} - 2(1+\gamma^2) K^j_{[a} K^k_{b]}\right) E^a_j E^b_k \tag{2.70}$$

$$= C_E[N] + C_K[N], \tag{2.71}$$

where $C_E[N]$, the Euclidean part of Hamiltonian constraint, contains curvature F, and $C_K[N]$ contains extrinsic curvature K. I present here quantization of Euclidean part, and will comment on the second part at the end of this subsection. The operator analogue of this operator cannot be made trivially by replacing the classical quantities by their corresponding quantum operators; there is no quantum operator representing F. Instead, there is a way, due to Thiemann [19], to express $C_E[N]$ as a Poisson bracket of basic fields,

$$C_E[N] = \int N tr\left(F\{V, A\}\right), \tag{2.72}$$

and then to quantize it by replacing Poisson brackets by ($\frac{-i}{\hbar}$ times) commutators.

Now, expand holonomy $h(A, \gamma_{x,u})$ at point x along the path $\gamma_{x,u}$ with length ϵ and tangent vector u at x as

$$h(A, \gamma_{x,u}) = 1 + \epsilon u^a A_a(x) + O(\epsilon^2). \tag{2.73}$$

In the same way, we can expand holonomy $h(A, \alpha_{x,uv})$ along a triangle loop formed by curves with tangents u and v at x, as

$$h(A, \alpha_{x,uv}) = 1 + \frac{1}{2}\epsilon^2 u^a u^b F_{ab}(x) + O(\epsilon^3). \tag{2.74}$$

Thus, at the limit $\epsilon \to 0$, $C_E[N]$ can be written as a Riemann sum, by deviding space into regions $\mathcal{R}_m$ with volume ϵ^3

$$C_E[N] = \lim_{\epsilon\to 0} \frac{1}{\epsilon^3} \sum_m \epsilon^3 N_m \epsilon^{ijk} tr\left(h_{\gamma^{-1}_{x_m,u_k}} h_{\alpha_{x_m,u_i u_j}} \left\{V(\mathcal{R}_m), h_{\gamma_{x_m,u_k}}\right\}\right), \tag{2.75}$$

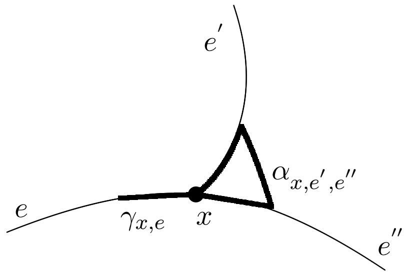

Figure 2.6: Edges emerging from a node and forming a triangle loop

where $h_{\gamma_{x_m,u_k}} \equiv h(A, \gamma_{x_m,u_k})$, and $u_i (i = 1, 2, 3)$ are three tangent vectors at x. From 2.75, the Hamiltonian constraint operator can be achieved by replacing holonomies and volume by their corresponding well-defined operators. It turns out that such an operator has non-vanishing eigenvalues only when acting on nodes of a spin network, which is resulted from the fact that it contains $\hat{V}$. This implies that the points x_m in 2.75 must be the positions of nodes x_n for a non-vanishing $C_E[N]$ acting on $|S\rangle$

$$\hat{C}_E[N] \, |S\rangle = \frac{-i}{\hbar} \lim_{\epsilon \to 0} \sum_n N_n \epsilon^{ijk} tr \left(\hat{h}_{\gamma^{-1}_{x_n,u_k}} \hat{h}_{\alpha_{x_n,u_i u_j}} \left[\hat{V}(\mathcal{R}_n), \hat{h}_{\gamma_{x_n,u_k}} \right] \right) |S\rangle \,, \tag{2.76}$$

Having known that x in above calculation is x_n, the location of nodes, there is now a natural choice for γ_{x,u_i}, u_i, and α. They are respectively, the edges with length ϵ emerging from a node, tangent vectors to the edges, and the triangle made by connecting two of edges at their endpoints. See figure 2.6.

Defining the Hamiltonian constraint operator as 2.76, only makes sense if the limit exists. So far, we have defined it on spin network states and we have made use of coordinates of nodes which manifestly breaks diffeomorphism invariance. At this level, the limit does not exist; removing regulator ϵ leads to divergences. However, It can be shown that the limit taken at the diffeomorphism invariant states does exist. For $\epsilon < \epsilon_{max}$, making the regulator smaller does not change the diffeomorphism invariant equivalence class of the graph, and hence the action of operator remains unchanged. Intuitively this happens because making ϵ smaller than Planck scale does not change anything, since there is no physics below Planck scale.

The action of $\hat{C}_E[N]$ on a general s-knot states happens in the following way:

1. Since it acts only on nodes, it gives a sum of terms for each node: $\sum_n N_n$;

2. For each node n, it gives a sum of two terms

   i for each triplet of edges (e, e', e'') emerging from nodes,

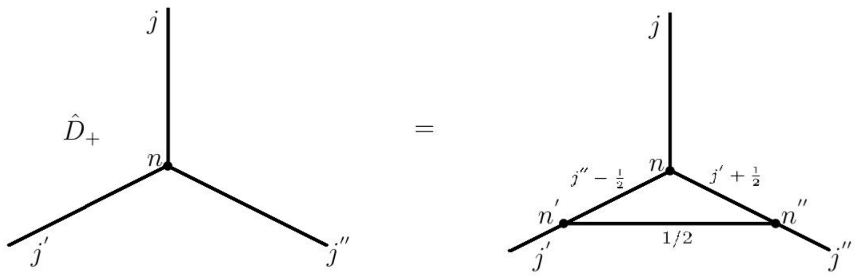

Figure 2.7: Action of Hamiltonian constraint on a node

ii for each permutation of those three edges $\sum_{e,e',e''}$.

Each of these 3 terms changes the s-knot in the following way:

1. It creates two new nodes n' and n'' along edges e' and e'';
2. It creates a new edge connecting n' and n'' labeled with a $j = 1/2$ representation;
3. It changes the j-representation label of new edges connecting n' and n'' by $\pm 1/2$;
4. It changes the intertwiner label of the node n to a new intertwiner between representations of new edges.

The above steps can be summurized in the form

$$\hat{C}_E[N]\,|S\rangle = \sum_n N_n \sum_{e,e',e'',r} \sum_{\epsilon',\epsilon''=\pm} \hat{H}_{n,e',e'',\epsilon',\epsilon''} \hat{D}_{n,e',e'',r,\epsilon',\epsilon''}\,|S\rangle \tag{2.77}$$

where $\hat{D}$ acts around the node n according to the mentioned steps, and $\hat{H}$ acts as a finite matrix at the space of intertwiners at n. See figure 2.7.

Comments

- Had we chosen $\gamma = i$, as in the original formulation of Ashtekar, the whole Hamiltonian constraint would have been the Euclidean part $C_E[N]$. However, one would have had to deal with difficulties in realizing the reality condition for the basic fields (A, E). Therefore, the Euclidean part of the Hamiltonian constraint, introduced in this section, can be used to define the dynamics too. Nevertheless, quantization of the second part, $C_K[N]$, can also be done in a similar but more subtle way which is discuussed in [20].
- Different variants of Hamiltonian constraint operator with the same classical limit can be constructed. They can result from (i) choosing another ordering of volume and

holonomies operators, (ii) choosing another irreducible representation of holonomies, and (iii) choosing a symmetric operator $\hat{C}_{sym} = \frac{1}{2}\left(\hat{C} + \hat{C}^{\dagger}\right)$.

- The remarkable fact about Hamiltonian constraint operator, however, is its finiteness. This, together with finiteness of volume and area operators, is a first indication of being free of UV divergences for a background indipendence QFT on the quantum space-time. For a more comprehensive discussion of avoidance of UV divergences in the presence of matter see [21].

2.3 Overview of Applications

The quantum geometrical picture presented by LQG opens up new windows toward describing nature in the quantum gravitational regime. The key tool in applying LQG to areas which has not been acsessible so far for QFT and classical GR lies in the discrete nature of spectum of area and volume operators. Below, I briefly review such application to the black hole entropy and loop quantum cosmology. In chapter 5, I will give a more detailed account of resolution of charged black hole singularity.

2.3.1 Statistical Origin of Black Hole Entropy

The problem of finding the statistical origin of black hole entropy, described in the introduction chapter, is cured in LQG to a fairly acceptable extent. The idea is simple: The event horizon of the black hole, has some certaint area A. The area in LQG is an observable and hence subject to quantum fluctuations. Such fluctuations are the microstates responsible for statistical origin of the black hole entropy.

Put it more precise, we take as our macrostate the macroscopic metric of a black hole. Then microstates corresponding to such macroscopic state which are responsible for entropy must be those that can affect energy exchange with exterior; those that can be distinguished form exterior. Observing from outside, these are the quantum geometrical states of the horizon. The laws of black hole mechanics tell us that a change of black hole energy occurs when the area of horizon changes. Therefore, as in statistical mechanics, we take our statistical ensemble as the quantum geometries for a given horizon area A.

Based on such considerations, we proceed by counting $N(A)$, number of microstates for a given macrostate A, and calculating entropy through the relation

$$S = k_b \ln N(A). \tag{2.78}$$

Consider the surface S of the event horizon. The possible area eigenstates on S are those which have an intersection with the edges of s-knot states. Every such intersecting edge is labeled by an irreducible representation j. As the sipmlest case, suppose the dominant contribution to the representations of intersecting edges comes from $j = 1/2$. For $j = 1/2$, the smallest possible area is $A_0 = 4\pi\gamma\hbar G\sqrt{3}$, and hence the number of possible quanta of area for a given area A is $\frac{A}{A_0}$. However, since the $j = 1/2$ representation is two dimensional, the total number of microstates, i.e. the total number of possible area eigenstates, becomes

$$N(A) = 2^{A/A_0}. \tag{2.79}$$

The black hole entropy, then, can be obtained via

$$S_{BH} = k_b \ln 2^{A/4\pi\gamma\hbar G\sqrt{3}} = \frac{\ln 2}{\gamma\pi\sqrt{3}} \frac{k_b c^{3/2} A}{4G\hbar}. \tag{2.80}$$

This is precisely the Bekenstein-Hawking entropy 1.7, provided we take the Immirzi parametre γ to be

$$\gamma = \frac{\pi\sqrt{3}}{\ln 2}. \tag{2.81}$$

This is in fact where the free parameter of the theory can be fixed.

A much more accurate calculation of microstates by considering contributions from all other representations has been done [22] which results in the same form of S_{BH} with $\gamma = 0.2375$.

2.3.2 Loop Quantum Cosmology

Quantum cosmology is a framework to study quantum mechanics of cosmological solutions of Einstein's equations. Initiated by Misner [23], it gained interest in 1980's and became a semi-classical aprroach toward quantum effects at big bag. The cosmological solutions in GR are characterized by their high degrees of symmetry such as homogeniety and isotropy. Applying such semmetries, thus, truncate the degrees of freedom of the theory and hence the whole gravitational field can be expressed in terms of finite number of parameters instead of the infinite number of degrees of freedom characterizing a fleld. Such reduced models, are called minisuperspaces. Quantization of minisuperspaces based on ADM variables and studying the Wheeler-DeWitt equation is what is now called the conventional quantum cosmology.

Loop quantum cosmology (LQC) [24], on the other hand, is the quantization of a minisuperspace by methods advocated by LQG. One applies symmetries at the classical level and finds the general form of the symmetric pair (A, E). Then, one quantizes the (A, E), consisting of finite number of parameters, using the quantization procedure discussed in the last section.

Below, I list some of the interesting results achieved in LQC without furthur explanation. The way two of such minisuperspaces are built and quantized are reviewd in chapter 4, and derivation of some results regarding singularity issue is similar to what I will present in chapter 5, for singularity resolution of the charged black holes.

1. The initial singularity is shown to be resolved locally: the inverse scale factor is bounded and the energy density is not infinite at $t = 0$, it has the value $\rho = 0.41\rho_{Pl}$,
2. The initial singularity is shown to be resolved globally: the evolution equation is not singular at $t = 0$; wave packets which are sent backward in time survive at classical singularity and can transverse it to the other side. The "big bang" is replaced with a "big bounce",
3. The Friedmann equation becomes modifyed $\left(\frac{\dot{a}}{a}\right)^2 = (8\pi G\rho/3)\left(1 - \frac{\rho}{\sqrt{3}/32\pi^2\gamma^3 G^2\hbar}\right)$,
4. Evolution of early universe leads naturally to the inflation.

This completes the outline of main ideas and tools used in LQG, and its application to black hole thermodynamics and cosmology. We are now ready to see how the established methods can be employed to investigate the singular behavior of black holes.

CHAPTER 3

Black Holes and Singularities in Classical General Relativity

THE term "Black hole" was first coined by John Wheeler in 1967. While one of the novel implications of Einstein's general theory of relativity, the existence of a massive body whose "gravitational attraction does not allow any of it rays to arrive us", and hence, are "invisible" was conjectured by Laplace back in 1798. Black holes are usually thought of as the final state of a collapsing star whose gravitational attraction, as a result of their heavy mass, overweighs all other repulsive forces, due to degenerate Fermi gas pressure. Their definition, thus, makes it impossible to be detected directly. However, astronomers believe that they can be detected, by means of indirect evidence such as X-ray emission from the infalling materials into a black hole, gravitational radiation, and Hawking radiation .

While such an astrophysical definition of a black hole is intuitively illuminating, its precise geometrical definition and characteristics, such as horizons and singularities, are discussed within the mathematical framework of GR. In what follows, I will briefly review generic definitions of black holes and singularities, and will illustrate some of their features for the case of Schwarzschild and Reissner-Nordström black holes.

3.1 Black Holes

A black hole is a region in space-time where nothing, including light, can escape from. That is why it is called "black" hole. To define such a region in a more precise way, we need some preliminary definitions [25, 26].

Let (M, g_{ab}) be a time orientable manifold (space-time).

- A *future directed timelike curve* $\lambda(t)$ is a smooth curve whose tangent vector is a future directed timelike vector for all points p.
- $I^+(p)$: The *chronological future* of a point $p \in M$ is the set of events that can be reached by a future directed timelike curve starting from p.
- $I^+(S)$: The chronological future of a subset $S \in M$ is defined as: $I^+(S) = \cup_{p \in S} I^+(p)$.
- $I^-(p)$ and $I^-(S)$ which are called *chronological past* are defined analogously.
- $\mathcal{I}^+$: The *future null infinity* is a null hypersurface which is in fact an idealization of faraway observers who can receive radiation from the isolated gravitating system.

Based on the above definitions, the black hole region, $\mathcal{B}$, of an asymptotically flat space-time (M, g_{ab}) is defined as:

$$\mathcal{B} = M - I^-(\mathcal{I}^+). \tag{3.1}$$

The event horizon $\mathcal{H}$ of a black hole is, then, defined as the boundary of $\mathcal{B}$.

We will now consider the above abstract definitions in the special case of Schwarzschild and Reissner-Nordström black holes.

3.1.1 Schwarzschild Black Hole

By applying spherical symmetry to the Einstein's equations in vacuum, $R_{\mu\nu} = 0$, one finds the metric describing the space-time outside a spherically symmetric mass M

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.2}$$

Basic Features

1. *Asymptotic flatness:* At the limit $r \to \infty$ the line element 3.2 becomes the flat Minkowski line element;

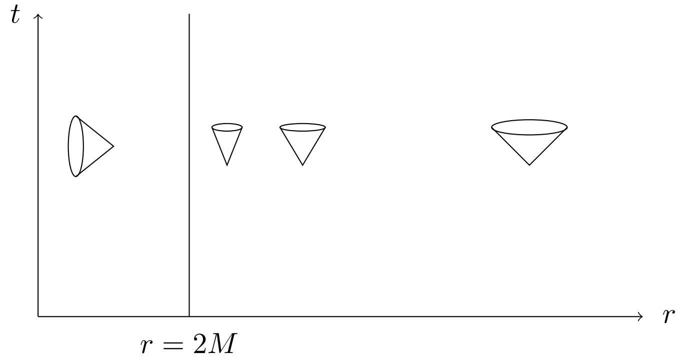

Figure 3.1: Local light cones in Schwarzschild coordinates

2. *Stationary:* since it has a timelike Killing vector field, it does not depend on the coordinate t, it is stationary;

3. *Static:* since the time-like Killing vector field, ∂_t, is orthogonal to the $t = const.$ family of hypersurfaces, i.e. there is no cross term $dtdx^a$ in the metric, it is static;

4. *Uniquness:* Brikhoff's theorem [27] guarntees that the space-time due to a spherically symmetric source is uniquely the Schwarzschild space-time.

Local Causal Structures and Horizons

To study causal structure and consequently to identify the event horizon let's consider the radial null geodesics characterized by $ds^2 = d\Omega^2 = 0$. This leads to the null geodesic equation

$$\frac{dt}{dr} = \pm\frac{r}{r-2m} \tag{3.3}$$

which determines the local light cones (See figure 3.1)

Note that the line element 3.2 is written in terms of coordinates measured by an observer at infinity (Minkowski space-time). At each point p of space-time the region within the light cone is causally connected to p. As can be seen from the table, the light cones become closer as one approaches the surface $r = 2M$. On the surface $r = 2M$ the light cone is totally closed; signals sent from it will remain on the surface and will not reach infinity. In the region within the surface, $r < 2M$, the light cone is directed toward the center which means signals never reach the horizon. Such a region satisfies the definition 3.1: it is the part of space-time excluding the region which is chronologically connected to the future null infinity (Minkowski space). The surface $r = 2M$, the boundary of Schwarzschild black hole, is the event horizon i.e. the last surface from which light can scape to future null infinity. On this surface $g_{00} = 0$, or put it the other way, $g_{00} = 0$ determines the event horizon surface.

The event horizon $r = 2M$ partitions the Schwarzschild space-time into two portions: the $r > 2M$ exterior region and the $r < 2M$ interior region. For $r < 2M$ the coefficients of dt^2 and dr^2 in 3.2 change sign. This means inside the black hole the coordinate r becomes timelike while the coordinate t becomes spacelike. Therefore the line element 3.2 becomes

$$ds^2 = -\left(\frac{2M}{t} - 1\right)^{-1} dt^2 + \left(\frac{2M}{t} - 1\right) dr^2 + t^2 d\Omega^2 \qquad (r < 2M). \tag{3.4}$$

Such a line element represents a kind of manifold known as Kantowski-Sachs space-time which is a homogeneous manifold with a general metric of the form

$$ds^2 = -dt^2 + A(t)dr^2 + B(t)d\Omega^2. \tag{3.5}$$

Coordinate Singularity

A look at line element 3.2 shows that the Schwarzschild metric becomes meaningless (singular) at $r = 0$ and $r = 2M$. However, this does not necessarily mean that the space-time becomes singular at such points. Under a suitable coordinate transformations, one can observe that the metric components are not infinite at (transformed values corresponding to) $r = 2M$.

For instance, consider the Kruskal-Szekerz coordinates (U, V, θ, ϕ). These are obtained by a transformation of the Schwarzschild coordinates (t, r, θ, ϕ) in the form

$$V = e^{v/4M}, \tag{3.6}$$

$$U = -e^{-u/4M}, \tag{3.7}$$

where

$$v = t + r + 2M \ln(\frac{r}{2M} - 1), \tag{3.8}$$

$$u = t - (r + 2M \ln(\frac{r}{2M} - 1)). \tag{3.9}$$

The line element 3.2 in this coordinate system takes the form

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2, \tag{3.10}$$

which is obviously well defined at $r = 2M$. Therefore, the singular behavior of the metric 3.2 at $r = 2M$ is just an artifact of the special coordinate system chosen and is called a *coordinate singularity.*

3.1.2 Reissner-Nordström Black Hole

To find the metric of a space-time produced by a spherically symmetric source of mass M and electric charge Q, one must seek a solution of Einstein-Maxwell equations. Einstein-Maxwell equations are Euler-Lagrange equations of the action

$$S[g_{\mu\nu}, A_\mu] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R - F_{\mu\nu}F^{\mu\nu}], \tag{3.11}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.12}$$

being the electromagnetic field strength tensor of the vector potential A_μ. Variation with respect to $g_{\mu\nu}$ and A_μ gives

$$R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = 2\left(-g^{\rho\lambda}F_{\mu\rho}F_{\nu\lambda} + \frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}\right); \tag{3.13}$$

$$\nabla_\mu F^{\mu\nu} = 0; \tag{3.14}$$

3.13 is in fact the Einstein's equations $R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = 8\pi G T^{EM}_{\mu\nu}$ with energy-momentum tensor of the electromagnetic field produced by a spherically symmetric charge distribution $T^{EM}_{\mu\nu} = \frac{1}{4\pi}\left(-g^{\rho\lambda}F_{\mu\rho}F_{\nu\lambda} + \frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}\right)$, and 3.14 is the covariant form of Maxwell equation in the vacuum.

Solving spherically symmetric Einstein-Maxwell equation leads to the following line element

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{3.15}$$

which is known as the Reissner-Nordström metric.

Basic Features

1. *Asymptotic flatness:* At the limit $r \to \infty$ the line element 3.15 represents the flat Minkowski space-time;

2. *Zero Charge limit:* At the limit $Q \to 0$ the Reissner-Nordström line element 3.15 reduces to Schwarzschild 3.2;

3. *Stationary* and *Static:* similar to the Schwarzschild case;

4. *Uniquness:* There is an analogue of the Brikhoff's theorem that guarantees any spherically symmetric solution of the Einstein-Maxwell field equations must be stationary and asymptotically flat.

Horizons

To find the event horizon(s) we seek solutions to $g_{00} = 0$:

$$g_{00} = 0 \Rightarrow r^2 - 2Mr + Q^2 = 0 \Rightarrow \begin{cases} \text{2 horizons} & r_\pm = r \pm \sqrt{M^2 - Q^2} & M^2 > Q^2 \\ \text{1 horizon} & r & M^2 = Q^2 \\ \text{No horizon} & & M^2 < Q^2 \end{cases} \tag{3.16}$$

In the physical case of $M^2 > Q^2$, the space-time is partitioned into 3 portions:

$$M^2 > Q^2 \Rightarrow \begin{cases} \text{Region I} & r > r_+ & \text{spherically symmetric} \\ \text{Region II} & r_- < r < r_+ & \text{homogeneous (Kantowski-Sachs)} \\ \text{Region III} & r < r_- & \text{spherically symmetric} \end{cases} \tag{3.17}$$

The region which lies between two event horizons is again of Kantowski-Sachs type since space and time exchange their roles in that region

$$ds^2 = -\left(\frac{2M}{t} - \frac{Q^2}{t^2} - 1\right)^{-1} dt^2 + \left(\frac{2M}{t} - \frac{Q^2}{t^2} - 1\right) dr^2 + t^2 d\Omega^2 \qquad (r_- < r < r_+). \tag{3.18}$$

Coordinate Singularities

At $g_{00} = 0$ and $g_{11} = 0$ the line element **??** becomes singular. However, as for the case of Schwarzschild, suitable coordinate transformations reveal that they are just coordinate singularities.

Define $\kappa_\pm$ and r^* as

$$\kappa_\pm \equiv \frac{r_\pm - r_\mp}{2r_\pm^2}, \tag{3.19}$$

$$dr^* = \frac{dr}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} \tag{3.20}$$

$$\Rightarrow \quad r^* = r + \frac{1}{2\kappa_+} \ln\left(\frac{|r - r_+|}{r_+}\right) \frac{1}{2\kappa_-} \ln\left(\frac{|r - r_-|}{r_-}\right) + const. \tag{3.21}$$

$$\tag{3.22}$$

We can transform the (t, r, θ, ϕ) coordinates to $(U^\pm, V^\pm, \theta, \phi)$ defined by:

$$U^\pm = -e^{-\kappa_\pm(t - r^*)} \qquad V^\pm = e^{\kappa_\pm(t + r^*)}. \tag{3.23}$$

The original Reissner-Nordström line element 3.15 in the new coordinates takes the form

$$ds^2 = -\frac{r_+ r_- e^{-2\kappa_+ r}}{r^2 \kappa_+^2} \left(\frac{r_-}{r - r_-}\right)^{\frac{\kappa_\pm}{\kappa_-} - 1} dU^+ dV^+ + r^2 d\Omega^2 \qquad (r_- < r), \tag{3.24}$$

$$ds^2 = -\frac{r_+ r_- e^{-2\kappa_- r}}{r^2 \kappa_-^2} \left(\frac{r_+}{r_+ - r} \right)^{\frac{\kappa_-}{\kappa_+} - 1} dU^- dV^- + r^2 d\Omega^2 \qquad (r \leq r_-). \tag{3.25}$$

This coordinate system is manifestly well-behaved at r_+ and r_+ indicating the fact that singularities of such points are artifacts of coordinate system.

3.2 Irremovable Singularities

Despite coordinate singularities of Schwarzschild and charged black holes, the $r = 0$ singularity is still not avoided by the coordinate transformation. In fact, calculation of the Kretschmann scalar curvature for a Schwarzschild black hole gives

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}. \tag{3.26}$$

For the Reissner-Nordström space-time, such a scalar curvature turns out to be

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48M^2 r^2 - 96MQ^2 r + 56Q^2}{r^8}. \tag{3.27}$$

At $r = 0$ they both diverge, and since are scalars remain the same in all coordinate systems. Therefore, $r = 0$ is a curvature singularity for both black holes. Singularities for which no coordinate system exist to remove them are called *irremovable* or *intrinsic*; they are the genuine singularities of space-time.

Nevertheless, divergent of scalar curvature is not necessary and sufficient for definition of a singularity. One difficulty arises from the fact that to speak of singularity as “a point” at which curvature blows up does not make sense in a theory in which the metric of space-time, determining position of points, is itself dynamical and subject to change. On the other hand, there might exist irremovable singularities for which the curvature scalar does not blow up. Consider the singularity at the tip of a cone formed by rolling up a sheet of paper. All curvature invariants remain finite as the singularity is approached; in fact, in this two-dimensional example the curvature tensor is everywhere zero. If we could assign a curvature to the singular point at the tip of the cone it would have to be infinite but, strictly speaking, we cannot include this point as part of the manifold since there is no coordinate chart that covers it. This situation suggests removing such a point from the manifold and hence leaving a “hole” in it. The incompleteness of geodesics reaching the hole, i.e. being inextendable of geodesics in at least one direction but having only a finite range of affine parameter, can serve as a definition.

In the case of the Schwarzschild vacuum solution, a particle on an ingoing radial geodesics will reach the coordinate singularity at r = 2M at finite affine parameter but, as we have seen, this geodesic can be continued into region II by an appropriate change of coordinates. Its continuation will then approach the curvature singularity at r = 0, coming arbitrarily close for finite affine parameter. The excision of any region containing r = 0 will therefore lead to an incompleteness of the geodesic. The vacuum Schwarzschild solution is therefore singular.

Therefore, we consider geodesic completeness, i.e. the existence of at least one incomplete geodesic, as the *global* criterion for identifying a singularity and classify three types of *local* criteria as follows:

i **curvature singularity** a scalar constructed polynomially from $R_{\mu\nu\rho\sigma}$ and its covariant derivatives blows up along the geodesic;

ii **paralelly propagated curvature singularity** no such scalar blows up, but a component of $R_{\mu\nu\rho\sigma}$ or its covariant derivative in a parallely propagated tetrad blows up along a geodesic;

iii **non-curvature singularity** no such scalar or component blows up.

CHAPTER 4

Symmetry Reduced Models of Quantum Geometry

Based upon the general framework of LQG introduced in chapter 2, I will now illustrate truncations of the theory restricted to specific spatial symmetries. To study particular gravitational configurations, such as cosmological models or black holes, characterized by a spatial symmetry, such as homogeneity, isotropy etc. requires implementing the desired symmetry to the theory and consequently working on a symmetric sector of the theory. For the case of quantum behavior of a Reisner-Nordström black hole, for instance, study of the spherically symmetric sector of LQG is required. What does this mean?

Consider the example of hydrogen atom in QM. We know that quantum states of the theory are elements of a Hilbert space. On the other hand, the symmetry that the hydrogen atom possesses, is spherical symmetry since the interactions is governed by the Columb potential. Therefore, one can suppose, at the *kinematical level*, that the wave function corresponding to the hydrogen atom must also possess spherical symmetry and be of the form of, say a spherical harmonic. However, the true quantum states of the hydrogen atom are those which are solutions to the spherically symmetric Schrödinger equation containing Columb potential. Such solutions turn out to be spherical harmonics as well. Nevertheless, note that the symmetry is implemented at the *dynamical level* (Scrödinger equation).

The true spherically symmetric sector of LQG must, as well, be determined as the solutions to the Hamiltonian constraint suitably adapted to the desired symmetry. Such adaptation of symmetry is not trivial since the macroscopic symmetry we know intuitively does not necessarily imply the same picture of symmetry at a microscopic level, i.e. for distribution of nodes of spin networks.

A symmetric sector of LQG, as defined above, does not exist so far. Nevertheless, many attempts has been made to construct a model realizing the quantum structure of a cosmological model using LQG methods [28, 29, 30]. In such models one implements the desired symmetry at the classical level, and obtains a symmetric pair of conjugate fields (A, E) satisfying the symmetry, or alternatively one might implement the symmetry at the kinematical level of the quantum theory. Then, one can quantize the symmetric phase space by methods developed in LQG. The degrees of freedom of the resulting symmetric phase space will be reduced, as is described below. Such reduction of degrees of freedom is termed *mini-superspace* model if the degrees of freedom are finite, and *midi-superspace* model if the degrees of freedom are still infinite i.e. a lower dimensional field theory. For instance, while the homogeneous models possess finite degrees of freedom (mini-superspace) and hence loose field theoretical aspects of gravity, the spherically symmetric models keeps the infinite degrees of freedom of the field (midi-superspace).

In this chapter, I will briefly outline the basic ideas and main features of two types of a symmetry reduced models, homogeneous Kantowski-Sachs and spherically symmetric, which are discussed extensively in [31] and [32, 33] respectively. Based on them, I will investigate the state of the singularity of a charged black hole at quantum level in next chapter.

4.1 Homogeneous Kantowski-Sachs Quantum Geometry

4.1.1 Classical Phase Space

A symmetric pair (A, E) corresponding to a 3-space symmetry characterized by Killing vector fields ξ^a are defined to satisfy

$$\mathcal{L}_\xi A = DA, \tag{4.1}$$

$$\mathcal{L}_\xi E = [E, \Lambda], \tag{4.2}$$

where Λ^i are generators of local $SU(2)$ gauge transformations and D denotes covariant derivative in gauge group.

Consider the Kantowski-Sachs space-time with the general metric 3.5

$$ds^2 = -dt^2 + A(t)dr^2 + B(t)d\Omega^2. \tag{4.3}$$

The $t = const.$ 3-space Σ with topology $\mathbb{R} \times S^2$ carry the Kantowski-Sachs symmetry group

$G = SO(3)$, with Killing vectors

$$\xi^1 = \sin\phi \partial_\theta + \cot\theta \cos\phi \partial_\phi \tag{4.4}$$

$$\xi^2 = -\cos\phi \partial_\theta + \cot\theta sin\phi \partial_\phi \tag{4.5}$$

$$\xi^3 = \partial_\phi \tag{4.6}$$

The symmetric conjugate pair satisfying 4.1 and 4.2 with the above Killing vectors have the general form:

$$A = c\tau_3 dr + (a\tau_1 + b\tau_2)d\theta + (-b\tau_1 + a\tau_2 sin\theta + \tau_3 cos\theta)d\phi \tag{4.7}$$

$$E = p_c\tau_3 sin\theta\partial_r + (p_a\tau_1 + p_b\tau_2)sin\theta\partial_\theta + (-p_b\tau_1 + p_a\tau_2)\partial_\phi, \tag{4.8}$$

where $\tau_i = \frac{i}{2}\sigma_i$ are generators of $SU(2)$, with σ_i being the Pauli matrices, and a, b, c, p_a, p_b, p_c are constant functions of coordinates representing homogenity.

Such a pair automatically satisfies the diffeomorphism constraint. However, two global gauge freedoms are still remained. The first one is fixed by demanding the Gauss constraint to hold. It reads

$$G = ap_b - bp_a, \tag{4.9}$$

which by choosing $a = p_a = 0$ becomes zero. The second one comes from the residual gauge transformation $(b, p_b) \to (-b, -p_b)$ which is indeed a parity transformation in the p_b variable.

The gauge and diffeomorphism invariant variables b, c, p_b, p_c thus define the 4 dimensional phase spase with the simplectic structure

$$\Omega = \frac{1}{2\gamma G}(2db \wedge dp_b + dc \wedge dp_c). \tag{4.10}$$

The volume of an elementary cell $\mathcal{I} \times S^2$, with $\mathcal{I} \in \mathbb{R}$, is given by

$$V = \int d^3x \sqrt{\mid detE \mid} = 4\pi \sqrt{\mid p_c \mid} \mid p_b \mid . \tag{4.11}$$

The Hamiltonian constraint turns out to be:

$$c[N] = -\frac{8\pi N}{\gamma^2} \frac{sgnp_c}{p_b\sqrt{|p_c|}} \left[(b^2 + \gamma^2)p_b^2 + 2cp_c bp_b\right], \tag{4.12}$$

with N being a constant function for such a homogeneous model.

4.1.2 Quantization

Quantum Kinematics

As discussed in chapter 2, the basic fields in quantum level are holonomies of connection along curves in Σ and fluxes of traid through 2-surfaces in Σ. To define holonomies, consider three sets of curves:

i) along the $\mathbb{R}$ direction of Σ with oriented length τ;

ii) along the equator of S^2 with oriented length μ;

iii) along the longitudes of S^2 also with oriented length μ.

Then, holonomies of connection 4.7 along such curves take the form:

$$h_r^{(\tau)} = \exp \int_0^\tau dr c\tau_3 = \cos\frac{\tau c}{2} + 2\tau_3 \sin\frac{\tau c}{2}, \tag{4.13}$$

$$h_\phi^{(\mu)} = \exp - \int_0^\mu d\phi b\tau_1 = \cos\frac{\mu b}{2} - 2\tau_1 \sin\frac{\mu b}{2}, \tag{4.14}$$

$$h_\theta^{(\mu)} = \exp \int_0^\mu d\theta b\tau_2 = \cos\frac{\mu b}{2} + 2\tau_2 \sin\frac{\mu b}{2}. \tag{4.15}$$

They generate the algebra of almost periodic functions of b and c with the general form:

$$f(b,c) = \sum_{\mu,\nu} f_{\mu,\nu} e^{\frac{i}{2}(\mu b + \tau c)}, \tag{4.16}$$

where $f_{\mu,\nu} \in \mathbb{C}$, $\mu, \nu \in \mathbb{R}$. This algebra is the Kantowski-Sachs analogue of the algebra of cylindrical functions in the full theory (see section 2.2.1). The Hilbert space of this reduced theory, $\mathcal{H}$ is spanned by a basis $|\mu,\tau\rangle$:

$$\langle b, c|\mu,\tau\rangle = e^{\frac{i}{2}(\mu b + \tau c)}, \tag{4.17}$$

which is orthonormal:

$$\langle \mu',\tau'|\mu,\tau\rangle = \delta_{\mu',\mu}\delta_{\tau',\tau}. \tag{4.18}$$

As in full theory, a representation is chosen in which holonomy operators act by multiplication

$$\hat{h}_r^{(\tau)} |\mu,\tau\rangle = h_r^{(\tau)} |\mu,\tau\rangle, \tag{4.19}$$

$$\hat{h}_\phi^{(\mu)} |\mu,\tau\rangle = h_\phi^{(\mu)} |\mu,\tau\rangle, \tag{4.20}$$

$$\hat{h}_\theta^{(\mu)} |\mu,\tau\rangle = h_\theta^{(\mu)} |\mu,\tau\rangle, \tag{4.21}$$

and fluxes of triad along preferred 2-surfaces, which are given by components p_b, p_c of triad, by differentiation

$$\hat{p}_b = -i\gamma\ell_{Pl}^2\frac{\partial}{\partial b}, \qquad \hat{p}_c = -2i\gamma\ell_{Pl}^2\frac{\partial}{\partial c}. \tag{4.22}$$

They are diagonal on states $|\mu, \tau\rangle$

$$\hat{p}_b\,|\mu,\tau\rangle = \frac{1}{2}\gamma\ell_{Pl}^2\mu\,|\mu,\tau\rangle\,, \tag{4.23}$$

$$\hat{p}_c\,|\mu,\tau\rangle = \gamma\ell_{Pl}^2\tau\,|\mu,\tau\rangle\,. \tag{4.24}$$

Based on the above operators, the volume operator can be written by direct quantization of 4.11

$$\hat{V} = 4\pi|\hat{p}_b|\sqrt{|\hat{p}_c|} \tag{4.25}$$

which is diagonal on states $|\mu, \tau\rangle$

$$\hat{V}|\tau,\mu\rangle = V_{\tau\mu}|\tau,\mu\rangle, \tag{4.26}$$

with

$$V_{\tau\mu} = \frac{2\pi}{\gamma}^{3/2}\ell_{pl}^3 \mid \mu \mid \sqrt{\mid \tau \mid}. \tag{4.27}$$

Quantum Dynamics

The construction of the Hamiltonian constraint operator is more technically subtle. As is discussed in more details in [31], to be able to use holonomies in the above expression one uses the parameter δ to formally expand the holonomies along the homogeneous directions. δ^2 is analogous to the size of the loop $h_\theta h_\phi h_\theta^{-1} h_\phi^{-1}$. The Classical Hamiltonian constraint can then be written as

$$C = \lim_{\delta\to 0} C^{(\delta)}, \tag{4.28}$$

with

$$C^{(\delta)} = -2(\gamma^3G\delta^3)^{-1}\sum_{ijk}\epsilon^{ijk}Tr\left(h_{ij}^{(\delta)}h_k^{(\delta)}\{h_k^{(\delta)-1},V\}\right) + 4(\gamma G\delta)^{-1}Tr\left(\tau_3 h_r^{(\delta)}\{h_r^{(\delta)-1},V\}\right). \tag{4.29}$$

The quantum operator, then, is obtained by replacing holonomies and volume by their corresponding quantum operators:

$$\hat{C}^{(\delta)} = 2i(\gamma^3\delta^3\ell_{pl}^2)^{-1}Tr\left(\sum_{ijk}\epsilon^{ijk}\hat{h}_i^{(\delta)}\hat{h}_j^{(\delta)}\hat{h}_i^{(\delta)-1}\hat{h}_j^{(\delta)-1}\hat{h}_k^{(\delta)}[\hat{h}_k^{(\delta)-1},\hat{V}] + [2(\gamma^2\delta^2)\tau_3\hat{h}_r^{(\delta)}[\hat{h}_r^{(\delta)-1},\hat{V}]\right). \tag{4.30}$$

Its action on $|\mu, \tau\rangle$ states takes the form

$$\begin{aligned}
\hat{C}^{(\delta)}|\tau, \mu\rangle &= (2\gamma^3\delta^3\ell_{pl}^2)^{-1}\{(V_{\tau,\mu+\delta} - V_{\tau,\mu-\delta}) \\
&\times (|\tau+2\delta, \mu+2\delta\rangle - |\tau-2\delta, \mu+2\delta\rangle - |\tau+2\delta, \mu-2\delta\rangle + |\tau-2\delta, \mu-2\delta\rangle) \\
&+ (V_{\tau+\delta,\mu} - V_{\tau-\delta,\mu})(|\mu+4\delta, \tau\rangle - 2(1+2\gamma^2\delta^2)|\mu, \tau\rangle + |\mu-4\delta, \tau\rangle\Big\}.
\end{aligned} \tag{4.31}$$

The implication of the above expressions for avoidance of singularity in Schwarzschild black hole will be discussed in next chapter.

4.2 Spherically Symmetric Quantum Geometry

4.2.1 Classical Phase Space

Spherical symmetry, expressed in spherical coordinates (r, θ, ϕ), manifests itself in dependence of fields on coordinates; a spherically symmetric function of coordinates does not depend on azimuthal and inclination angles. It is only a function of radial distance r. The same consideration is applied to our classical spherically symmetric pair (A, E)

$$(A(\vec{x}), E(\vec{x})) \longrightarrow (A(r), E(r)). \tag{4.32}$$

Furthermore, the gauge group of such a reduced model is $U(1)$ instead of $SU(2)$ in the full theory.

The general form of the pair satisfying 4.1, 4.2 in the case of spherical symmetry turns out to be:

$$A = A_r(r)\tau_3 dr + (A_1(r)\tau_1 + A_2(r)\tau_2)d\theta + ((A_1(r)\tau_2 - A_2(r)\tau_1)sin\theta + \tau_3 cos\theta)d\phi, \tag{4.33}$$

$$E = E^r(r)\tau_3 sin\theta\partial_r + (E^1(r)\tau_1 + E^2(r)\tau_2)sin\theta\partial_\theta + (E^1(r)\tau_2 - E^2(r)\tau_1)\partial_\phi, \tag{4.34}$$

where $\tau_i = -\frac{i}{2}\sigma_i$ are generators of $su(2)$ algebra, with σ_i being the Pauli matrices. They satisfy canonical Poisson bracket

$$\{A_a^i(r), E_j^b(r')\} = 8\pi\gamma G\delta_j^i\delta_a^b\delta(r - r'), \tag{4.35}$$

or equivalently define the symplectic structure

$$\Omega = \frac{1}{2\gamma G}\int dx(dA_r \wedge dE^r + 2dA_1 \wedge dE^1 + 2dA_2 \wedge dE^2). \tag{4.36}$$

However, it turn out that performing a canonical transformation would simplify later calculations. Define new variables A_ϕ and E_ϕ as

$$A_\phi(r) \equiv \sqrt{A_1(r)^2 + A_2(r)^2}, \tag{4.37}$$

$$E_\phi(r) \equiv \sqrt{E_1(r)^2 + E_2(r)^2}. \tag{4.38}$$

These are gauge invariant quantities. Define also the internal directions Λ_ϕ^A and Λ_ϕ^E in the $\Lambda_1 - \Lambda_2$ plane as

$$\Lambda_\phi^A(r) \equiv (A_1\Lambda_2 - A_2\Lambda_1)/A_\phi, \tag{4.39}$$

$$\Lambda_\phi^E \equiv (E^1\Lambda_2 - E^2\Lambda_1)/E^\phi. \tag{4.40}$$

These new internal directions can also be parametrized by two angles $\alpha(r)$ and $\beta(r)$

$$\Lambda_\phi^A(r) = \Lambda_1 cos\beta(r) + \Lambda_2 sin\beta(r), \tag{4.41}$$

$$\Lambda_\phi^E(r) = \Lambda_1 cos(\alpha(r) + \beta(r)) + \Lambda_2 sin(\alpha(r) + \beta(r)). \tag{4.42}$$

The symplectic structure 4.36 in terms of new variables takes the form

$$\Omega = \frac{1}{2\gamma G}\int dr(dA_r \wedge dE^r + 2d(A_\phi cos\alpha) \wedge dE^\phi + 2d(\alpha+\beta) \wedge d(A_\phi E^\phi sin\alpha)), \tag{4.43}$$

which suggests the new set of mutually conjugate variables:

$$\{A_r(r), E^r(r)\}, \tag{4.44}$$

$$\{\bar{A}(r) \equiv 2A_\phi cos\alpha, E^\phi\}, \tag{4.45}$$

$$\{\eta(r) \equiv \alpha + \beta, P^\eta(r) \equiv 2A_\phi E^\phi sin\alpha\}. \tag{4.46}$$

Recalling the relation between fluxes and triads, $E_i^a = (\det e)e_i^a$, the 3 triad fields corresponding to 4.34 becomes

$$e = sgn(E^r)\frac{E^\phi}{\sqrt{|E^r|}}\Lambda_3 dr + \sqrt{|E^r|}\Lambda_E^\theta d\theta + \sqrt{|E^r|}\Lambda_E^\phi sin\theta d\phi, \tag{4.47}$$

with prime denoting differentiation with respect to r, and

$$\Lambda_\theta^A = -\Lambda_1 sin\beta(r) + \Lambda_2 cos\beta(r), \tag{4.48}$$

$$\Lambda_\theta^E = -\Lambda_1 sin(\alpha(r)+\beta(r)) + \Lambda_2 cos(\alpha(r)+\beta(r)). \tag{4.49}$$

The spin connection of the triad 4.47 becomes

$$\bar{\Gamma} = -\eta'(r)\Lambda_3 dr + \frac{E^{r\prime}}{2E^\phi}\Lambda_E^\phi d\theta - \frac{E^{r\prime}}{2E^\phi}\Lambda_E^\theta sin\theta d\phi + \Lambda_3 cos\theta d\phi. \tag{4.50}$$

Recall the relation between A and Γ 2.40

$$A = \bar{\Gamma} + \gamma K, \tag{4.51}$$

which leads to

$$\begin{aligned} A_\phi cos\alpha &= A_\phi \Lambda^A_\phi \cdot \Lambda^\phi_E && (4.52)\\ &= (\bar{\Gamma}_\phi \Lambda^\theta_E + \gamma K_\phi \Lambda^\phi_E) \cdot \Lambda^\phi_E && (4.53)\\ &= \gamma K_\phi. && (4.54) \end{aligned}$$

This means our final form of conjugate variables are

$$\{A_r(r), E^r(r)\}, \tag{4.55}$$

$$\{\gamma K_\phi(r), E^\phi(r)\}, \tag{4.56}$$

$$\{\eta(r), P^\eta(r)\}. \tag{4.57}$$

In terms of above variables, the Gauss and Hamiltonian constraints take the form:

$$G[\lambda] = \int dr \lambda (E^{r\prime} + P^\eta), \tag{4.58}$$

$$C[N] = -\frac{1}{2G}\int drN(r)\frac{1}{\sqrt{|E^r|}}\left((1 - \bar{\Gamma}^2_\phi + K^2_\phi)E^\phi + \frac{2}{\gamma}K_\phi E^r(A_r + \eta'(r)) + 2E^r\bar{\Gamma}'_\phi\right). \tag{4.59}$$

4.2.2 Quantization

Quantum Kinematics

Along the standard lines of constructing basic operators and states in the kinematical Hilbert space of LQG, one starts with holonomies of the connections. Holonomies of A_r along curves γ in R are defined as $h^{(\gamma)} \equiv exp\left(\frac{i}{2}\int_\gamma A_r(r)\right)$ which are elements in $U(1)$. For A_ϕ point holonomies $exp\,(i\mu A_\phi(r))$ are used and point holonomies of $\eta \in S^1$, have the form $exp\,(i\eta(r))$ which are elements of $U(1)$.

The kinematical Hilbert space of the present reduced theory is the space spaned by spin network state $T_{g,k,\mu}$:

$$T_{g,k,\mu} = \prod_{e\in g} exp\left(\frac{i}{2}k_e\int_e drA_r(r)\right)\prod_{\nu\in V(g)} exp\,(i\mu_\nu\gamma K_\phi(\nu))\, exp\,(ik_\nu\eta(\nu))\,. \tag{4.60}$$

For a given graph g, these are cylindrical functions of holonomies along edges e of g. Such edges are labeled by irreducible representations of $U(1)$.

Holonomies act on spin network states by multiplication. Their corresponding momenta, on the other hand, act by differentiation:

$$\hat{E}^r(r)T_{g,k,\mu} = \gamma\frac{\ell_p^2}{2}(k_{e^+(r)} + k_{e^-(r)})T_{g,k,\mu}, \tag{4.61}$$

$$\int dr\hat{E}^\phi(r)T_{g,k,\mu} = \gamma\ell_p^2\sum_v \mu_\nu T_{g,k,\mu}, \tag{4.62}$$

$$\int dr\hat{E}^\eta(r)T_{g,k,\mu} = 2\gamma\ell_p^2\sum_v k_\nu T_{g,k,\mu}. \tag{4.63}$$

The volume operator can be express as the operator $\hat{V} = 4\pi\int dr|\hat{E}^\phi(r)|\sqrt{|\hat{E}^r(r)|}$ which is diagonal in spin network representation:

$$\hat{V}T_{g,k,\mu} = V_{k,m}T_{g,k,\mu} \tag{4.64}$$

$$V_{k,m} = 4\pi\gamma^{3/2}\ell_p^3\sum_\nu|\mu_\nu|\sqrt{\frac{1}{2}|k_{e^+(r)} + k_{e^-(r)}|}. \tag{4.65}$$

Implementing the Gauss constraint as an operator on spin networks to select the gauge invariant states, leads to a restriction on labels:

$$\hat{G}[\lambda]T_{g,k,\mu} = \gamma\ell_p^2\sum_\nu \lambda(\nu)(k_{e^+(r)} - k_{e^-(r)} + 2k_\nu)T_{g,k,\mu} \tag{4.66}$$

$$\hat{G}[\lambda]T_{g,k,\mu} = 0 \Longrightarrow k_\nu = -\frac{1}{2}(k_{e^+(r)} - k_{e^-(r)}), \tag{4.67}$$

which by imposing 4.67 on 4.60 results in:

$$T_{g,k,\mu} = \prod_{e\in g} exp\left(\frac{i}{2}k_e\int_e dr(A_r(r) + \eta'(r))\right)\prod_{\nu\in V(g)} exp\left(i\mu_\nu\gamma K_\phi(\nu)\right). \tag{4.68}$$

Quantum Dynamics

To quantize the Hamiltonian constraint 4.59, the conventional techniques to obtain an anomaly free Hamiltonian will be used through expressing product of triads in terms of Poisson brackets of volume and connection and promoting Poisson brackets to commutators:

$$\hat{C}_\nu \propto \sum_{\nu,r,\theta,\phi} N(\nu)\epsilon^{r\theta\phi}tr(\hat{h}_r\hat{h}_\theta\hat{h}_r^{-1}\hat{h}_\theta^{-1}\hat{h}_\phi\left[\hat{h}_\phi^{-1},\hat{V}\right]). \tag{4.69}$$

The main task of realizing such an operator is done and extensively discussed in [33]. To be able to use holonomies in the above expression, one uses the parameter δ to formally expand the holonomies along the homogeneous direction. δ^2 is analogous to the size of the loop $h_\theta h_\phi h_\theta^{-1}h_\phi^{-1}$. It turns out that $\hat{C}_\nu$ can be written as the sum of three components

$$\hat{C}_\nu = \hat{C}_L + \hat{C}_C + \hat{C}_R, \tag{4.70}$$

and by acting on a vertex ν, it changes the vertex label, μ, its two adjacent vertex $\mu_{\pm}$, and the edge labels $k_{\pm}$ connecting $\mu_{\pm}$ to μ. Schematically spin network states, upon which $\hat{H}_{\nu}$ is acting can be represented

$$|\mu_{-}, k_{-}, \mu, k_{+}, \mu_{+}\rangle = \cdots \underset{\mu_{-}}{\bullet} \overset{k_{-}}{\text{———}} \underset{\mu}{\bullet} \overset{k_{+}}{\text{———}} \underset{\mu_{+}}{\bullet} \cdots \tag{4.71}$$

The action of different components of $\hat{C}_{\nu}$ on spin network states then becomes:

$$\begin{aligned}
\hat{C}_C\,|\mu_{-}, k_{-}, \mu, k_{+}, \mu_{+}\rangle \;=\; & \frac{\ell_{pl}}{2\sqrt{2}G\gamma^{3/2}\delta^2}\Big(|\mu|\Big(\sqrt{|k_{+}+k_{-}+1|}-\sqrt{|k_{+}+k_{-}-1|}\Big) \\
& \times(|\mu_{-}, k_{-}, \mu+2\delta, k_{+}, \mu_{+}\rangle + |\mu_{-}, k_{-}, \mu-2\delta, k_{+}, \mu_{+}\rangle) \\
& -2(1+2\gamma^2\delta^2(1-\Gamma_{\phi}^2))\,|\mu_{-}, k_{-}, \mu, k_{+}, \mu_{+}\rangle) \\
& -4\gamma^2\delta^2 sgn_{\delta/2}(\mu)\sqrt{|k_{+}+k_{-}|}\Gamma'_{\phi}\,|\mu_{-}, k_{-}, \mu, k_{+}, \mu_{+}\rangle\Big)\,;
\end{aligned} \tag{4.72}$$

$$\begin{aligned}
\hat{C}_R\,|\mu_{-}, k_{-}, \mu, k_{+}, \mu_{+}\rangle \;=\; & \frac{\ell_{pl}}{4\sqrt{2}G\gamma^{3/2}\delta^2} sgn_{\delta/2}(\mu)\sqrt{|k_{+}+k_{-}|} \\
\times\; & \Big(|\mu_{-}, k_{-}, \mu+\tfrac{1}{2}\delta, k_{+}+2, \mu_{+}+\tfrac{1}{2}\delta\rangle - |\mu_{-}, k_{-}, \mu+\tfrac{1}{2}\delta, k_{+}+2, \mu_{+}-\tfrac{1}{2}\delta\rangle \\
+\; & |\mu_{-}, k_{-}, \mu-\tfrac{1}{2}\delta, k_{+}+2, \mu_{+}+\tfrac{1}{2}\delta\rangle - |\mu_{-}, k_{-}, \mu-\tfrac{1}{2}\delta, k_{+}+2, \mu_{+}-\tfrac{1}{2}\delta\rangle \\
-\; & |\mu_{-}, k_{-}, \mu+\tfrac{1}{2}\delta, k_{+}-2, \mu_{+}+\tfrac{1}{2}\delta\rangle - |\mu_{-}, k_{-}, \mu+\tfrac{1}{2}\delta, k_{+}-2, \mu_{+}-\tfrac{1}{2}\delta\rangle \\
-\; & |\mu_{-}, k_{-}, \mu-\tfrac{1}{2}\delta, k_{+}-2, \mu_{+}+\tfrac{1}{2}\delta\rangle - |\mu_{-}, k_{-}, \mu-\tfrac{1}{2}\delta, k_{+}-2, \mu_{+}-\tfrac{1}{2}\delta\rangle\Big)\,,
\end{aligned} \tag{4.73}$$

where

$$sgn_{\delta/2}(\mu) \equiv \frac{1}{\delta}(|\mu+\delta/2| - |\mu-\delta/2|). \tag{4.74}$$

The action of $\hat{C}_L$ takes place in the same way except for changing k_{-} and μ_{-} instead of k_{+} and μ_{+}.

I will make use of the above equation in the next chapter to illustrate how charged black hole singularity can be avoided globally.

CHAPTER 5

Results

HAVING presented the general settings of LQG and its symmetry reduced models, I will now consider the cases of Schwarzschild and Reisner-Nordström black holes, construct their classical phase space prepared for loop quantization, and discuss the resolution of their classically irremovable singularities within the framework of LQG.

Intuitively, singularity resolution occurs as a result of fundamental discreteness of space; while in a classical continuum, divergences emerge as distance goes to zero, there is no room for divergences in quantum level since there is no zero distance below the Planck length. Put it slightly different, in the same manner quantum mechanical model for the hydrogen atom prevents a classical electron from collapsing to the nucleus, and hence energy divergence, by putting a lower bound on the spectrum of energy, quantum geometrical model put an upper bound on the classically divergent curvature components and prevent black holes from forming a singularity.

As was defined in chapter 3, a singularity is characterized by two criteria: global and local. The irremovable singularity of both Schwarzschild and charged black holes are of the curvature singularity type; the scalar curvature diverges at $r = 0$. The idea of investigating whether the local criterion still holds at quantum level or not is to construct, based on a suitable symmetry reduced model, the quantum operator corresponding to the classically divergent field and inquire whether or not its spectrum is bounded above. Globally, the absence of singularity at quantum level manifests in quantum evolution equation; the evolution of quantum gravitational field in an *internal time* parameter, generated by Hamiltonian constraint operator, must not come to a halt at singularity.

In this chapter, by explicit calculations, I illustrate the above statements more vividly for the desired black holes. During the following sections, I will proceed based on the following steps:

1. The Schwarzschild black hole

   i given the metric of space-time 3.2, the classical phase space (Ashtekar variables (A, E)) is constructed for both regions I and II;

   ii the classically divergent field is identified;

   iii *local singularity resolution*: since classical irremovable singularity lies in region I, the techniques developed in 4.1 for Kantowski-Sachs reduced model is employed to construct the quantum analogue of the divergent quantity, and its spectrum is shown to be bounded above;

   ii the internal time parameter is identified;

   iv *global singularity resolution*: the evolution equation is shown to be well behaved at classical singularity.

2. The Reisner-Nordström black hole

   i given the metric of space-time 3.15, the classical phase space is constructed for regions I, II, and III;

   ii the classically divergent field is identified;

   iii *local singularity resolution*: since classical irremovable singularity lies in region III, using the techniques developed in 4.2 for spherically symmetric models the quantum analogue of the divergent quantity is constructed, and its spectrum is shown to be bounded above;

   ii the internal time parameter is identified;

   iv *global singularity resolution*: the evolution equation is shown to be smooth while traversing the singularity.

5.1 Calculation of Ashtekar Variables for the Schwarzschild Black Hole

5.1.1 Region $r < 2m$

Inside the horizon, the Schwarzschild line element reads:

$$ds^2 = -\left(\frac{2m}{t} - 1\right)^{-1} dt^2 + \left(\frac{2m}{t} - 1\right) dr^2 + t^2 d\Omega^2. \tag{5.1}$$

In a local inertial frame, one can choose 4 orthogonal tetrad one form $e^I(x) = e^I_\mu(x)dx^\mu$:

$$g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu, \tag{5.2}$$

with inverse tetrad:

$$e^\mu_I = g^{\mu\nu} \eta_{IJ} e^J_\nu. \tag{5.3}$$

The tetrad field in turn determines its unique compatible spin connection; an anti-symmetric $\mathfrak{so}(3,1)$ valued one form $\omega^{IJ}(x) = \omega^{IJ}_\mu(x)dx^\mu$, such that:

$$de^I + \omega^I_J \wedge e^J = 0. \tag{5.4}$$

According to definition 5.2, the tetrad fields can be determined only up to a Lorentz transformation. This leaves us with an $SO(3,1)$ freedom in choosing tetrads. It is manifest in the fact that, given the metric 5.1, we are free to choose their signs and minkowski indices, which can be viewed as a sort of labeling for 4 tetrad fields. However, in order to serve as the fundamental fields for constructing the conjugate pair (A, E), a particular labeling must be chosen. The reason for such a choice will be clear below.

We choose 4 particular minkowski indices but do not fix the signs:

$$e^0 = \pm\left(\frac{2m}{t} - 1\right)^{-1/2} dt \ ; \ e^1 = \pm t sin\theta d\phi \ ; \ e^2 = \pm t d\theta \ ; \ e^3 = \pm\left(\frac{2m}{t} - 1\right)^{1/2} dr. \tag{5.5}$$

Components of the spin connection compatible with the above set of tetrads become:

$$\omega^{30} = -\omega^{03} = -\frac{m}{t^2} dr, \tag{5.6}$$

$$\omega^{20} = -\omega^{02} = \left(\frac{2m}{t} - 1\right)^{1/2} d\theta, \tag{5.7}$$

$$\omega^{10} = -\omega^{01} = \left(\frac{2m}{t} - 1\right)^{1/2} sin\theta d\phi, \tag{5.8}$$

$$\omega^{12} = -\omega^{21} = cos\theta d\phi. \tag{5.9}$$

Now the A field 2.31 can be constructed out of 5.8:

$$A^i = \frac{1}{2}\epsilon^i_{jk}\omega^{jk} + \gamma\omega^{0i}, \tag{5.10}$$

$$A^3_r = \mp\frac{\gamma m}{t^2}, \tag{5.11}$$

$$A^2_\theta = \pm\gamma\left(\frac{2m}{t} - 1\right)^{1/2}, \tag{5.12}$$

$$A^1_\phi = \pm\gamma\left(\frac{2m}{t} - 1\right)^{1/2} sin\theta, \tag{5.13}$$

$$A^3_\phi = \pm cos\theta. \tag{5.14}$$

To construct the E field, we choose a gauge in which $e^0 = 0$, as is usually done in the full theory (see section 2.1.3). This way we are in fact breaking the $SO(3,1)$ symmetry into $SO(3)$ on a hypersurface with topology $\Sigma_{in} = \mathbb{R} \times S^2$. The 3 triad fields on Σ_{in} become:

$$e^1 = \pm t sin\theta d\phi \ ; \ e^2 = \pm t d\theta \ ; \ e^3 = \pm\left(\frac{2m}{t} - 1\right)^{1/2} dr, \tag{5.15}$$

with determinant,

$$\det e = \pm t^2\left(\frac{2m}{t} - 1\right)^{1/2} sin\theta. \tag{5.16}$$

and the inverse triad:

$$e_1 = \pm\frac{1}{t sin\theta}\partial_\phi \ ; \ e_2 = \pm\frac{1}{t}\partial_\theta \ ; \ e_3 = \pm\left(\frac{2m}{t} - 1\right)^{-1/2}\partial_r. \tag{5.17}$$

3 triad fields can define their $\mathfrak{su}(2)$ valued compatible spin connection Γ^{jk}:

$$\bar\Gamma^{12} = -\bar\Gamma^{21} = cos\theta d\phi. \tag{5.18}$$

and $\bar\Gamma^i = \frac{1}{2}\epsilon^i_{jk}\bar\Gamma^{jk}$:

$$\Gamma^3_\phi = \pm cos\theta. \tag{5.19}$$

This gives rise to components of the extrinsic curvature 2.40:

$$\gamma K^i_a = A^i_a - \bar\Gamma^i_a, \tag{5.20}$$

$$K^3_r = \mp\frac{m}{t^2}, \tag{5.21}$$

$$K^2_\theta = \pm\left(\frac{2m}{t} - 1\right)^{1/2}, \tag{5.22}$$

$$K^1_\phi = \pm\left(\frac{2m}{t} - 1\right)^{1/2} sin\theta, \tag{5.23}$$

which will be used to calculate the Hamiltonian constraint. From 5.16 and 5.17, components of the gravitational electric field 4.47 turn out to be:

$$E_i^a = (\det e) e_i^a, \tag{5.24}$$

$$E_1 = \pm t \left(\frac{2m}{t} - 1\right)^{1/2} \partial_\phi, \tag{5.25}$$

$$E_2 = \pm t \left(\frac{2m}{t} - 1\right)^{1/2} sin\theta \partial_\theta, \tag{5.26}$$

$$E_3 = \pm t^2 sin\theta \partial_r. \tag{5.27}$$

Note that had we chosen other Minkowski indices for tetrad 5.5 we would not have obtained the conjugate pair(A, E) with correct indices satisfying $\{A_a^i(x), E_j^b(x')\} = \delta_j^i \delta_a^b \delta(x - x')$.

So far, the phase space variables are determined up to a sign freedom. By demanding E and A to satisfy the diffeomorphism, Gauss and Hamiltonian constraints, their signs can be fixed relative to each other. They are calculated below.

Curvature Components

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon_{jk}^i A_a^j A_b^k. \tag{5.28}$$

Non-zero components:

$$F_{r\theta}^1 = \partial_r A_\theta^1 - \partial_\theta A_r^1 + \epsilon_{32}^1 A_r^3 A_\theta^2 = -\frac{m\gamma^2}{t^2} \left(\frac{2m}{t} - 1\right)^{1/2}, \tag{5.29}$$

$$F_{r\phi}^2 = \partial_r A_\theta^1 - \partial_\theta A_r^1 + \epsilon_{32}^1 A_r^3 A_\theta^2 = -\frac{m\gamma^2}{t^2} \left(\frac{2m}{t} - 1\right)^{1/2} \sin\theta, \tag{5.30}$$

$$F_{\theta\phi}^3 = \partial_\theta A_\phi^3 - \partial_\phi A_\theta^3 + \epsilon_{21}^3 A_\theta^2 A_\phi^1 = -\gamma^2 \left(\frac{2m}{t} - 1\right)^{1/2} \cos\theta. \tag{5.31}$$

Gauss Constraint

$$G_i = \partial_a E_i^a + \epsilon_{ijk} A_a^j E^{ak} = 0. \tag{5.32}$$

Components:

$$\begin{aligned} G_1 &= \partial_\phi E_1^\phi + \epsilon_{123} A_r^2 E^{r3} + \epsilon_{132} A_r^3 E^{r2} = 0, && (5.33) \\ , G_2 &= \partial_\theta E_2^\theta + \epsilon_{213} A_r^1 E^{r3} + \epsilon_{231} A_\phi^3 E^{\phi 1}, \\ &= t \left(\frac{2m}{t} - 1\right)^{1/2} \cos\theta (sgn(E_2^\theta) + sgn(A_\phi^3 E^{\phi 1})), && (5.34) \\ , G_3 &= D_a E_3^a = \partial_r E_3^r + \epsilon_{312} A_\phi^1 E^{\phi 2} + \epsilon_{321} A_\theta^2 E^{\theta 1} = 0. && (5.35) \end{aligned}$$

Diffeomorphism Constraint

$$H_a = F^i_{ab} E^b_i = 0. \tag{5.36}$$

Components:

$$\begin{aligned} H_r &= F^2_{r\theta} E^\theta_2 + F^1_{r\phi} E^\phi_1 = 0, & (5.37)\\ H_\theta &= F^3_{\theta r} E^r_3 + F^1_{\theta\phi} E^\phi_1 = \gamma t \left(\frac{2m}{t} - 1\right) cos\theta \left\{ sgn(A^1_\phi) + sgn(A^2_\theta A^3_\phi) \right\}, & (5.38)\\ H_\phi &= F^3_{\phi r} E^r_3 + F^2_{\phi\theta} E^\theta_2 = 0. & (5.39) \end{aligned}$$

Hamiltonian Constraint

$$\begin{aligned} C &= \epsilon_{ijk} F^i_{ab} E^a_j E^b_k - 2(1+\gamma^2) K^i_{[a} K^j_{b]} E^a_i E^b_j \\ &= \epsilon_{132} F^1_{r\theta} E^r_3 E^\theta_2 + \epsilon_{231} F^2_{r\phi} E^r_3 E^\phi_1 + \epsilon_{321} F^3_{\theta\phi} E^\theta_2 E^\phi_1 \\ &\quad -2(1+\gamma^2) \left(K^3_{[r} K^2_{\theta]} E^r_3 E^\theta_2 + K^3_{[r} K^1_{\phi]} E^r_3 E^\phi_1 + K^2_{[\theta} K^1_{\phi]} E^\theta_2 E^\phi_1 \right) \\ &= t \left(\frac{2m}{t} - 1\right) sin^2\theta \left\{ sgn(E^\theta_2) + sgn(E^\phi_1) \right\}. \end{aligned} \tag{5.40}$$

For the above constraints to be zero, we must have:

$$sgn(E^\theta_2) = -sgn(E^\phi_1), \tag{5.41}$$

$$sgn(A^3_\phi) = +1, \tag{5.42}$$

$$sgn(A^1_\phi) = -sgn(A^2_\theta). \tag{5.43}$$

This leaves us with two alternatives:

$$\boxed{\begin{aligned} &\text{I} \begin{cases} A^i_a = c\tau_3 dr + b\tau_2 d\theta + (cos\theta\tau_3 - bsin\theta\tau_1) d\phi \\ E^a_i = p_c \tau_3 sin\theta \partial_r + p_b \tau_2 sin\theta \partial_\theta - p_b \tau_1 \partial_\phi, \end{cases} & (5.44)\\ &\text{II} \begin{cases} A^i_a = c\tau_3 dr - b\tau_2 d\theta + (cos\theta\tau_3 + bsin\theta\tau_1) d\phi \\ E^a_i = p_c \tau_3 sin\theta \partial_r - p_b \tau_2 sin\theta \partial_\theta + p_b \tau_1 \partial_\phi, \end{cases} & (5.45) \end{aligned}}$$

where,

$$b = \pm\gamma \left(\frac{2m}{t} - 1\right)^{1/2} \; ; \; c = \mp \frac{\gamma m}{t^2} \tag{5.46}$$

$$p_c = \pm t^2 \; ; \; p_b = t \left(\frac{2m}{t} - 1\right)^{1/2}. \tag{5.47}$$

Two alternatives correspond to the residual gauge freedom $(b, p_b) \to (-b, -p_b)$, mentioned in 4.1.

This final form shows that the phase space is 4 dimensional, (b, p_b, c, p_c), with symplectic structure:

$$\Omega = \frac{1}{8\pi\gamma G}(2db \wedge dp_b + dc \wedge dp_c). \tag{5.48}$$

5.1.2 Region $r > 2m$

Outside the horizon, the Schwarzschild line element reads:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{5.49}$$

The suitable choice for labeling 4 orthogonal frame fields turns out to be:

$$e^0 = \pm\left(1 - \frac{2m}{r}\right)^{1/2} dt \ ; \ e^1 = \pm r sin\theta d\phi \ ; \ e^2 = \pm r d\theta \ ; \ e^3 = \pm\left(1 - \frac{2m}{r}\right)^{-1/2} dr. \tag{5.50}$$

Similar to the calculations of the previous section, one finds:

The A field can be constructed using spin connections:

$$A^3_r = \mp\frac{\gamma m}{r^2}, \tag{5.51}$$

$$A^2_\theta = \pm\gamma\left(1 - \frac{2m}{r}\right)^{1/2}, \tag{5.52}$$

$$A^1_\phi = \pm\gamma\left(1 - \frac{2m}{r}\right)^{1/2} sin\theta, \tag{5.53}$$

$$A^3_\phi = \pm cos\theta. \tag{5.54}$$

Choosing $e^0 = 0$ gauge, the E fields become:

$$E_1 = \pm r\left(1 - \frac{2m}{r}\right)^{-1/2} \partial_\phi, \tag{5.55}$$

$$E_2 = \pm r\left(1 - \frac{2m}{r}\right)^{-1/2} sin\theta\partial_\theta, \tag{5.56}$$

$$E_3 = \pm r^2 sin\theta\partial_r. \tag{5.57}$$

Demanding the constraints to be held, we obtain two alternatives:

$$\boxed{\begin{array}{l} \tilde{I}\begin{cases} \tilde{A}^i_a = \tilde{c}\tau_3 dr + \tilde{b}\tau_2 d\theta + (cos\theta\tau_3 - \tilde{b}sin\theta\tau_1)d\phi \\ \tilde{E}^a_i = \tilde{p}_c\tau_3 sin\theta\partial_r + \tilde{p}_b\tau_2 sin\theta\partial_\theta - \tilde{p}_b\tau_1\partial_\phi, \end{cases} \quad (5.58) \\ \tilde{II}\begin{cases} \tilde{A}^i_a = \tilde{c}\tau_3 dr - \tilde{b}\tau_2 d\theta + (cos\theta\tau_3 + \tilde{b}sin\theta\tau_1)d\phi \\ \tilde{E}^a_i = \tilde{p}_c\tau_3 sin\theta\partial_r - \tilde{p}_b\tau_2 sin\theta\partial_\theta + \tilde{p}_b\tau_1\partial_\phi, \end{cases} \quad (5.59) \end{array}}$$

where,

$$\tilde{b} = \pm\gamma\left(1 - \frac{2m}{r}\right)^{1/2} \ ; \ \tilde{c} = \mp\frac{\gamma m}{r^2} \tag{5.60}$$

$$\tilde{p}_c = \pm r^2 \ ; \ \tilde{p}_b = r\left(1 - \frac{2m}{r}\right)^{-1/2}. \tag{5.61}$$

5.2 Singularity Resolution of Schwarzschild Black Holes

5.2.1 Local Singularity Resolution

Recall the Kretschmann scalar curvature of the Schwarzschild black hole 3.26:

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}. \tag{5.62}$$

The irremovable singularity occurs at $r = 0$ through appearing $\frac{1}{r}$ in such diverging curvature components. It lies in the Kantowski-Sachs region I of the Schwarzschild space-time where space and time interchange their role. Our goal is to show that the quantum operator corresponding to $\frac{1}{r}(=\frac{1}{t})$, on the other hand, exhibits a spectrum which is bounded above. To construct such an operator, we need the classical function to be expressed suitably in terms of well-defined operators in the reduced theory. Comparing the gravitational electric field E, 5.44 and 5.45, and the general form 4.8, one finds that $\frac{sgn(p_c)}{\sqrt{|p_c|}} = \frac{1}{t}$ is a candidate to serve as the desired classical function. However, one cannot naively replace the inverse square root of triad with it's operator analouge since, as can be seen in 4.24, it has zero eigenvalue. Nevertheless, being expressed as a Poisson bracket of functions having well-defined operators, it's corresponding quantum operator can be realized by replacing Poisson bracket with ($\frac{-i}{\hbar}$ times) commutator.

Calculating following quantity on the classical phase space

$$\begin{aligned}\left\{c, \sqrt{|p_c|}\right\} &= 2\pi\gamma G\left(\frac{\partial c}{\partial c}\frac{\partial\sqrt{|p_c|}}{\partial p_c} - \frac{\partial c}{\partial p_c}\frac{\partial\sqrt{|p_c|}}{\partial c}\right)\\ &= 2\pi\gamma G\left(\frac{sgn(p_c)}{\sqrt{|p_c|}}\right),\end{aligned}$$

presents the desired divergent function $\mathcal{R}$

$$\mathcal{R} \equiv \frac{1}{2\pi\gamma G}\left\{c, \sqrt{|p_c|}\right\} = \frac{sgn(p_c)}{\sqrt{|p_c|}} = \frac{1}{t}. \tag{5.63}$$

As was discussed in 2.2.1, the connection components are not quantized directly in LQG. Instead, their holonomy along curves in 3-space, 4.13, has an operator analogue

$$h_r^{(\tau)} = \exp\int_0^\tau drc\tau_3 = \cos\frac{\tau c}{2} + 2\tau_3\sin\frac{\tau c}{2}. \tag{5.64}$$

Following the standard methods presented in 2.2.1, expand the holonomy

$$h_r^{(\tau)} = 1 + \epsilon\int_0^\tau drc\tau_3 + \mathcal{O}(\epsilon^2), \tag{5.65}$$

and write $\mathcal{R}$ as

$$\mathcal{R} = \frac{1}{2\pi\gamma G}tr\left(\tau_3 h_r^{(\tau)}\left\{h_r^{(\tau)-1}, \sqrt{|p_c|}\right\}\right). \tag{5.66}$$

Now, quantization is done in a strightforward maner;

1. holonomy along the $\mathbb{R}$ direction of Σ_{in}, $h_r^{(\tau)}$, is replaced by its quantum operator $\hat{h}_r^{(\tau)}$ 4.19;

2. $\sqrt{|p_c|}$ is replaced by the operator $\sqrt{|\hat{p}_c|}$: this is the operator whose eigenvalues on $|\mu, \nu\rangle$ are square root of (absolute value of) eigenvalues of $\hat{p}_c$ 4.24;

3. the Poisson bracket $\left\{h_r^{(\tau)-1}, \sqrt{|p_c|}\right\}$ will be replaced by the commutator $\frac{i}{\hbar}\left[h_r^{(\tau)-1}, \sqrt{|p_c|}\right]$.

$$\hat{\mathcal{R}} = \frac{1}{2\pi\gamma\ell_{Pl}^2} tr\left(\tau_3 \hat{h}_r^{(\tau)}\left[\hat{h}_r^{(\tau)-1}, \sqrt{|\hat{p}_c|}\right]\right). \tag{5.67}$$

with $\ell_{Pl} = \sqrt{G\hbar}$ being the Planck length 1.1.

We now investigate the action of $\hat{\mathcal{R}}$ on a general state $|\tau, \mu\rangle$. Using 5.64 we have

$$\begin{aligned}
\hat{h}_r^{(\tau)}\left[\hat{h}_r^{(\tau)-1}, \sqrt{|\hat{p}_c|}\right] &= \left(\cos\frac{\tau c}{2} + 2\tau_3 \sin\frac{\tau c}{2}\right)\left[\cos\frac{\tau c}{2} - 2\tau_3 \sin\frac{\tau c}{2}, \sqrt{|\hat{p}_c|}\right] \\
&= \sqrt{|\hat{p}_c|} - cos(\frac{\tau c}{2})\sqrt{|\hat{p}_c|}cos(\frac{\tau c}{2}) - sin(\frac{\tau c}{2})\sqrt{|\hat{p}_c|}sin(\frac{\tau c}{2}) \\
&\quad + 2\tau_3\left(cos(\frac{\tau c}{2})\sqrt{|\hat{p}_c|}sin(\frac{\tau c}{2}) - sin(\frac{\tau c}{2})\sqrt{|\hat{p}_c|}cos(\frac{\tau c}{2})\right).
\end{aligned} \tag{5.68}$$

Therefore,

$$\begin{aligned}
\hat{\mathcal{R}} &= \frac{1}{2\pi\gamma\ell_{Pl}^2} tr\left(\tau_3 \hat{h}_r^{(\tau)}\left[\hat{h}_r^{(\tau)-1}, \sqrt{|\hat{p}_c|}\right]\right) \\
&= \frac{1}{2\pi\gamma\ell_{Pl}^2}\left(cos(\frac{\tau c}{2})\sqrt{|\hat{p}_c|}sin(\frac{\tau c}{2}) - sin(\frac{\tau c}{2})\sqrt{|\hat{p}_c|}cos(\frac{\tau c}{2})\right)
\end{aligned} \tag{5.69}$$

To see the action of $\hat{\mathcal{R}}$, with $\tau = 1$, on $|\tau, \mu\rangle$ we need the following calculations:

$$\begin{aligned}
&\left(sin\frac{c}{2}\sqrt{|\hat{p}_c|}cos\frac{c}{2}\right)e^{\frac{i}{2}(b\mu+c\tau)} \\
=\ &\left(sin\frac{c}{2}\sqrt{|\hat{p}_c|}\right)cos\frac{c}{2}e^{\frac{i}{2}(b\mu+c\tau)} \\
=\ &\sqrt{\gamma}\ell_{Pl}\left(sin\frac{c}{2}\sqrt{|\hat{p}_c|}\right)\frac{1}{2}\left[e^{\frac{i}{2}(b\mu+c(\tau+1))} + e^{-\frac{i}{2}(b\mu+c(\tau-1))}\right] \\
=\ &\sqrt{\gamma}\ell_{Pl}\left(sin\frac{c}{2}\right)\frac{1}{2}\left[\sqrt{|\tau+1|}e^{\frac{i}{2}(b\mu+c(\tau+1))} + \sqrt{|\tau-1|}e^{-\frac{i}{2}(b\mu+c(\tau-1))}\right] \\
=\ &\sqrt{\gamma}\ell_{Pl}\frac{1}{2}\left[\sqrt{|\tau+1|}sin\frac{c}{2}e^{\frac{i}{2}(b\mu+c(\tau+1))} + \sqrt{|\tau-1|}sin\frac{c}{2}e^{-\frac{i}{2}(b\mu+c(\tau-1))}\right] \\
=\ &\sqrt{\gamma}\ell_{Pl}\frac{1}{2}\left[\sqrt{|\tau+1|}\frac{i}{2}\left(e^{\frac{i}{2}(b\mu+c\tau)} - e^{\frac{i}{2}(b\mu+c(\tau+2))}\right) + \sqrt{|\tau-1|}\frac{i}{2}\left(e^{\frac{i}{2}(b\mu+c(\tau-2))} - e^{\frac{i}{2}(b\mu+c\tau)}\right)\right] \\
=\ &\sqrt{\gamma}\ell_{Pl}\frac{i}{4}\left[(\sqrt{|\tau+1|} - \sqrt{\tau-1})e^{\frac{i}{2}(b\mu+c\tau)} - \sqrt{|\tau+1|}e^{\frac{i}{2}(b\mu+c(\tau+2))} + \sqrt{|\tau-1|}e^{\frac{i}{2}(b\mu+c(\tau-2))}\right].
\end{aligned} \tag{5.70}$$

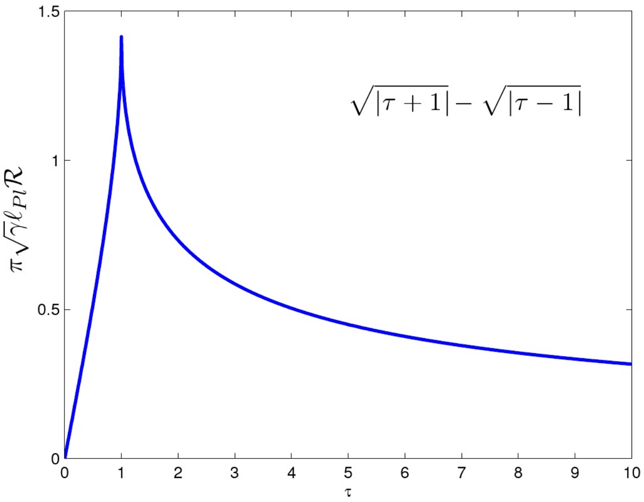

Figure 5.1: The upper bounded spectrum of $\frac{1}{r}$ for Schwarzschild black hole

Similarly,

$$\begin{aligned}&\left(cos\frac{c}{2}\sqrt{|\hat{p}_c|}sin\frac{c}{2}\right)e^{\frac{i}{2}(b\mu+c\tau)}\\ =\;&\sqrt{\gamma}\ell_{Pl}\frac{i}{4}\left[-(\sqrt{|\tau+1|}-\sqrt{|\tau-1|})e^{\frac{i}{2}(b\mu+c\tau)}-\sqrt{|\tau+1|}e^{\frac{i}{2}(b\mu+c(\tau+2))}+\sqrt{|\tau-1|}e^{\frac{i}{2}(b\mu+c(\tau-2))}\right].\end{aligned} \tag{5.71}$$

From 5.69, 5.70, and 5.71 we then have:

$$\hat{\mathcal{R}}\,|\tau,\mu\rangle=\frac{1}{2\pi\sqrt{\gamma}\ell_{Pl}}(\sqrt{|\tau+1|}-\sqrt{|\tau-1|})\,|\tau,\mu\rangle \tag{5.72}$$

As can be seen in the figure 5.1, such an operator has a bounded spectrum with maximum value of $\frac{1}{\sqrt{2}\pi\sqrt{\gamma}\ell_{Pl}}$. This means that the scalar curvature 3.26, which is classically divergent, at quantum level has a maximum value of:

$$(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})_{max}=\left(\frac{48M^2}{r^6}\right)_{max}=\frac{6M^2}{\gamma^3\pi^6\ell_{Pl}^6}. \tag{5.73}$$

Furetheremore, the operator 5.72 is diagonal on spin network states and hence commute with all operators at the kinematical Hilbert space of the spherically symmetric quantum geometry.

These facts show that the classical singularity of Schwarzschild black hole are locally avoided in the kinematical level of a truncation of LQG to sphericall symmetry. The global criterion, originally drived in [31], is reviewed in the next section.

5.2.2 Global Singularity Resolution

To study the dynamics, we must encode time evolution coded in Hamiltonian constraint. To that end, one must assign an "internal time" parameter to the theory. This is usually realized via coupling a scalar field to the theory. However, since in the region I inside the horizon, space and time interchange their role, a look at components of the gravitational electric field 5.44 and 5.45 reveals that the monotonic function $P_c = \pm t^2$ can be considered as the desired internal time parameter. At quantum level, thus, the eigenvalues of $\hat{p}_c$ play the role of discrete internal time. The question of whether or not the classical irremovable singularity is avoided globally in quantum level , therefore, can be answered by enquiring the Hamiltonian constraint acting on a general state $|\mu, \nu\rangle$ and see whether or not the evolution stops at $\tau = 0$. This issue has been studied in [31] which leads to the following.

From equation 4.31 for the Hamiltonian constraint of a Kantowski-Sachs reduced model, one can form the symmetric Hamiltonian constraint $\hat{C}^{(\delta)}_{symm} = \frac{1}{2}\left(\hat{C}^{(\delta)} + \hat{C}^{(\delta)\dagger}\right)$ and look at its action on $|\tau, \mu\rangle$:

$$\begin{aligned} \hat{C}^{(\delta)}_{symm}\,|\tau,\mu\rangle \;=\;& 2\delta\left(\sqrt{|\tau+2\delta|}+\sqrt{|\tau|}\right)\left(|\tau+2\delta,\mu+2\delta\rangle-(|\tau+2\delta,\mu-2\delta\rangle\right) \\ & +\left(\sqrt{|\tau+\delta|}+\sqrt{|\tau-\delta|}\right)\left((\mu+2\delta)\,|\tau,\mu+4\delta\rangle\right. \\ -\;& (1+2\gamma^2\delta^2)\,|\tau,\mu\rangle-(\mu-2\delta)\,|\tau,\mu-4\delta\rangle) \\ & +2\delta\left(\sqrt{|\tau-2\delta|}+\sqrt{|\tau|}\right)\left(|\tau-2\delta,\mu-2\delta\rangle-|\tau-2\delta,\mu+2\delta\rangle\right). \end{aligned} \tag{5.74}$$

As can be seen in the above equation, quantum evolution is regular through traversing singularity. In other words, if we start at some finite positive τ and carry out a backward evolution using 5.74. The backward evolution continues only as long as the coefficient of the states at $\tau - 2\delta$, the other side of singularity, is non-zero. From the above equation, this coefficient is $\left(\sqrt{|\tau-2\delta|}+\sqrt{|\tau|}\right)$ which never vanishes. Therefore, the backward evolution remais well-defined and determines the wave function not only for $\tau > 0$ but also for $\tau \leq 0$. In this sense, the classical singularity can be bypassed by the quantum evolution.

5.3 Calculation of Ashtekar Variables for a Charged Black Holes

5.3.1 Region II

According to the devision 3.17, in region II the metric of space-time takes the form

$$ds^2 = -\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{-1} dt^2 + \left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right) dr^2 + t^2 d\Omega^2. \tag{5.75}$$

The suitable choice for labeling 4 orthogonal frame fields turns out to be:

$$e^0 = \pm\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{-1/2} dt \ ; \ e^1 = \pm t sin\theta d\phi \ ; \ e^2 = \pm t d\theta \ ; \ e^3 = \pm\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} dr, \tag{5.76}$$

which leads to compatible spin connection components:

$$\omega^{30} = -\omega^{03} = \left(\frac{Q^2}{t^3} - \frac{m}{t^2}\right) dr, \tag{5.77}$$

$$\omega^{20} = -\omega^{02} = \left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} d\theta, \tag{5.78}$$

$$\omega^{10} = -\omega^{01} = \left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} sin\theta d\phi, \tag{5.79}$$

$$\omega^{12} = -\omega^{21} = cos\theta d\phi. \tag{5.80}$$

The A field can be constructed using spin connections:

$$A^3 = \mp\gamma\left(\frac{Q^2}{t^3} - \frac{m}{t^2}\right) dr, \tag{5.81}$$

$$A^2 = \pm\gamma\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} d\theta, \tag{5.82}$$

$$A^1 = \pm\gamma\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} sin\theta d\phi, \tag{5.83}$$

$$A^3 = \pm cos\theta d\phi. \tag{5.84}$$

To construct the E field, we break the $SO(3,1)$ symmetry into $SO(3)$ on a hypersurface with topology $\Sigma_{II} = R \times S^2$ by choosing a gauge in which $e^0 = 0$.

The 3 triad fields are then:

$$e^1 = \pm t sin\theta d\phi \ ; \ e^2 = \pm t d\theta \ ; \ e^3 = \pm\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} dr, \tag{5.85}$$

with determinant

$$\det e = t^2 sin\theta \left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2}, \tag{5.86}$$

and inverse triad

$$e_1 = \pm\frac{1}{tsin\theta}\partial_\phi \ ; \ e_2 = \pm\frac{1}{t}\partial_\theta \ ; \ e_3 = \pm\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{-1/2}\partial_r, \tag{5.87}$$

And the E fields become:

$$E_1 = \pm t\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2}\partial_\phi, \tag{5.88}$$

$$E_2 = \pm t\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} sin\theta\partial_\theta, \tag{5.89}$$

$$E_3 = \pm t^2 sin\theta\partial_r. \tag{5.90}$$

Similar to the region I of the Schwarzschild case, the 3 triad fields 5.85 define their compatible spin connection, $\bar{\Gamma}^{ij} \wedge e^j + de^i = 0$

$$\bar{\Gamma}^{12} = -\bar{\Gamma}^{21} = cos\theta d\phi, \tag{5.91}$$

$$\bar{\Gamma}^3 = \frac{1}{2}\left(\epsilon^{312}\bar{\Gamma}^{12} + \epsilon^{321}\bar{\Gamma}^{21}\right) = cos\theta d\phi. \tag{5.92}$$

Extrinsic curvature is related to A via $\gamma K = A - \bar{\Gamma}$

$$K_r^3 = \frac{1}{\gamma}A_r^3 = \mp\left(\frac{Q^2}{t^3} - \frac{m}{t^2}\right)dr, \tag{5.93}$$

$$K_\theta^2 = \frac{1}{\gamma}A_\theta^2 = \pm\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2}, \tag{5.94}$$

$$K_\phi^1 = \frac{1}{\gamma}A_\phi^1 = \pm\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} sin\theta. \tag{5.95}$$

To fix the relative signs of the (A, E) fields we damand them to satisfy the Gauss, diffeomorphism and Hamiltonian constraints calculated below.

Curvature Components

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon_{jk}^i A_a^j A_b^k. \tag{5.96}$$

Non-zero components:

$$F_{r\theta}^1 = \partial_r A_\theta^1 - \partial_\theta A_r^1 + \epsilon_{32}^1 A_r^3 A_\theta^2 = \gamma^2\left(\frac{Q^2}{t^3} - \frac{m}{t^2}\right)\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2}, \tag{5.97}$$

$$F^2_{r\phi} = \partial_r A^2_\phi - \partial_\phi A^2_r + \epsilon^2_{31} A^3_r A^1_\phi = -\gamma^2 \left(\frac{Q^2}{t^3} - \frac{m}{t^2}\right)\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} \sin\theta, \tag{5.98}$$

$$F^3_{\theta\phi} = \partial_\theta A^3_\phi - \partial_\phi A^3_\theta + \epsilon^3_{21} A^2_\theta A^1_\phi = -\gamma^2 \left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} \pm \cos\theta. \tag{5.99}$$

Gauss Constraint

$$G_i = \partial_a E^a_i + \epsilon_{ijk} A^j_a E^{ak} = 0. \tag{5.100}$$

Components:

$$\begin{aligned}
G_1 &= \partial_\phi E^\phi_1 + \epsilon_{123} A^2_r E^{r3} + \epsilon_{132} A^3_r E^{r2} = 0, && (5.101)\\
G_2 &= \partial_\theta E^\theta_2 + \epsilon_{213} A^1_r E^{r3} + \epsilon_{231} A^3_\phi E^{\phi 1}, \\
&= t\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} \cos\theta(sgn(E^\theta_2) + sgn(A^3_\phi E^{\phi 1})), && (5.102)\\
G_3 &= D_a E^a_3 = \partial_r E^r_3 + \epsilon_{312} A^1_\phi E^{\phi 2} + \epsilon_{321} A^2_\theta E^{\theta 1} = 0. && (5.103)
\end{aligned}$$

Diffeomorphism Constraint

$$H_a = F^i_{ab} E^b_i = 0. \tag{5.104}$$

Components:

$$\begin{aligned}
H_r &= F^2_{r\theta} E^\theta_2 + F^1_{r\phi} E^\phi_1 = 0, && (5.105)\\
H_\theta &= F^3_{\theta r} E^r_3 + F^1_{\theta\phi} E^\phi_1 = \gamma t \left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right) cos\theta \left\{sgn(A^1_\phi) + sgn(A^2_\theta A^3_\phi)\right\}, && (5.106)\\
H_\phi &= F^3_{\phi r} E^r_3 + F^2_{\phi\theta} E^\theta_2 = 0. && (5.107)
\end{aligned}$$

Hamiltonian Constraint

$$\begin{aligned}
C &= \epsilon_{ijk} F^i_{ab} E^a_j E^b_k - 2(1+\gamma^2) K^i_{[a} K^j_{b]} E^a_i E^b_j \\
&= \epsilon_{132} F^1_{r\theta} E^r_3 E^\theta_2 + \epsilon_{231} F^2_{r\phi} E^r_3 E^\phi_1 + \epsilon_{321} F^3_{\theta\phi} E^\theta_2 E^\phi_1 \\
&\quad -2(1+\gamma^2)\left(K^3_{[r} K^2_{\theta]} E^r_3 E^\theta_2 + K^3_{[r} K^1_{\phi]} E^r_3 E^\phi_1 + K^2_{[\theta} K^1_{\phi]} E^\theta_2 E^\phi_1\right) \\
&= t\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right) sin^2\theta \left\{sgn(E^\theta_2) + sgn(E^\phi_1)\right\}. && (5.108)
\end{aligned}$$

These constraints are zero if:

$$sgn(E_2^\theta) = -sgn(E_1^\phi), \tag{5.109}$$

$$sgn(A_\phi^3) = +1, \tag{5.110}$$

$$sgn(A_\phi^1) = -sgn(A_\theta^2). \tag{5.111}$$

Therefore, there are two alternatives for phase space variables:

$$\mathrm{I}\begin{cases} A_a^i = c\tau_3 dr + b\tau_2 d\theta + (cos\theta\tau_3 - bsin\theta\tau_1)d\phi \\ E_i^a = p_c\tau_3 sin\theta\partial_r + p_b\tau_2 sin\theta\partial_\theta - p_b\tau_1\partial_\phi, \end{cases} \tag{5.112}$$

$$\mathrm{II}\begin{cases} A_a^i = c\tau_3 dr - b\tau_2 d\theta + (cos\theta\tau_3 + bsin\theta\tau_1)d\phi \\ E_i^a = p_c\tau_3 sin\theta\partial_r - p_b\tau_2 sin\theta\partial_\theta + p_b\tau_1\partial_\phi, \end{cases} \tag{5.113}$$

where

$$b = \pm\gamma\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2} \; ; \, c = \mp\gamma\left(\frac{Q^2}{t^3} - \frac{m}{t^2}\right)t^2 \tag{5.114}$$

$$p_c = \pm t^2 \; ; \, p_b = t\left(\frac{2m}{t} - \frac{Q^2}{t^2} - 1\right)^{1/2}. \tag{5.115}$$

Two alternatives correspond to the residual gauge freedom $(b, p_b) \rightarrow (-b, -p_b)$.

Threfore, the 4 dimensional phase space consists of (b, p_b, c, p_c).

5.3.2 Regions I and III

The line element in regions I and II reads:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2. \tag{5.116}$$

The suitable choice for labeling 4 orthogonal frame fields turns out to be:

$$e^0 = \pm\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{1/2}dt \; ; \, e^1 = \pm r sin\theta d\phi \; ; \, e^2 = \pm r d\theta \; ; \, e^3 = \pm\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1/2}dr, \tag{5.117}$$

which leads to compatible spin connection components:

$$\omega^{30} = -\omega^{03} = -\frac{m}{r^2}dt \; ; \, \omega^{20} = -\omega^{02} = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{1/2}d\theta, \tag{5.118}$$

$$\omega^{10} = -\omega^{01} = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{1/2}sin\theta d\phi \; ; \, \omega^{12} = -\omega^{21} = cos\theta d\phi. \tag{5.119}$$

The A field can be constructed using spin connections:

$$A_r^3 = \mp\frac{\gamma m}{r^2}, \tag{5.120}$$

$$A^2_\theta = \pm\gamma \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{1/2}, \tag{5.121}$$

$$A^1_\phi = \pm\gamma \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{1/2} sin\theta, \tag{5.122}$$

$$A^3_\phi = \pm cos\theta. \tag{5.123}$$

Choosing $e^0 = 0$ gauge, the E fields become:

$$E_1 = \pm r \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1/2} \partial_\phi, \tag{5.124}$$

$$E_2 = \pm r \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1/2} sin\theta\partial_\theta, \tag{5.125}$$

$$E_3 = \pm r^2 sin\theta\partial_r. \tag{5.126}$$

Demanding the constraints to be held, we obtain two alternatives:

$$\boxed{\begin{aligned} &\tilde{I}\begin{cases} \tilde{A}^i_a = \tilde{c}\tau_3 dr + \tilde{b}\tau_2 d\theta + (cos\theta\tau_3 - \tilde{b}sin\theta\tau_1)d\phi \\ \tilde{E}^a_i = \tilde{p}_c\tau_3 sin\theta\partial_r + \tilde{p}_b\tau_2 sin\theta\partial_\theta - \tilde{p}_b\tau_1\partial_\phi, \end{cases} && (5.127) \\ &\tilde{II}\begin{cases} \tilde{A}^i_a = \tilde{c}\tau_3 dr - \tilde{b}\tau_2 d\theta + (cos\theta\tau_3 + \tilde{b}sin\theta\tau_1)d\phi \\ \tilde{E}^a_i = \tilde{p}_c\tau_3 sin\theta\partial_r - \tilde{p}_b\tau_2 sin\theta\partial_\theta + \tilde{p}_b\tau_1\partial_\phi, \end{cases} && (5.128) \end{aligned}}$$

where

$$\tilde{b} = \pm\gamma \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{1/2} \; ; \tilde{c} = \mp\frac{\gamma m}{r^2} \tag{5.129}$$

$$\tilde{p}_c = \pm r^2 \; ; \tilde{p}_b = r \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1/2}. \tag{5.130}$$

5.4 Singularity Resolution of Charged Black Holes

5.4.1 Local Singularity Resolution

A look at Kretschmann scalar curvature of the Reissner-Nordström black hole 3.27

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48M^2r^2 - 96MQ^2r + 56Q^2}{r^8}, \tag{5.131}$$

reveals that the irremovable curvature singularity of classical Reissner-Nordström black hole occurs at $r = 0$. It lies in the spherically symmetric region III of the Reissner-Nordström space-time. In this section, we will see how the quantum operator corresponding to $\frac{1}{r}$, on the contrary, has an upper bounded spectrum. Comparing the gravitational electric field E, 5.127, 5.128, and the general form 4.34, $\frac{sgn(E^r)}{\sqrt{|E^r(r)|}} = \frac{1}{r}$ turns out to be the suitable choice to serve as the desired classical function. It can be expressed as the Poisson bracket of well-defined operators in the spherically symmeteric reduced model of LQG 4.2.

Consider the following quantity on the classical phase space

$$\begin{aligned} \left\{A_r(r), \sqrt{|E^r(r)|}\right\} &= \frac{1}{2\pi\gamma G}\sum_{i=1}^{3}\frac{\partial A_r(r)}{\partial A_i(r)}\frac{\partial\sqrt{|E^r(r)|}}{\partial E^i(r)} - \frac{\partial A_r(r)}{\partial E^i(r)}\frac{\partial\sqrt{|E^r(r)|}}{\partial A_i(r)} \\ &= \frac{1}{2\pi\gamma G}\frac{\partial A_r(r)}{\partial A_r(r)}\frac{\partial\sqrt{|E^r(r)|}}{\partial E^r(r)} \\ &= \frac{1}{2\pi\gamma G}\frac{sgn(E^r)}{\sqrt{|E^r(r)|}}. \end{aligned} \tag{5.132}$$

Therefore, $\mathcal{R}$ is a suitable classically divergent quantity to be quantized:

$$\mathcal{R} \equiv \frac{sgn(E^r)}{\sqrt{|E^r(r)|}} = \frac{1}{2\pi\gamma G}\left\{A_r(r), \sqrt{|E^r(r)|}\right\}. \tag{5.133}$$

Since $\hat{E}^r(r)$ is diagonal on spin network states 4.60, $\sqrt{|\hat{E}^r(r)|}$ can be defined as a diagonal operator whose eigenvalues are square roots of (absolute value of) those of $\hat{E}^r(r)$.

Along the inhomogeneous direction r, we can expand holonomy $h_r(r)$ as:

$$h_r = exp\left(i\int_e drA_r(r)\right) \approx 1 + i\epsilon A_r(\nu), \tag{5.134}$$

where $\epsilon = \nu_+ - \nu$ is the coordinate distance between two vertices ν and ν_+ connected by the edge e. This enables us to, as is usuall in LQG and was used in 2.75, express $\mathcal{R}$ to the order of ϵ^2 as:

$$\mathcal{R} = \frac{1}{2\pi\gamma G}tr\left(\tau_3 h_r\{h_r^{-1}, \sqrt{|E^r(r)|}\}\right), \tag{5.135}$$

and consequently it's coresponding operator as:

$$\hat{\mathcal{R}} = \frac{1}{2\pi\gamma\ell_{Pl}^2}tr\left(\tau_3\hat{h}_r\left[\hat{h}_r^{-1}, \sqrt{|\hat{E}^r(r)|}\right]\right). \tag{5.136}$$

To calculate the above expression, we write holonomy of $A_r(r)$ along the paths in the r direction as:

$$h_r = exp\left(i\int_e drA_r(r)\right) = cos(\frac{1}{2}\int A_r) + 2\tau_3 sin(\frac{1}{2}\int A_r), \tag{5.137}$$

and therefore,

$$\begin{aligned}\hat{h}_r\left[\hat{h}_r^{-1}, \sqrt{|\hat{E}^r(r)|}\right] &= \sqrt{|\hat{E}^r(r)|} - cos(\frac{1}{2}\int A_r)\sqrt{|\hat{E}^r(r)|}cos(\frac{1}{2}\int A_r)\\ &-sin(\frac{1}{2}\int A_r)\sqrt{|\hat{E}^r(r)|}sin(\frac{1}{2}\int A_r)\\ &+2\tau_3\left(cos(\frac{1}{2}\int A_r)\sqrt{|\hat{E}^r(r)|}sin(\frac{1}{2}\int A_r)\right.\\ &\left.-sin(\frac{1}{2}\int A_r)\sqrt{|\hat{E}^r(r)|}cos(\frac{1}{2}\int A_r)\right).\end{aligned}$$

This leads to

$$\hat{\mathcal{R}} = \frac{1}{2\pi\gamma\ell_{Pl}^2}\left(cos(\frac{1}{2}\int A_r)\sqrt{|\hat{E}^r(r)|}sin(\frac{1}{2}\int A_r) - sin(\frac{1}{2}\int A_r)\sqrt{|\hat{E}^r(r)|}cos(\frac{1}{2}\int A_r)\right). \tag{5.138}$$

To investigate it's action on spin network states, one can use identities expressing $\sin r$ and $\cos r$ in terms of exponentials and write:

$$\begin{aligned}\hat{\mathcal{R}}T_{g,k,\mu} &= \frac{1}{2\pi\gamma\ell_{Pl}^2}\left((e^{\frac{i}{2}\int A_r} + e^{-\frac{i}{2}\int A_r})\sqrt{|\hat{E}^r(r)|}(e^{\frac{i}{2}\int A_r} - e^{-\frac{i}{2}\int A_r})\right.\\ &\left.-(e^{\frac{i}{2}\int A_r} - e^{-\frac{i}{2}\int A_r})\sqrt{|\hat{E}^r(r)|}(e^{\frac{i}{2}\int A_r} + e^{-\frac{i}{2}\int A_r})\right)\\ &\times\prod_{e,\nu} e^{\left(\frac{ik_e}{2}\int_e(A_r+\eta')\right)}e^{\left(i\mu_\nu\gamma K_\phi\right)},\end{aligned}$$

which gives rise to the spectrum

$$\hat{\mathcal{R}}T_{gk\mu} = \frac{1}{2\pi\sqrt{\gamma}\ell_{Pl}}\left(\sqrt{\frac{1}{2}|k_{e^+(r)} + k_{e^-(r)} + 2|} - \sqrt{\frac{1}{2}|k_{e^+(r)} + k_{e^-(r)} - 2|}\right)T_{gk\mu}. \tag{5.139}$$

Such an operator is diagonal on spin network states and hence commute with all operators at the kinematical Hilbert space of the spherically symmetric quantum geometry. Moreover, it has a bounded spectrum (see figure 5.2) with maximum value of $\frac{1}{\sqrt{2}\pi\sqrt{\gamma}\ell_{Pl}}$. Thus, the scalar curvature 5.131, which is classically divergent, at quantum level has a maximum value of:

$$\begin{aligned}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})_{max} &= \left(\frac{48M^2r^2 - 96MQ^2r + 56Q^2}{r^8}\right)_{max}\\ &= \frac{1}{\gamma^3\pi^6\ell_{Pl}^6}\left(6M^2 - \frac{96MQ^2}{\sqrt{2}\pi\sqrt{\gamma}\ell_{Pl}} + \frac{28Q^2}{\pi^2\gamma\ell_{Pl}^2}\right).\end{aligned} \tag{5.140}$$

This reflects the local singularity avoidance of charged black holes. Note that the above value reduces to 5.73 at the limit $Q \to 0$.

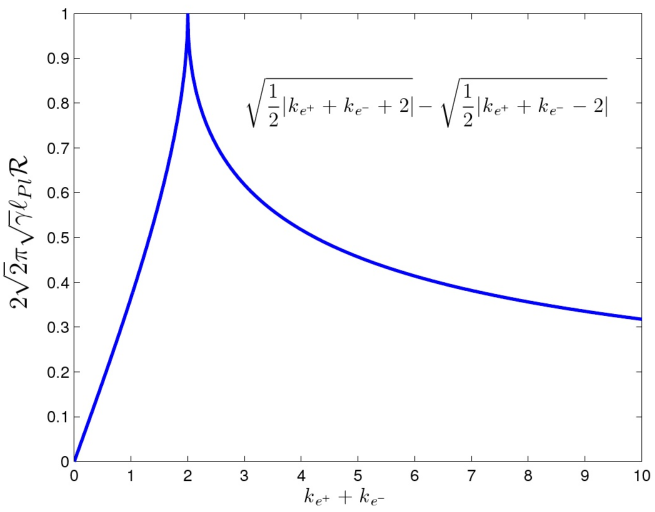

Figure 5.2: The upper bounded spectrum of $\frac{1}{r}$ for a charged black hole

5.4.2 Global Singularity Resolution

The global condition for singularity avoidance in quantum level, is studied by considering the quantum Hamiltonian constraint. As for the case of Schwarzschild black hole, we therefore need a notion of internal time to evolve a wave function backward in it and see whether the evolution can pass the singularity or not. In section 5.2.2, we argued that $\pm t^2$ can be considered as an internal time for Kantowski-Sachs models. Recall that the region II in the Reissner-Nordstorm space-time also is of the Kantowski-Sacks type, and hence, $p_c = \pm t^2$ 5.115 can serve for this purpose. However, the singularity of a charged black hole lies in region III. It seems resonable, for the sake of continuity, to take the monotonic function $E^r(r) = \pm r^2 (= \pm t^2)$ as the internal time in this region and the eigenvalues of its quantum operator as the discrete time parameter in quantum level. Our task is, then, to study the Hamiltonian constraint acting on an arbitrary spin network state and see if the coefficients of such states vanishes while evloving through singularity.

It can be seen from 4.72 and 4.73, the coefficients of states $|\mu_-, k_-, \mu, k_+, \mu_+\rangle$ when acted by Hamiltonian constraint, are not non-vanishing. However, as is discussed in [29, 32], among the various options to choose an ordering for Hamiltonian constraint, the symmetric one, $\hat{C} = \hat{C}_\nu^\dagger + \hat{C}_\nu$, turns out to be the one with non-vanishing coefficients and can serve as the criteria of non-singular evolution equation.

CHAPTER 6

Conclusion and Outlook

In search for constructing a theory describing nature, simplified models have played crucial roles to make the bridge between phenomenological results and the fundamental rigorous theories. The semi-classical ad hoc postulates of Bohr model for hydrogen atom served as such a bridge between observed atomic spectral lines and the fundamental theory of quantum mechanics formulated later by Heisenberg and Schrödinger in 1926. The situation for the present day theoretical physics is much worse; in the absence of any observational evidence for quantum gravity, existence of a consistence picture to describe some ill-defined aspects of the classical theory can play the role of phenomenological results, and simplified models to resolve such ill-defined aspects can bridge between expected results and the tentative fundamental theory.

One of such problematic aspects of the classical theory of gravity is the existence of singularities, where theory results in infinities. Such singular behavior of classical general relativity is widely believed to be cured in the quantum theory. Among the different approaches toward quantum theory of GR, LQG provides the ground to examine such a quest at least as simplified toy models. These are called symmetry reduced models in which one applies desired symmetries characterizing a black hole, or a cosmological model, at the classical level and then, one can quantize the reduced theory by the methods developed in LQG. In reducing the classical phase space, the field aspect of gravity could be lost or weakened. Based on such models, one can examine the two criteria characterizing a classical singularity at quantum level and inspect whether or not they still hold. The local criterion is the unboundedness of curvature scalar at $r = 0$, and the global one simply states that evolution equations stop while encountering the singularity.

For the Schwarzschild black hole, the interesting region to study is the interior region of the event horizon, where the irremovable singularity lies. Such a region is a homogeneous one of the Kantowski-Sachs type. The symmetry reduced model used to study this case reduces the infinite degrees of freedom of the gravitational field to a finite number (minisuperspace). Based on loop quantization of such a model, the operator analogue of a divergent classical quantity, out of which the classical divergent curvature is made, is constructed and is shown to have a bounded operator. This local criterion together with the global one, non-singular quantum evolution equation, indicates the non-singular behavior of Schwarzschild black hole in LQG.

For a Reissner-Nordström space-time, characterizing the space-time of a charged black hole, the spherically symmetric reduced model is a $U(1)$ gauge theory, instead of $SU(2)$ for the full theory, and hence spin network states are characterized by irreducible representations of $U(1)$ group. In such a reduced model, which is a midisuperspace and hence still keeps the field theoretical aspects of gravity, the corresponding operator to the infinite quantity in the classical theory is shown to be bounded from above, and the quantum Einstein equations is shown to be smooth while traversing singularity.

Although still an open problem in the full theory, the avoidance of singularity in reduced and simplified models of LQG is well-established for many homogeneous and inhomogeneous cases. This will shed light on our true understanding of nature of space and time at the quantum scales in the full theory of quantum gravity. Specifically, the non-singular behavior of space-time can develop a way to settle the mysterious paradox of information loss in black holes: the space-time does not *end* at singularity, it can be traverse using quantum evolution which opens up a new space-time region. Consequently, information approaching singularity is not lost and can be retained on the other side of the singularity.

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APPENDIX A

Mathematical Preliminaries

In this appendix, I will review the basic defintions and the implications of some mathematical tools used throughout the thesis, for the sake of being self-contained. In doing so I have mainly made use of references [25], [34], and [35].

A.1 Elements of Differential Geometry and Topology

Topological Space. *A topological space $(X, \mathcal{J})$ consists of a set X together with a collection $\mathcal{J}$ of subsets of X satisfying the following three properties:*

1. *if $O_\alpha \in \mathcal{J}$ for all α, then $\cup_\alpha \in \mathcal{J}$;*

2. *if $O_1, \dots, O_n \in \mathcal{J}$, then $\cap_{i=1}^n O_i \in \mathcal{J}$;*

3. *the entire set X and the empty $\varnothing$ set are in $\mathcal{J}$.*

The collection $\mathcal{J}$ is called a topology on X, and the subsets O_α are called open sets. A prime example of a topological space is the real line $\mathbb{R}$, with open intervals (a, b) $\forall a, b \in \mathbb{R}$ as open sets.

Contineous map. *If $(X, \mathcal{J})$ and $(Y, \mathcal{K})$ are two topological spaces, a map $f : X \to Y$ is said to be contineous if the inverse image, $f^{-1}[O] \equiv \{x \in X | f(x) \in O\}$, of any open set O in Y is an open set in X.*

Homeomorphism. *If f is a contineous, one-to-one and onto map between $(X, \mathcal{J})$ and $(Y, \mathcal{K})$, and its inverse is contineous, it is called a homeomorphism, and the two topological spaces are called homeomorphic.*

Manifold. *An n dimensional manifold M is a topological space which is locally homeomorphic to $\mathbb{R}^n$, i.e. for all open sets $U \in M$, there is a homeomorphism $\phi : O_\alpha \to \mathbb{R}^n$. ϕ is called a coordinate system on M.*

A familiar example of a manifold is the surface of the 2-dimensional sphere which is locally homeomorphic with $\mathbb{R}^2$.

Differentiable function. *A function $f : M \to \mathbb{R}$ from an n-dimensional manifold M to the reals is differentiable if and only if $f o\phi^{-1}$ is differentiable for any coordinate $\phi : U \to \mathbb{R}^n$.*

Differentiable Manifold. *A manifold is called differentiable if for any two coordinates $\phi_\alpha : O_\alpha \to \mathbb{R}^n$ and $\phi_\beta : O_\beta \to \mathbb{R}^n$, $\phi_\alpha o \phi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable.*

Vector field. *A vector field v on a differentiable manifold M is defined to be a function from $C^\infty(M)$ to $C^\infty(M)$ satisfying the following properties:*

$$
\begin{aligned}
v(f+g) &= v(f) + v(g) \\
v(\alpha f) &= \alpha v(f) \\
v(fg) &= v(f)g + f v(g),
\end{aligned}
$$

for all $f, g \in C^\infty(M)$ and $\alpha \in \mathbb{R}$.

Denoting the vector space of all vector fields on M by $Vect(M)$, a natural basis for $Vect(\mathbb{R}^n)$ is the set of partial derivatives $\{\partial_\mu\} = \{\partial_1, \ldots \partial_n\}$. Then every vector field on $Vect(\mathbb{R}^n)$ can be written as $V = V^\mu \partial_\mu$. V^μ are called components of V.

Tangent space. *The tangent space T_pM is the vector space of all vector fields at a point p (tangent vectors).*

One-form. *A one-form ω on a manifold M is a linear functional from $Vect(M)$ to $C^\infty(M)$ sasisfying*

$$
\begin{aligned}
\omega(v+w) &= \omega(v) + \omega(w) \\
\omega(gv) &= g\omega(v),
\end{aligned}
$$

for all $g \in C^\infty(M)$ and $v, w \in Vect(M)$.

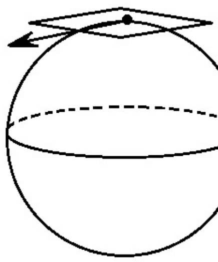

Figure A.1: A tangent space of S^2 and a tangent vector in it

Cotangent space. *The cotangent space T_p^*M is the vector space of all linear maps from T_pM to $\mathbb{R}$ (cotangent vector).*

Tensor field. *A rank (k,l) tensor is function from tensor product of k copies of T_pM and l copies of T_p^*M to $\mathbb{R}$.*

• Let M and N be two differentiable manifolds of dimension m and n, and coordinates x^μ and x^α, respectively. And consider a map $\phi : M \to N$ and a function $f : N \to \mathbb{R}$.

Pullback. *The pullback of ϕ by f is defined as: $\phi_* f \equiv f o\phi : M \to \mathbb{R}$.*

Pushforward. *Consider $V(p)$ a vector at a point p on M. The pushforward vector $\phi^* V$ at the point $\phi(p)$ on N by giving its action on functions on N is defined by: $(\phi^* V)(f) = V(\phi_* f)$.*

Consider the natural basis $\{\partial_\mu\}$ and $\{\partial_\alpha\}$ on M and N. To see the relation between components of $V = V^\mu \partial_\mu$ and its push forward $\phi^* V = (\phi^* V)^\alpha \partial_\alpha$ we write:

$$\begin{aligned} (\phi^* V)^\alpha \partial_\alpha &= V^\mu \partial_\mu (\phi_* f) \\ &= V^\mu \partial_\mu (f o \phi) \\ &= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f. \end{aligned}$$

The pullback $\phi_* \omega$ of a one-form ω on N can be seen as its action on a vactor field V on M by

$$(\phi_* \omega)(V) = \omega(\phi^* V). \tag{A.1}$$

Diffeomorphism. *A diffeomorphism between manifolds M and N is a map $\phi : M \to N$ which is invertible, and both ϕ and ϕ^{-1} are smooth.*

Using a diffeomorphism we can define the pullback and pushforward of both vectors and forms using ϕ and ϕ^{-1}. Consequently for a (k,l) tensor field T on M, the pushforward by means of a diffeomorphism ϕ is defined by

$$(\phi^* T)(\omega^{(1)} \dots \omega^{(k)}, V^{(1)} \dots V^{(l)}) = T(\phi_* \omega^{(1)} \dots \phi_* \omega^{(k)}, [\phi^{-1}]^* V(1) \dots [\phi^{-1}]^* V^{(l)}). \quad \text{(A.2)}$$

In components this becomes

$$(\phi^* T)^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{\nu_l}}{\partial y^{\beta_l}} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}. \quad \text{(A.3)}$$

Although it looks like the familiar transformation law for tensor fields, they are, indeed, not the same. The transformation law tells us how components of a tensor field change under a *passive* transformation of *coordinates.* However, the above relation tells us how components change under an *active* transformation of *points* of a manifold or a diffeomorphism.

• Using diffeomorphism, we can construct a derivative operator by comparing the difference between the value of a tensor at some point p and its value at $\phi(p)$ pulled back to p. To do so, we consider a family of diffeomorphisms ϕ_t as curve whose tangents are the vector V (flow of V):

$$\frac{d\phi_t}{dt} = V. \quad \text{(A.4)}$$

V is called the generator of diffeomorphism.

Lie Derivative. *The Lie derivative of a tensor $T^{\mu_1 \cdots \mu_k}_{\nu_1 \dots \nu_l}$ along the vector field V is defiend via:*

$$\mathcal{L}_V T^{\mu_1 \cdots \mu_k}_{\nu_1 \dots \nu_l} \equiv \lim_{t \to 0} \frac{1}{t} \left((\phi_t)_* [T^{\mu_1 \cdots \mu_k}_{\nu_1 \dots \nu_l}(\phi_t(p))] - T^{\mu_1 \cdots \mu_k}_{\nu_1 \dots \nu_l}(p) \right). \quad \text{(A.5)}$$

Lie derivative evaluate the change of tensorfields along the flow of a vector field. It possesses the following properties for all tensors T and S and constants a, b.

$$\begin{aligned} \mathcal{L}_V(aT + bS) &= a\mathcal{L}_V T + b\mathcal{L}_V S && \text{(linearity)}, \\ \mathcal{L}_V(T \otimes S) &= (\mathcal{L}_V T) \otimes S + T \otimes (\mathcal{L}_V S) && \text{(Leibniz rule)}. \end{aligned}$$

The Lie derivative of a vector field Y along another one X turns out to be their Lie bracket or commutator:

$$\mathcal{L}_X Y = [X, Y]\,. \quad \text{(A.6)}$$

For a scalar f it acts like ordinary partial derivative and consequently for one-form:

$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + (\partial_\mu V^\nu)\omega_\nu. \quad \text{(A.7)}$$

• Diffeomorphism can also be used to define the notion of symmetry. We say that a diffeomorphism ϕ is a symmetry of some tensor T if the tensor is invariant after being pulled

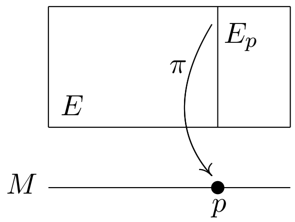

Figure A.2: bundle $\pi : E \to M$.

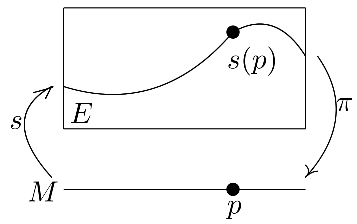

Figure A.3: A section s of a bundle $\pi : E \to M$.

back under ϕ: $\phi_* T = T$. If the diffeomorphism is generated by a vector field V, this is equal to $\mathcal{L}_V = 0$. Of particular interest is the symmetry of the metric tensor, $\phi_* g_{\mu\nu} = g_{\mu\nu}$, which is called an isometry.

Killing Vector Fields. *A killing vector field ξ is the generator of the one-parameter family of isometries*

$$\mathcal{L}_\xi g_{\mu\nu} = 0. \tag{A.8}$$

Bundle. *A bundle is a structure consisting of a manifold E, total space, a manifold M, base space, and an onto map $\pi : E \to M$, projection map.*

For each $p \in M$, the space $E_p = \{q \in E : \pi(q) = p\}$ is called the fiber over p, and the total space E is the union of all the fibers: $E = \cup_{p \in M} E_p$.

Tangent/Cotangent Bundle. *The tangent (cotangent) bundle TM of a manifold M is a bundle whose total space is the union of all the tangent (cotangent) spaces of M: $TM = \cup_{p \in M} T_p M$ ($T^* M = \cup_{p \in M} T_p^* M$).*

Section. *A section of a bundle $\pi : E \to M$ is a function $s : M \to E$ such that for any $p \in M$, $s(p) \in E_p$.*

The set of all sections of E is denoted by $\Gamma(E)$.

In other words, a section assign to each point in the base space a vector in the fiber over that point. Consequently, a section of the tangent bundle is a vector field and a section of a cotangent bundle is a one-form.

• To define the notion of differentiation for sections is not trivial, since a section of a bundle assigns to each point in the base a vector in the fiber over that point and there is no canonical way to add or subtract vectors in different bundles.

Covariant Derivative. *Given any sectrion s and vector field v, the covariant derivative of s in the direction v, $D_v s$ is a function from $\Gamma(E)$ to $\Gamma(E)$ such that*

$$\begin{aligned} D_v(\alpha s) &= \alpha D_v s, \\ D_v(s+t) &= D_v s + D_v t, \\ D_v(fs) &= v(f)s + f D_v s, \\ D_{v+w}(s) &= D_v s + D_w s, \\ D_{fv}(s) &= f D_v s. \end{aligned}$$

• Let $\{x^\mu\}$ be coordinates on an open set $U \in M$, and let $\{e_i\}$ be a basis of sections of E over U. Then, we can express $D_\mu e_j$ uniquely as a linear combination of the sections e_i, with functions on U as the coefficients. This defines $A^i_{\mu j}$, the components of **connection** or **vector potential**:

$$D_\mu e_j = A^i_{\mu j} e_i. \tag{A.9}$$

For an arbitrary section s, the covariant derivative turns out to be

$$D_\mu s = \partial_\mu s + A_\mu s, \tag{A.10}$$

where

$$A_\mu = A^i_{\mu j} e_j \otimes e^i, \tag{A.11}$$

$$A = A_\mu dx^\mu. \tag{A.12}$$

Curvature. *Given two vector fields v and w an M, the curvature $F(v,w)$ is defined as the operator on sections s of E via*

$$F(v,w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s. \tag{A.13}$$

The curvature of a connection measures the failure of covariant derivatives to commute. With $\{x^\mu\}$ being coordinates on an open set $U \in M$

$$F_{\mu\nu} \equiv F(\partial_\mu, \partial_\nu) = [D_\mu, D_\nu], \tag{A.14}$$

since partial derivatives commute. Acting on $\{e_i\}$, a basis of sections of E over U, it gives

$$\begin{aligned} F_{\mu\nu}e_i &= D_\mu D_\nu e_i - D_\nu D_\mu e_i \\ &= D_\mu(A^j_{\nu i}e_j) - D_\nu(A^j_{\mu i}e_j) \\ &= (\partial_\mu A^j_{\nu i})e_j + A^k_{\mu j}A^j_{\nu i}e_k - (\partial_\nu A^j_{\mu i})e_j - A^k_{\nu j}A^j_{\mu i}e_k, \end{aligned}$$

or more compactly

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{A.15}$$

• Consider a space-time M, with coordinates $\{\partial_\mu\}$, and a bundle P over it whose fibers are Minkowski spaces with coordinates $\{\partial_I\}$.

Frame Fileds (tetrad). *A frame field e^I is a one-form field, $e^I(x) = e^I_\mu(x)dx^\mu$, which maps tangent vectors of M to Lorentz vectors (vectors with values in the Minkowski space)*

$$e : TM \to P.$$

The set of three space-like orthonormal frame fields is also called *triad*.

Spin Connection. *Spin connection is the connection defined on the bundle P. It is a one form with values in the Lorentz algebra $\mathfrak{so}(3,1)$ (see A.2)*

$$\omega^I_J(x) = \omega^I_{J\mu}(x)dx^\mu,$$

and defines a covariant derivative of the sections of the bundle:

$$Du^I = du^I + \omega^I_J \wedge u^J.$$

It is called compatible with e^I, in the sense that

$$De^I = 0 \Rightarrow de^I + \omega^I_J \wedge e^J = 0.$$

The Curvature R of ω. *This is a Lorentz algebra valued two-form*

$$R^I_J = R^I_{J\mu\nu}dx^\mu dx^\nu$$

defined by

$$R^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J.$$

• Let $\gamma(t) : [0, T] \to M$ be a smooth path from point p to q, and suppose that for $t \in [0, T]$, $u(t)$ is a vector in the fiber of E over $\gamma(t)$.

Parallel transport. *A vector $u(t)$ is said to be parallely transported along $\gamma(t)$, if*

$$D_{\gamma'(t)}u(t) = 0. \tag{A.16}$$

Holonomy. *Holonomy $h_\gamma(A)u$ of a vector u along path γ is the result of parallel transporting u from p to q along path γ with the covariant derivative made up of vector potential A.*

Holonomy is, in fact, the solution to the differential equation

$$D_{\gamma'(t)}u(t) = \frac{d}{dt}u(t) + A(\gamma'(t))u(t). \tag{A.17}$$

The solution to the above differential equation by fixing an initial condition $u(0) = u$, is found by irritation and turns out to be:

$$u(t) = h_\gamma(A)u, \tag{A.18}$$

with

$$h_\gamma(A) = \mathcal{P}\exp\left(\int_0^t A(\gamma'(s))ds\right), \tag{A.19}$$

where $\mathcal{P}$ denotes the path ordered product.

A.2 Elements of Lie Groups and their Representations

Basic Definitions

Group. *A group G is a set together with a binary operator $\times : G \times G \to G$ satisfying the following properties:*

1. *for all two elements $g, h, k \in G$, the binary operation is associative i.e. $g \times (h \times k) = (g \times h) \times k$;*
2. *There is an identity element $1 \in G$ such that for any element $g \in G$, $g = g$;*
3. *for any element $g \in G$, there is an inverse element g^{-1} such that $g \times g^{-1} = 1$.*

Homomorphism. *Given two groups G and H, a map $\rho : G \to H$ is a homomorphism if*

$$\rho(g \times h) = \rho(g) \times \rho(h), \tag{A.20}$$

for all $g, h \in G$.

Isomomorphism. *A homomorphism which is one-to-one and onto is called an isomorphism. for all $g, h \in G$.*

In a wide range of appliactions in mathematics and physics, we are interested in realizing elements of a certain group, as defined above in an abstract manner, as transformations on some objects. For instance, consider the rotation group. One might be interested in finding out how three dimensional vectors in 3-space transform under rotation. The other interest might arise as how wavefunctions in quantum mechanics transform under rotation, or how spinors of a dirac field are transformed. In any case, the realization of abstract notion of "rotation" can be different depending on which mathematical objects we are intented to investigate the effect of rotation on. For 3-vectors, rotations group elements are certain 3×3 matrices, for spinors certain 2×2 matrices and so forth. Such a realization is called a representation of the abstract group. Therefore, a representation can be considered as a map assigning to each element of an abstract group a transformation on , generally, a vector space.

Representation. *A representation ρ of a group G on a vector space V is a homomorphism from G to the group $GL(V)$ of all invertible linear transformations of a vector space V.*

$$\rho : G \rightarrow GL(V), \tag{A.21}$$

$$\rho(g \times h)v = \rho(g) \times \rho(h)v, \tag{A.22}$$

for all $g, h \in G$ and $v \in V$.

Invariant subspace. *A subspace V' of a vector space V is called invariant, if $\rho(g)v \in V'$, for all $g \in G$, $v \in V$, and a representation ρ. for all $g, h \in G$.*

Irreducible representation. *A representation ρ of a group G on a vector space V, is called irreducible if it does not have any invariant subspace exept the trivial ones 0 and V.*

• Let G be a group, ρ be a representation of G on V, and $\rho^{'}$ be a representation of G on $V^{'}$

Direct sum of representations. *$\rho \oplus \rho^{'}$ is the representation of G on $V \oplus V^{'}$:*

$$(\rho \oplus \rho^{'})(g)(v, v^{'}) = (\rho(g)v, \rho^{'}(g)v^{'}),$$

for all $v \in V, v^{'} \in V^{'}$, and is called the direct sun representation of ρ and $\rho^{'}$.

Tensor product of representations. *$\rho \otimes \rho^{'}$ is the representation of G on $V \otimes V^{'}$:*

$$(\rho \otimes \rho^{'})(g)(v \otimes v^{'}) = \rho(g)v \otimes \rho^{'}(g)v^{'},$$

for all $v \in V, v^{'} \in V^{'}$, and is called the tensor product representation of ρ and $\rho^{'}$.

So far, there has been no notion of contineuity or differentability of elements of the group. However, the familiar groups in physics, such as rotation or Lorentz group, are all contineous, in the sense that they depend on a contineous parameters, angle velocity etc. Such groups are examples of Lie groups.

Lie group. *A Lie group is a group which is also a differentiable manifold.*

Recall that an algebra is a vector space equipped with a bilinear multiplication operation which is distributive.

Lie algebra. *A Lie algebra is the tangent space of the identity element of a Lie group. It can also be defined, in a more abstract way, as any vector space $\mathfrak{g}$ equipped with a bilinear map $[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that the following identities hold:*

1. $[v, w] = -[w, v]$;
2. $[u, \alpha v + \beta w] = \alpha[u, v] + \beta[u, w]$;
3. $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$,

for all $u, v, w \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{R}$.

Lie algebra isomorphism. *A map f between two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is called isomorphism if*

$$f([v, w]) = [f(v), f(w)], \tag{A.23}$$

for all $v, w \in \mathfrak{g}$.

Useful Lie Groups in Physics

If p, q are non-negative integers with $p + q = n$, let g be a metric on $\mathbb{R}^n$ of signature (p, q):

$$g(v, w) = v^1 w^1 + \ldots + v^p w^p - v^{p+1} w^{p+1} - \ldots - v^{p+q} w^{p+q}. \tag{A.24}$$

The orthogonal group $O(p, q)$ is the set of $n \times n$ matrices T that preserve the metric

$$g(Tv, Tw) = g(v, w), \tag{A.25}$$

for all $v, w \in \mathbb{R}^n$.

The special orthogonal group, $SO(p, q)$, is the set of matrices in $O(p, q)$ that also have determinant 1.

The complex analoge of orthogonal group is unitary group $U(n)$. This group consists of all unitary $n \times n$ complex matrices, that is, those that preserve the usual inner product on $\mathbb{C}^n$: $\langle v, w \rangle = \sum_{i=1}^{n} v^{*i} w^i$.

$SU(n)$ is the subgroup of $U(n)$ whose elements have determinant 1.

Below, I will present some of the above abstract definitions, explicitly for groups that have been used in the present thesis.

U(1)

The Lie group $U(1)$ sonsists of unitary complex 1×1 matrices which are, in fact, complex numbers with moduli 1.

$$U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}. \tag{A.26}$$

It shows that, as a manifold, $U(1)$ is the unit circle in the complex plane. Elements of $U(1)$ are, in fact, phases in quantum mechanics, and a $U(1)$ transformation amounts for a multiplication of wave function with a phase. This group, has an irreducible representation ρ_n on $\mathbb{C}$ given by

$$\rho_n(e^{i\theta})v = e^{in\theta}v, \forall v \in \mathbb{C}. \tag{A.27}$$

SO(2)

The Lie group $SO(2)$ consists of 2×2 orthogonal matrices with determinant 1.

$$SO(2) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}\}. \tag{A.28}$$

$U(1)$ is isomorphic to $SO(2)$, with an isomorphism being given by

$$\rho(e^{i\theta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{A.29}$$

This isomorphism reflects the fact that rotations of 2-dimensional real vectors in $\mathbb{R}^2$ are the same as rotations of the complex plane $\mathbb{C}$.

SO(3): Rotation Group

The Lie group $SO(2)$ consists of 3×3 orthogonal matrices with determinant 1. These are, in fact, rotation matrices in 3-dimensions characterized by 3 parameters (3 rotation angles). For instance, consider rotation around z axis by angle θ

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.30}$$

This can be thought of as a path in the $SO(3)$ manifold. The corresponding element of $\mathfrak{so}(3)$ algebra, which is the tangent space of $\mathbb{1} = R_z(0)$, is the tangent vector to this path at identity (or at $\theta = 0$)

$$\frac{dR_z(\theta)}{d\theta}\bigg|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.31}$$

the corresponding elements of algebra for rotations around y and z turns out to be

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.32}$$

They form a basis for $\mathfrak{so}(3)$ algebra and are called generators of $\mathfrak{so}(3)$. Elements of group are, then, exponentials of algebra elements:

$$R_i(\theta) = e^{J_i\theta} \tag{A.33}$$

SU(2)

• GROUP

The Lie group $SU(2)$ consists of 2×2 unitary matrices with determinant 1.

$$SU(2) = \{\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}. \tag{A.34}$$

Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.35}$$

These are traceless matrices satisfying

$$\sigma_i^2 = 1, \sigma_i\sigma_j = -\sigma_j\sigma_i = i\sigma_k(\text{cyclic permutation}), [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{A.36}$$

Elements of $SU(2)$ can be written as linear combinations of identity matrix $\mathbb{1}$ and $-i\sigma_i$:

$$SU(2) = \{a\mathbb{1} + b(-\sigma_1) + c(-\sigma_2) + d(-\sigma_3) : a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 = 1\}, \tag{A.37}$$

which shows that as a manifold, $SU(2)$ is S^3, the unit sphere in quaternions.

• REPRESENTATIONS

Irreducible representations of $SU(2)$ are labeled by $j/2$ for integer j, are denoted by U_j and are called spin-j representations. To specify them we need to give an explicit form for the homomorphism

$$\rho : G \to GL(V). \tag{A.38}$$

In the case of $SU(2)$

$$U_j : SU(2) \to GL(\mathcal{H}_j) \tag{A.39}$$

$\mathcal{H}_j$ is the space of polynomial functions on $(x, y) \in \mathbb{C}^2$ that are homogeneous of degree $2j$, that is linear combinations of $f(x, y) = x^p y^q$, with $p + q = 2j$. $\mathcal{H}_j$ has dimension $2j + 1$ since it has a basis given by $\{x^{2j}, x^{2j-1}y, x^{2j-2}y^2, \ldots, y^{2j}\}$. Now, for any $g \in SU(2)$, $U_j(g)$ is a linear transformation of $\mathcal{H}_j$ given by

$$(U_j(g)f)(x, y) = f(g^{-1}(x, y)). \tag{A.40}$$

Spin-$\frac{1}{2}$ Representation

Dimension of $\mathcal{H}_{\frac{1}{2}}$ is $2(\frac{1}{2}) + 1 = 2$. Therefore, it consists of polynomials of degree 2: $\begin{pmatrix} a \\ b \end{pmatrix}$, for all $a, b \in \mathbb{C}$. A basis for this polynomial can be chosen, in Dirac ket notation of QM states,

$$|\frac{1}{2}, \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\frac{1}{2}, -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.41}$$

The spin-$\frac{1}{2}$ representation of $SU(2)$ is thus consists of a set of 2×2 matrices, satisfying A.40, which act on $\begin{pmatrix} a \\ b \end{pmatrix}$. Such representations are spaned by the Pauli matrices A.35.

Spin-1 Representation

Dimension of $\mathcal{H}_1$ is $2(1) + 1 = 3$. Therefore, it consists of polynomials of degree 3: $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, for all $a, b, c \in \mathbb{C}$. A basis for this polynomial can be chosen as

$$|1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A.42}$$

The spin-1 representation of $SU(2)$ is thus consists of a set of 3×3 matrices spaned by

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{A.43}$$

Intertwiners. *An intertwiner i between representations $j_1, \ldots, j_N$ is an element of an orthonormal basis of the Hilbert space $\mathcal{H}^0_{j_1,\ldots,j_N}$, which is an invariant subspace of $\mathcal{H}_{j_1,\ldots,j_N} = \mathcal{H}_{j_1} \otimes \ldots \otimes \mathcal{H}_{j_N}$, where $\mathcal{H}_{j_l}$ is the Hilbert space of irreducible representation j_l. Therefore, an intertwiner can be represented by $i = v^{\beta_l}_{\alpha_l}$.*

- Algebra

The Lie algebra $\mathfrak{so}(3)$, is the 3-dimensional real algebra spaned by $-i\sigma_i$ which are generators of $\mathfrak{so}(3)$. The $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ algebras are isomorphic, in the sence that $\frac{i}{2}\sigma_i \to J_i$ is a Lie algebra isomorphism. However, $SU(2)$ is the double cover of $SO(3)$. This means that to every element of $SO(3)$ there correspond two elements of $SU(2)$. For instance, $e^{\theta\sigma_i}$ and $-e^{\theta\sigma_i}$ both correspond to $e^{\theta J_i}$.

SO(3,1): Lorentz Group

The Lie group $SO(3,1)$ consists of 4×4 matrices describing Lorentz transformations (boost and rotation). For pure boost along x direction with velocity $\frac{v}{c} = \tanh\phi$, they have the form:

$$L_x(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.44}$$

With the same method for $SU(2)$, one can find K_i, and J_i, the generators of boost in x^j direction and rotation around x^j axix, and observe that they satisfy:

$$[K_i, K_j] = i\epsilon_{ijk}J_k, \tag{A.45}$$

$$[K_i, J_j] = 0, \tag{A.46}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{A.47}$$

A.45 shows that pure boosts don't form a group, and A.47 is the expected algebra of rotation. Defining $A_i \equiv \frac{1}{2}(J_i + iK_i)$, and $B_i \equiv \frac{1}{2}(J_i - iK_i)$, the above commutation relations become

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \tag{A.48}$$

$$[A_i, B_j] = 0, \tag{A.49}$$

$$[B_i, B_j] = i\epsilon_{ijk}B_k. \tag{A.50}$$

These are two $SU(2)$s which commute with each other

$$SO(3,1) = SU(2) \otimes SU(2). \tag{A.51}$$

The complex Lorentz Algebra, on the other hand, can be written as the sum of two complex rotation algebra:

$$\mathfrak{so}(3,1,\mathbb{C}) = \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C}). \tag{A.52}$$

APPENDIX B

Functional Representation of Quantum Field Theory

Non-relativistic quantum mechanics in Scrödinger representation is formulated by specifying states and observables of the system. The former are vectors in a Hilbert space while the latter are self-adjoint operators acting on the states in Hilbert space. Choosing a representation $|x\rangle$ in which position operator is diagonal, one can define the wave function $\psi(x)$ as the coefficient of expansion of an arbitrary state, $|\psi\rangle$, in the chosen basis

$$|\psi\rangle = \sum_x |x\rangle \langle x|\psi\rangle \Longrightarrow \psi(x) = \langle \psi|x\rangle \,. \tag{B.1}$$

The commutation relation between coordinates and momenta conjugate pairs, $[\hat{x}, \hat{p}] = i\hbar$, can, then, be implemented by defining operators as:

$$\hat{x}\psi(x) = x\psi(x). \tag{B.2}$$

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x). \tag{B.3}$$

The Hilbert space of a relativistic quantum field turns out to be the Fock space $\mathcal{F}$. Motivated by the fact that number of particles in a relativistic system is not fixed, the physically interested representation well suited for particle physics phenomena is the one in which particle numbers is sharply defined and therefore the particle number operator, $\hat{N}$, is diagonal

$$\hat{N}\,|N\rangle = N\,|N\rangle \,. \tag{B.4}$$

Nevertheless, there is another way to choose a representation which is the natural generalization of the non-relativistic wave “functions” to wave “functionals”. In such a functional representation of QFT, one considers states $|\phi\rangle$ on which field operator $\hat{\phi}(x)$ acts diagonally

$$\hat{\phi}(x)\,|\phi\rangle = \phi(x)\,|\phi\rangle\,, \tag{B.5}$$

the wave functional is defined

$$\Psi[\phi] = \langle\Psi|\phi\rangle \tag{B.6}$$

for an arbitrary state, $|\Psi\rangle$ in the fock space $\mathcal{F}$. Defining the momentum field, $\hat{\pi}(x)$, via acting as functional derivatives

$$\hat{\pi}(x)\Psi[\phi] = -i\hbar\frac{\delta}{\delta\phi(x)}\Psi[\phi], \tag{B.7}$$

the canonical field commutation relation is fulfilled

$$\left[\hat{\phi}(x), \hat{\pi}(y)\right] = i\hbar\delta(x-y). \tag{B.8}$$

Time evolution of the theory, generated by the Hamiltonian operator $\hat{H}$, is governed by the functional Scrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi,t] = \hat{H}\Psi[\phi,t]. \tag{B.9}$$

Free Scalar Field

As the simplest example, I look at the quantization of free scalar field in functional representation and illustrate the connection with conventional QFT. The Hamiltonian of a scalar field has the form

$$H = \frac{1}{2}\int d^3x(\pi^2 + |\nabla\phi|^2 + m^2\phi^2). \tag{B.10}$$

Expressing momentum as functional derivative, the Schrödinger equation takes the form of a functional differential equation

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi,t] = \frac{1}{2}\int d^3x(-\hbar^2\frac{\delta^2}{\delta\phi^2(x)} + |\nabla\phi|^2 + m^2\phi^2)\Psi[\phi,t]. \tag{B.11}$$

Since the Hamiltonian does not explicitly depend on time, we can write the time dependence of the wave functional as a phase

$$\Psi[\phi,t] = e^{-iEt/\hbar}\Psi[\phi], \tag{B.12}$$

where $\Psi[\phi]$ will satisfy the time-independent Scrödinger equation

$$\frac{1}{2}\int d^3x(-\hbar^2\frac{\delta^2}{\delta\phi^2(x)} + |\nabla\phi|^2 + m^2\phi^2)\Psi[\phi,t] = E\Psi[\phi,t]. \tag{B.13}$$

To find a general solution for the wave functional, one can start with the ground energy state E_0. Using Standard methods in functional calculus, the ground state functional Ψ_0 turns out to be [36]

$$\Psi_0[\phi] = Nexp\left(-\frac{1}{2\hbar}\int d^3k\omega(k)\phi(k)\phi(k)\right), \tag{B.14}$$

where $\phi(k)$ in the Fourier expansion coefficient of the field, and $\omega^2(k) = k^2 + m^2$.

Relation with Particle Number Representation

One can use the Fourier expansion of the field, $\phi(k) = (2\pi)^{-3/2}\int d^3xe^{ik.x}\phi(x)$, to write B.10 as

$$H = \frac{1}{2}\int d^3k\left(\pi^2(k) + \omega^2(k)\phi^2(k)\right). \tag{B.15}$$

By defining

$$a(k) = \frac{i}{\sqrt{2\omega}}\pi(k) + \sqrt{\frac{\omega}{2}}\phi(k), \tag{B.16}$$

$$a^\dagger(k) = \frac{-i}{\sqrt{2\omega}}\pi(-k) + \sqrt{\frac{\omega}{2}}\phi(-k), \tag{B.17}$$

Hamiltonian takes the form

$$H = \int d^3k\omega(k)a^\dagger(k)a(k). \tag{B.18}$$

In conventional QFT, we write the field operator in terms of a and $a^\dagger$, and consequently the field operator would not be diagonal on states $|n\rangle$. In functional representation quantization, on the other hand, $a(k)$ and $a^\dagger(k)$ become operators according to B.5 and B.7

$$\hat{a}(k) = \frac{\hbar}{\sqrt{2\omega}}\frac{\delta}{\delta\phi(k)} + \sqrt{\frac{\omega}{2}}\phi(k), \tag{B.19}$$

$$\hat{a}^\dagger(k) = \frac{-i}{\sqrt{2\omega}}\frac{\delta}{\delta\phi(-k)} + \sqrt{\frac{\omega}{2}}\phi(-k). \tag{B.20}$$

Now, defining the ground state by requiring Hamiltonian, and therefore $\hat{a}$ to be zero on it, $\hat{a}(k)\,|0\rangle = 0$, the ground state wave functional, $\Psi_0[\phi] = \langle\phi|0\rangle$, must satisfy

$$\frac{\hbar}{\sqrt{2\omega}}\frac{\delta}{\delta\phi(k)}\Psi_0[\phi] + \sqrt{\frac{\omega}{2}}\phi(k)\Psi_0[\phi] = 0. \tag{B.21}$$

This functional differential equation can be simply solved by

$$\Psi_0[\phi] = Nexp\left(-\frac{1}{2\hbar}\int d^3k\omega(k)\phi(k)\phi(k)\right), \tag{B.22}$$

which is the same as the solution to the the Schrödinger equation B.14. The one-particle state with momentum k is created by $a^\dagger(k)$:

$$a^\dagger(k)\Psi_0[\phi] \equiv \Psi_k[\phi] = \langle\phi|k\rangle = \sqrt{2\omega}\phi(k)\Psi_0[\phi]. \tag{B.23}$$

Now for a general one-particle state $|k\rangle$ in Fock basis representation,

$$|f\rangle = \int \frac{d^3k}{\sqrt{2\omega}} f(k) \, |k\rangle \, , \tag{B.24}$$

have the corresponding wave functional, $\Psi_f[\phi]$, defined by

$$\Psi_f[\phi] = \langle \phi | f \rangle = \int d^3k f(k) \phi(k) \Psi_0[\phi]. \tag{B.25}$$