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ASYMPTOTICS OF WAVE FUNCTIONS OF THE STATIONARY SCHRÖDINGER EQUATION IN THE WEYL CHAMBER

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We study stationary solutions of the Schrödinger equation with a monotonic potential U in a polyhedral angle (Weyl chamber) with the Dirichlet boundary condition. The potential has the form $U(\mathbf{x}) = \sum_{j=1}^{n} V(x_j)$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, *with a monotonically increasing function* $V(y)$. *We construct semiclassical asymptotic formulas for eigenvalues and eigenfunctions in the form of the Slater determinant composed of Airy functions with arguments depending nonlinearly on* x_j. *We propose a method for implementing the Maslov canonical operator in the form of the Airy function based on canonical transformations.*

Keywords: stationary Schrödinger equation, boundary value problem, Weyl-chamber-type polyhedral angle, spectrum, quantization condition, Maslov canonical operator, Airy function

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1. Introduction

The two-dimensional and three-dimensional Ising models in which a droplet of a phase contacts the wall of the containing box were studied in [1]. To describe the contact surface, it was necessary to consider a special boundary value problem for the n-dimensional Schrödinger equation

$$-\frac{h^2}{2}\triangle\Psi + U(\mathbf{x})\Psi = \mathcal{E}\Psi, \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{1.1}$$

with the potential $U(\mathbf{x}) = \sum_{i=1}^{n} V(x_i)$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, and a monotonically increasing $V(x)\colon V(0) = 0$, $V'(x) > 0$, $V \in C^2$. The Schrödinger operator was considered in the domain $\Omega = \{x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$ (Weyl chamber) with Dirichlet boundary conditions on $\partial\Omega$,

$$\psi|_{\partial\Omega} = 0. \tag{1.2}$$

An important observation was that the eigenfunctions of problem (1.1), (1.2) are given by the Slater determinant. Namely, we have the following assertion.

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Lemma 1. *Let E_k and $\psi_k(x)$ be the eigenvalues and the corresponding eigenfunctions of the one-dimensional problem*

$$-\frac{h^2}{2}\psi''(x) + V(x)\psi = E\psi, \quad x \geq 0, \quad \psi(0) = 0. \tag{1.3}$$

We construct the Slater determinant composed of the functions $\psi_j(x_k)$:

$$\Psi^{\mathrm{SK}}_{\mathbf{k}}(x) = \frac{1}{n}\det\begin{pmatrix}\psi_{k_1}(x_1) & \cdots & \psi_{k_1}(x_n)\\ \vdots & \ddots & \vdots\\ \psi_{k_n}(x_1) & \cdots & \psi_{k_n}(x_n)\end{pmatrix}. \tag{1.4}$$

Then for the multi-index $\mathbf{k} = \{k_1, \dots, k_n\}$, $k_i \neq k_j$, $i, j = 1, \dots, n$, such a determinant is an eigenfunction of problem (1.1), (1.2) *in the domain Ω corresponding to the eigenvalue*

$$\mathcal{E}_{\mathbf{k}} = \sum_{j=1}^{n} E_{k_j}. \tag{1.5}$$

This fact holds only for a Weyl chamber and is inapplicable to general polyhedrons, which indicates that there is a certain “integrability” structure in the considered problem that allows reducing the study of semiclassical asymptotic formulas for the multidimensional problem to a set of one-dimensional problems and, moreover, allows constructing exact solutions of the studied problem for a potential of the form $V(x) = ax$ or $V(x) = ax^2$. In the first case, the eigenfunctions $\psi_k(x)$ and the eigenvalues E_k of one-dimensional problem (1.3) can be expressed in terms of the Airy function and its zeros $z_k < 0$, $\mathrm{Ai}(z_k) = 0$, $z_{k+1} < z_k$, $k = 0, 1, \dots$,

$$\psi_k = \mathrm{Ai}\biggl(x\sqrt[3]{\frac{2a}{h^2}} + z_k\biggr), \qquad E_k = |z_k|\frac{(ah)^{2/3}}{2^{1/3}}.$$

It is hardly possible to construct exact solutions for other functions V, and the asymptotic methods can be used here. On the other hand, a semiclassical approximation, as a rule, does not work very effectively in boundary value problems, because a variation of the domain Ω destroys the “integrability,” and there are very few cases of boundary value problems that can be studied explicitly. It is intuitively clear that this fact is connected with the integrability in the theory of billiards (see [2]). We therefore believe that under the assumption that h is a small positive parameter, it would be rather interesting to consider this boundary value problem in the Weyl chamber using a semiclassical approximation.

Here, we construct asymptotic solutions of problem (1.1), (1.2) for an arbitrary monotonically increasing smooth potential $V(x)$, $V'(x) > 0$, $x > 0$. In Sec. 2, we present formulas for these solutions and discuss the relation between the eigenvalues and the Bohr–Sommerfeld quantization condition. In Sec. 3, we present their “geometric” derivation based on the Maslov canonical operator and its properties. To complete the presentation, we discuss the required properties of the change of variables in the canonical operator in the appendix.

2. Asymptotic eigenfunctions and eigenvalues of the one-dimensional problem: Bohr–Sommerfeld quantization conditions

We consider problem (1.3) with an arbitrary smooth monotonic potential $V(x)$: $V'(x) > 0$, $x \geq 0$. In this case, the asymptotic wave functions are expressed in terms of the Airy function $\mathrm{Ai}(z)$, and the asymptotic eigenvalues are expressed in terms of the zeros of $\mathrm{Ai}(z)$ using quantization conditions similar to the Bohr–Sommerfeld quantization conditions. We first write the corresponding formulas and then give their constructive proof.

We let $x^*(E)$ denote a positive solution of the equation $V(x) = E$ and introduce the functions

$$\mathcal{Y}(x, E) = \operatorname{sgn}(x - x^*)\left|\frac{3}{2}\int_{x^*}^{x} \sqrt{|2(V - E)|}\, dx\right|^{2/3} \tag{2.1}$$

and

$$R(x, E) = \frac{\mathcal{Y}(x, E)}{V(x) - E}. \tag{2.2}$$

It is easy to see that the constructed functions $\mathcal{Y}(x, E)$ and $R(x, E)$ are infinitely differentiable for $E > 0$ and $\mathcal{Y}_x = \partial\mathcal{Y}(x, E)/\partial x$ does not vanish (and of course does not go to infinity) on any finite interval of the line $\mathbb{R}_x$.

We recall that we let z_k, $z_{k+1} < z_k$, $k = 0, 1, \ldots$, denote the zeros of the Airy function $\mathrm{Ai}(z)$. We determine the numbers E_k, $k = 0, 1, 2, \ldots$, from the equation

$$\frac{1}{h}\int_0^{x^*(E_k)} \sqrt{2(E_k - V)}\, dx = \frac{2}{3}|z_k|^{3/2}. \tag{2.3}$$

We introduce the functions

$$\psi_k(x) = \frac{C\sqrt[4]{|R(x, E_k)|}}{\sqrt[6]{h}}\,\mathrm{Ai}\left(\frac{\mathcal{Y}(x, E_k)}{h^{2/3}}\right). \tag{2.4}$$

Proposition 1. 1. *The functions $\psi_k(x)$ and the numbers E_k, $k = 0, 1, 2, \ldots$, determine the respective asymptotic eigenfunctions and eigenvalues of problem* (1.3) *up to $O(h^{5/6})$ and $O(h^2)$.*

2. *In determinant* (1.4), *the functions ψ_k are taken in form* (2.4) *with the numbers $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, and the numbers E_k in formula* (1.5) *are obtained from quantization condition* (2.3). *These functions and numbers then approximate the respective eigenfunctions and eigenvalues of the operator up to $O(h)$ and $O(h^2)$.*

Without conditions (2.3), formula (2.4) is contained in [3] and was obtained using the method of reference equations. We derive it differently in Sec. 3 and now discuss equality (2.3). We recall (see, e.g., [4]) that for large negative z, the asymptotic formula

$$\mathrm{Ai}(z) = \frac{1}{\sqrt{\pi}\sqrt[4]{-z}}\sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right) + O((-z)^{3/2})$$

holds for the Airy function. Under the condition $\mathcal{Y}(x, E_k)/h^{2/3} \ll -1$, this implies that

$$\psi_k(x) = \frac{C}{\sqrt[4]{(E - V)}}\sin\left(\frac{1}{h}\int_{x^*}^{x} \sqrt{|2(E_k - V)|}\, dx + \frac{\pi}{4}\right). \tag{2.5}$$

This function naturally coincides with the WKB asymptotic form [5], [6] of the functions $\psi_k(x)$ to the left of the focal point $x = x^*$ (outside its neighborhood). The zeros of this function are determined by the relation

$$\frac{1}{h}\int_0^{x^*(E_k)} \sqrt{2(E_k - V)}\, dx = \pi\left(\frac{3}{4} + k\right), \quad k = 0, 1, 2, \ldots, \tag{2.6}$$

which can be rewritten as follows. It is well known (see [6] and the next section) that the semiclassical asymptotic form of the eigenfunctions $\psi_k(x)$ are related to the curves on the phase plane $\mathbb{R}^2_{px}$ (one-dimensional Lagrangian manifolds) defined by the equations

$$\widetilde{\Lambda}(E) = \left\{p, x \in \mathbb{R}^2_{p,x}\colon\ \frac{p^2}{2} + V(x) = E,\ x \geq 0\right\}.$$

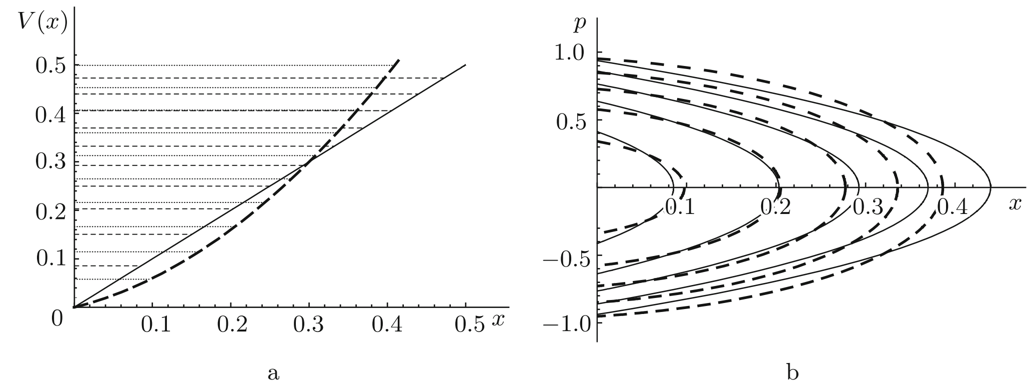

Fig. 1. (a) Potentials $V_1(y) = y$ (solid line) and $V_2 = 2(y+0.1)^2 - 0.02$ (dashed line)). (b) Lagrangian manifolds in the phase space $\Lambda_k \in \mathbb{R}^2_{p,y}$ corresponding to the energy E_k, $k = 0, 2, 4, 6, 8$, in Example 1 (solid lines) and in Example 2 (dashed lines).

In the studied case, it is reasonable to add a vertical segment to the arc $\widetilde{\Lambda}(E)$,

$$\Gamma(E) = \{p, x \in \mathbb{R}^2_{p,x} \colon -\sqrt{E - V(0)} \le p \le \sqrt{E - V(0)},\ x = 0\},$$

and construct a closed curve $\Lambda(E) = \widetilde{\Lambda}(E) \cup \Gamma(E)$ (see Fig. 1). Equality (2.6) can then be written as the Bohr–Sommerfeld quantization condition

$$\frac{1}{2\pi} \oint_{\Lambda(E_k)} \sqrt{2(E_k - V)}\, dx = h\left(\frac{3}{4} + k\right). \tag{2.7}$$

The number 3/4 is $\operatorname{Ind} \Lambda(E_k)/4$, where $\operatorname{Ind} \Lambda(E_k)$ is the Maslov index of a closed path on $\Lambda(E_k)$ coinciding with $\Lambda(E_k)$. For smooth curves $\Lambda(E_k)$, it is equal to 2, but the curve $\Lambda(E_k)$ here is not smooth because of the boundary condition, and its index is equal to 3. In a more general case, such manifolds ("horned spheres") and the quantization condition on them were described in [7]. If we now return to the original n-dimensional problem and introduce the corresponding $2n$-dimensional phase space, then the product of the one-dimensional manifolds $\Lambda(E_{k_j})$ corresponding to different x_j gives an n-dimensional nonsmooth Lagrangian manifold of such a type, and the quantization conditions on it give the spectrum of the original n-dimensional Schrödinger operator.

Finally, we note that the Bohr–Sommerfeld quantization condition generally holds for large k, namely, for $k \sim 1/h$. Replacing condition (2.3) with quantization condition (2.7) is in fact replacing zeros of the Airy function with the zeros of its asymptotic approximation and gives a good approximation even for small k. For example, the first root is $z_0 \approx -2.338$, and its asymptotic approximation is $\tilde{z}_0 \approx -2.320$, i.e., the error is less than 1%. Therefore, to determine E_k even for $k = 0, 1, \dots$, we can use condition (2.7) instead of condition (2.3).

We consider the following two examples:

1. $V(x) = x$ and
2. $V(x) = 2(x + 0.1)^2 - 0.02$ for $h = 0.01$.

We show plots of the potentials and the first ten eigenvalues and also the corresponding Lagrangian manifolds in Fig. 1.

In Figs. 2 and 3, we show the asymptotic behavior of the wave functions $\psi_k(x)$ and the asymptotic behavior of the functions $\Psi^{\mathrm{SK}}_{\{k,m\}}(x, y)$ for the two-dimensional angle $(x, y) \in \mathbb{R}^2_+$, $x \le y$.

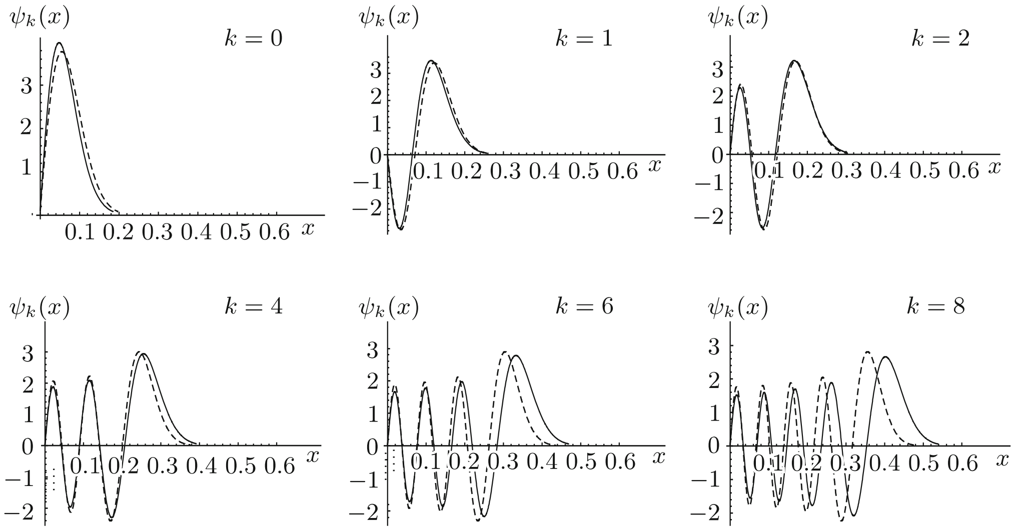

Fig. 2. Asymptotic behavior of the wave functions $\psi_k(x)$, $k = 0, 1, 2, 4, 6, 8$, for Example 1 (solid lines) and Example 2 (dashed lines).

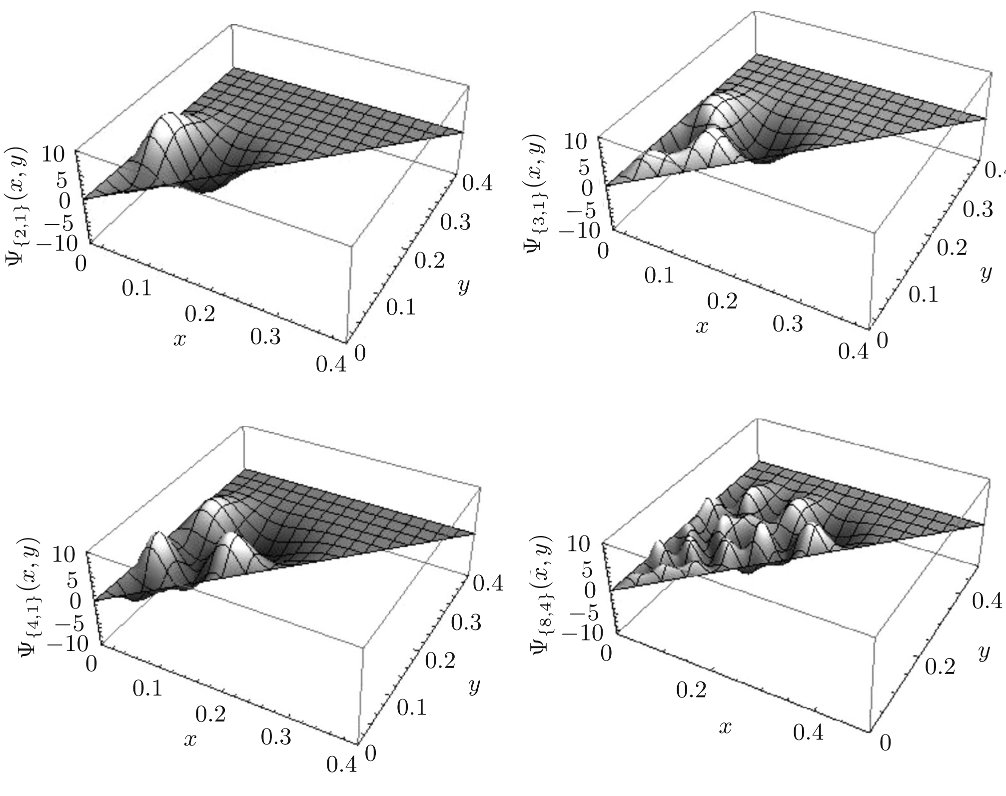

Fig. 3. Asymptotic behavior of the wave functions $\Psi^{\mathrm{SK}}_{\{k,m\}}(x, y)$ in the angle for $k = 2, 3, 4$, $m = 1$ and $k = 8$, $m = 4$ in Example 2.

3. Derivation of asymptotic formulas for wave functions by canonical changes of variables in the Maslov canonical operator

Proof of Proposition 1. Formula (2.3) obviously follows from (2.4). Formulas (2.4) can in principle be obtained using the Langer method and reference equations (see [3], [8], and also [9]). Here, we show how they can be derived directly from the Maslov canonical operator and its general properties.

We construct (formal) asymptotic approximations in the form of the canonical operator [6] that approximate the wave functions in the one-dimensional case. The asymptotic approximations are (locally) related to a family of invariant curves, i.e., Lagrangian manifolds Λ on the phase plane $\mathbb{R}^2_{y,p}$, which are similar to horizontal parabolas and are trajectories of the Hamiltonian system with the Hamiltonian $H(x,p) = p^2/2+V(x)$ at the energy level $H = E$:

$$\Lambda = \{p = P(t,E),\ x = X(t,E),\ t \in [-T,T]\} \equiv \{H(x,p) = E,\ x \geq 0\}.$$

The functions $p = P(t,E)$ and $x = X(t,E)$ are solutions of the Hamiltonian system

$$\dot{p} = -H_x = -v_x, \qquad \dot{x} = H_p = p, \qquad p|_{t=0} = 0, \qquad x|_{t=0} = x^*(E). \tag{3.1}$$

Here, $x^*(E)$ is a solution of the equation $V(x) = E$, x^* is a turning point on the curves $\Lambda(E)$, the time t, i.e., the coordinate on the curve, is chosen such that $t = 0$ corresponds to the turning point, and the time $t = \pm T$ corresponds to the point $x = 0$ on the axis x. Positive values of t are associated with positive values of $P(t,E)$, negative values are associated with negative values, and $P(t,E) = \pm\sqrt{E - V(X(t))}$ in this case. It is natural to take dt as the measure on the curves $\Lambda(E)$. The Jacobians of projections of points on the curve $\Lambda(E)$ on the axes x and p are respectively equal to $J(t,E) = \dot{X} \equiv P$ and $\widetilde{J} = \dot{P} \equiv -V_x$. The Jacobian $J(t,E)$ vanishes at the turning point $t = 0$, and the Jacobian $\widetilde{J} \neq 0$ for all t.

The manifold $\Lambda(E)$ can be covered by a single chart, and the asymptotic solution of Eq. (1.1) (still disregarding the Dirichlet condition for $x = 0$) can be written as the canonical operator $K^h_{\Lambda(E)}$ applied to a function equal to unity by the (single) integral

$$\psi_{\mathrm{as}}(x,E) = K^h_{\Lambda(E)} 1 \equiv \frac{C}{\sqrt{2\pi h}} \int_{-\infty}^{\infty} \frac{e(t)}{\sqrt{|\widetilde{J}(t)|}} \exp\left[\frac{i}{h}\left(\int_0^t P\,dX + p(x - X(t))\right)\right]\Big|_{t=t(p)} dp, \tag{3.2}$$

where $e(t)$ is a smooth cutoff function equal to unity on the interval $[-T,T]$ and to zero outside a neighborhood of this interval, $t(p)$ is a solution of the equation $P(t) = p$, C is a complex constant, and the canonical operator ψ_{as} is defined up to $O(h)$ and is independent of the choice of the partition of unity $e(t)$. The value of the spectral parameter E is chosen from the boundary condition $\psi(0,E) = 0$. A drawback of this formula is that it is ineffective: in particular, it does not allow obtaining simple formulas for the spectrum of the original problem. The goal in the further argument is to obtain an effective representation of function (3.2) in the form of the Airy function with a complicated argument and an amplitude depending on x.

Integral (3.2) can in principle be represented in the form of the Airy function using the approach proposed in [4], but we believe that the "geometric" approach proposed below turns out to be simpler and more visual and, in addition, can be used in other situations. Namely, we assume that Λ is a Lagrangian manifold on the phase plane, $d\mu$ is a measure, and A is a function on it. We assume that instead of the variable x in the configuration space, we choose the variable $y = Y(x)$, this change is nondegenerate, and $\partial Y/\partial x > 0$ for definiteness. This change induces a canonical change of variables on the phase plane $x \to y$, $p \to q$: $y = \mathcal{Y}(x)$, $q = p/\mathcal{Y}'(x)$. Conversely, $x = \mathcal{X}(y)$, $p = q/\mathcal{X}'$. Because the change of coordinates is canonical, the Hamiltonian $H - E$ in the new coordinates (denoted by $\mathcal{H}(q,y)$) satisfies the formula $\mathcal{H}(q,y,E) = H(\mathcal{P}(q,y), \mathcal{X}(y)) - E$. We show that the change of variables can be chosen such that $\mathcal{H}(q,y) = g(y)(q^2 + y)$, where $g(y) = G(\mathcal{X}(y)) \neq 0$ is a smooth function. Indeed, we have

$$\mathcal{H}(q,y,E) = \left(\frac{\partial\mathcal{Y}}{\partial x}\right)^2 \frac{q^2}{2} + V - E = \frac{V - E}{\mathcal{Y}}\left[\left(\frac{\partial\mathcal{Y}}{\partial x}\right)^2 \frac{\mathcal{Y}}{2(V - E)} q^2 + y\right].$$

We now choose the function $\mathcal{Y}$ from the condition

$$\left(\frac{\partial \mathcal{Y}}{\partial x}\right)^2 \frac{\mathcal{Y}}{2(V-E)} = 1 \quad \Longleftrightarrow \quad \left(\frac{\partial \mathcal{Y}}{\partial x}\right)^2 \mathcal{Y} = 2(V-E). \tag{3.3}$$

We supplement this equation with the initial condition $\mathcal{Y}(x^*) = 0$ and integrate it to obtain

$$\mathcal{Y}(x,E) = \operatorname{sgn}(x-x^*)\left(\frac{3}{2}\int_{x^*}^{x} \sqrt{|2(V-E)|}\,dx\right)^{2/3}. \tag{3.4}$$

It is easy to see that the constructed function $\mathcal{Y}(x,E)$ is infinitely differentiable and $\mathcal{Y}_x = \partial\mathcal{Y}(x,E)/\partial x$ does not vanish (and naturally does not go to infinity) on any finite interval of the line $\mathbb{R}_x$. This function thus determines a one-to-one transformation $x \to y$, and

$$R(x) = \frac{\mathcal{Y}(x)}{V(x)-E} \equiv \frac{2}{\mathcal{Y}_x^2} \tag{3.5}$$

is also a smooth nonvanishing function. Because the Hamiltonian is

$$\mathcal{H} = \frac{\mathcal{Y}}{V-E}(q^2+y) \equiv R(q^2+y),$$

the curve Λ in the coordinates (q,y) is defined by the relation

$$\mathcal{H} = 0 \quad \Longleftrightarrow \quad q^2 + y = 0. \tag{3.6}$$

Because the function $\mathcal{H}$ is zero on the trajectories, the Hamiltonian system in the variables (q,y) becomes

$$\dot{q} = -\frac{1}{R}, \qquad \dot{y} = 2\frac{q}{R}.$$

If we introduce a new time τ by the formula $d\tau = dt/R$, then this system and its solutions $q = Q(\tau)$, $y = Y(\tau)$ take the simplest form (which is naturally consistent with (3.6)):

$$\frac{dq}{d\tau} = -1, \qquad \frac{dy}{d\tau} = 2q$$

and

$$Q = -\tau, \qquad Y = -\tau^2. \tag{3.7}$$

We now construct the Maslov canonical operator $[\widetilde{K}^h_{\Lambda(E)}A](y)$ acting on the function A on the curve Λ in the new coordinate y in the configuration space (coordinates (q,y) in the phase space) and the coordinate τ on the curve Λ with the measure $d\tau$. Using formula (3.2) in this case with regard to the relation $\tau = -q$, we obtain

$$[\widetilde{K}^h_{\Lambda(E)}A](y) = \frac{e^{i\pi/4}}{\sqrt{2\pi h}}\int_{-\infty}^{\infty} A(\tau)e(\tau)\exp\left[\frac{i}{h}\left(\frac{2\tau^3}{3} - \tau(y+\tau^2)\right)\right]\bigg|_{\tau=-q} dq.$$

We can now use the properties of the canonical operator with respect to changes of variables. Namely, the general properties of the canonical operator (see [6], [7], and the appendix) imply the relation

$$K^h_{\Lambda(E)}1 = \sqrt{\left|\frac{\partial\mathcal{Y}(x)}{\partial x}\right|}\left[\widetilde{K}^h_{\Lambda(E)}\sqrt{\left|\frac{dt}{d\tau}\right|}\right](y)\bigg|_{y=\mathcal{Y}(x)} = \sqrt[4]{\frac{2}{R}}[\widetilde{K}^h_{\Lambda(E)}\sqrt{R}](y)|_{y=\mathcal{Y}(x)}$$

up to terms of higher order in the parameter h (more precisely, up to $O(h^{5/6})$). We now note that the function $|R(\mathcal{X}(Y(\tau)))|$ under the action of the canonical operator is obtained from the smooth function $|R(\mathcal{X}(y))|^{-1/2}$ defined in the configuration space. We can therefore interchange it with the canonical operator and obtain

$$K^h_{\Lambda(E)}1 = \sqrt[4]{2|R(x)|}[\widetilde{K}^h_{\Lambda(E)}1](y)|_{y=\mathcal{Y}(x)} =$$
$$= c\frac{\sqrt[4]{|R(x)|}}{\sqrt{h}}\int_{-\infty}^{\infty} e(-q)\exp\left[\frac{i}{h}\left(\frac{1}{3}q^3+\mathcal{Y}(x)q\right)\right]dq,$$

$c = \sqrt[4]{2}e^{i\pi/4}/\sqrt{2\pi}$, with formulas (3.3) and (3.5) and the definition of canonical operator taken into account. The integral in the last expression is equal to $2\pi h^{1/3}\,\mathrm{Ai}(\mathcal{Y}(x)/h^{2/3})$ up to $O(h^\infty)$, where $\mathrm{Ai}(y)$ is the Airy function. This immediately implies (2.4). This formula gives a formal asymptotic approximation for the wave function, i.e., an asymptotic approximation that leads to a small discrepancy after its substitution in the original one-dimensional equation. The proof that (2.4) determines the "true" asymptotic approximation of the eigenfunction practically repeats the reasoning in [10].

Appendix : Changes of variables in the Maslov canonical operator

We here present several facts in a compact form that are useful for practical calculations [6], [7]. We assume that the change of coordinates $x = \mathcal{X}(y) \iff y = \mathcal{Y}(x)$ is given on the real line $x \in \mathbb{R}$ and $\partial\mathcal{X}/\partial y > 0$. This change induces the canonical change of variables $(p, x) \to (q, y)$, $p = q/\mathcal{X}_y$ on the phase plane $\mathbb{R}^2_{p,x}$ with the coordinates (p, x). We assume that a smooth curve (Lagrangian manifold) $\Lambda = \{p = P(\alpha),\ x = X(\alpha),\ \alpha \in \mathbb{R}\}$, where α are the coordinates on the curve Λ, is given on $\mathbb{R}^2_{p,x}$. We also assume that other coordinates β related to α by the equation $\alpha = \mathcal{A}(\beta) \iff \beta = \mathcal{B}(\alpha)$, $\partial\mathcal{A}/\partial\beta > 0$ are given on Λ. We finally assume that a function $A(\alpha)$ is given on Λ. We use the Maslov canonical operator on the curve Λ with the central point α^0 to define the function $\psi = [K_\Lambda A(\alpha)](x)$. We pass from the coordinates (p, x) on the phase plane to the coordinates (q, y). This transition gives a curve $\widetilde{\Lambda}$ on the phase plane with the coordinates (q, y). Choosing the coordinate β on $\widetilde{\Lambda}$, we can write $\widetilde{\Lambda} = \{q = Q(\beta),\ y = Y(\beta)\}$ and $\Lambda = \{p = P(\alpha) \equiv Q(B(\alpha))/\mathcal{X}_y(Y(B(\alpha))),\ x = X(\alpha) \equiv \mathcal{X}(Y(B(\alpha)))\}$. We now construct the Maslov canonical operator on the curve $\widetilde{\Lambda}$ with the coordinates β assuming that the central point is $\beta^0 = \mathcal{B}(\alpha^0)$. We then have

$$[K_\Lambda A(\alpha)](x) = \frac{1}{\sqrt{\mathcal{X}_y(y)}}\big[K_\Lambda\big(\sqrt{\mathcal{A}_\beta(\beta)}A(\alpha(\mathcal{A}(\beta)))\big)\big](y)\big|_{y=\mathcal{Y}(x)}. \tag{A.1}$$

We assume that a smooth function (symbol) $\Phi(p, x)$ and the corresponding pseudodifferential operator $\Phi(\overset{2}{x}, -ih\,\overset{1}{\partial/\partial x})$ are given in the phase space. We act with this operator on $[K_\Lambda A(\alpha)](x)$ and obtain the formula of commutation of the pseudodifferential operator and the canonical operator

$$\Phi\bigg(\overset{2}{x}, -ih\,\overset{1}{\frac{\partial}{\partial x}}\bigg)[K_\Lambda A(\alpha)](x) = [K_\Lambda(\Phi|_\lambda A(\alpha) + O(h))](x),$$

where $\Phi|_\lambda$ is the restriction of the function Φ to Λ. Of course, the inverse formula also holds, which allows moving a part of the amplitude A or even the whole amplitude out of the canonical operator. In this case, if Φ is polynomial in p, then $\Phi(\overset{2}{x}, -ih\,\overset{1}{\partial/\partial x})$ is a differential operator, and if Φ is independent of p, then this simply a function. Therefore, if $\sqrt{\mathcal{A}_\beta(\beta)}$ in (A.1) can be represented as $\Phi(y)$, then (A.1) becomes

$$[K_\Lambda A(\alpha)](x) = \frac{\Phi(y)}{\sqrt{\mathcal{X}_y(y)}}[K_\Lambda A(\alpha(\mathcal{A}(\beta)))](y)\big|_{y=\mathcal{Y}(x)}. \tag{A.2}$$

This relation means that we can construct the Maslov canonical operator using the new coordinates q, y in the phase space, introducing the new coordinates β on the Lagrangian manifold, and then returning to the coordinates x by formula $y = \mathcal{Y}(x)$. All formulas given above also hold in the multidimensional case if the derivatives $\mathcal{X}_y(y)$ and $\mathcal{A}_\beta$ are respectively replaced with $\det \mathcal{X}_y(y)$ and $\det \mathcal{A}_\beta(\beta)$.

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