A Theory of Long-Period Magnetic Pulsations
1. Steady State Excitation of Field Line Resonance

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A theory of long-period (Pc 3 to Pc 5) magnetic pulsations is presented based on the idea of a steady state oscillation of a resonant local field line that is excited by a monochromatic surface wave at the magnetosphere. A coupled wave equation between the shear Alfvén wave representing the field line oscillation and the surface wave is derived and solved for the dipole coordinates. The theory gives the frequency, the sense of polarizations, orientation angle of the major axis, and the ellipticity as a function of magnetospheric parameters. It also clarifies some of the contradicting ideas and observations in relation to the sense of polarization and excitation mechanism. At lower latitude it is shown that the orientation angle rather than the sense of rotation is a more critical parameter in finding the direction of wave propagation in the azimuthal coordinate and hence in finding the evidence of wave excitation at the magnetospheric surface by the solar wind.

Theories of magnetic pulsations can be categorized basically into two groups. One group of pulsations primarily concerns those with the excitation mechanism, which is the 'active' aspect of the problem, and the other group of pulsations concerns those with the resonance mechanism, which is the 'passive' aspect of the problem. For the long-period pulsations ($\tau \geq 45$ s) the former group considers excitation by the Kelvin-Helmholtz instability of the outer magnetospheric plasmas due to the solar wind [*Dungey*, 1955; *Parker*, 1958; *Sen*, 1963; *Southwood*, 1968; *Boller and Stolov*, 1970] or by the local plasma instabilities such as two-stream [*Nishida*, 1964; *Kimura and Matsumoto*, 1968], drift wave [*Swift*, 1967; *Hasegawa*, 1971a, b], or mirror [*Hasegawa*, 1969] instabilities.

The theories that treat the passive aspect of the problem are mostly centered around the idea of the magnetospheric cavity resonance [*Watanabe*, 1961; *Dungey*, 1963; *Radoski*, 1966] or the resonance of the local field line (in the limit of a large wave number in the azimuthal direction [*Radoski*, 1967a; *Dungey*, 1968; *Orr*, 1973]).

Observationally, these long-period continuous pulsations can be characterized by the following properties.

1. At higher latitude the mostly circularly polarized pulsations reverse their sense of polarization near noon [*Samson et al.*, 1971], whereas there is no such systematic reversal at lower latitude [*Lanzerotti et al.*, 1972].

2. Polarization reversal also occurs at a different latitude [*Samson et al.*, 1971]. The demarcation line coincides with the line of peak wave amplitude and linear polarization.

3. Systematic change of the orientation angle in the H-D plane occurs from the second to the first (the first to the second) quadrant near noon in the northern (southern) hemisphere [*Van-Chi et al.*, 1968; *Lanzerotti et al.*, 1972]. (As is illustrated in Figure 1, the tendency is less obvious in the afternoon sector at Siple.)

4. For each of the events the frequency is independent of latitude, but when it is averaged over many events, the frequency of the peak amplitude is a decreasing function of latitude [*Samson and Rostoker*, 1972].

When the consequences of the theoretical predictions are

compared with these properties of pulsations, it can easily be seen that none of the foregoing theories are satisfactory enough to explain the observed phenomena consistently.

For example, although the cavity or the field line resonance ideas can explain either the latitude independent or dependent pulsations by choosing a small or large azimuthal wave number, they fail to explain a change of the sense of polarization or the orientation angle near noon. Nor do they explain why most of the pulsations are elliptically polarized with its major axis always tilted in the H-D plane, i.e., not aligned with the H or D axis. On the other hand, the Kelvin-Helmholtz theory, which is successful in explaining the switch of the sense of polarization near noon, fails to explain latitude dependent frequency and the lower-latitude pulsations that do not reveal systematic switches of the sense of polarization.

The local instability theories may apply to some specific cases, but again they are difficult to apply to the majority of the phenomena that seem to have dependency on a global effect such as dawn-dusk asymmetry.

A natural step to take here is therefore to combine the active and passive theories and to treat the problem of coupling between the active and the passive modes. Because many of the high-latitude pulsations seem to switch their sense of polarization near noon, for the active mode it may be natural to take the Kelvin-Helmholtz mode or a similar surface mode excited by the solar wind friction. Whereas for a passive mode it is natural to take the local field line oscillation by the shear Alfvén wave because of the observed localization of the pulsations. A coupling between these modes is being studied independently by *Southwood* [1973], using a straight field line model. We employ the dipole field, which is crucial in deciding the orientation angle of the major axis of the polarization ellipse as will be seen.

Method of Approach

Here we introduce the basic method of the theoretical approach to the problem. We consider the oscillation of the local field line excited by a surface wave. The surface wave is assumed to be monochromatic and is excited by the Kelvin-Helmholtz instability at the magnetospheric bound-
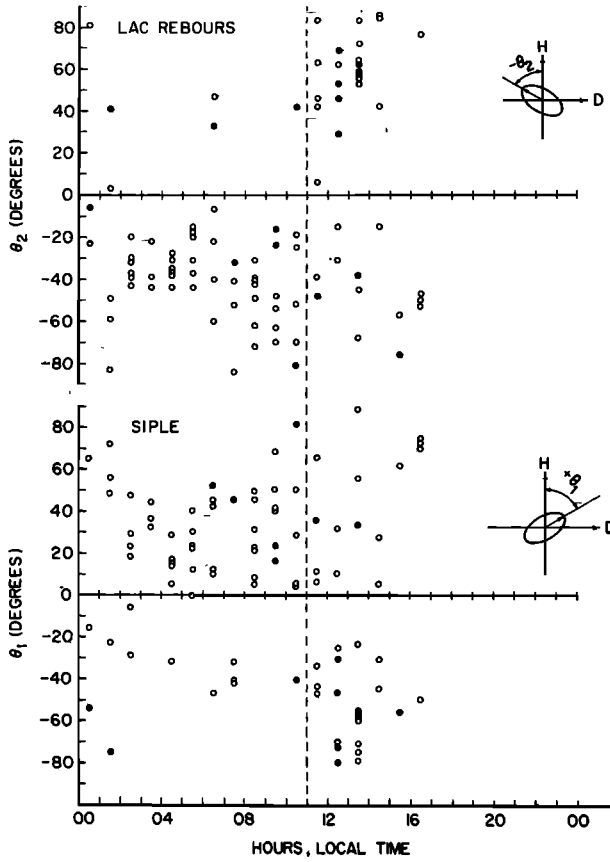
Fig. 1. Orientation angles of the major axes in the H-D plane polarization ellipses at Lac Rebours (47°52'N, 72°27'W geographic) and Siple (76°55'S, 83°55'W). The dashed line illustrates the dawn-dusk asymmetry [after *Lanzerotti et al.*, 1972].

ary or by some other type of friction between the solar wind and the magnetopause. We start with a review of the Kelvin-Helmholtz instability. In a uniform plasma the surface wave equation that may be excited by the Kelvin-Helmholtz instability is given by [e.g., *Sen*, 1963]

$$\nabla^2[p + (\mathbf{b}\cdot\mathbf{B}/\mu_0)] = 0 \qquad (1)$$

where p is the perturbed plasma pressure, $\mathbf{b}$ and $\mathbf{B}$ are the perturbed and unperturbed magnetic flux density, and μ_0 is the vacuum magnetic permeability. Because this is a Laplace equation, the general solution associated with the surface perturbation is of the form

$$\exp\left[i(k_\perp x + k_\parallel z) - (k_\perp^2 + k_\parallel^2)^{1/2}\,|y|\right]$$

where x and z are parallel to the surface and y is perpendicular to the surface. Note that the exponentiating distance is comparable to the wavelength parallel to the surface. Just inside the magnetospheric boundary, x, y, and z correspond approximately to the azimuthal coordinate, the radial (inward) coordinate, and the coordinate parallel to the magnetic field, respectively.

Because the magnetospheric boundary is not under a dynamical equilibrium, a dissipative layer must be assumed there in general [*Eviatar and Wolf*, 1968]. This assumption modifies the simple theory of the Kelvin-Helmholtz instability. However, we assume here that the ordinary result of the Kelvin-Helmholtz theory still holds if the perturbed

wavelength is much larger than the proton Larmor radius, which is a typical size of such a dissipative layer.

When incompressibility is assumed for simplicity, the condition of the instability is well known (see, for example, *Boller and Stolov* [1970]) and is given by

$$(\mathbf{k}\cdot\mathbf{v}_s)^2 > \left(\frac{1}{N_s} + \frac{1}{N_m}\right)\times[N_s(\mathbf{k}\cdot\mathbf{v}_A)_s^2 + N_m(\mathbf{k}\cdot\mathbf{v}_A)_m^2] \quad (2)$$

where subscripts s and m stand for quantities for the solar wind and the magnetosphere, $\mathbf{v}_s$ is the velocity of the solar wind, $\mathbf{v}_A$ is the Alfvén velocity whose direction is taken to be parallel to the unperturbed magnetic field, and N is the unperturbed number density of the plasma. Because $|v_A| \propto BN^{-1/2}$, the magnetic flux density inside the magnetosphere is much larger than that outside, and $N_s \gg N_m$, (2) can in most cases be simplified to

$$\mathbf{k}\cdot\mathbf{v}_s \sim k_\perp v_s > (\mathbf{k}\cdot\mathbf{v}_A)_m = k_\parallel v_A \qquad (2')$$

where $k_\parallel$ is the wave number parallel to the magnetic field and v_A is the magnitude of the Alfvén velocity within the magnetosphere. When the compressed geomagnetic field at the dayside ($\sim 100\ \gamma$), $v_A \sim 2 \times 10^3$ km/s, and $v_s \sim 4 \times 10^2$ km/s are taken into account, (2') is satisfied with a large perpendicular (azimuthal) wave number $k_\perp \gtrsim 5\ k_\parallel$, and the minimum value of $k_\parallel$ is decided by the local length of the field line s by $k_\parallel = \pi/s$.

At the threshold of the instability the excited frequency is given by

$$\omega = \mathbf{k}\cdot\mathbf{v}_s\,\frac{N_s}{N_s + N_m} \sim \mathbf{k}\cdot\mathbf{v}_s \sim k_\parallel v_A \qquad (3)$$

Therefore it is conceivable that a set of discrete monochromatic frequencies corresponding to the $(2n - 1)\pi$ ($n = 1$, 2, $\cdots$) modes in the standing shear Alfvén wave at the magnetospheric boundary is preferentially excited.

Now we consider how this surface wave can couple to a local field line oscillation deep within the magnetosphere ($L \sim 4$). We assume that a one-fluid MHD equation can sufficiently describe the Alfvén wave perturbation. With this assumption we ignore a possible interaction with electrostatic drift waves. After the electric field is eliminated, the relevant linearized MHD equations become

$$\rho_m\ddot{\boldsymbol{\xi}} = \frac{1}{\mu_0}(\nabla \times \mathbf{b}) \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{b} - \nabla p \qquad (4)$$

$$\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \qquad (5)$$

where $\boldsymbol{\xi}$ is the displacement vector defined by

$$\partial\boldsymbol{\xi}/\partial t = \mathbf{v} \qquad (6)$$

and $\mathbf{v}$ is the perturbed fluid velocity. In (4), ρ_m and p are mass density ($= m_i N$) and the perturbed plasma pressure. Because we are interested in the oscillation of the local field line, which is possible only in the shear Alfvén wave perturbation, we seek for an incompressible perturbation. The mass conservation gives

$$n + N\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla N = 0 \qquad (7)$$

where n is the perturbed number density. From (7) we can see immediately that the ordinary incompressibility assumption, $n \sim 0$, $\nabla\cdot\boldsymbol{\xi} \sim 0$, does not hold because of the nonuniform plasma density, $\nabla N \neq 0$. The incompressible perturbation

$(n \sim 0)$ will have a finite $\nabla \cdot \boldsymbol{\xi} \sim -\kappa_N \cdot \boldsymbol{\xi}$, where $\kappa_N \sim \nabla (\ln N)$; otherwise, a finite density perturbation $n \sim \boldsymbol{\xi} \cdot \nabla N \sim (\kappa_N \cdot \boldsymbol{\xi}) N$ results. This fact indicates a coupling between the shear Alfvén wave of our concern and a compressional mode.

In association with the variation of the number density, the plasma pressure also changes. Using the adiabatic assumption and (7), we have for the pressure perturbation

$$p = -NT(\kappa_P \cdot \boldsymbol{\xi} + \gamma \nabla \cdot \boldsymbol{\xi}) \qquad (8)$$

where T is the plasma temperature (in energy units) and $\kappa_P \sim \nabla (\ln NT)$. The set of equations (4), (5), (7), and (8) contains three basic MHD waves: the ion acoustic wave, the magnetosonic (compressional or isotropic Alfvén) wave, and the shear (anisotropic) Alfvén wave. (The incompressional surface wave of (1), as will be shown, becomes an evanescent compressional mode in the presence of compressibility.) Because we are looking at a wave having a large perpendicular wave number in consistency with the Kelvin-Helmholtz perturbations, the magnetosonic wave will have a frequency much higher than the frequency of our interest (which is of the order of the shear Alfvén wave resonance of the local field line). This fact enables us to reduce the coupling between the shear Alfvén wave and the rest of the mode by using two small parameters ϵ and β, where

$$\epsilon \sim k_\parallel / k_\perp \sim \kappa_N / k_\perp \sim \kappa_B / k_\perp \sim \kappa_P / k_\perp \ll 1 \qquad (9)$$

$$\beta = \frac{NT}{B^2 / 2\mu_0} \qquad (10)$$

with $\kappa_B = \nabla \ln B$.

With these preparations we introduce here the basic scheme of the coupling and the theoretical approach to the problem.

Let us first see how the surface wave (1) is modified owing to the nonuniformity. After simple vector operations in (4) using $\nabla \cdot \mathbf{b} = \nabla \cdot \mathbf{B} = 0$ we have

$$\nabla(\mu_0 p + \mathbf{b} \cdot \mathbf{B}) = 2(\mathbf{b} \cdot \nabla)\mathbf{B} - \mu_0 \rho_m \ddot{\boldsymbol{\xi}} + \nabla \times (\mathbf{b} \times \mathbf{B}) \qquad (4')$$

Hence if we take the divergence of both sides, we have

$$\nabla^2(p + \mathbf{b} \cdot \mathbf{B}/\mu_0) = 2\nabla \cdot [(\mathbf{b} \cdot \nabla)\mathbf{B}]/\mu_0 - \nabla \cdot (\rho_m \ddot{\boldsymbol{\xi}}) \qquad (11)$$

If the plasma is uniform and if the incompressibility ($\nabla \cdot \boldsymbol{\xi} = 0$) is assumed, we recover the wave equation of the surface wave given in (1). If the plasma density were uniform, we could assume an incompressible perturbation, but $\nabla^2(p + \mathbf{b} \cdot \mathbf{B}/\mu_0) \neq 0$ if the magnetic field were nonuniform. This finding shows that the surface wave can couple to the shear Alfvén wave through the nonuniform magnetic field. The coupling would vanish if the perturbed magnetic field were linearly polarized in the azimuthal direction (because then $(\mathbf{b} \cdot \nabla)\mathbf{B} = 0$), but because the surface wave is circularly polarized, that cannot be the case.

We now look at the modification of the shear Alfvén wave by the surface wave. From (4),

$$\rho_m \ddot{\boldsymbol{\xi}} = -\nabla[p + (\mathbf{b} \cdot \mathbf{B}/\mu_0)] + \frac{1}{\mu_0}[(\mathbf{b} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{b}] \qquad (4'')$$

whereas from (5),

$$\mathbf{b} = -\mathbf{B}(\nabla \cdot \boldsymbol{\xi}) + (\mathbf{B} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{B} \qquad (5')$$

Substituting (5') into (4''), we obtain

$$\mu_0 \rho_m \ddot{\boldsymbol{\xi}} - (\mathbf{B} \cdot \nabla)^2 \boldsymbol{\xi} = -\mu_0 \nabla[p + (\mathbf{B} \cdot \mathbf{b}/\mu_0)] + \mathbf{C} \qquad (12)$$

where

$$\mathbf{C} = (\mathbf{b} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)[(\boldsymbol{\xi} \cdot \nabla)\mathbf{B} + \mathbf{B}(\nabla \cdot \boldsymbol{\xi})] \qquad (13)$$

with $\mathbf{b}$ given by (5'). Equation 12 illustrates very nicely the coupling between the wave equation of the shear Alfvén wave, representing the oscillation of the field line (right-hand side is equal to 0), and the surface wave (equation 1) through the coupling coefficient $\mathbf{C}$, which appears in consequence of the nonuniform magnetic field.

However, if we evaluate the orders of magnitude of both sides of (12) using our small parameters ϵ and β, the right-hand side is found to involve terms larger than the left-hand side by a factor of ϵ^{-1}. This result implies that the field line resonance of the shear Alfvén wave is completely wiped out unless a particular choice of $\boldsymbol{\xi}$ vector is made such that it reduces the size of these large terms on the right-hand side. The property of this unique $\boldsymbol{\xi}$ vector depends on the actual shape of the unperturbed magnetic field, and a concrete form will be shown in the next section.

WAVE EQUATION IN DIPOLE COORDINATES

In this section we elaborate on the approach described in the preceding section to derive the coupled wave equation for a dipole field. We adopt here the dipole coordinates (ν, μ, φ) used by Radoski [1967b] (see Figure 2). These coordinates and their respective scale factors are related to the spherical coordinates (r, θ, φ) as follows:

$$\nu = \sin^2 \theta / r \qquad \mu = \cos \theta / r^2 \qquad \varphi = \varphi$$

$$h_\nu = (r^2/\sin \theta)(1 + 3 \cos^2 \theta)^{-1/2}$$

$$h_\varphi = r \sin \theta \qquad h_\mu = h_\nu h_\varphi = M/B$$

where M is the earth's magnetic dipole moment, and B is the field strength. The unit vector $\boldsymbol{\mu}$ is directed along the field line, $\boldsymbol{\varphi}$ is in the azimuthal direction, and $\mathbf{v} = \boldsymbol{\mu} \times \boldsymbol{\varphi}$ is normal to the field line and pointing toward the earth (projected to the northern (southern) hemisphere, φ and ν correspond to D and $-H$ ($+H$) direction). We assume perturbations of the following form:

$$\boldsymbol{\xi}(\mu, \varphi, \nu, t) = e^{-i\omega t} e^{im\varphi} \boldsymbol{\xi}(\mu, \nu)$$

Here $|m| \sim 1/\epsilon \gg 1$, and $\boldsymbol{\xi}(\mu, \nu)$ satisfies the MHD equations as well as the boundary conditions. Equation of motion (12)
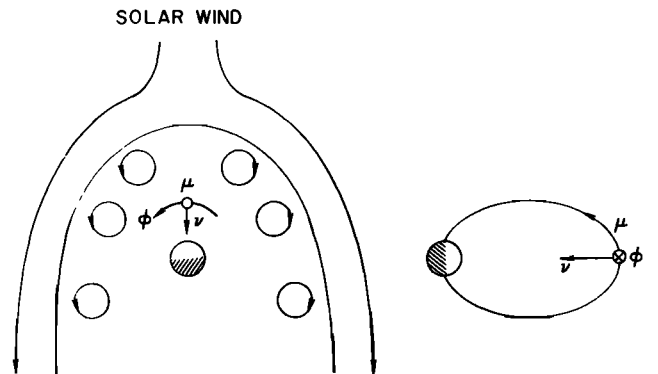

Fig. 2. Illustrations of the dipole (ν, μ, φ) coordinate used in the text.

then becomes for each component

$$K_{A\varphi}^{2}(\mu, \nu)\xi_{\varphi} = \frac{im}{h_{\varphi}}\left[\frac{\mu_0 p + b_\mu B}{B^2}\right] \qquad (14)$$

$$K_{A\nu}^{2}(\mu, \nu)\xi_{\nu} = \frac{1}{h_{\nu}}\frac{\partial}{\partial \nu}\left[\frac{\mu_0 p + b_\mu B}{B^2}\right] + 2\kappa_{B\nu}\frac{\mu_0 p}{B^2} \qquad (15)$$

$$\omega^2 \rho_m \xi_\mu = \frac{1}{h_\mu}\frac{\partial p}{\partial \mu} \qquad (16)$$

Here

$$K_{Aj}^{2}\xi_j = \left[\frac{\omega^2}{v_A^2} + \frac{1}{Bh_\mu}\frac{\partial}{\partial \mu}\left(\frac{B}{h_\mu}\frac{\partial}{\partial \mu}\right)\right]\xi_j$$

$$- \frac{\xi_j}{h_j Bh_\mu}\frac{\partial}{\partial \mu}\left(\frac{B}{h_\mu}\frac{\partial h_j}{\partial \mu}\right) \qquad j = \varphi, \nu \qquad (17)$$

Thus (17) with $K_{Aj}^{2} = 0$ corresponds to the guided wave equation for the toroidal ($\xi_\nu = 0$) or poloidal ($\xi_\varphi = 0$) mode. Because we are primarily interested in the coupling problem in the ν direction around the equatorial plane, we do not solve the detailed boundary value problem in the μ direction; instead, we assume that a suitable solution in the μ direction is obtained in a WKB form. Then operators K_{Aj}^{2} when operated to $\xi(\mu, \nu)$ become a function of μ and ν. Particularly near the equator, where $\mu \sim 0$, K_{Aj}^{2} can be regarded as a function of ν only. The perturbed magnetic field b is related to the fluid displacement ξ through (5'), which in the dipole coordinates becomes

$$\frac{b_\mu}{B} = -\left[\frac{im}{h_\varphi}\xi_\varphi + \frac{1}{h_\nu}\frac{\partial \xi_\nu}{\partial \nu} - \xi_\nu\left(\frac{1}{h_\nu^2}\frac{\partial h_\nu}{\partial \nu}\right)\right] \qquad (18)$$

$$\frac{b_\mu}{B} = -(\nabla \cdot \xi) + \frac{1}{h_\mu}\frac{\partial \xi_\mu}{\partial \mu} - \kappa_{B\mu}\xi_\mu - 2\kappa_{B\nu}\xi_\nu \qquad (18')$$

$$\frac{b_\varphi}{B} = \frac{1}{h_\mu}\frac{\partial \xi_\varphi}{\partial \mu} - \left(\frac{1}{h_\varphi h_\mu}\frac{\partial h_\varphi}{\partial \mu}\right)\xi_\varphi \qquad (19)$$

$$\frac{b_\nu}{B} = \frac{1}{h_\mu}\frac{\partial \xi_\nu}{\partial \mu} - \left(\frac{1}{h_\nu h_\mu}\frac{\partial h_\nu}{\partial \mu}\right)\xi_\nu \qquad (20)$$

Equation of state (8) relates the perturbed pressure p to ξ; i.e.,

$$\mu_0 p / B^2 = -\beta[\kappa_P \cdot \xi + \gamma(\nabla \cdot \xi)] \qquad (8')$$

In general, the above set of MHD equations is difficult to solve and only special cases where the plasma is cold and the compressional component vanishes have been treated (see Orr [1973] and references therein).

As was described in the preceding section, we approach this problem by using two small parameters, ϵ and β. First, we discuss the physical implications of the two small parameters from ordering arguments. From (8'), (18), and (18') we obtain the following order estimates:

$$\mu_0 p / B^2 \sim \beta(Bb_\mu/B^2) \qquad (21)$$

$$\frac{K_{Aj}^{2}\xi_j}{k_\perp} \sim \epsilon^2 \frac{Bb_\mu}{B^2} \qquad j = \varphi, \nu \qquad (22)$$

The above order estimates show that the shear Alfvén wave resonance, which is represented by $K_{Aj}^{2} = 0$ in (14) or (15), is attained only for a particular orientation of ξ vector such that the compressional magnetic field component b_μ is small ($\sim\beta$, ϵ^2), because otherwise the right-hand side of (14) or

(15) dominates over the left-hand side, and the oscillation of the field line becomes 'forced' oscillation rather than 'natural' oscillation. Thus from (14) we can solve b_μ and p in terms of ξ by successive approximations. That is, we put to zeroth order

$$[Bb_\mu/B^2]^{(0)} = 0 \qquad (23)$$

or from (18) and (18'),

$$\xi_\varphi^{(0)} = \frac{ih_\varphi}{mh_\nu}\frac{\partial \xi_\nu^{(0)}}{\partial \nu} - \left(\frac{ih_\varphi}{mh_\nu^2}\frac{\partial h_\nu}{\partial \nu}\right)\xi_\nu^{(0)} \qquad (24)$$

$$(\nabla \cdot \xi)^{(0)} = \frac{1}{h_\mu}\frac{\partial \xi_\mu^{(0)}}{\partial \mu} - \kappa_{B\mu}\xi_\mu^{(0)} - 2\kappa_{B\nu}\xi_\nu^{(0)} \qquad (25)$$

Equation 24 shows the orientation of ξ such that the local oscillation of the shear Alfvén wave can occur. The first-order value of b_μ then is

$$\left[\frac{b_\mu B}{B^2}\right]^{(1)} = -\frac{\mu_0 p^{(0)}}{B^2} - \frac{ih_\varphi}{m}K_{A\varphi}^{2}\xi_\varphi^{(0)} \qquad (26)$$

where

$$\mu_0 p^{(0)} / B^2 = -\beta[\kappa_P \cdot \xi^{(0)} + \gamma(\nabla \cdot \xi)^{(0)}] \qquad (27)$$

Combining (25), (27), and (16), we can see that $\xi_\mu^{(0)} \sim \beta\xi_\nu^{(0)}$; (25) and (27) then become

$$(\nabla \cdot \xi)^{(0)} \simeq -2\kappa_B \xi_\nu^{(0)} \qquad (25')$$

$$\mu_0 p^{(0)} / B^2 \simeq -\beta[\kappa_P \xi_\nu^{(0)} + \gamma(\nabla \cdot \xi)^{(0)}] \qquad (27')$$

In (25') and (27') we have omitted the subscript ν from κ_B and κ_P to simplify the notation. Substituting (24), (25'), (26), and (27') into the right-hand side of (15), we arrive at the wave equation for ξ_ν only:

$$\frac{1}{h_\nu^2}\frac{\partial^2 \xi_\nu}{\partial \nu^2} + \left[\frac{1}{h_\nu^2 K_{A\varphi}^{2}}\frac{\partial}{\partial \nu}\left(\frac{h_\varphi^2}{h_\nu}K_{A\varphi}^{2}\right) - \frac{1}{h_\nu^2}\frac{\partial h_\nu}{\partial \nu}\right]\frac{1}{h_\nu}\frac{\partial \xi_\nu}{\partial \nu}$$

$$+ \left[\frac{4\gamma\beta m^2 \kappa_B^{2}}{h_\varphi^2 K_{A\varphi}^{2}}\left(1 - \frac{\kappa_P}{2\gamma\kappa_B}\right)\right.$$

$$\left. - \frac{1}{h_\nu^2 K_{A\varphi}^{2}h_\nu}\frac{\partial}{\partial \nu}\left(\frac{h_\varphi^2 K_{A\varphi}^{2}}{h_\nu^2}\frac{\partial h_\nu}{\partial \nu}\right) - \frac{m^2 K_{A\nu}^{2}}{h_\varphi^2 K_{A\varphi}^{2}}\right]\xi_\nu = 0$$

$$(28)$$

Here we have omitted the superscript (0) to simplify the notation. Note, for uniform plasmas $K_{A\varphi}^{2} = K_{A\nu}^{2}$ (=const), then (28) becomes

$$K_{A\varphi}^{2}\left(\frac{1}{h_\nu^2}\frac{\partial^2}{\partial \nu^2} - \frac{m^2}{h_\varphi^2}\right)\xi_\nu = 0$$

i.e., the shear Alfvén wave ($K_{A\varphi}^{2} = 0$) and the compressional surface wave ($k_\nu^2 + k_\varphi^2 = 0$) are decoupled. Note also that in a nonuniform case as $m \to \infty$, $K_{A\nu}^{2} = 0$ is a solution of (28). This mode corresponds to a poloidal oscillation obtained by Dungey [1968]. However, this mode does not have a resonant coupling with the surface wave and hence may not be excited strongly. Equation 28 thus indicates the coupling between the surface wave and the shear Alfvén wave given by $K_{A\varphi}^{2} = 0$ due to nonuniformities as well as field line curvatures. Roughly speaking, because $m^2 \sim \epsilon^{-2} \gg 1$ and $K_{A\nu}^{2}/K_{A\varphi}^{2} \sim 1$, the coupling is weak ($\sim\epsilon^2$) except near $K_{A\varphi}^{2} = 0$. We therefore expect an existence of a surface wave away from the resonant field line where $K_{A\varphi}^{2} = 0$ and a shear Alfvén wave near the resonant

field line. Because in the equatorial region the surface wave excited by the solar wind has a maximum amplitude and the coupled wave equation assumes a simplest form, we shall confine our treatment in this region and thus reduce the problem to one dimension in order to study the nature of coupling in more detail. Near the equator ($\mu \simeq 0$), (28) then becomes

$$\frac{d^2\xi_\nu}{d\nu^2} + \left[\frac{1}{K_{A\varphi}{}^2} \frac{dK_{A\varphi}{}^2}{d\nu} + \frac{2}{\nu} \right] \frac{d\xi_\nu}{d\nu}$$

$$+ \left[\frac{A}{\nu K_{A\varphi}{}^2} \frac{dK_{A\varphi}{}^2}{d\nu} - \frac{m^2}{\nu^2} \frac{K_{A\nu}{}^2}{K_{A\varphi}{}^2} - \frac{2}{\nu^2} \right]\xi_\nu = 0 \qquad (29)$$

$$1/L_b R_E \le \nu \le 1/R_E$$

where

$$A = 2 + 4\gamma\beta m^2 \kappa_B{}^2 \frac{1 - (\kappa_P/2\gamma\kappa_B)}{\nu \, dK_{A\varphi}{}^2/d\nu}$$

and L_b is the L value of the magnetopause. It is understood here that all quantities correspond to values at $\mu \simeq 0$. In particular, $K_{A\varphi}$ and $K_{A\nu}$ are not operators but are functions of ν. Although the values of $K_{A\nu}{}^2$ and $K_{A\varphi}{}^2$ can be rather close in low latitudes ($\partial h_\mu/\partial\mu$, $\partial B/\partial\mu \propto \cos\theta \simeq 0$), we take them to be different. Specifically, we assume $K_{A\nu}{}^2 \ne 0$ at $K_{A\varphi}{}^2 = 0$. However, it becomes apparent in later analyses that the qualitative picture of the coupling is insensitive to the exact relation between $K_{A\varphi}{}^2$ and $K_{A\nu}{}^2$. Equation 29 plus the appropriate boundary conditions completely determines ξ_ν. Wave polarization in the φ-ν plane then can be found from the ratio b_φ/b_ν (see the appendix); i.e., near the equator,

$$\frac{b_\varphi}{b_\nu} = \alpha + i\delta \simeq \frac{\partial\xi_\varphi/\partial\mu}{\partial\xi_\nu/\partial\mu}$$

$$\simeq \frac{1}{m\nu} \left[\frac{\nu^2}{\xi_\nu} \frac{d\xi_\nu}{d\nu} + 2\nu + \nu^2 \frac{\partial\ln k_\mu}{\partial\nu} \right] \qquad (30)$$

Here

$$ik_\mu \equiv \left[\frac{\partial\xi_\nu/\partial\mu}{\xi_\nu} \right]_{\mu=0}$$

In the next section we discuss the solution and the associated wave polarization near the resonant field line.

NEAR THE RESONANT FIELD LINE

When it is assumed that $K_{A\varphi}{}^2 = 0$ at $\nu = \nu_0$, then near the resonant field line we have

$$K_{A\varphi}{}^2 \simeq (\nu - \nu_0)\left(\frac{dK_{A\varphi}{}^2}{d\nu} \right)_0 \qquad (31)$$

Here we use the subscript 0 to denote quantities evaluated at $\nu = \nu_0$, and $(dK_{A\varphi}{}^2/d\nu)_0$ is taken to be nonzero. Substituting (31) into (29), we immediately see that the solution has a singularity at $\nu = \nu_0$. In fact, as will be shown later, this singularity is logarithmic. As is realized also by Southwood [1973], the nature of this singularity is the same as that of the singularities encountered in studying wave propagations in an inhomogeneous medium where the refractive index becomes infinite [Budden, 1961; Ginzburg, 1967]. The wave amplitude becomes infinite there, and the energy is accumulated in the vicinity of the pole. This physically unjustifiable result indicates the inappropriateness of treating

the wave propagation as a steady state process [Ginzburg, 1967]. To eliminate this difficulty, in accordance with the causal relation of the Laplace transformation in time, we have to assume a small positive imaginary part in ω:

$$\omega = \omega_r + i\omega_i \qquad \omega_i > 0 \qquad |\omega_i/\omega_r| \ll 1$$

With $\omega_i \ne 0$, we then have near $\nu = \nu_0$, where $\text{Re}(K_{A\varphi}{}^2) = 0$,

$$K_{A\varphi}{}^2 \simeq \left[\frac{d(K_{A\varphi}{}^2)^R}{d\nu} \right]_0 \nu_0 \left[\frac{\nu}{\nu_0} - 1 + i\eta \right] \qquad (32)$$

where

$$(K_{A\varphi}{}^2)^R = \text{Re}(K_{A\varphi}{}^2)$$

$$\eta = \frac{2\omega_i\omega_r}{v_{A0}{}^2\nu_0} \left[\frac{d(K_{A\varphi}{}^2)^R}{d\nu} \right]_0^{-1} \qquad (33)$$

Note $|\eta| \sim O(\omega_i/\omega_r) \ll 1$, but in the presence of a loss (such as that due to ionospheric dissipation), ω_i is not infinitely small but rather positive finite to overcompensate the effect of loss; hence, in general, η has a finite positive value. By defining the new variable

$$z \equiv \left(\frac{\nu}{\nu_0} - 1 \right) + i\eta \equiv s + i\eta \equiv |z| \, e^{i\Psi}$$

(29) becomes, near $s = 0$,

$$\frac{d^2\xi_\nu}{dz^2} + \left[\frac{1}{z} + 2 \right] \frac{d\xi_\nu}{dz} + \left(\frac{G_0}{z} + D_0 \right)\xi_\nu = 0 \qquad (34)$$

$$|z| \ll 1$$

Here

$$G_0 = A_0 - \left[\frac{(m^2/\nu)K_{A\nu}{}^2}{d(K_{A\varphi}{}^2)^R/d\nu} \right]_0 \sim \epsilon^{-2}$$

and $D_0 = -2$. With $|z| \ll \epsilon^2$ we can solve (34) by series expansions, and the general solution is

$$\xi_\nu = C_1\xi_{1\nu} + C_2\xi_{2\nu} \qquad (35)$$

Here

$$\xi_{1\nu} = 1 + a_1 z + a_2 z^2 + O(a_1{}^3 z^3) \qquad (36)$$

$$\xi_{2\nu} = \xi_{1\nu} \ln z + b_1 z + b_2 z^2 + O(b_1{}^3 z^3) \qquad (37)$$

$$a_1 = -G_0 \sim \epsilon^{-2}$$

$$a_2 = \tfrac{1}{4}[G_0(2 + G_0) - D_0] \simeq G_0{}^2/4 \sim \epsilon^{-4}$$

$$b_1 = 2(G_0 - 1) \simeq 2G_0$$

$$b_2 = -\tfrac{1}{4}[b_1(2 + G_0) + 4a_2 + 2a_1] \simeq -\tfrac{3}{4}G_0{}^2$$

Here C_1 and C_2 are two constants to be determined from boundary conditions. Unless $|C_2/C_1|$ is vanishingly small, we can approximate (35) with $|z| \ll 1$ to be

$$\xi_\nu \simeq C_2\xi_{1\nu} \ln z \qquad (35')$$

For $|z| \ll \epsilon^2$, $\xi_{1\nu} \simeq 1$, and ξ_ν can be further approximated to be

$$\xi_\nu \simeq C_2 \ln z = C_2[\xi_\nu{}^R + i\xi_\nu{}^I]$$

$$= C_2[1 + i\Psi/\ln|z|] \ln |z| \qquad (38)$$

Meanwhile, we have $d\xi_{1r}/dz \simeq a_1$, and from (35')

$$\frac{d\xi_r}{dz} \simeq C_2\left[\left(\frac{d\xi_r}{dz}\right)^R + i\left(\frac{d\xi_r}{dz}\right)^I\right] \tag{39}$$

where

$$\left(\frac{d\xi_r}{dz}\right)^R \simeq \frac{s}{|z|^2} + a_1 \ln|z| \tag{40}$$

$$\left(\frac{d\xi_r}{dz}\right)^I \simeq -\frac{\eta}{|z|^2} + a_1\Psi \tag{41}$$

Equation 30 then gives the ratio b_φ/b_r, which decides the wave polarization:

$$\frac{b_\varphi}{b_r} = \alpha + i\delta \simeq \frac{i}{m}\left[\frac{d\xi_r/dz}{\xi_r} + 2 + \frac{d}{dz}\ln k_\mu\right]$$

or because the first term $(\sim 1/z \ln z)$ dominates the last two terms (~ 1),

$$\alpha \simeq \frac{1}{m\ln|z|}\left[\left(\frac{d\xi_r}{dz}\right)^R\left(\frac{\xi_r^I}{\ln|z|}\right) - \left(\frac{d\xi_r}{dz}\right)^I\right] \tag{42}$$

$$\delta \simeq \frac{1}{m\ln|z|}\left[\left(\frac{d\xi_r}{dz}\right)^R + \left(\frac{d\xi_r}{dz}\right)^I\left(\frac{\xi_r^I}{\ln|z|}\right)\right] \tag{43}$$

In the following three subsections we discuss separately the values of α and δ as well as the associated wave polarizations for $|s/\eta| \ll 1$, $|s/\eta| \simeq 1$, and $|s/\eta| \gg 1$.

Case 1: $|s/\eta| \ll 1$. Here $\Psi \simeq \pm\pi/2$, the upper sign being for $\eta > 0$ and the lower sign for $\eta < 0$. With $|\eta| \ll 1$ we obtain from the above equations

$$\alpha \simeq -\frac{1}{m\ln|\eta|}\left(\frac{d\xi_r}{dz}\right)^I \simeq \frac{1}{m\eta\ln|\eta|} \tag{44}$$

$$\delta \simeq \frac{1}{m\eta\ln|\eta|}\left[\frac{s}{\eta} - \frac{\Psi}{\ln|\eta|}\right] \tag{45}$$

Thus the sign of α depends on that of $m\eta$; i.e., sign $(\alpha) = -$sign $(m\eta)$. With $\eta > 0$, $\alpha > 0$ for $m < 0$, and the major axis of the ellipse lies inside the first quadrant of the φ-ν plane. For $m < 0$, $\alpha < 0$, and the major axis lies inside the second quadrant. With $\eta < 0$, the results are opposite. As for the sense of polarization, it depends on the sign of δ, which is independent of the sign of η. For $|s/\eta| < |\pi/2 \ln|\eta||$ (e.g., exactly at the resonant field line, $s = 0$), sign $(\delta) = -$sign (m). Hence wave polarization is on the left-hand side for $m < 0$ and on the right-hand side for $m > 0$. A little away from the resonance, where $|s/\eta| > |\pi/2 \ln|\eta||$, sign $(\delta) = -$sign (s/m). Thus for $m < 0$, the wave is left-hand-polarized

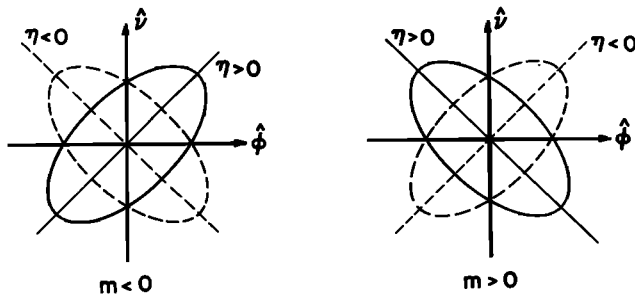**Fig. 3.** Tilt of the major axis near the resonant field line at the equatorial plane. When φ and ν are projected along the field lines to the northern (southern) hemisphere, φ (E-W) is in the D direction, and ν (N-S) is in the $-H$ ($+H$) direction.
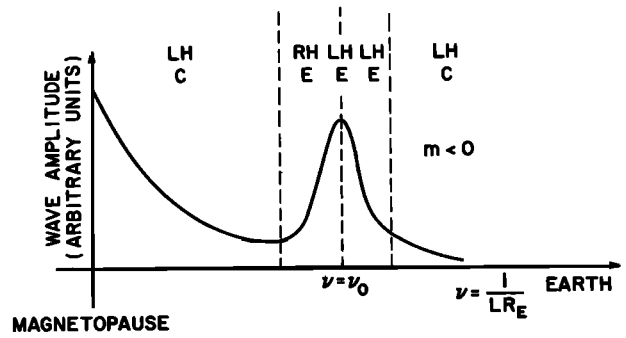

Fig. 4. Sketch of the wave amplitude and sense of polarization versus radius in the equatorial plane for $m < 0$ (local morning). When it is projected along the field line to the ground, it can be viewed as a plot versus latitude. The C and E stand for circular and elliptical polarization, respectively. The results are opposite for $m > 0$ (local afternoon). The ν_0 is the location of the resonant field line. The ellipticity is large (linear polarization) near the resonant field line and decreases away.

for $s > 0$ (closer to the earth) and right-hand-polarized for $s < 0$ (farther from the earth). For $m > 0$, the results are opposite. Since $|\delta/\alpha| \ll 1$, the ellipticity is large (close to a linear polarization) with the minimum value, $\mathcal{E}_M \simeq 1 - |\delta/\alpha|$ at $\alpha^2 \simeq 1$.

Case 2: $|s/\eta| \simeq 1$. Here $\Psi \sim O(1)$, and we obtain

$$\alpha \simeq 1/2\eta m \ln|\eta| \tag{46}$$

$$\delta \simeq s/2\eta^2 m \ln|\eta| \tag{47}$$

Therefore the major axis has the same properties as the preceding case does. Again, the sense of polarization is independent of η, sign $(\delta) = -$sign (s/m), and has been discussed in the preceding case. Since $|\delta/\alpha| \simeq 1$, the ellipticity is smaller than that of the preceding case with $\mathcal{E}_M \simeq 0.59$ at $\alpha^2 \simeq 0.5$.

Case 3: $|s/\eta| \gg 1$. In this case we have

$$\Psi \simeq \pm|\eta/s| \qquad s > 0$$

$$\Psi \simeq \pm\pi \qquad s < 0$$

Again, the upper sign is for $\eta > 0$, and the lower sign is for $\eta < 0$. Then α and δ become

$$\alpha \simeq \frac{1}{ms\ln|s|}\frac{\eta}{s} \qquad s > 0$$

$$\alpha \simeq \frac{1}{ms\ln|s|}\left(\pm\frac{\pi}{\ln|s|} + \frac{\eta}{s}\right) \qquad s < 0 \tag{48}$$

$$\delta \simeq 1/ms\ln|s| \tag{49}$$

Note here that for $s < 0$, η/s and $\pm\pi/\ln|s|$ have the same sign. Hence sign $(\alpha) = -$sign (η/m); and the major axis has the same properties as given in the previous two cases. Also, sign $(\delta) = -$sign (ms), which indicates that the wave polarization has properties similar to those of the preceding case. With $|\delta/\alpha| \gg 1$ the ellipticity is smaller (close to a circular polarization) with $\mathcal{E}_M \simeq 0$ at $\delta \simeq 1$. The above results are summarized in Figures 3 and 4.

Finally, let us assume $K_{A_r}{}^2 = 0$ at $\nu = \nu_0$, where $K_{A\varphi}{}^2 = 0$, and discuss the resonance coupling. It is apparent that the only changes needed in the above analyses are now $G_0 = A_0 \sim m^2\beta \sim \epsilon^{-1}$ (when it is assumed that $\beta \sim \epsilon$)

and $D_0 = -(m^2 + 2) \sim \epsilon^{-2}$. Thus $a_1 \sim \epsilon^{-1}$, and $a_2 \sim \epsilon^{-2}$; in other words, all the above results are expected to hold for $|z| \ll \epsilon$ instead of $|z| \ll \epsilon^2$ as they do in the case with $K_{A\varphi}{}^2 \neq K_{A\nu}{}^2$. The changes are therefore only quantitative. That is, for $|z| \ll \epsilon^2$, we have

$$|b_\varphi/b_\nu| \sim |\alpha + i\delta| \sim \left|\frac{1}{mz \ln z}\right| \gtrsim \left|\frac{1}{\epsilon \ln \epsilon}\right| \gg 1$$

and for $|z| \ll \epsilon$,

$$|b_\varphi/b_\nu| \gtrsim \left|\frac{1}{\ln \epsilon}\right| \ll 1$$

Thus in the former case the waves tend to incline toward the φ direction. In the latter case, however, the waves have no particular tendency of inclination. Orr and Matthew [1971] compute the resonance frequencies for the poloidal and toroidal oscillations and find that they are almost identical (1-3% difference) except for the fundamental harmonic (30% difference); that is, the latter case may be more common in practical situations.

AWAY FROM THE RESONANT FIELD LINE

In this case, because

$$\frac{(dK_A{}^2/d\nu)}{K_A{}^2} \sim \nu^{-1}$$

and $K_{A\varphi}{}^2/K_{A\nu}{}^2 \sim 1$, the wave equation (29) can be approximated to be

$$\frac{d^2\xi_\nu}{d\nu^2} - \frac{m^2}{\nu^2} \xi_\nu \simeq 0 \qquad (50)$$

$$\nu_0 < \nu \leq \frac{1}{R_E} \qquad \frac{1}{L_b R_E} \leq \nu < \nu_0$$

This is a surface wave type equation (i.e., $k_\nu{}^2 + k_\varphi{}^2 \simeq 0$). In fact, it describes the evanescent compressional mode. The general solution is

$$\xi_\nu \simeq B_1 \nu^{|m|} + B_2 \nu^{-|m|} \qquad (51)$$

The values of B_1 and B_2 depend on the boundary conditions as well as the actual position of the resonant field line. If we assume that the fields vanish near the earth's surface, we then have approximately

$$\xi_\nu \simeq B_2 \nu^{-|m|} \qquad (52)$$

From (30) and (51) we see that the wave polarization is mainly circular and that the sense of polarization depends on sign of m; i.e., the polarization is left-handed for $m < 0$ and is right-handed for $m > 0$. These off-resonance results are also summarized in Figure 4.

SUMMARY AND DISCUSSIONS

We have shown through general formulations that owing to nonuniformities as well as field line curvatures, surface waves excited by the solar wind at the magnetopause can couple to the shear Alfvén waves (guided waves) of the resonant local field lines inside the magnetosphere. For surface waves with fast azimuthal (E-W) variations (small ϵ) and in low β plasmas the coupling that retains the natural field line oscillation can occur only for particular orientation of the ξ vector. (Otherwise, the nature of the field line oscillation is dominated by the nature of the

exciting source.) We then apply this property to derive a coupled wave equation in the dipole coordinates. This wave equation is then solved for the equatorial region. The solution exhibits surface waves away from the resonant field line. Near the resonant field line, shear Alfvén wave occurs, and the solution has a logarithmic singularity. This unphysical property is removed by introducing finite dissipations. On the basis of the solutions we then examine the characteristics of the wave polarization, i.e., the tilt of the major axis, the sense of polarization, and the ellipticity. These results are summarized in Figures 3 and 4. Let us now discuss these theoretical results in the light of experimental observations. Since the surface wave is excited by the solar wind, a natural choice is $m < 0$ for local morning and $m > 0$ for local afternoon.

Tilt of major axis (Figure 3). Theory indicates that the tilt depends on the signs of η and m. When it is assumed that $\eta > 0$, the theory then predicts that in local morning ($m < 0$) the major axis lies in the second (first) quadrant of the H-D plane for most latitudes in the northern (southern) hemisphere. This prediction assumes that the major axis direction does not change significantly along the direction of the field line. This assumption is supported by conjugacy observation by Lanzerotti et al. [1972]. Furthermore, the tilt is predicted to switch across the local noon. These predictions are consistent with the low-latitude conjugate observations of Lanzerotti et al. [1972] and Van-Chi et al. [1968]. Samson [1972], using a string of seven stations located at different latitudes in the northern hemisphere (59°-77°N) has also observed similar phenomena.

Let us now discuss the physical implications of $\eta > 0$ as suggested by the observations. Since $\omega_\iota > 0$ to take account of dissipations, $\eta > 0$ then implies that

$$[d(K_{A\varphi}{}^2)^R/d\nu]_0 > 0$$

Because

$$(K_{A\varphi}{}^2)^R \simeq \frac{\omega^2}{v_A{}^2} - k_\parallel{}^2 \sim \frac{\omega^2}{v_A{}^2} - \nu^2$$

then

$$[d(K_{A\varphi}{}^2)/d_\nu]_0$$

is generally negative except in regions (e.g., the plasmapause) where V_A decreases with ν due to rapid density increase (i.e., faster than B^2). The present theory thus suggests that pulsations tend to occur in regions where density increases faster than B^2.

Recently, Uberoi [1972] has pointed out the close analogy between the problem treated here and the problem of electrostatic oscillations in a cold inhomogeneous plasma. By using the small parameter ϵ and looking at the steady state solution we have ignored in the present paper the possibility of exciting the weakly damped collective (position independent) mode due to a sudden jump in v_A as well as the related initial value aspect of this problem [Sedláček, 1971]. These considerations are currently under study and are reported in the companion paper.

Wave amplitude, sense, and ellipticity of polarization (Figure 4). The theory predicts that the wave amplitude peaks at the resonant field line. The polarization is linear near the peak, and the ellipticity gradually decreases away
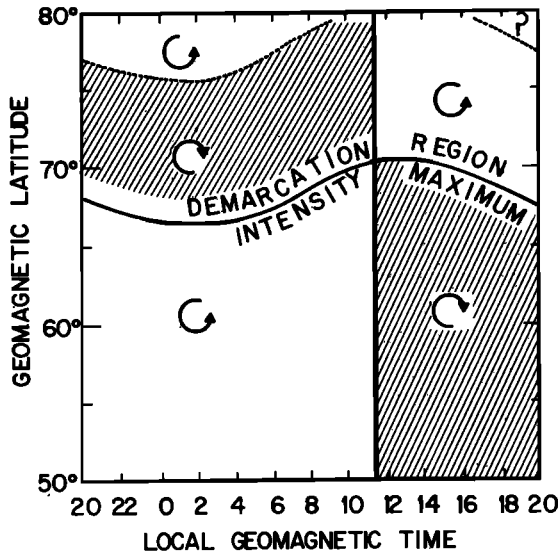
Fig. 5. The diurnal variation of the polarization sense for quasimonochromatic micropulsations of frequency ~5 mHz [after *Samson et al.*, 1971].

from the peak. Furthermore, the sense of polarization is opposite in the two sides of the resonant peak. The theory also predicts the reversal of sense of polarization across local noon. All these theoretical predictions agree with the observation of *Samson et al.* [1971] (see Figure 5). The demarcation line in Figure 5 then corresponds to the location of the resonant field lines. Let us now explain why in low latitudes no clear dawn-dusk asymmetry is observed. For a station located at low latitudes we can expect the observed pulsations to be near the resonant field line where the wave amplitude peaks. Otherwise, the wave signal may be too low to be detected. Another indication of excitation of shear Alfvén waves is that in a low latitude most of the observed pulsations have periods of the order of the local first harmonic shear Alfvén period ~40 s (Figure 6). Because the sense of polarization
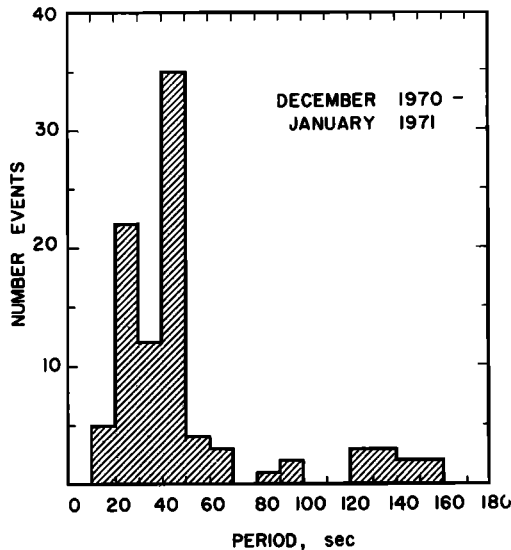

Fig. 6. Plot of number events versus period at the two conjugate points, Siple and Lac Rebours. It indicates that most of the events have frequency near the local shear Alfvén frequency (plot provided by L. J. Lanzerotti).

reverses not only across local noon but also across the resonant peak, which has diurnal variations, the sense of polarization thus cannot be expected to exhibit a clear diurnal pattern. Finally, let us note that the above results are also predicted by *Southwood* [1973] based on similar ideas and a straight field line model as well as by *McClay* [1970] based on the coupling of symmetric modes and the Hall effects.

Latitudinal dependence of frequency. Because the resonant frequency increases as the latitude decreases, our theory thus satisfactorily explains the observations of *Samson and Rostoker* [1972]; i.e., for each individual event the frequency is the same for all latitudes (observing the Kelvin-Helmholtz surface wave), and, however, the latitude of the intensity peak decreases as the pulsation frequency increases (observing the oscillation of the local field line). In conclusion, we remark here that the theoretical approach of studying long-period magnetic pulsations as coupling between surface waves excited by the solar wind at the magnetopause and the shear Alfvén waves of local resonant field lines has yielded results consistent with many aspects of the observational results.

APPENDIX: GENERAL CONSIDERATION OF WAVE POLARIZATION

In this appendix we study the characteristics of the wave polarization given $b_\varphi/b_\nu = \alpha + i\delta$.

With the convention taken as $e^{-i\omega t}$, b_φ and b_ν can be written as

$$b_\nu = \cos \omega t \qquad b_\varphi = \Omega \cos (\omega t - \lambda) \qquad \text{(A1)}$$

where $\Omega e^{i\lambda} \equiv \alpha + i\delta$. From (A1) we obtain

$$b_\varphi^2 - 2\alpha b_\varphi b_\nu + (\alpha^2 + \delta^2)b_\nu^2 - \delta^2 = 0 \qquad \text{(A2)}$$

By using a new pair of axes (b_φ', b_ν'), which makes an angle θ_0 in the counterclockwise direction with (b_φ, b_ν), (A2) can be reduced to the standard elliptic equation

$$\frac{b_\varphi'^2}{(\delta^2/A)} + \frac{b_\nu'^2}{(\delta^2/C)} = 1 \qquad \text{(A3)}$$

The angle θ_0 is given by

$$\tan 2\theta_0 = \frac{-2\alpha}{1 - (\alpha^2 + \delta^2)} \qquad \text{(A4)}$$

and

$$A = (\cos \theta_0 - \alpha \sin \theta_0)^2 + \delta^2 \sin^2 \theta_0$$
$$C = (\sin \theta_0 + \alpha \cos \theta_0)^2 + \delta^2 \cos^2 \theta_0 \qquad \text{(A5)}$$

Thus $0 \leq \theta_0 \leq \pi/2$, and the orientation of the major axis is decided by the relative value of A and C. It can be shown that

$$A - C = \left(-\frac{1}{2\alpha}\right) \sin 2\theta_0[(1 - \alpha^2 - \delta^2)^2 + 4\alpha^2] \qquad \text{(A6)}$$

Hence, for $\alpha > 0$, $A < C$, and b_φ' is the major axis; for $\alpha < 0$, $A > C$, and b_ν' is the major axis. That is, for $\alpha > 0$, the major axis lies in the first quadrant of φ-ν plane; for $\alpha < 0$, it is in the second quadrant.

From the above discussions and from (A1) it is easy to see that the sense of polarization is entirely decided by the sign of δ, i.e., left-hand polarization for $\delta > 0$ and right-hand polarization for $\delta < 0$. Here the convention

is taken with the thumb pointing along the B_0 direction.

As for the ellipticity $\mathcal{E}$, it is given by

$$\mathcal{E} = 1 - (A/C)^{1/2}$$

if $A < C$ or $\alpha > 0$;

$$\mathcal{E} = 1 - (C/A)^{1/2}$$

if $A > C$ or $\alpha < 0$. In terms of a fixed ratio of $|\delta/\alpha|$ we have the following minimum ellipticities. For $|\delta/\alpha| \ll 1$,

$$\mathcal{E}_M \simeq 1 - |\delta|/2$$

at $|\alpha| \simeq 1$; for $|\delta/\alpha| \simeq 1$,

$$\mathcal{E}_M \simeq 0.59$$

at $|\alpha| \simeq 2^{-1/2}$; for $|\delta/\alpha| \gg 1$,

$$\mathcal{E}_M \simeq 0$$

at $|\delta| \simeq 1$.

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* * *

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