Thus, collecting our results together and defining the constants $\mu = GM/c^2$ and $q^2 = GQ^2/(4\pi\epsilon_0 c^4)$, the line element for the spacetime outside a static spherically symmetric body of mass M and charge Q has the form

$$\boxed{ds^2 = c^2\left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),} \tag{12.44}$$

from which one may read off the metric coefficients $g_{\mu\nu}$ that determine the gravitational field of the object. The resulting solution is known as the *Reissner–Nordström* geometry. The electromagnetic $F_{\mu\nu}$ of the field of the object is given by (12.36) with

$$\boxed{E(r) = \frac{Q}{4\pi\epsilon_0 r^2}.}$$

12.6 The Reissner–Nordström geometry: charged black holes

The Reissner–Nordström (RN) metric (12.44) is only valid down to the surface of the charged object. As in our discussion of the Schwarzschild solution, however, it is of interest to consider the structure of the full RN geometry, namely the solution to the coupled Einstein–Maxwell field equations for a charged *point* mass located at the origin $r = 0$, in which case the RN metric is valid for all positive r.

Calculation of the invariant curvature scalar $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ shows that the only intrinsic singularity in the RN metric occurs at $r = 0$. In the 'Schwarzschild-like' coordinates (t, r, θ, ϕ), however, the RN metric also possesses a coordinate singularity wherever r satisfies

$$\Delta(r) \equiv 1 - \frac{2\mu}{r} + \frac{q^2}{r^2} = 0, \tag{12.45}$$

with $\Delta(r) = -1/g_{11}(r) = g_{00}(r)/c^2$. Multiplying (12.45) through by r^2 and solving the resulting quadratic equation, we find that the coordinate singularities occur on the surfaces $r = r_\pm$, where

$$\boxed{r_\pm = \mu \pm (\mu^2 - q^2)^{1/2}.} \tag{12.46}$$

It is clear that there exist three distinct cases, depending on the relative values of μ^2 and q^2; we now discuss these in turn.

- *Case 1*: $\mu^2 < q^2$ In this case $r_\pm$ are both imaginary, and so no coordinate singularities exist. The metric is therefore regular for all positive values of r. Since the function $\Delta(r)$ always remains positive, the coordinate t is always timelike and r is always spacelike. Thus, the intrinsic singularity at $r = 0$ is a timelike line, as opposed to a spacelike line in the Schwarzschild case. This means that the singularity does not necessarily lie in the future of timelike trajectories and so, in principle, can be avoided. In the absence of any event horizons, however, $r = 0$ is a *naked singularity*, which is visible to the outside world. The physical consequences of a naked singularity, such as the existence of closed timelike curves, appear so extreme that Penrose has suggested the existence of a *cosmic censorship hypothesis*, which would only allow singularities that are hidden behind an event horizon. As a result, the case $\mu^2 < q^2$ is not considered physically realistic.

- *Case 2*: $\mu^2 > q^2$ In this case, $r_\pm$ are both real and so there exist *two* coordinate singularities, occurring on the surfaces $r = r_\pm$. The situation at $r = r_+$ is very similar to the Schwarzschild case at $r = 2\mu$. For $r > r_+$, the function $\Delta(r)$ is positive and so the coordinates t and r are timelike and spacelike respectively. In the region $r_- < r < r_+$, however, $\Delta(r)$ becomes negative and so the physical natures of the coordinates t and r are interchanged. Thus, a massive particle or photon that enters the surface $r = r_+$ from outside must necessarily move in the direction of decreasing r, and thus $r = r_+$ is an event horizon. The major difference from the Schwarzschild geometry is that the irreversible infall of the particle need only continue to the surface $r = r_-$, since for $r < r_-$ the function $\Delta(r)$ is again positive and so t and r recover their timelike and spacelike properties. Within $r = r_-$, one may (with a rocket engine) move in the direction of either positive or negative r, or stand still. Thus, one may avoid the intrinsic singularity at $r = 0$, which is consistent with the fact that $r = 0$ is a timelike line. Perhaps even more astonishing is what happens if one then chooses to travel back in the direction of positive r in the region $r < r_-$. On performing a *maximal analytic extension* of the RN geometry, in analogy with the Kruskal extension for the Schwarzschild geometry discussed in Section 11.9, one finds that one may *re-cross* the surface $r = r_-$, but this time from the inside. Once again one is moving from a region in which r is spacelike to a region in which it is timelike, but this time the sense is reversed and one is forced to move in the direction of *increasing* r. Thus $r = r_-$ acts as an 'inside-out' event horizon. Moreover, one is eventually forceably ejected from the surface $r = r_+$ but, according to the maximum analytic extension, the particle emerges into a asymptotically flat spacetime *different* from that from which it first entered the black hole. As discussed in Section 11.9, however, such matters are at best highly speculative, and we shall not pursue them further here.

- *Case 3*: $\mu^2 = q^2$ In this case, called the *extreme* Reissner–Nordström black hole, the function $\Delta(r)$ is positive everywhere *except* at $r = \mu$, where it equals zero. Thus, the coordinate r is everywhere spacelike except at $r = \mu$, where it becomes null, and hence $r = \mu$ is an event horizon. The extreme case is basically the same as that considered in case 2, but with the region $r_- < r < r_+$ removed.

We may illustrate the properties of the RN spacetime in more detail by considering the paths of photons and massive particles in the geometry, which we now go on to discuss. Since the case $\mu^2 > q^2$ is the most physically reasonable RN spacetime, we shall restrict our discussion to this situation.

12.7 Radial photon trajectories in the RN geometry

Let us begin by investigating the paths of radially incoming and outgoing photons in the RN metric for the case $\mu^2 > q^2$. Since $ds = d\theta = d\phi = 0$ for a radially moving photon, we have immediately from (12.44) that

$$\frac{dt}{dr} = \pm\frac{1}{c} = \left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)^{-1} = \pm\frac{1}{c}\frac{r^2}{(r-r_-)(r-r_+)}, \tag{12.47}$$

where, in the second equality, we have used the result (12.46); the plus sign corresponds to an *outgoing* photon and the minus sign to an *incoming* photon. On integrating, we obtain

$$ct = r - \frac{r_-^2}{r_+ - r_-}\ln\left|\frac{r}{r_-} - 1\right| + \frac{r_+^2}{r_+ - r_-}\ln\left|\frac{r}{r_+} - 1\right| + \text{constant} \qquad \text{(outgoing)},$$

$$ct = -r + \frac{r_-^2}{r_+ - r_-}\ln\left|\frac{r}{r_-} - 1\right| - \frac{r_+^2}{r_+ - r_-}\ln\left|\frac{r}{r_+} - 1\right| + \text{constant} \qquad \text{(ingoing)}.$$

We will concentrate in particular on the ingoing radial photons. To develop a better description of infalling particles in general, we may construct the equivalent of the advanced Eddington–Finkelstein coordinates derived for the Schwarzschild metric in Section 11.5. Once again this coordinate system is based on radially infalling photons, and the trick is to use the integration constant as the new coordinate, which we denote by p. As before, p is a null coordinate and it is more convenient to work instead with the timelike coordinate t' defined by $ct' = p - r$. Thus, we have

$$ct' = ct - \frac{r_-^2}{r_+ - r_-}\ln\left|\frac{r}{r_-} - 1\right| + \frac{r_+^2}{r_+ - r_-}\ln\left|\frac{r}{r_+} - 1\right|. \tag{12.48}$$

On differentiating, or from (12.47) directly, one obtains

$$c\,dt' = dp - dr = c\,dt + \left[\frac{1}{\Delta(r)} - 1\right] dr, \tag{12.49}$$

where $\Delta(r)$ is defined in (12.45). Using the above expression to substitute for c in (12.44), one quickly finds that

$$ds^2 = c^2\Delta\, dt'^2 - 2(1-\Delta)\, dt'\, dr - (2-\Delta)\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

which is the RN metric in advanced Eddington–Finkelstein coordinates. In particular, we note that this form is regular for all positive values of r and has an instrinsic singularity at $r = 0$.

From (12.47) and (12.49), one finds that, in advanced Eddington–Finkelstein coordinates, the equation for ingoing radial photon trajectories is

$$ct' + r = \text{constant}, \tag{12.50}$$

whereas the trajectories for outgoing radial photons satisfy the differential equation

$$c\frac{dt'}{dr} = \frac{2-\Delta}{\Delta}. \tag{12.51}$$

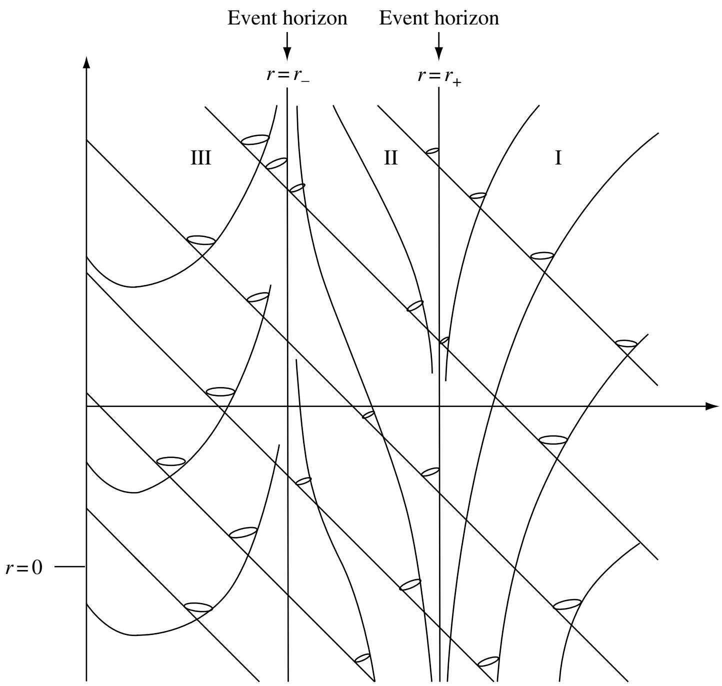

Figure 12.1 Spacetime diagram of the Reissner–Nordström solution in advanced Eddington–Finkelstein coordinates. The straight diagonal lines are ingoing photon worldlines whereas the curved lines correspond to outgoing photon worldlines.

We may use these equations to determine the light-cone structure of the RN metric in these coordinates. For ingoing radial photons, the trajectories (12.50) are simply straight lines at 45° in a spacetime diagram. For outgoing radial photons, (12.51) gives the gradient of the trajectory at any point in the spacetime diagram, and so one may sketch these without solving (12.51) explicitly. This resulting spacetime diagram is shown in Figure 12.1. It is worth noting that the light-cone structure depicted confirms the nature of the event horizon at $r = r_+$. Moreover, the light-cones remain tilted over in the region $r_- < r < r_+$, indicating that any particle falling into this region must move inwards until it reaches $r = r_-$. Once in the region $r < r_-$, the lightcones are no longer tilted and so particles need not fall into the singularity $r = 0$. As was the case in Section 11.5 for the Schwarzschild metric, however, this spacetime diagram may be somewhat misleading. For an outward-moving particle in the region $r < r_-$, Figure 12.1 suggests that it can only reach $r = r_-$ asymptotically, but by peforming an analytic extension of the RN solution one can show that the particle can cross the surface $r = r_-$ in finite proper time.

12.8 Radial massive particle trajectories in the RN geometry

We now consider the trajectories of radially moving massive particles for the case $\mu^2 > q^2$. To simplify our discussion, we will assume that the particles are electrically neutral. In this case, the particles will follow geodesics. In the more general case of an electrically charged particle, one must also take into account the Lorenz force on the particle produced by the electromagnetic field of the black hole. The equation of motion for the particle is then given by (6.13).

For a radially moving particle, the 4-velocity has the form

$$[u^\mu] = (u^0, u^1, 0, 0) = (\dot{t}, \dot{r}, 0, 0),$$

where the dots denote differentiation with respect to the proper time τ of the particle. The geodesic equations of motion, obeyed by neutral particles in the RN metric, are most conveniently written in the form (3.56):

$$\dot{u}_\sigma = \tfrac{1}{2}(\partial_\sigma g_{\mu\nu}) u^\mu u^\nu.$$

Since the metric coefficients in the RN line element (12.44) do not depend on t, we immediately obtain

$$\boxed{u_0 = g_{00}\dot{t} = \text{constant.}}$$

The radial equation of motion may then be obtained using the normalisation condition $g_{\mu\nu} u^\mu u^\nu = c^2$, which gives

$$g_{00}(u^0)^2 + g_{11}(u^1)^2 = c^2.$$

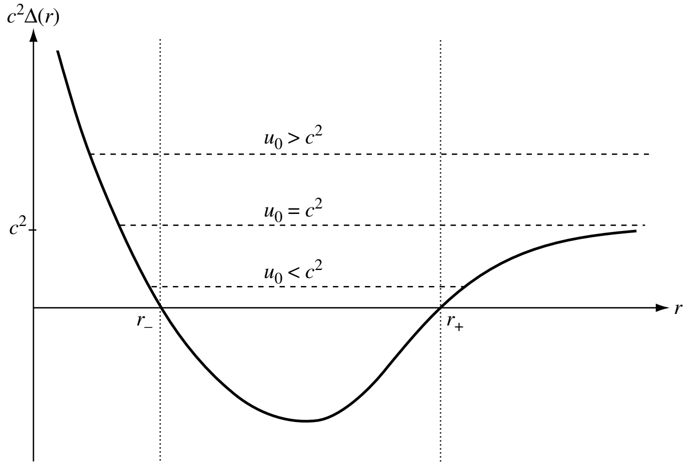

Figure 12.2 The limits of radial motion for a neutral massive particle in the Reissner–Nordström geometry.

Using the fact, from (12.45), that $\Delta(r) = g_{00}/c^2 = -1/g_{11}$, one finds that

$$\boxed{\dot{r}^2 + c^2\Delta(r) = \frac{u_0^2}{c^2}.} \tag{12.52}$$

This clearly has the form of an 'energy' equation, in which $c^2\Delta(r)$ plays the role of a potential. Qualitative information on the properties of the radial trajectories can be obtained directly from (12.52) by simply plotting the function $c^2\Delta(r)$; this plot is shown in Figure 12.2. The radial limits of the motion depend on the choice of the constant u_0, as indicated. The case $u_0 = c^2$ corresponds to the particle being released from rest at infinity. In all cases, there exists an inner radial limit that is greater than zero. This indicates that a neutral particle moving freely under gravity *cannot reach* the central intrinisic singularity at $r = 0$ but is instead repelled once it has approached to within some finite distance. As mentioned in Section 12.6. performing a maximum analytic extension the RN metric suggests that the particle passes back through $r = r_-$ and $r = r_+$ and ultimately emerges in a different asymptotically flat spacetime.

Exercises

12.1 For a general static diagonal metric, show that the 4-velocity of a perfect fluid in the spacetime must have the form

$$[u^\mu] = \frac{c}{\sqrt{g_{00}}}(1, 0, 0, 0).$$

12.2 Calculate the gravitational binding energy $E = \widetilde{M} - M$ of a spherical star of constant density ρ and coordinate radius R. Compare your answer with the corresponding Newtonian result and interpret your findings physically.

12.3 Derive the Oppenheimer–Volkoff equation from the Einstein equations for a static spherically symmetric perfect-fluid distribution, and show that it reduces to the standard equation for hydrostatic equilibrium in the Newtonian limit.

12.4 In Newtonian gravity, show directly that the equation for hydrostatic equilibrium is

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}.$$

12.5 Show that, in the Newtonian limit, the equation before (12.15) reduces to

$$\frac{d\Phi(r)}{dr} = \frac{Gm(r)}{r},$$

where $\Phi(r)$ is the Newtonian gravitational potential.

12.6 For a spherical star of uniform density ρ and central pressure p_0, verify that the Oppenheimer–Volkoff equation requires $p(r)$ to satisfy

$$\frac{\rho c^2 + 3p(r)}{\rho c^2 + p(r)} = \frac{\rho c^2 + 3p_0}{\rho c^2 + p_0}\left(1 - \frac{8\pi G}{3c^2}\rho r^2\right)^{1/2},$$

and hence show that

$$p(r) = \rho c^2 \frac{(1 - 2\mu r^2/R^3)^{1/2} - (1 - 2\mu/R)^{1/2}}{3(1 - 2\mu/R)^{1/2} - (1 - 2\mu r^2/R^3)^{1/2}},$$

where R is the coordinate radius of the star.

12.7 In Newtonian gravity, obtain the expression for $p(r)$ for a spherical star of uniform density ρ, central pressure p_0 and radius R. Compare your result with that obtained in Exercise 12.6.

12.8 Show that, for a spherical star of uniform density ρ,

$$R^2 < \frac{c^2}{3\pi G\rho} \quad \text{and} \quad M^2 < \frac{16c^6}{243\pi\rho G^3}.$$

If a photon is emitted from the star's surface and received by a stationary observer at infinity, show that the observed redshift must obey the constraint $z < 2$. Show also, however, that the observed redshift for a photon emitted from the star's centre can be arbitrarily large.

12.9 For a spherical star of uniform density ρ, show that in order for the star not to lie within its own Schwarzschild radius, one requires

$$M^2 < \frac{3c^6}{32\pi\rho G^3}.$$

Compare this limit with that derived in Exercise 12.8.

12.10 For a spherical uniform-density star of mass M and coordinate radius R, show that the line element of spatial sections with $t =$ constant can be written in the form

$$d\sigma^2 = \frac{Rc^2}{2GM}\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right].$$

12.11 Consider a static infinitely long cylindrical configuration of matter that is invariant to translations and Lorentz boosts along the axis of symmetry (a cosmic string). Adopting 'cylindrical polar' coordinates (ct, r, ϕ, z), show that a self-consistent solution to the Einstein field equations may be obtained if the stress-energy tensor for the matter is of the form

$$[T^{\mu\nu}] = \mathrm{diag}(\rho c^2, 0, 0, -\rho c^2),$$

such that there is a negative pressure (or tension) along the string, and the line element is of the form

$$ds^2 = c^2\,dt^2 - dr^2 - B(r)\,d\phi^2 - dz^2,$$

where $B(r)$ satisfies

$$\frac{B''}{2B} - \frac{(B')^2}{4B^2} = -\kappa\rho c^2.$$

Show further that $b(r) = \sqrt{B(r)}$ satisfies $b'' = -\kappa c^2 \rho b$.
Hint: You may find your answers to Exercises 8.9, 9.28 and 9.29 useful.

12.12 Suppose that the matter distribution in a cosmic string has a uniform density across the string, such that

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r \le r_0, \\ 0 & \text{for } r > r_0. \end{cases}$$

By demanding that $g_{\phi\phi} \to -r^2$ as $r \to 0$, so that the spacetime geometry is regular on the axis of the string, show that the line element for $r \le r_0$ is

$$ds^2 = c^2\,dt^2 - dr^2 - \left(\frac{\sin\lambda r}{\lambda r}\right)^2 d\phi^2 - dz^2,$$

where $\lambda = \sqrt{\kappa\rho_0 c^2}$. By demanding that $g_{\phi\phi}$ and its derivative with respect to r are both continuous at $r = r_0$, show that the line element for $r > r_0$ is

$$ds^2 = c^2\,dt^2 - dr^2 - \left[\frac{\sin\lambda r_0}{\lambda r} + (r - r_0)\cos\lambda r_0\right]^2 d\phi^2 - dz^2.$$

For the interesting case in which $\lambda r_0 \ll 1$, show that for $r \gg r_0$ the line element takes the form

$$ds^2 = c^2\,dt^2 - dr^2 - \left(1 - \frac{8G\mu}{c^2}\right) r^2\,d\phi^2 - dz^2,$$

where $\mu = \pi r_0^2 \rho_0$ is the 'mass per unit length' of the string. Interpret this line element physically.

12.13 Show that the electromagnetic field tensor outside a static spherically symmetric charged matter distribution has the form

$$[F_{\mu\nu}] = E(r) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $E(r)$ is some arbitrary function. Hence show that, if the line element outside the matter distribution has the form

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

the energy–momentum tensor of the electromagnetic field in this region is given by

$$[T_{\mu\nu}] = \frac{1}{2}c^2\epsilon_0 E^2\,\mathrm{diag}\left(\frac{1}{B}, -\frac{1}{A}, \frac{r^2E^2}{AB}, \frac{r^2E^2\sin^2\theta}{AB}\right).$$

12.14 Calculate the invariant curvature scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ for the Reissner–Nordström geometry and hence show that the only intrinsic singularity occurs at $r = 0$.

12.15 Show that the worldlines of radially moving photons in the Reissner–Nordström geometry are given by

$$ct = r - \frac{r_-^2}{r_+ - r_-}\ln\left|\frac{r}{r_-} - 1\right| + \frac{r_+^2}{r_+ - r_-}\ln\left|\frac{r}{r_+} - 1\right| + \text{constant} \qquad \text{(outgoing)},$$

$$ct = -r + \frac{r_-^2}{r_+ - r_-}\ln\left|\frac{r}{r_-} - 1\right| - \frac{r_+^2}{r_+ - r_-}\ln\left|\frac{r}{r_+} - 1\right| + \text{constant} \qquad \text{(ingoing)}.$$

12.16 Show that, by introducing the advanced Eddington–Finkelstein timelike coordinate

$$ct' = ct - \frac{r_-^2}{r_+ - r_-}\ln\left|\frac{r}{r_-} - 1\right| + \frac{r_+^2}{r_+ - r_-}\ln\left|\frac{r}{r_+} - 1\right|,$$

the Reissner–Nordström line element takes the form

$$ds^2 = c^2\Delta\,dt'^2 - 2(1-\Delta)\,dt'\,dr - (2-\Delta)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

where $\Delta \equiv \Delta(r) = 1 - 2\mu/r + q^2/r^2$. Hence show that the worldlines of radially moving photons in advanced Eddington–Finkelstein coordinates are given by

$$ct' + r = \text{constant} \quad \text{(incoming)}, \qquad c\frac{dt'}{dr} = \frac{2-\Delta}{\Delta} \quad \text{(outgoing)}.$$

What is the significance, if any, of the fact that $c\,dt'/dr = 0$ at $\Delta(r) = 2$ for outgoing radially moving photons?

12.17 For a particle of mass m and charge e in geodesic motion in the Reissner–Nordström geometry, show that the quantity

$$k = m\left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)\frac{dt}{d\tau} + \frac{eq}{r}$$

is conserved, and interpret this result physically.

12.18 An observer is in a circular orbit of coordinate radius $r = R$ in the Reissner–Nordström geometry. Find the components of the magnetic field measured by the observer.