then we obtain a $D(2,1;\alpha=-\frac{h+2}{2})$-invariant model, with

$$g_{ab}=\left(\delta_a^c\delta_b^d+I_a^{rc}I_b^{rd}\right)\partial_c\partial_d L(X), \tag{4.39a}$$

$$D^a=\frac{2}{h}X^a, \tag{4.39b}$$

$$K=\frac{h+2}{2h}L, \tag{4.39c}$$

and c_{abc} given by equation (4.31d). In an $\mathcal{N}=2$ superspace formalism, X^a is a superfield obeying certain constraints [29, 33] and the potential L is the superspace integrand [13, 32, 30].

5. The Quantum Mechanics of a Test Particle in a Reissner-Nordström Background

Our goal is to apply the results on superconformal quantum mechanics to the quantum mechanics of a collection of supersymmetric black holes. As a warm-up in this section we consider the problem of a quantum test particle moving in the black hole geometry. The four-dimensional case was treated in [11], which will be followed and adapted to five dimensions in this section.

Consider a five-dimensional extremal Reissner-Nordström black hole of charge Q. The geometry of such a black hole is described by the metric

$$ds^2=-\frac{dt^2}{\psi^2}+\psi d\vec{x}^2, \tag{5.1a}$$

and the gauge field

$$A=\psi^{-1}dt, \tag{5.1b}$$

where $\vec{x}$ is the $\mathbb{R}^4$ coordinate, and $\psi=1+\frac{Q}{|\vec{x}^2|}$. We have set $M_p=L_p=1$. The horizon in these coordinates is at $|\vec{x}|=0$.

Introduce a test particle with mass m and charge q. The particle action is

$$S=-m\int d\tau+q\int A. \tag{5.2}$$

Parametrize the particle's trajectory as $\vec{x}=\vec{x}(t)$. Eventually we will require the test particle to be supersymmetric (by imposing $q=m$). A supersymmetric test particle at rest at a fixed distance from the black hole, remains at rest, so it is sensible to consider a test particle that moves slowly. Accordingly we shall assume $|\dot{\vec{x}}|\ll 1$. In this parametrization, we can make the following substitution:

$$d\vec{x}=\dot{\vec{x}}dt, \tag{5.3}$$

which allows us to rewrite (5.1) to obtain the metric

$$ds^2 = -\frac{dt^2}{\psi^2} + \psi|\dot{\vec{x}}|^2 dt^2. \tag{5.4}$$

Now we solve the equation $ds^2 = -d\tau^2$ and find

$$d\tau = \frac{dt}{\psi} - \frac{1}{2}\psi^2|\dot{\vec{x}}|^2 dt + \mathcal{O}(\dot{x}^4), \tag{5.5}$$

which is substituted into (5.2) to obtain the action,

$$S = -m\int (\frac{dt}{\psi} - \frac{1}{2}\psi^2|\dot{\vec{x}}|^2 dt) + q\int \frac{dt}{\psi}. \tag{5.6}$$

For a supersymmetric test particle, $m = q$, this action reduces to

$$S = \frac{m}{2}\int \psi^2|\dot{\vec{x}}|^2 dt. \tag{5.7}$$

If the particle is near the horizon, at distances $r \ll \sqrt{Q}$, then we can approximate $\psi = \frac{Q}{|\vec{x}|^2}$, so that

$$S_p = \frac{mQ^2}{2}\int dt\frac{|\dot{\vec{x}}|^2}{|\vec{x}|^4}, \tag{5.8}$$

or, if we define a new quantity $\vec{y} = \frac{\vec{x}}{|\vec{x}|^2}$, then we see that we are actually in flat space:

$$S_p = \frac{mQ^2}{2}\int dt|\dot{\vec{y}}|^2. \tag{5.9}$$

Far from the black hole, spacetime and the moduli space look flat once again. Thus the moduli space can be described as two asymptotically flat regions connected by a wormhole whose radius scales as $\sqrt{Q}$. At low energies (relative to $M_p/\sqrt{Q}$) the wavefunctions spread out and do not fit into the wormhole. Hence the quantum mechanics is described by near and far superselection sectors that decouple completely at low energies.

This geometry leads to a problem. Consider the near horizon quantum theory. Given any fixed energy level E, there are infinitely many states of energy less than E. This suggests that there are infinitely many states of a test particle localized near the horizon of a black hole, which appears problematic for black hole thermodynamics. The possibility of such states arises from the large redshift factors near the horizon of a black hole. Similar problems have been encountered in studies of ordinary quantum fields in a black hole geometry.

The new observation of [11] is that this problem is in fact equivalent to the problem encountered by DFF [10] in their analysis of conformal quantum mechanics. To see this equivalence let ρ denote the radial coordinate $|\vec{y}|$. The Hamiltonian corresponding to (5.9) is

$$H = \frac{1}{2mQ^2}(p_\rho^2 + \frac{4}{\rho^2}J^2). \tag{5.10}$$

This is the DFF Hamiltonian of (2.1) with $g = \frac{4}{mQ^2}J^2$. The coordinate ρ grows infinite at the horizon. Thus this potential pushes a particle to the horizon whenever J^2 is nonzero. Our problem of infinitely many states at low energies is just the problem discussed by DFF.

Applying the DFF trick, as discussed in section 2, provides the solution to this problem. We work in terms of $H+K$ rather than H, since the former has a discrete spectrum of normalizable eigenstates. There is an $SL(2,\mathbb{R})$ symmetry generated by H, D and K, where D and K are defined to be

$$D = \tfrac{1}{2}(\rho p_\rho + p_\rho \rho); \tag{5.11a}$$

$$K = \tfrac{1}{2}mQ^2\rho^2. \tag{5.11b}$$

These generators satisfy equations (2.3).

The appearance of the $SL(2,\mathbb{R})$ symmetry was not an accident. It arises from the geometry of our spacetime. Near the horizon, we find that

$$ds^2 \to -\frac{r^4}{Q^2}dt^2 + \frac{Q}{r^2}dr^2 + Qd\Omega_3^2. \tag{5.12}$$

We recognize this metric as that of $AdS_2 \times S^3$. Introduce new coordinates $t^\pm = t \pm \frac{Q}{4r^2}$ on AdS_2. Now the metric can be written in the form

$$ds_2^2 = -\frac{Qdt^+dt^-}{(t^+ - t^-)^2}. \tag{5.13}$$

The $SL(2,\mathbb{R})$ isometry generators are then

$$h = \frac{\partial}{\partial t^+} + \frac{\partial}{\partial t^-}, \tag{5.14a}$$

$$d = t^+\frac{\partial}{\partial t^+} + t^-\frac{\partial}{\partial t^-}, \tag{5.14b}$$

$$k = (t^+)^2\frac{\partial}{\partial t^+} + (t^-)^2\frac{\partial}{\partial t^-}. \tag{5.14c}$$

Here h shifts the time coordinate, and d rescales all coordinates.

The $SL(2,\mathbb{R})$ symmetry of the near-horizon particle action reflects the $SL(2,\mathbb{R})$ isometry group of the near-horizon AdS_2 geometry. As pointed out in [11, 34], the trick of DFF to replace H by $H+K$ has a nice interpretation in AdS_2. To understand it, we must first review AdS_2 geometry.

INTERLUDE: ADS_2 GEOMETRY

On AdS_2, introduce global coordinates $u^{\pm}$ defined in terms of the coordinates of (5.13) by the relation

$$t^{\pm} = \tan u^{\pm}. \tag{5.15}$$

Then the AdS_2 metric takes the form

$$ds^2 = -\frac{Q}{4}\frac{du^+du^-}{\sin^2(u^+ - u^-)}. \tag{5.16}$$

In these coordinates, the global time generator is

$$h + k = \frac{\partial}{\partial u^+} + \frac{\partial}{\partial u^-}. \tag{5.17}$$

In figure 2 it is seen that the time coordinate conjugate to h is not a good global time coordinate on AdS_2, but the time coordinate conjugate to $h+k$ is. In fact, the generators h and d preserve the horizon, while $h+k$ preserves the boundary $u^+ = u^- + \pi$ (the right boundary in figure 2).

So in conclusion the DFF trick has a beautiful geometric interpretation in the black hole context. It is simply a coordinate transformation to "good" coordinates on AdS_2.

6. Quantum Mechanics on the Black Hole Moduli Space

6.1. THE BLACK HOLE MODULI SPACE METRIC

In this section we will consider five-dimensional $\mathcal{N} = 1$ supergravity with a single $U(1)$ charge coupled to the graviphoton and no vector multiplets.[13] We will use units with $M_p = L_p = 1$. The action is

$$S = \int d^5x\sqrt{g}\,[R - \frac{3}{4}F^2] + \frac{1}{2}\int A\wedge F\wedge F + \text{fermions}. \tag{6.1}$$

[13]Adding neutral hypermultiplets would not affect the discussion, since they decouple. Since these lectures were given, the case with additional vector multiplets was solved in [35], and the four-dimensional case was solved in [23]. The supersymmetry of cases with more than eight initial supersymmetries [36, 37] has not been worked out.

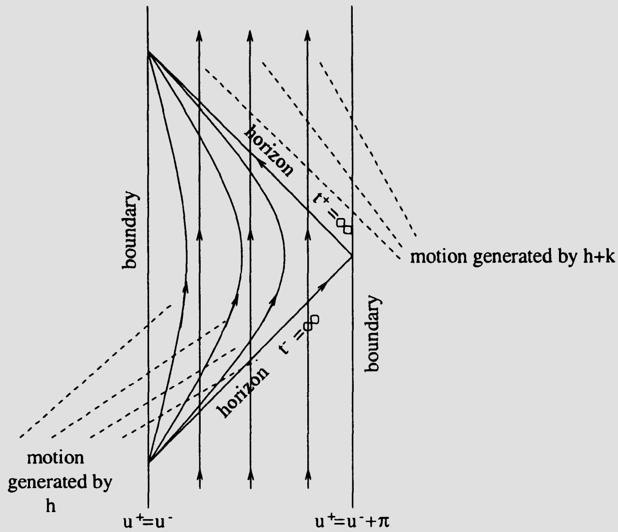

Figure 2. The geometry of AdS_2. The time conjugate to $h+k$ is a good global coordinate.

We can also get this from M-theory compactified on a Calabi-Yau with $b_2 = 1$ (the simplest example of such a threefold is the quintic). The black holes are then M2-branes wrapping Calabi-Yau two-cycles.

This system has a solution describing N static extremal black holes

$$ds^2 = -\psi^{-2}dt^2 + \psi d\vec{x}^2, \tag{6.2a}$$

$$A = \psi^{-1}dt, \tag{6.2b}$$

(*cf.* equation (5.1)) where ψ is the harmonic function on $\mathbb{R}^4$

$$\psi = 1 + \sum_{A=1}^{N} \frac{Q_A}{|\vec{x} - \vec{x}_A|^2}, \tag{6.2c}$$

and $\vec{x}_A$ is the $\mathbb{R}^4$ coordinate of the $A^{\underline{\text{th}}}$ black hole, whose charge is Q_A. Another picture of these holes is M2-branes wrapping Calabi-Yau cycles. The space of solutions is called the moduli space, which is parametrized by the $4N$ collective coordinates $\vec{x}_A$. The slow motion of such black holes is governed by the moduli space metric G_{AB}, so that the low energy effective action takes the form

$$S = \frac{1}{2}\int dt\dot{x}^A\dot{x}^B G_{AB}. \tag{6.3}$$

Note that due to the no-force condition there is no potential term in the action, and since $|\dot{\vec{x}}_A| \ll 1$, the higher order corrections can be neglected.

The first calculation of the moduli space metric of the four-dimensional Reissner-Nordström black holes was performed in [3, 4] and was generalized to dilaton black holes in [38]. The metric on the moduli space for the five-dimensional black holes (6.2) was derived in [14]. In order to find this metric, one starts with the following ansatz describing the linear order perturbation of the black hole solution (6.2)

$$ds^2 = -\psi^{-2}dt^2 + \psi d\vec{x}^2 + 2\psi^{-2}\vec{R}\cdot d\vec{x}dt, \tag{6.4a}$$

$$A = \psi^{-1}dt + (\vec{P} - \psi^{-1}\vec{R})\cdot d\vec{x}, \tag{6.4b}$$

where $\vec{P}$ and $\vec{R}$ are quantities that are first order in velocities. In equation (6.2c), $\vec{x}_A$ is replaced with $\vec{x}_A + \vec{v}_A t$. This is the most general Galilean-invariant ansatz to linear order. Then (roughly) one uses the equations of motion to solve for $\vec{P}$ and $\vec{R}$. Inserting this into the five-dimensional supergravity action gives the following result [14] for the action:

$$S = \frac{1}{2}\int dt\dot{x}^A\dot{x}^B G_{AB} = \frac{1}{4}\int dt\dot{x}^{Ak}\dot{x}^{Bl}(\delta^i_k\delta^j_l + I^{ri}_k I^{rj}_l)\partial_{Ai}\partial_{Bj}L, \tag{6.5}$$

where

$$L = -\int d^4x\psi^3, \tag{6.6}$$

with

$$\psi = 1 + \sum_{A=1}^{N}\frac{Q_A}{|\vec{x}-\vec{x}_A|^2}, \tag{6.7}$$

and I^r is a triplet of self-dual complex structures on $\mathbb{R}^4$ obeying equation (4.29). This Lagrangian has $\mathcal{N} = 4$ supersymmetry when Hermitian fermions $\lambda^{Ai} = \lambda^{Ai\dagger}$ are added.

6.2. THE NEAR-HORIZON LIMIT

6.2.1. *Spacetime geometry*

Taking the near-horizon limit of (6.2a) corresponds to neglecting the constant term in (6.2c). In figure 3 we have illustrated the resultant spatial geometry at a moment of fixed time for three black holes. Before the limit is taken (figure 3a), the geometry has an asymptotically flat region at large $|\vec{x}|$. Near the limit (figure 3b), as the origin is approached along a spatial trajectory, a single "throat" approximating that of a charge $\sum Q_A$ black hole is encountered. This throat region is an $AdS_2 \times S^3$ geometry with radii of

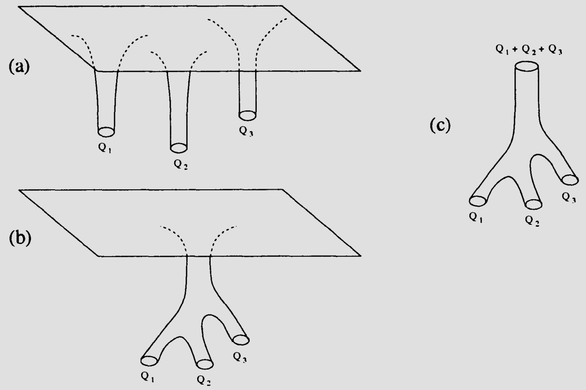

Figure 3. (a) Widely separated black holes. (b) Near-coincident black holes. (c) The near-horizon limit.

order $\sqrt{\sum Q_A}$. As one moves deeper inside the throat towards the horizon, the throat branches into smaller throats, each of which has smaller charge and correspondingly smaller radii. Eventually there are N branches with charge Q_A. At the end of each of these branches is an event horizon. When the limit is achieved (figure 3c), the asymptotically flat region moves off to infinity. Only the charge $\sum Q_A$ "trunk" and the many branches remain.

6.2.2. *Moduli space geometry*

It is also interesting to consider the near-horizon limit of the moduli space geometry. The metric is again given by (6.5), where one should neglect the constant term in the harmonic function (6.7). This is illustrated in figure 4 for the case of two black holes. Near the limit there is an asymptotically flat $\mathbb{R}^{4N}$ region corresponding to all N black holes being widely separated. This is connected to the near-horizon region where the black holes are strongly interacting, by tubelike regions which become longer and thinner as the limit is approached. When the limit is achieved, the near-horizon region is severed from the tubes and the asymptotically flat region.

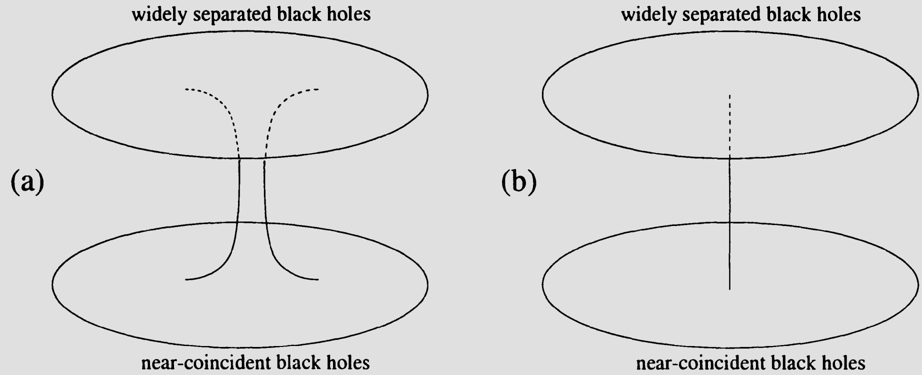

Figure 4. (a) Regions of the two-black hole moduli space. (b) The near-horizon limit.

6.3. CONFORMAL SYMMETRY

The near-horizon quantum mechanics has an $SL(2,\mathbb{R})$ conformal symmetry. The dilations D and special conformal transformations K are generated by

$$D = -\frac{1}{2}(x^{Ai}P_{Ai} + h.c.), \tag{6.8}$$

$$K = 6\pi^2 \sum_{A\neq B}^{N} \frac{Q_A^2 Q_B}{|\vec{x}_A - \vec{x}_B|^2}. \tag{6.9}$$

By splitting the potential L appearing in the metric (6.5) into pieces representing the 1-body, 2-body and 3-body interactions, one can show [14] that the conditions (4.34) and (4.36) are satisfied. Thus the $SL(2,\mathbb{R})$ symmetry can be extended to the full $D(2,1;0)$ superconformal symmetry as was described in section 4.4. This group is the special case of the $D(2,1;\alpha)$ superconformal groups for which there is an $SU(1,1|2)$ subgroup (in fact, $D(2,1;0) \cong SU(1,1|2) \rtimes SU(2)$), in agreement with [39].

So we have seen that there are noncompact regions of the near-horizon moduli space corresponding to coincident black holes. These regions are eliminated by the potential K in the modified Hamiltonian $L_0 = \frac{1}{2}(H+K)$, which is singular at the boundary of the noncompact regions. L_0 has a well defined spectrum with discrete eigenstates. A detailed description of the quantum states of this system remains to be found [40].

7. Discussion

Let us recapitulate. We have found that at low energies the quantum mechanics of N black holes divides into superselection sectors. One sector describes the dynamics of widely separated, non-interacting black holes. The other "near horizon" sector describes highly redshifted, near-coincident black holes and has an enhanced superconformal symmetry. Since they completely decouple from widely separated black holes, states of the near horizon theory are multi-black hole bound states.

It is instructive to compare this to an M-theoretic description of these black holes. In Calabi-Yau compactification of M theory to five dimensions, the black holes are described by M2-branes multiply wrapped around holomorphic cycles of the Calabi-Yau. In principle all the black hole microstates are described by quantum mechanics on the M2-brane moduli space, which at low energies should be the dual CFT_1 living on the boundary of AdS_2 [12]. In practice so far this problem has not been tractable. This moduli space has what could be called (in a slight abuse of terminology) a Higgs branch and a Coulomb branch. This Higgs branch is a sigma model whose target is the moduli space of a single multiply wrapped M2-brane worldvolume in the Calabi-Yau. In the Coulomb branch the M2-brane has fragmented into multiple pieces, and the branch is parametrized by the M2-brane locations. At finite energy the Coulomb branch connects to the Higgs branch at singular points where the M2-brane worldvolume degenerates.

At first one might think that the considerations of this paper correspond to the Coulomb branch, since the multi-black hole moduli space is parametrized by the black hole locations. However it is not so simple. The fact that the near horizon sector decouples from the sector describing non-interacting black holes strongly suggests that it is joined to the Higgs branch. Indeed in the D1/D5 black hole, there is a similar near-horizon region of the Coulomb branch which is not only joined to but is in fact a dual description of the singular regions of the Higgs branch [41, 42, 43, 44, 45]. We conjecture there is a similar story here: the near-horizon, multi-black hole quantum mechanics is dual to at least part of the Higgs branch of multiply wrapped M2-branes. Near-horizon microstates should therefore account for at least some of the internal black hole microstates. Exactly how much of the black hole microstructure is accounted for in this way remains to be understood.

Acknowledgments

We thank F. Cachazo, R. Jackiw, J. Maldacena, A. Maloney, G. Papadopoulos, L. Thorlacius, P. Townsend and especially M. Spradlin for enlightening discussions and communication. The research reviewed here was supported in part by NSERC and DOE grant DE-FGO2-91ER40654. We thank the organizers and especially Lárus Thorlacius for an excellent school, the participants for their stimulating questions, and NATO for financial support

at the school.

A. Differential Geometry with Torsion

In this appendix, we give a brief summary of differential calculus with torsion, for the reader who is frustrated by the usual absence of such a discussion in most general relativity books.[14] Recall [47] that the covariant derivative of a tensor is given in terms of the (not necessarily symmetric) connection C^c_{ab}. The torsion $c^c{}_{ab}$ is just the antisymmetric part of the connection:

$$c^c{}_{ab} \equiv C^c_{[ab]} = \tfrac{1}{2}(C^c_{ab} - C^c_{ba}). \tag{A.1}$$

Either by direct computation, or by recalling that the difference between two connections is a tensor, one finds that the torsion is a true tensor. Of course, the torsion does contribute to the curvature tensor, and we remind the reader that many of the familiar symmetries of the curvature tensor are not obeyed in the presence of torsion. Also, if the symmetric part of the connection is given by the Levi-Civita connection, then the full connection annihilates the metric iff the fully covariant torsion tensor $c_{abc} = g_{ad}c^d{}_{bc}$ is completely antisymmetric.

Hopefully, the preceding paragraph was familiar. We now discuss the torsion in a tangent space formalism. As usual, the first step is to define the vielbein e^α_a, which is a basis of cotangent space vectors, labelled by $\alpha = 1, \ldots, n$, where n is the dimension of the manifold, obeying

$$\delta_{\alpha\beta} e^\alpha_a e^\beta_b = g_{ab}. \tag{A.2}$$

The vielbein e^α_a, and the inverse vielbein e^a_α which obeys

$$e^a_\alpha e^\beta_a = \delta^\beta_\alpha, \tag{A.3}$$

can then be used to map tensors into the tangent space; *e.g.* $V^\alpha \equiv V^a e^\alpha_a$.

The connection one-form $\Omega_a{}^\alpha{}_\beta$ is defined by demanding that the vielbein is covariantly constant:

$$\nabla_a e^\alpha_b \equiv \partial_a e^\alpha_b + \Omega_a{}^\alpha{}_\beta e^\beta_b - C^c_{ab} e^\alpha_c = 0. \tag{A.4}$$

Note that equation (A.4) is valid for any choice of connection, and does not imply that the metric is covariantly constant. The metric is covariantly constant iff $\delta_{\alpha\beta}$ is covariantly constant, which in turn holds iff the connection one-form $\Omega_{a\alpha\beta}$ is antisymmetric in the tangent space indices, where

[14]One excellent reference for physicists is [46].

we have lowered the middle index using the tangent space metric $\delta_{\alpha\beta}$. In other words, the familiar antisymmetry of the connection one-form [47] exists if and only if the metric is covariantly constant, whether or not there is torsion.

Equation (A.4) is easily solved for the connection one-form, giving

$$\Omega_a{}^\alpha{}_\beta = e^\alpha_b \partial_a e^b_\beta + C^c_{ab} e^\alpha_c e^b_\beta. \tag{A.5}$$

An immediate corollary of this, and the fact that the difference of two connections C^c_{ab} and C'^c_{ab} is a tensor, is that the difference between two connection one-forms is a tensor, and is, in fact, the same tensor as $C^c_{ab} - C'^c_{ab}$, but with the b and c indices lifted to the tangent bundle.[15]

The unique torsion-free connection one-form which annihilates the metric (*i.e.* that obtained from equation (A.4) using the Levi-Civita connection) is known as the spin connection, and is usually denoted $\omega_a{}^\alpha{}_\beta$. Given a completely antisymmetric torsion $c_{abc} = c_{[abc]}$, as in the first paragraph of this appendix, we define the connection one-form

$$\Omega^{+}_a{}^\alpha{}_\beta = \omega_a{}^\alpha{}_\beta + c^\alpha{}_{a\beta}, \tag{A.6}$$

where, of course, any required mapping between the tangent bundle and the spacetime is achieved by contracting with the vielbein.

As usual, spinors ψ are defined on the tangent bundle, and their covariant derivative is given by

$$\nabla_a \psi = \partial_a \psi - \frac{1}{4}\Omega_{a\alpha\beta}\gamma^{\alpha\beta}\psi, \tag{A.7}$$

where $\gamma^{\alpha\beta} \equiv \frac{1}{2}\left[\gamma^\alpha, \gamma^\beta\right]$ is a commutator of $SO(n)$ γ-matrices, which satisfy $\left\{\gamma^\alpha, \gamma^\beta\right\} = 2\delta^{\alpha\beta}$.

References

1. Hawking, S.W. (1975) Particle Creation by Black Holes, *Comm. Math. Phys.* **43**, 199–220.
2. Hawking, S.W. (1976) Black Holes and Thermodynamics, *Phys. Rev. D* **13**, 191–197.
3. Ferrell, F. and Eardley, D. (1987) Slow-Motion Scattering and Coalescence of Maximally Charged Black Holes, *Phys. Rev. Lett.* **59**, 1617–1620.
4. Gibbons, G.W. and Ruback, P.J. (1986) The Motion of Extreme Reissner-Nordstrom Black Holes in the Low Velocity Limit, *Phys. Rev. Lett.* **57**, 1492–1495.
5. Traschen, J. and Ferrell, R. (1992) Quantum Mechanical Scattering of Charged Black Holes, *Phys. Rev. D* **45**, 2628–2635.
6. Callan, C.G., Coleman, S. and Jackiw, R. (1970) A New Improved Energy-Momentum Tensor, *Ann. Phys. (NY)* **59**, 42–73.

[15] In this discussion, we are assuming that a vielbein has been chosen once and for all; we do not consider the effect of changing frames or coordinates.

7. Jackiw, R. (1972) Introducing Scale Symmetry, *Physics Today* **25**, 23–27.
8. Hagan, C.R. (1972) Scale and Conformal Transformations in Galilean-Covariant Field Theory, *Phys. Rev. D* **5**, 377–388.
9. Niederer, U. (1972) The Maximal Kinematical Invariance Group of the Free Schrödinger Equation, *Helv. Phys. Acta* **45**, 802–810.
10. de Alfaro, V., Fubini, S. and Furlan, G. (1976) Conformal Invariance in Quantum Mechanics, *Nuovo. Cim.* **34A**, 569–612.
11. Claus, P., Derix, M., Kallosh, R., Kumar, J., Townsend, P.K. and Van Proeyen, A. (1998) Black Holes and Superconformal Mechanics, *Phys. Rev. Lett.* **81**, 4553–4556.
12. Maldacena, J. (1998) The Large N Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys. **2**, 231–252.
13. Michelson, J. and Strominger, A. (1999) The Geometry of (Super) Conformal Quantum Mechanics, HUTP-99/A045, hep-th/9907191.
14. Michelson, J. and Strominger, A. (1999) Superconformal Multi-Black Hole Quantum Mechanics, *JHEP* **09**, 005.
15. Treiman, S.B., Jackiw, R. and Gross, D.J. (1972) *Lectures on current algebra and its applications*, Princeton University Press, Princeton.
16. Claus, P., Kallosh, R. and Van Proeyen, A. (1998) Conformal Symmetry on the World Volumes of Branes, KUL-TF-98/54, SU-ITP-98/67, hep-th/9812066.
17. Witten, E. (1981) Dynamical Breaking of Supersymmetry, *Nucl. Phys.* **B188**, 513–554.
18. Witten, E. (1982) Constraints on Supersymmetry Breaking, *Nucl. Phys.* **B202**, 253–316.
19. Witten, E. (1982) Supersymmetry and Morse Theory, *J. Diff. Geom.* **17**, 661–692.
20. Fubini, S. and Rabinovici, E. (1984) Superconformal Quantum Mechanics, *Nucl. Phys.* **B245**, 17–44.
21. Salomonson, P. and van Holten, J.W. (1982) Fermionic Coordinates and Supersymmetry in Quantum Mechanics, *Nucl. Phys.* **B196**, 509–531.
22. Gauntlett, J.P. (1993) Low-Energy Dynamics of Supersymmetric Solitons, *Nucl. Phys.* **B400**, 103–125.
23. Maloney, A., Spradlin, M. and Strominger, A. (1999) Superconformal Multi-Black Hole Moduli Spaces in Four Dimensions, HUTP-99/A055, hep-th/9911001.
24. Coles, R.A. and Papadopoulos, G. (1990) The Geometry of the One-Dimensional Supersymmetric Non-Linear Sigma Models, *Class. Quant. Grav.* **7**, 427–438.
25. Alvarez-Gaumé, L. (1983) Supersymmetry and the Atiyah-Singer Index Theorem, *Comm. Math. Phys.* **90**, 161–173.
26. Friedan, D. and Windey, P. (1984) Supersymmetric Derivation of The Atiyah-Singer Index and the Chiral Anomaly, *Nucl. Phys.* **B235 (FS11)**, 395–416.
27. Sevrin, A., Troost, W. and Van Proeyen, A. (1988) Superconformal Algebras in Two Dimensions with $N = 4$, *Phys. Lett.* **208B**, 447–450.
28. Gibbons, G.W., Papadopoulos, G. and Stelle, K.S. (1997) HKT and OKT Geometries on Soliton Black Hole Moduli Spaces, *Nucl. Phys.* **B508**, 623–658.
29. Gates, S.J. Jr., Hull C.M. and Roček, M. (1984) Twisted Multiplets and New Supersymmetric Non-linear σ-Models, *Nucl. Phys.* **B248**, 157–186.
30. Hull, C.M. (1999) The Geometry of Supersymmetric Quantum Mechanics, QMW-99-16, hep-th/9910028.
31. Grantcharov, G. and Poon, S.-Y. (1999) Geometry of Hyper-Kähler Connections with Torsion, math.DG/9908015.
32. Hellerman, S. and Polchinski, J. (1999) Supersymmetric Quantum Mechanics from Light Cone Quantization, NSF-ITP-99-101, hep-th/9908202.
33. Douglas, M., Polchinski, J. and Strominger, A. (1997) Probing Five-Dimensional Black Holes with D-Branes, *JHEP* **12**, 003.
34. Kallosh, R. (1999) Black Holes and Quantum Mechanics, hep-th/9902007.
35. Gutowski, J. and Papadopoulos, G. (1999) The Dynamics of Very Special Black Holes, hep-th/9910022.

36. Kaplan, D.M. and Michelson, J. (1997) Scattering of Several Multiply Charged Extremal $D = 5$ Black Holes, *Phys. Lett.* **B410** 125–130.
37. Michelson, J. (1998) Scattering of Four-Dimensional Black Holes, *Phys. Rev. D* **57** 1092–1097.
38. Shiraishi, K. (1993) Moduli Space Metric for Maximally-Charged Dilaton Black Holes, *Nucl. Phys.* **B402**, 399–410.
39. Gauntlett, J.P., Myers R.C. and Townsend, P.K. (1999) Black Holes of D=5 Supergravity, *Class. Quant. Grav.* **16**, 1–21.
40. Britto-Pacumio, R., Strominger A. and Volovich A., work in progress.
41. Maldacena, J. and Strominger, A. (1997) Semiclassical Decay of Near Extremal Fivebranes, *JHEP* **12**, 008.
42. Maldacena, J., Michelson, J. and Strominger, A. (1999) Anti-de Sitter Fragmentation, *JHEP* **03**, 011.
43. Seiberg, N. and Witten, E. (1999) The D1/D5 System and Singular CFT, *JHEP* **04**, 017.
44. Berkooz, M. and Verlinde, H. (1999) Matrix Theory, AdS/CFT and Higgs-Coulomb Equivalence, IASSNS-HEP-99/67, PUPT-1879, hep-th/9907100.
45. Aharony, O. and Berkooz, M. (1999) IR Dynamics of d=2, N=(4,4) Gauge Theories and DLCQ of "Little String Theories", PUPT-1886, RUNHETC-99-31, hep-th/9909101.
46. Nakahara, M. (1990) *Geometry, Topology and Physics*, Institute of Physics Publishing, Philadelphia.
47. Wald, R.M. (1984) *General Relativity*, The University of Chicago Press, Chicago.