A Theory of Long-Period Magnetic Pulsations
2. Impulse Excitation of Surface Eigenmode

LIU CHEN AND AKIRA HASEGAWA

Bell Laboratories, Murray Hill, New Jersey 07974

A theory of long-period magnetic pulsations (Pc 3 to Pc 5) is presented as an initial value problem to explain impulse-excited pulsations. By using a one-dimensional model a wave equation that shows a coupling between a surface wave and a shear Alfvén wave is derived. By solving this equation on the basis of initial value approach we conclude that there is a continuous spectrum with damping proportional to inverse power of time and that there are weakly damped discrete eigenmodes (surface eigenmodes) due to sharp variations in the plasma parameters. The frequency ω_r and the damping rate γ of the surface eigenmode are given approximately by $\omega_r = k_\parallel[(B_I^2 + B_{II}^2)/\mu_0(\rho_I + \rho_{II})]^{1/2}$ and $\gamma/\omega_r = |k_\perp/\nabla(\ln v_A^2)|$ respectively, where $v_A(=B/(\mu_0\rho)^{1/2})$ is the Alfén speed, $k_\parallel$ and $k_\perp$ are wave numbers parallel and perpendicular to the magnetic field, and subscripts I and II refer to quantities associated with each side of the surface. The result is used to explain recent observations of plasmapause-associated magnetic pulsations as well as magnetic pulsations excited by sudden commencements and sudden impulses near the magnetopause.

In the companion paper [*Chen and Hasegawa*, 1974] we presented a theory of long-period magnetic pulsations based on the idea of a steady state resonance coupling between a monochromatic surface wave excited at magnetopause due to Kelvin-Helmholtz instability (and/or other mechanisms) and a shear Alfvén wave associated with local field line oscillations. We have shown that this theory can unify many contradictory observations and predict, among other things, the dawn-dusk asymmetries in the sense of polarization as well as the tilt of the major axis of the polarization ellipse. However, recently, *Lanzerotti et al.* [1973], using latitudinal data near the plasmapause, have observed weakly damped coherent oscillations at $L \simeq 3.2$, which are accompanied by impulselike perturbations at higher latitudes, $L \simeq 4.4$ and 4.0. The steady state approach, which shows latitude independent frequency, obviously cannot explain these observations. A natural step to take here is to apply an initial value approach and study effects of sharp changes in magnetospheric plasma parameters. The present paper describes this approach. First, we use a one-dimensional model and ideal MHD equations to derive a coupled wave equation. We then obtain the solution of this equation by constructing its Green function. For this purpose we note the analogy of the present problem to that of an electrostatic oscillation in a nonuniform cold plasma. From this analogy we can conclude the existence of a continuous spectrum (noncollective modes) and a discrete spectrum (collective modes). Since the noncollective modes damp away as inverse power of time owing to phase mixing, we concentrate on the collective modes, where damping can be rather weak if sharp discontinuities in the plasma parameters exist. We then calculate the oscillation frequency and damping rate in terms of magnetospheric parameters. Finally, we apply these theoretical results to the recent observations [*Lanzerotti et al.*, 1973] of plasmapause-associated magnetic pulsations as well as magnetic pulsations excited by the sudden commencements and sudden impulses at the magnetopause [*Saito and Matsushita*, 1967] and show how the magnetospheric parameters can be inferred by using the observed frequency and damping rate in time and in space.

MODEL AND THE WAVE EQUATION

We assume the plasma to be an ideal MHD fluid with its equilibrium properties (density ρ, pressure P, and confining magnetic field $\mathbf{B} = B\mathbf{z}$) varying only in the y direction. (In the magnetosphere, x and y roughly correspond to W-E and radially inward directions, respectively.) Here P and $\mathbf{B}$ satisfy the equilibrium condition $d/dy\ (P + B^2/2\mu_0) = 0$. (If we include the effect of the curved field line, this condition is modified. In particular, one can have a situation in which both P and B increase (or decrease) with radius.) Linearizing the standard set of MHD equations, we have the equation of motion as

$$\mu_0\rho\ddot{\boldsymbol{\xi}} - (\mathbf{B}\cdot\nabla)^2\boldsymbol{\xi} = -\mu_0\nabla\tilde{p} - \mathbf{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi}) \quad (1)$$

Here $\boldsymbol{\xi}$ is the fluid displacement vector, $\partial\boldsymbol{\xi}/\partial t = \mathbf{v}$; $\tilde{p} = p + \mathbf{B}\cdot\mathbf{b}/\mu_0$ is the perturbed total pressure; $\mathbf{b}$, the perturbed magnetic field, is expressed in terms of $\boldsymbol{\xi}$ as

$$\mathbf{b} = (\mathbf{B}\cdot\nabla)\boldsymbol{\xi} - \mathbf{B}(\nabla\cdot\boldsymbol{\xi}) - (dB/dy)\,\xi_y\mathbf{z} \quad (2)$$

For a compressible fluid the adiabatic equation of state and the continuity equation relate the perturbed p and $\boldsymbol{\xi}$:

$$p = -\xi_y\, dP/dy - \gamma P(\nabla\cdot\boldsymbol{\xi}) \quad (3)$$

(For an incompressible fluid ($\nabla\cdot\boldsymbol{\xi} = 0$), one has a similar equation [*Uberoi*, 1972]. However, as is shown in the companion paper [*Chen and Hasegawa*, 1974], the nonuniformity does not generally allow incompressible perturbation.) We then take the Laplace time and the Fourier space transforms of the perturbed quantities; i.e.,

$$\bar{\boldsymbol{\xi}}(y, k_\parallel, k_\perp, \omega) = \int_0^\infty dt \int_{-\infty}^\infty dx \int_{-\infty}^\infty dz\ \boldsymbol{\xi}(x, y, z, t) \cdot \exp\,[i(\omega t - k_\perp x - k_\parallel z) \qquad \mathrm{Im}\,\omega \geq 0$$

$$\boldsymbol{\xi}(x, y, z, t) = \frac{1}{(2\pi)^2}\int_c d\omega \int_{-\infty}^\infty dk_\perp \int_{-\infty}^\infty dk_\parallel\ \bar{\boldsymbol{\xi}}(y, k_\parallel, k_\perp, \omega) \cdot \exp\,[i(k_\perp x + k_\parallel z - \omega t)]$$

where the integration path c runs parallel to the real axis of the ω plane above all singularities of $\bar{\xi}$. Using the above representations and (2) and (3) in (1), we obtain, after straightforward calculations, the following coupled wave equation in $\bar{\xi}_y$:

$$\frac{d}{dy}\left[\frac{\epsilon\alpha B^2}{\epsilon - \alpha B^2 k_\perp^2}\frac{d\bar{\xi}_y}{dy}\right] + \epsilon\bar{\xi}_y = \bar{S}(y, \omega) \qquad (4)$$

Here

$$\epsilon(y) = \omega^2\mu_0\rho(y) - k_\parallel^2 B^2(y)$$

$$\alpha(y) = 1 + \beta + \beta^2 k_\parallel^2 B^2/(\omega^2\mu_0\rho - \beta k_\parallel^2 B^2)$$

$$\beta(y) = \gamma\mu_0 P/B^2$$

and $\bar{S}$ corresponds to initial conditions. For waves with $|k_\perp| \gg |k_\parallel|$ we can assume $|\alpha B^2 k_\perp^2| \gg |\epsilon|$, and (4) becomes

$$\frac{d^2\bar{\xi}_y}{dy^2} + \frac{d\ln\epsilon}{dy}\frac{d\bar{\xi}_y}{dy} - k_\perp^2\bar{\xi}_y = \bar{S}'(y, \omega) \qquad (4')$$

Equation 4′ is the model wave equation that we shall use. The problem now is to construct the Green function $G(y, s)$ of (4′), from which the complete solution can be obtained. As *Uberoi* [1972] has pointed out, (4′) has an exact analogy with the wave equation describing an electrostatic oscillation in a nonuniform cold plasma, which has been extensively studied by *Barston* [1964] and *Sedláček* [1971*a*, *b*]. We shall briefly discuss this analogy in the next section.

ANALOGY WITH ELECTROSTATIC OSCILLATIONS IN NONUNIFORM COLD PLASMAS

The oscillation spectrum of a nonuniform cold plasma has been studied by many authors. These oscillations can be described by a wave equation similar to (4′), $\bar{\xi}_y$ being replaced by the potential ψ and ϵ being replaced by $\epsilon' = \omega^2 - \omega_p^2(y)$. Note that the boundary conditions $\bar{\xi}_y(\psi) \rightarrow 0$ as $y \rightarrow \pm\infty$ are the same. Furthermore, since the matching conditions are also the same, i.e., $\bar{\xi}_y(\psi)$ is continuous and

$$\bar{p} = -i\frac{\epsilon}{k_\perp}\bar{\xi}_x = -\frac{\alpha B^2}{\epsilon - \alpha B^2 k_\perp^2}\epsilon\frac{d\bar{\xi}_y}{dy}\left(\epsilon'\mathbf{E} = -\epsilon'\frac{d\psi}{dy}\right)$$

is continuous, we can expect the Green functions to have identical structures. From this analogy we can draw the following qualitative pictures [*Sedláček*, 1971*a*].

First, there exists a continuous spectrum,

$$\{\omega^2 \mid \min\omega_A^2 \le \omega^2 \le \max\omega_A^2\} \qquad \omega_A^2(y) = k_\parallel^2 B^2/\mu_0\rho$$

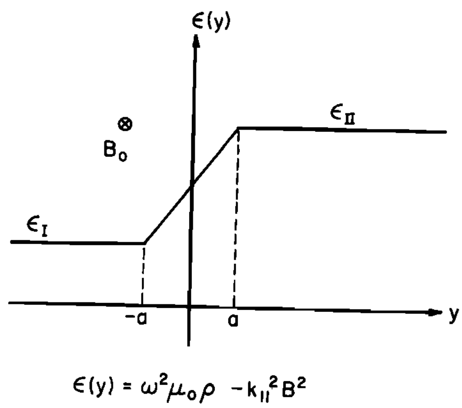

Fig. 1. Profile of $\epsilon(y)$ used in the text.

which asymptotically corresponds to noncollective oscillations with position dependent frequency and damping proportional to the inverse power of time.

Second, there also exist collective eigenoscillations (position independent) with exponential dampings. The damping is weak if ϵ has sharp discontinuities.

When we apply the above results to the magnetosphere, one sees the interesting possibility of exciting weakly damped collective modes due to the sharp jump in density and/or magnetic field at the magnetopause, plasmapause, and perhaps at the edge of the ring current. In the absence of such a jump the eigenmode in the magnetosphere is a diffused damped mode corresponding to the noncollective oscillation, and a monochromatic oscillation is excited only by a monochromatic source function as discussed in the companion paper. In the next section we adopt a simple model for $\epsilon(y)$ and calculate the eigenoscillation frequency and the damping rate of the collective modes. The analysis is closely parallel to that of *Sedláček* [1971*a*].

CALCULATIONS OF THE COLLECTIVE MODES

The collective modes come in through the Green function, which can be written as

$$G(y, y'; \omega) = J^{-1}[\psi_1(y, \omega)\psi_2(y', \omega)H(y' - y) + \psi_2(y, \omega)\psi_1(y', \omega)H(y - y')] \qquad (5)$$

where dependences on $k_\perp$ and $k_\parallel$ are understood. Here H is the unit step function, ψ_1 and ψ_2 are two homogeneous solutions to (4′) satisfying boundary conditions at $y = -\infty$ and $y = +\infty$, respectively, J is the conjunct of ψ_1 and ψ_2,

$$J(k_\perp, k_\parallel; \omega) = \epsilon(y)\left[\frac{d\psi_2}{dy}\psi_1 - \frac{d\psi_1}{dy}\psi_2\right] \qquad (6)$$

and J is independent of y and y'.

Let us take $\epsilon(y)$ to be (see Figure 1)

$$\epsilon(y) = \epsilon_\mathrm{I} = \omega^2\mu_0\rho_\mathrm{I} - k_\parallel^2 B_\mathrm{I}^2 \qquad y \le -a$$

$$\epsilon(y) = \delta y + \eta \qquad -a \le y \le a \qquad (7)$$

$$\epsilon(y) = \epsilon_\mathrm{II} = \omega^2\mu_0\rho_\mathrm{II} - k_\parallel^2 B_\mathrm{II}^2 \qquad y \ge a$$

where $\delta = (\epsilon_\mathrm{II} - \epsilon_\mathrm{I})/2a$ and $\eta = (\epsilon_\mathrm{II} + \epsilon_\mathrm{I})/2$, the subscripts I and II indicating each side of the surface. (We have also analyzed this problem with $\epsilon(y)$ nowhere constant. As might be expected, the results are similar as long as $|k_\perp L| \gg 1$, where L is the scale length of $\epsilon(y)$ variation at $|y| = a^+$.) We also assume that for $|y| \le a$, ω_A^2 monotonically decreases with y. Then J becomes

$$J(k_\perp, k_\parallel; \omega) = \delta Y_\mathrm{I} Y_\mathrm{II} D(k_\perp, k_\parallel; \omega) \qquad (8)$$

where $Y_\mathrm{I} = k_\perp\epsilon_\mathrm{I}/\delta$, $Y_\mathrm{II} = k_\perp\epsilon_\mathrm{II}/\delta$, and D, the 'dispersion function,' is

$$D(k_\perp, k_\parallel; \omega) = -[I_0(Y_\mathrm{II}) + I_1(Y_\mathrm{II})][K_0(Y_\mathrm{I}) + K_1(Y_\mathrm{I})] + [I_0(Y_\mathrm{I}) - I_1(Y_\mathrm{I})][K_0(Y_\mathrm{II}) - K_1(Y_\mathrm{II})] \qquad (9)$$

Here I and K are modified Bessel functions, and D is a multivalued function due to the logarithmic terms in K. This gives rise to branch cuts and therefore Riemann sheets (Figure 2). The complete solution $\xi_y(y, t)$ can be obtained by performing inverse Laplace transform and integrating over the initial conditions; i.e.,

$$\xi_y(y, t) = \frac{1}{2\pi} \int dy' \int_c d\omega\, e^{-i\omega t} G(y, y'; \omega) \bar{S}'(y', \omega) \quad (10)$$

In performing the ω integration one closes the integration path by extending it into the lower half of the ω plane. The two major contributions are from integrations along the branch cuts (on the real axis of the ω plane) of G, which corresponds to the continuous spectrum, and from simple poles of the analytic continuation of D into the lower half of the ω plane, which corresponds to the collective modes. This finding is similar to the theory of Landau damping in warm plasma oscillations [*Sedláček*, 1971*b*]. We are only interested in the least-damped modes, which exist in the $n = \pm 1$ Riemann sheets when $|k_\perp a| \ll 1$. Assuming $|k_\perp a| \ll 1$ or, equivalently, $|Y_{\rm I}|$, $|Y_{\rm II}| \ll 1$, we have the following 'dispersion relation' on the $n = \pm 1$ sheet:

$$D_{+1}(k, \omega) \simeq -\left(\frac{1}{Y_{\rm I}} + \frac{1}{Y_{\rm II}}\right) + \pi i = 0 \quad (11)$$

From (11) we obtain the oscillation frequency ω_r and the damping rate γ:

$$\omega_r \simeq \left[\frac{k_\parallel^2(B_{\rm I}^2 + B_{\rm II}^2)}{\mu_0(\rho_{\rm I} + \rho_{\rm II})}\right]^{1/2} \quad (12)$$

$$\frac{\gamma}{\omega_r} \simeq -\frac{\pi}{2}(k_\perp 2a) \cdot \frac{\mu_0 \rho_{\rm I} \rho_{\rm II}(V_{A\rm I}^2 - V_{A\rm II}^2)}{(\rho_{\rm I} + \rho_{\rm II})(B_{\rm I}^2 + B_{\rm II}^2)} \quad (13)$$

Note that owing to the difference in ϵ between this case and the electrostatic case a simple substitution of ω_p in electrostatic case by $k_\parallel v_A$ does not lead to (12) and (13). On the $n = -1$ sheet we have $\omega_r' = -\omega_r$ and $\gamma' = \gamma$. Note also that $\omega_{A\rm II}^2 \le \omega_r^2 \le \omega_{A\rm I}^2$.

At the plasmapause, $\rho_{\rm II} \gg \rho_{\rm I}$ and $B_{\rm I}^2 \simeq B_{\rm II}^2$; we then have

$$\omega_r \simeq 2^{1/2} k_\parallel v_{A\rm II} \quad (14)$$

$$\gamma/\omega_r \simeq -(\pi/4)\,|k_\perp/\kappa| \quad (15)$$

Here $\kappa^{-1} = 2a \simeq [d \ln \rho/dy|_{y=0}]^{-1}$ is the scale length of density variation at the plasmapause.

At the magnetopause, however, $\rho_{\rm II} \gg \rho_{\rm I}$ and $B_{\rm II}^2 \ll B_{\rm I}^2$. Here subscripts I and II stand for the earthside and the sunside, respectively. We then have

$$\omega_r \simeq k_\parallel B_{\rm I}/(\mu_0 \rho_{\rm II})^{1/2} \quad (16)$$

$$\gamma/\omega_r \simeq -(\pi/2)\,|k_\perp/\kappa'| \quad (17)$$

Here $(\kappa')^{-1}$ is the scale length of density and magnetic field variation at the magnetopause.

Physically, this collective mode corresponds to the eigenmode of the surface wave excited at the boundary due to the discontinuity in ϵ. The sharper the discontinuity is, i.e., the smaller $|k_\perp/\kappa|$ is, the weaker the damping is. The number of the collective modes is approximately equal to the number of discontinuities [*Barston*, 1964]. As the profile becomes smooth, these modes become heavily damped, and one is left only with the continuous spectrum. Finally, as can be expected from a surface wave, let us note that the collective mode has a peak at the jump and decays away exponentially as $\exp[-|k_\perp y|]$.

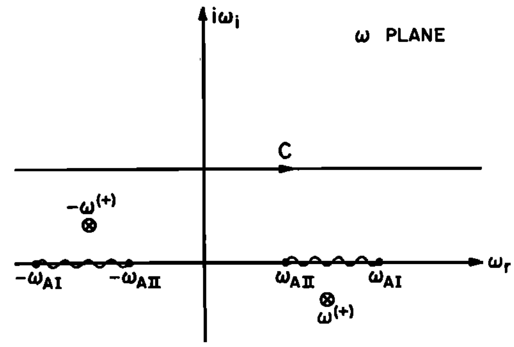

Fig. 2. The branch cuts and the least-damped roots of the dispersion function on its $n = +1$ Riemann sheet; C is the integration path for the inverse Laplace transform.

SUMMARY AND DISCUSSION

In the previous sections we have presented a theory of long-period micropulsations as an initial value problem. Using a one-dimensional model and the Laplace transform, we derive a wave equation that shows the coupling between the surface wave (i.e., the evanescent compressional Alfvén wave) and the shear Alfvén wave. We then note the analogy of this problem to that of electrostatic oscillations in nonuniform cold plasmas. From this analogy we can conclude that there exists first, a continuous spectrum

$$\{\omega^2 \mid \min \omega_A^2 \le \omega^2 \le \max \omega_A^2\}$$

which corresponds to noncollective modes with, asymptotically, position dependent frequency $\omega_A(y)$ and damping proportional to inverse power of time, and second, a discrete spectrum, which corresponds to collective modes (surface eigenmodes) with position independent frequency and exponential damping. The damping can be rather weak if the plasma parameters have sharp variations. Expressions of the frequency ω_r and damping rate γ of the surface eigenmodes are then derived in terms of plasma parameters. As we have shown in the companion paper, the dynamics of MHD plasmas in a dipole magnetic field can be described by a coupled wave equation with a structure similar to the one used here. Therefore we can expect that the fundamental physics obtained here, i.e., the continuous spectrum and the surface eigenmodes, also exists in the more complicated and realistic case. There are, however, other effects due to the presence of the ionosphere-earth boundary. With the boundary located at $y = c$, the boundary condition then becomes $\xi_y = 0$ at $y = -\infty, c$ instead of at $y = \pm\infty$. An additional damping introduced by this effect is negligible, however, if $k_\perp c \gg 1$, because γ/ω due to this effect may be estimated to be $\exp[-k_\perp c]$, even if the ionosphere is totally absorbing. The other effect is a damping due to the ionospheric dissipations associated with the motion of the field line in the ionosphere. The order of magnitude of this damping rate can be roughly estimated by $\gamma \simeq P/W$. Here P is the dissipated power, and W is the total wave energy. If we take the skin depth of the ionosphere and the ground λ as the dissipative layer, γ/ω then may be deduced to be $\sim\lambda/l$, where l is the length of the field line. Consequently, damping due to this effect is also negligible.

Let us now discuss the above theoretical results with regard to recent observations of the plasmapause-associated micropulsations mentioned earlier [*Lanzerotti et al.*,

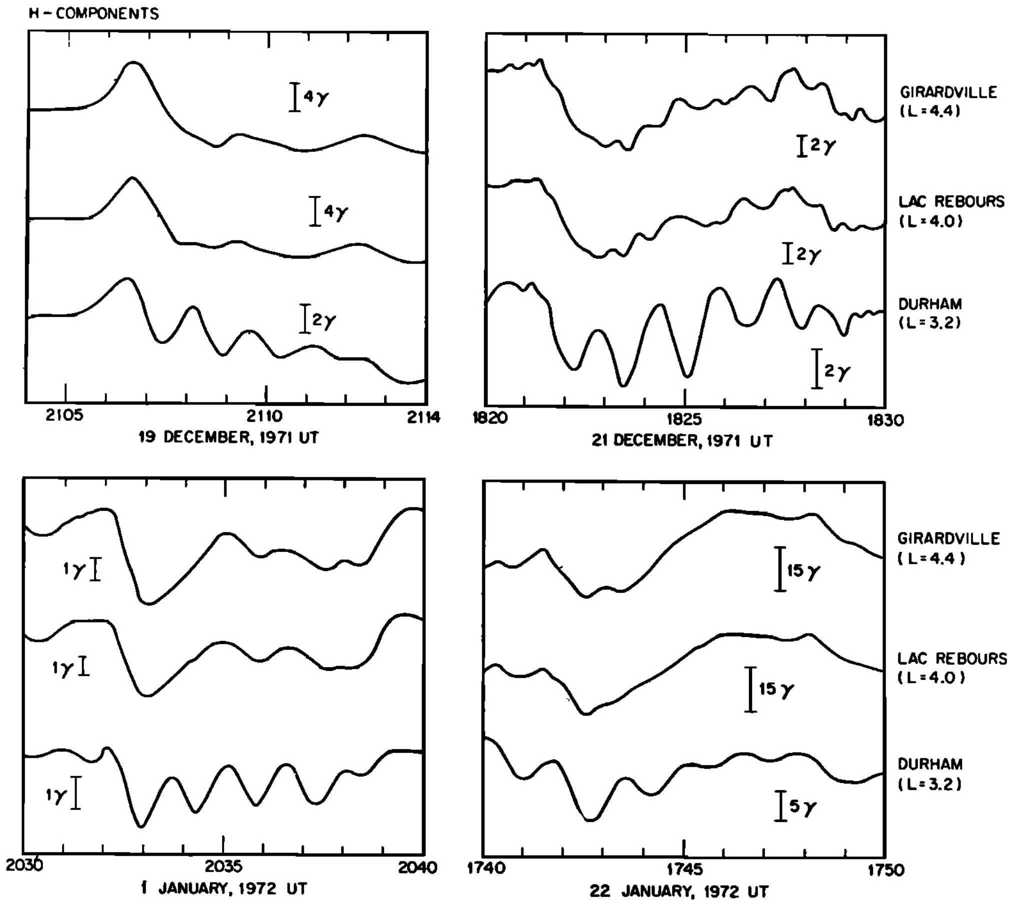

Fig. 3. Latitude dependence of magnetic variations during four time periods when sinusoidal oscillations were observed at the lowest latitude. All three stations are located at the same magnetic longitude [after *Lanzerotti et al.*, 1973].

1973]. Four typical events are shown in Figure 3. The damped oscillations observed at $L = 3.2$ are assumed to be the excitations of weakly damped surface eigenmodes due to a sharp density jump. The disappearance of such oscillations at higher latitudes ($L = 4.0$ and 4.4) is consistent with the theoretical result that the surface eigenmode has its peak magnitude at the jump and decays away exponentially with a scale of $k_\perp^{-1}$. These observations then suggest that during these events the plasmapause was located near $L = 3.2$. Away from the plasmapause (e.g., at $L = 4.0$), only modes corresponding to the continuous spectrum were excited and, consequently, damped away owing to spatial phase mixing. Furthermore, using the observed oscillation frequency ($\omega \sim 0.07$ rad/s), damping rate ($\gamma/\omega \sim 0.08$), and maximum decay length (i.e., $k_{\perp,\min} \sim 1/0.8\ R_E$) in (14) and (15), we obtain for the fundamental mode $V_{AII} \simeq 650$ km/s and $\kappa_{\max}^{-1} \simeq 0.05\ R_E$. These results are consistent with satellite observations [*Burton et al.*, 1970; *Chappell et al.*, 1970].

The surface eigenmode may also explain the long-period damped type pulsations often associated with sudden commencements and sudden impulses; i.e., Psc 4 and Psc 5 [*Saito and Matsushita*, 1967]. Although detailed information is lacking, we can roughly check the frequency range. Near the magnetopause ($L \simeq 8$), from (16) and the balance of momentum flux ($B_I^2/2\mu_0 = \rho_{II} V_S^2$), the eigenmode frequency is (if the Doppler shift by the solar wind motion is neglected)

$$\omega_r \simeq 2^{1/2} k_\parallel V_S \qquad (18)$$

Here V_S is the solar wind speed. Taking $V_S \simeq 400$ km/s, we then have for the fundamental mode a period $T \simeq 400$ s, which corresponds to Psc 5 predominantly observed in auroral zone. Finally, we emphasize the significance of using micropulsation data as a possible diagnostic tool for the dynamics of the plasmapause and, perhaps, the magnetopause and solar wind, and we note that damped Alfvén waves attributed as surface eigenmodes have also been observed in laboratory theta pinch experiments [*Grossmann and Tataronis*, 1973].

Acknowledgments. The authors acknowledge valuable discussions with H. Fukunishi and L. J. Lanzerotti.

* * *

The Editor thanks J. C. Samson and D. J. Southwood for their assistance in evaluating this paper.

REFERENCES

Barston, E. M., Electrostatic oscillations in inhomogeneous cold plasmas, *Ann. Phys.*, *29*, 282, 1964.

Burton, R. K., C. T. Russell, and C. R. Chappell, The Alfvén velocity in the magnetosphere and its relationship to ELF emissions, *J. Geophy. Res.*, *75*, 5582, 1970.

Chappell, C. R., K. K. Harris, and G. W. Sharp, A study of the influence of magnetic activity on the location of the plasmapause as measured by Ogo 5, *J. Geophys. Res.*, *75*, 50, 1970.

Chen, L., and A. Hasegawa, A theory of long-period magnetic pulsations, 1, Steady state excitation of field line resonance, *J. Geophys. Res.*, *79*, this issue, 1974.

Grossmann, W., and J. Tataronis, Decay of MHD waves by phase mixing, 2, The theta-pinch in cylindrical geometry, *Z. Phys.*, *261*, 217, 1973.

Lanzerotti, L. J., H. Fukunishi, A. Hasegawa, and L. Chen, Excitation of the plasmapause at ULF frequencies, *Phys. Rev. Lett., 31,* 624, 1973.
Saito, T., and S. Matsushita, Geomagnetic pulsations associated with sudden commencements and sudden impulses, *Planet. Space Sci., 15,* 573, 1967.
Sedláček, Z., Electrostatic oscillations in cold inhomogeneous plasma, 1, Differential equation approach, *J. Plasma Phys., 5,* 239, 1971*a*.
Sedláček, Z., Electrostatic oscillations in cold inhomogeneous plasma, 2, Integral equation approach, *J. Plasma Phys., 6,* 187, 1971*b*.
Uberoi, C., Alfvén waves in inhomogeneous magnetic fields, *Phys. Fluids, 15,* 1673, 1972.

(Received July 26, 1973;
accepted November 13, 1973.)