Research Article

Olivier Goubet* and Emna Hamraoui

Blow-up of solutions to cubic nonlinear Schrödinger equations with defect: The radial case

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Abstract: In this article we investigate both numerically and theoretically the influence of a defect on the blow-up of radial solutions to a cubic NLS equation in dimension 2.

Keywords: Cubic NLS equation, blow-up, defect

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1 Introduction

The issue of the existence of blow-up solutions for nonlinear Schrödinger (NLS) equations in $\mathbb{R}^2$ has widely been investigated in the literature (see [3, 16] and the references therein). These equations read

$$iu_t = \Delta u + |u|^{2\sigma} u,$$

supplemented with initial data in $H^1(\mathbb{R}^2)$. For $\sigma < 1$ the solutions are global in time. Then the so-called cubic NLS equation $\sigma = 1$ is critical in H^1. In fact, there exists solutions of the cubic NLS equation that blow up in finite time. This can be established for instance by the so-called Glassey's virial method [9]. Conversely, a famous result of Weinstein [17] asserts that any solution whose mass is less than the mass of the ground state is global in time. For the existence and properties of the ground state see [1, 4, 5, 13]. Actually, consider C_{GN} the best constant in the Gagliardo–Nirenberg inequality

$$\|u\|_{L^4}^4 \le C_{\mathrm{GN}} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

Then $C_{\mathrm{GN}} = 2/\|Q\|_{L^2}^2$, where $\|Q\|_{L^2}$ is the mass of the ground state and if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution starting from u_0 cannot blow up in finite time ($\|Q\|_{L^2(\mathbb{R}^2)}^2 = 2\pi \times 1.86225\dots$, see [17]). This is easy to check observing that the NLS equation has two invariants that are respectively the mass $\|u(t)\|_{L^2}$ and the energy

$$E(t) = \|\nabla u\|_{L^2}^2 - \frac{1}{2}\|u\|_{L^4}^4. \tag{1}$$

A point defect has been introduced and studied for NLS equations in dimension 2 in [7, 8, 11, 14]. In this article we are concerned with the blow-up of radial solutions to a cubic nonlinear Schrödinger equation with

***Corresponding author: Olivier Goubet:** Laboratoire Amiénois de Mathématique Fondamentale et Appliquée, CNRS UMR 7352, Université de Picardie Jules Verne, 80039 Amiens, France, e-mail: olivier.goubet@u-picardie.fr
Emna Hamraoui: Laboratoire Amiénois de Mathématique Fondamentale et Appliquée, CNRS UMR 7352, Université de Picardie Jules Verne, 80039 Amiens, France; and Unité de Recherche Multifractales et Ondelettes, UR11ES53, Université de Monastir, 5000 Monastir, Tunisia, e-mail: emna.hamraoui@gmail.com

a radial defect, located on the sphere of radius r_0. The equation reads

$$\begin{aligned} i\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r} - Zv\delta_{r_0} - |v|^2 v = 0, \qquad & r > 0,\ t \geq 0, \\ v(0, r) = v_0(r), \quad & r \geq 0, \\ \frac{\partial v}{\partial r}(t, 0) = 0, \qquad & t \geq 0. \end{aligned} \tag{2}$$

Here the unknown v depends only on the distance r to the origin, and the defect is modeled by the term $Zv\delta_{r_0}$. The real number Z is the amplitude of the defect, and δ_{r_0} is the usual delta measure at $r = r_0$. Moreover, the scaling $v(t, r) = r_0 w(r_0^2 t, r_0 r)$ (changing accordingly the value of $|Z|$) allows us to focus on the case $r_0 = 1$.

The rest of the article is organized as follows. In a second section we introduce the mathematical framework associated with equation (2) and we investigate the initial value problem, discussing the role of the Gagliardo–Nirenberg inequality in our case. In a third section we revisit the virial method. In a fourth section, we investigate numerically how a defect can affect the behavior of explosive solutions.

We now introduce some notations. We denote by L^2_{rad} (or simply L^2) the set of functions $v : [0, +\infty[\to \mathbb{C}$ measurable such that

$$\|v\|_{L^2}^2 = \int_0^{+\infty} r|v(r)|^2 dr < +\infty.$$

We denote by H^1_{rad} (or simply H^1) the set of radial functions such that

$$\|v\|_{H^1}^2 = \int_0^{+\infty} r(|v|^2 + |v_r|^2)\, dr < +\infty.$$

We also define here the invariants respectively for the mass as

$$M(t) = \int_0^{+\infty} r|v(r, t)|^2\, dr,$$

and respectively for the energy as

$$E(t) = \left\|\frac{\partial v}{\partial r}(t)\right\|_{L^2_{\text{rad}}}^2 - Z|v(t, 1)|^2 - \frac{1}{2}\|v(t)\|_{L^4_{\text{rad}}}^4.$$

We have divided by 2π the quantities defined above (see (1)) for the sake of convenience. Let us observe that there is an extra term while $Z \neq 0$. Moreover, if a function v is in H^1_{rad}, then v is continuous in $(0, +\infty)$ and $v(1) = \langle v, \delta_1\rangle$ makes sense. This is valid due to the following lemma

Lemma 1.1. *Any v in $H^1_{\text{rad}}(\mathbb{R}^2)$ is a continuous function for $r > 0$ that satisfies*

$$\sqrt{r}|v(r)| \leq \|v\|_{H^1_{\text{rad}}}. \tag{3}$$

Proof. Consider first v a smooth compactly supported radial function. Equality (3) holds true integrating $\partial_r|v|^2 = 2\,\text{Re}(\bar{v}v_r)$ between r and $+\infty$ and using the Cauchy–Schwarz inequality. We then conclude by a density argument: if $v_k \in C^\infty_{\text{rad},0}$ converges towards v in H^1, then the sequence $\sqrt{r}v_k(r)$ converges uniformly towards $\sqrt{r}v(r)$. □

2 The initial value problem

In this section, we address the issue of the existence of solutions to (2) in $C([0, T), H^1_{\text{rad}}) \cap C^1([0, T); H^{-1}_{\text{rad}})$.

2.1 The mathematical framework

We now introduce a mathematical setting that allow us to address the defect as a transmission problem.

Let a_1 be the bilinear form in H^1_{rad} defined as

$$a_1(v,w) = \mathrm{Re}\Big(\int_0^{+\infty} \frac{\partial v}{\partial r}(t)\overline{\frac{\partial w}{\partial r}} r\,dr\Big) - Z\,\mathrm{Re}(v(t,1)\overline{w}(1)) \quad \text{for all } v,w \in H^1_{\mathrm{rad}}.$$

Then we state and prove the following lemma.

Lemma 2.1. *The bilinear form $a_1(\cdot,\cdot)$ is continuous and symmetric in H^1_{rad}.*

Proof. For all $v, w \in H^1_{\mathrm{rad}}$,

$$|a_1(v,w)| \le \Big\|\frac{\partial v}{\partial r}\Big\|_{L^2_{\mathrm{rad}}}\Big\|\frac{\partial w}{\partial r}\Big\|_{L^2_{\mathrm{rad}}} + |Z||w(t,1)||v(t,1)|.$$

We recall that for any fixed r we have

$$r|v(r)|^2 \le \|v\|^2_{H^1_{\mathrm{rad}}}.$$

We then have

$$|a_1(v,w)| \le \Big\|\frac{\partial v}{\partial r}\Big\|_{L^2_{\mathrm{rad}}}\Big\|\frac{\partial w}{\partial r}\Big\|_{L^2_{\mathrm{rad}}} + |Z|\|v\|_{H^1_{\mathrm{rad}}}\|w\|_{H^1_{\mathrm{rad}}} \le (1+|Z|)\|v\|_{H^1_{\mathrm{rad}}}\|w\|_{H^1_{\mathrm{rad}}}.$$

This completes the proof of the lemma. □

Proposition 2.2. *There exists A_1 a unbounded self-adjoint operator in H^1_{rad} such that*

$$a_1(v,w) = \langle A_1 v, w\rangle_{H^{-1}_{\mathrm{rad}},H^1_{\mathrm{rad}}}.$$

Proof. Due to the proof of Lemma 2.1 for $\lambda > 0$ large enough the bilinear for $b_1(v,w) = a_1(v,w) + \lambda(v,w)_{L^2_{\mathrm{rad}}}$ is coercive, continuous and symmetric. The Lax–Milgram theorem applies and for any $f \in H^{-1}_{\mathrm{rad}}$ there exists a unique $v \in H^1_{\mathrm{rad}}$ such that $b_1(v,w) = \langle f, w\rangle$ for all $w \in H^1$. We define B as the maximal monotone operator such that $b_1(v,w) = \langle Bv, w\rangle$ and define A_1 as $A_1 = B - \lambda\,\mathrm{Id}$. Then we also have

$$a_1(v,w) = \langle A_1 v, w\rangle_{H^{-1}_{\mathrm{rad}},H^1_{\mathrm{rad}}}.$$

This completes the proof of the proposition. □

We now characterize the domain of A_1. We state:

Proposition 2.3. *The domain of A_1 is*

$$D(A_1) = \Big\{v \in H^1_{\mathrm{rad}} : \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big) \in L^2_{\mathrm{rad}}(0,1) \cap L^2(1,+\infty) \text{ and } \frac{\partial v}{\partial r}(t,1^+) - \frac{\partial v}{\partial r}(t,1^-) = -Zv(t,1)\Big\}.$$

Proof. Consider a test radial function $w \in C_0^\infty$. We seek v in H^1 such that for any such w,

$$\Big|\mathrm{Re}\Big(\int_0^{+\infty} r\frac{\partial v}{\partial r}(t)\overline{\frac{\partial w}{\partial r}}\,dr\Big) - Z\,\mathrm{Re}(v(t,1)\overline{w}(1))\Big| \le c\|w\|_{L^2_{\mathrm{rad}}}.$$

Consider first w that vanishes at a neighborhood of $r = 1$. We then have

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big) \in L^2_{\mathrm{rad}}(0,1) \cap L^2(1,+\infty).$$

Therefore the derivative of v has traces at $r = 1, r < 1$ and $r = 1, r > 1$ (see [2]). We consider now a general $w \in H^1_{\mathrm{rad}}$. Integrating by parts, we have

$$\int_0^1 \frac{\partial v}{\partial r}(t)\overline{\frac{\partial w}{\partial r}} r\,dr = -\int_0^1 \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\overline{w} r\,dr + \frac{\partial v}{\partial r}(t,1^-)w(t,1),$$

and

$$\int_1^{+\infty} \frac{\partial v}{\partial r}(t)\overline{\frac{\partial w}{\partial r}} r\,dr = -\int_1^{+\infty} \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\overline{w} r\,dr - \frac{\partial v}{\partial r}(t,1^+)w(t,1).$$

Introducing

$$[v_r]_1 = \frac{\partial v}{\partial r}(t,1^+) - \frac{\partial v}{\partial r}(t,1^-),$$

we thus obtain

$$\Big|-\mathrm{Re}\Big(\int_0^{+\infty} \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\overline{w} r\,dr - \overline{w}(t,1)(Zv(1) + [v_r]_1)\Big)\Big| \le C\|w\|_{L^2_{\mathrm{rad}}}.$$

Since this is valid for any w, we infer the transmission condition

$$Zv(1) + [v_r]_1 = 0. \tag{4}$$

The proof of the proposition is complete. □

We now have enough material to handle the Initial Value Problem.

Proposition 2.4. *For any v_0 in H^1_{rad} there exist a $T > 0$ and a unique solution of nonlinear Schrödinger equation (2) in $C([0,T); H^1_{\mathrm{rad}}) \cap C^1([0,T); H^{-1})$. If moreover v_0 belongs to $D(A_1)$, then the solution remains in $D(A_1)$ for $t < T$.*

Proof. For the uniqueness of solutions, we rely on a famous argument due to Vladimirov. To begin with, we recall the Trudinger inequality (written here for radial functions) [2]. For $M > 0$, there exist $\mu, K > 0$ such that if $\|v\|_{H^1_{\mathrm{rad}}(\mathbb{R}^2)} < M$, then

$$\int_0^{+\infty} (\exp(\mu|v(r)|^2) - 1)^2 r\,dr \leqslant K^2.$$

Let $v(t)$ and $\tilde{v}(t)$ be two solutions of (2) starting from $v(0)$. Introduce $M = 8\sup_{[0,T_0]}(\|v\|_{H^1} + \|\tilde{v}\|_{H^1})$ for $T_0 < T$. Setting $w(t) = v(t) - \tilde{v}(t)$, we see that $w(t)$ satisfies

$$\begin{aligned} i\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r} - Zw\delta_{r_0} - (|v|^2 v - |\tilde{v}|^2\tilde{v}) = 0, \quad & r > 0,\ t \ge 0, \\ w(0,r) = 0, \quad & r \ge 0. \end{aligned} \tag{5}$$

Considering the scalar product of (5) with $iw(t)$, we then have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2_{\mathrm{rad}}} = \mathrm{Im}\int_0^{+\infty} r(|v|^2 v - |\tilde{v}|^2\tilde{v})\overline{w(t)}\,dr \le \int_0^{+\infty} r|\tilde{v}|(|\tilde{v}| + |v|)|w|^2\,dr,$$

so

$$\frac{d}{dt}\|w(t)\|^2_{L^2_{\mathrm{rad}}} \le 4\int_0^{+\infty} r(|\tilde{v}|^2 + |v|^2)|w|^2\,dr.$$

Let us introduce the function $h^2(x) = |\tilde{v}(x)|^2 + |v(x)|^2$. Thanks to the Hölder inequality for $p > 2$, we then have

$$\frac{d}{dt}\|w(t)\|^2_{L^2_{\mathrm{rad}}} \le 4\int_0^{+\infty} |h|^2|w|^{\frac{4}{p}}|w|^{\frac{2p-4}{p}} r\,dr \le 4\|w\|^{\frac{2p-4}{2}}_{L^2_{\mathrm{rad}}}\Big(\int_0^{+\infty}(|h|^p|w|^{\frac{4}{p}}|w|^2 r\,dr\Big)^{\frac{2}{p}} \le 4\|w\|^{\frac{2p-4}{p}}_{L^2_{\mathrm{rad}}}\|w\|^{\frac{4}{p}}_{L^4_{\mathrm{rad}}}\|h\|^2_{L^{2p}_{\mathrm{rad}}}.$$

On one hand, using the elementary inequality $a^{2p} \le (\frac{p}{\mu})^p(\exp(\mu a^2) - 1)$, we have

$$\|h\|^2_{L^{2p}_{\mathrm{rad}}} \le \frac{p}{\mu}(K)^{\frac{2}{p}},$$

since the H^1-norm of h is bounded by M. On the other hand, due to the embedding $H^1_{\mathrm{rad}} \subset L^4_{\mathrm{rad}}$, we have

$$\|v\|_{L^4_{\mathrm{rad}}} \le c\|v\|_{H^1_{\mathrm{rad}}}.$$

So, gathering the previous inequalities, we obtain

$$\frac{d}{dt}\|w(t)\|^2_{L^2_{\rm rad}} \le CK^{\frac{2}{p}}\|w\|^{\frac{2p-4}{p}}_{L^2_{\rm rad}}. \tag{6}$$

By integrating (6) between 0 and T, we have

$$\|w(t)\|^2_{L^2_{\rm rad}} \le (\alpha T)^{\frac{p}{2}}.$$

For T small enough such that $\alpha T < 1$, $T < T_0$ and $p \to \infty$ we have

$$\|w(t)\|^2_{L^2_{\rm rad}} = 0$$

in $[0, T]$, and hence the uniqueness of solutions, since we can iterate this argument on $[T, 2T]$, and then in $[kT, (k+1)T]$ for any k.

For the existence result, the difficulty is that we cannot use Strichartz estimates due to the defect. We use instead the regularization method described in [3, Section 3.3]. Setting $F(v) = |v|^2 v$, let us recall that the equation reads in its abstract form

$$iv_t = Av + F(v).$$

First step: Shifting. The operator A is not positive. We overcome this difficulty considering

$$w(t) = \exp(-i\lambda t)v(t)$$

that is solution to

$$iw_t = (A + \lambda\,\mathrm{Id})w + F(w). \tag{7}$$

We know that, for λ large enough, $B = A + \lambda\,\mathrm{Id}$ is a positive symmetric unbounded operator such that

$$D(B^{\frac{1}{2}}) = H^1_{\rm rad}.$$

Second step: Regularizing the nonlinearity. We introduce for $\varepsilon > 0$ the operator $J_\varepsilon = (\mathrm{Id} + \varepsilon B)^{-1}$. We set $F_\varepsilon(v) = J_\varepsilon F(J_\varepsilon v)$. Then we have that F_ε is a locally Lipschitz map from $H^1_{\rm rad}$ into $H^1_{\rm rad}$ uniformly with respect to ε. Actually, if v and w belong to some bounded set of $H^1_{\rm rad}$,

$$\begin{aligned}\|F_\varepsilon(v) - F_\varepsilon(w)\|_{H^1} &\le \|F(J_\varepsilon v) - F(J_\varepsilon w)\|_{H^{-1}} \le c\|F(J_\varepsilon v) - F(J_\varepsilon w)\|_{L^{\frac{4}{3}}}\\ &\le K\|J_\varepsilon(v-w)\|_{L^4} \le K\|J_\varepsilon(v-w)\|_{H^1} \le K\|v-w\|_{H^1}.\end{aligned}$$

Third step: Construction of an approximate solution. We now perform a fixed point in $C([0,T]; H^1_{\rm rad})$ for the Duhamel's form of the equation that reads

$$w^\varepsilon(t) = e^{-itB}v_0 + \int_0^t e^{-i(t-s)B}F_\varepsilon(w^\varepsilon(s))\,ds. \tag{8}$$

This is standard and omitted for the sake of conciseness. It is worth to point out that since the nonlinearity is uniformly locally Lipschitz in $H^1_{\rm rad} = D(B^{\frac{1}{2}})$, the time T *does not depend on* ε. Moreover, the solution w^ε belongs to $C([0,T]; H^1_{\rm rad}) \cap C^1([0,T]; H^{-1}_{\rm rad})$ and satisfies, going back to $v^\varepsilon = \exp(i\lambda t)w^\varepsilon$,

$$iv^\varepsilon_t = Av^\varepsilon + F_\varepsilon(v^\varepsilon) = 0. \tag{9}$$

Fourth step: A priori estimates. We already know that the sequence v^ε is uniformly bounded in the space $C([0,T]; H^1_{\rm rad}) \cap C^1([0,T]; H^{-1}_{\rm rad})$. Since the embedding $H^1_{\rm rad} \subset L^4_{\rm rad}$ is compact (see [12]), we can extract a subsequence still denoted by v^ε that converges to v in $L^\infty([0,T]; H^1_{\rm rad})$ weak-star and strongly in $L^4(0,T; L^4_{\rm rad})$ and such that v^ε_t converges to v_t in $L^\infty([0,T]; H^{-1}_{\rm rad})$ weak-star. We also have that some invariants are conserved. Since $\mathrm{Im}(F_\varepsilon(v), v) = 0$, the mass $\|v^\varepsilon(t)\|_{L^2_{\rm rad}} = \|v_0\|_{L^2}$ is constant. We also have that the modified energy $E_\varepsilon(v) = (Av, v) - \frac{1}{2}\|J_\varepsilon v\|^4_{L^4_{\rm rad}}$ is conserved along the trajectories.

Fifth step: Passing to the limit. Observing that for any given v in H^1_{rad},

$$\|F_\varepsilon(v) - F(v)\|_{H^{-1}} \le c(\|F(v^\varepsilon) - F(v)\|_{L^{\frac{4}{3}}} + \|(J_\varepsilon - \text{Id})F(v)\|_{H^{-1}} \le K(\|J_\varepsilon v - v\|_{H^1} + \|(J_\varepsilon - \text{Id})F(v)\|_{H^{-1}},$$

it is standard to pass to the limit either in (9) and (8) to have a solution v in $L^\infty([0,T]; H^1_{\text{rad}})$ (and then continuous in time due to (8)) of the equation. We can also pass to the limit in the invariant.

Sixth step: Miscellaneous results. Proceeding as in [3, Section 3.3], we can prove that the solution depends continuously on the initial data and the existence of a maximal time of existence $T_{\max}$ such that if $T < +\infty$, then the solution blows up.

We complete the proof of the theorem by proving that if the initial data belongs to $D(A_1)$, then the solution remains in $D(A_1)$. Assume v_0 in $D(A_1)$. Consider a solution w of the equation that remains bounded by M in H^1_{rad} for t in $[0,T]$. Due to (3), the L^∞-norm of w outside a ball of radius $\frac{1}{2}$ remains bounded by C_M. We use the so-called Brezis–Gallouet inequality to have

$$\|w\|_{L^\infty(B(0,\frac{1}{2}))} \le C_M(1 + \log(1 + \|\Delta w\|^2_{L^2(B(0,\frac{3}{4})}))^{\frac{1}{2}}.$$

Going back to the equation, this inequality implies (the constant C_M varying from one line to one another)

$$\|w\|_{L^\infty(B(0,\frac{1}{2}))} \le C_M(1 + \log(1 + \|w_t\|^2_{L^2})^{\frac{1}{2}}. \tag{10}$$

We now differentiate equation (7) with respect to t to have a new equation for $Z = w_t$ that reads

$$iZ_t = BZ + 2\,\text{Re}(\overline{w}Z)w + |w|^2 Z. \tag{11}$$

Considering the scalar product of (11) with iZ leads to

$$\frac{d}{dt}\|Z\|^2_{L^2} \le c\|w\|^2_{L^\infty}\|Z\|^2_{L^2}.$$

Using (10), we then have

$$\frac{d}{dt}\|Z\|^2_{L^2} \le C_M\|Z\|^2_{L^2}(1 + \log(1 + \|Z\|^2_{L^2})).$$

We then infer from this that $\|Z(t)\|_{L^2} \le c(Z_0)\exp(\exp(C_M T))$. Going back to the equation, we have that Bw remains also bounded in L^2 for t in $[0,T)$. □

2.2 A sufficient condition for a solution to be global

At this stage we have a local solution that takes value in H^1. As for the case $Z = 0$, the solution is global in time if we can prove an inequality that reads

$$E(t) \ge c\|v_r\|^2_{L^2} - C.$$

We now define the generalized Gagliardo–Nirenberg constant as C_Z such that for all v in H^1_{rad},

$$\|v\|^4_{L^4} \le C_Z(\|v_r\|^2_{L^2} - Z|v(1)|^2)\|v\|^2_{L^2}. \tag{12}$$

In the case $Z < 0$, if $C_Z > C_{\text{GN}}$, then we can improve the sufficient condition for a solution to be global. This is not the case. We state and prove:

Proposition 2.5. *Assume $Z < 0$. Then we have $C_Z = C_{\text{GN}}$.*

Proof. For $Z < 0$, we have $C_Z > C_{\text{GN}}$. Let us take $v(r) = w(\mu r)$ in (12) with w in H^1_{rad}. Then we have, dividing the resulting equality by μ^2,

$$\|w\|^4_{L^4} \le C_Z(\|w_r\|^2_{L^2} - Z|w(\mu)|^2)\|w\|^2_{L^2}.$$

Due to (3), then $|w(\mu)|^2$ converges towards 0 and we are back to the usual Gagliardo–Nirenberg inequality. Then $C_{\text{GN}} \le C_Z$. □

Remark 2.6. It is worth to point out that for the proof of this proposition we have used that H^1 is invariant by dilations. The paradox is that $D(A_1)$ and the PDE under consideration are not invariant by dilations.

3 Revisiting the virial's method

We now introduce the very definition of the virial V and of the momentum q (see [6, 9, 15]) in the radial case as

$$V(t) = \operatorname{Im} \int_0^{+\infty} \Big(r^2 \frac{\partial}{\partial r}\Big) v(t,r)\bar{v}(t,r)\,dr, \quad q(t) = \int_0^{+\infty} r^3 |v(t,r)|^2\,dr.$$

3.1 The momentum identity

We first state and prove that if the solution above belongs in some weighted space for $t = 0$, it remains in the same weighted space.

Proposition 3.1. *Consider $v \in C([0,T[; H^1_{\mathrm{rad}})$ such that*

$$\int_0^{+\infty} r^3 |v_0|^2\,dr < \infty.$$

Then for all $t \in [0,T[$,

$$q(t) = \int_0^{+\infty} r^3 |v(t)|^2\,dr < \infty.$$

Proof. We first prove the identity assuming that the initial data is smooth, say in $D(A_1)$, and we then conclude by density. Let then $v \in C([0,T[; D(A_1))$ be a solution of (2). We define

$$q_R = \int_0^{+\infty} r^3 \exp\Big(-\frac{r}{R}\Big)|v(t,r)|^2\,dr$$

for $R > 1$. We first compute $\frac{\partial q_R}{\partial t}$, and then let $R \to +\infty$. We have

$$\begin{aligned}
\frac{\partial q_R}{\partial t} &= 2\operatorname{Re}\int_0^{+\infty} r^3 \exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial t}\bar{v}\,dr \\
&= 2\operatorname{Re}\int_0^{+\infty} r^3 \exp\Big(-\frac{r}{R}\Big)\Big(-\frac{i}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big) - i|v|^2 v\Big)\bar{v}\,dr \\
&= 2\operatorname{Im}\int_0^{+\infty} r^2 \exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\bar{v}\,dr \\
&= 2\operatorname{Im}\int_0^{1} r^2 \exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\bar{v}\,dr + 2\operatorname{Im}\int_1^{+\infty} r^2 \exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\bar{v}\,dr.
\end{aligned}$$

On one hand

$$\begin{aligned}
2\operatorname{Im}\int_0^1 r^2 \exp\Big(-\frac{r}{R}\Big)\bar{v}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)dr &= -4\operatorname{Im}\int_0^1 r^2 \exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\bar{v}\,dr + 2\operatorname{Im}\int_0^1 \frac{r^3}{R}\exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\bar{v}\,dr \\
&\quad + 2\operatorname{Im}\Big(\exp\Big(-\frac{1}{R}\Big)\frac{\partial v}{\partial r}(1^-)\bar{v}(1^-)\Big).
\end{aligned}$$

On the other hand

$$\begin{aligned}
2\operatorname{Im}\int_1^{+\infty} r^2 \exp\Big(-\frac{r}{R}\Big)\bar{v}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)dr &= -4\operatorname{Im}\int_1^{+\infty} r^2 \exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\bar{v}\,dr + 2\operatorname{Im}\int_1^{+\infty} \frac{r^3}{R}\exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\bar{v}\,dr \\
&\quad - 2\operatorname{Im}\Big(\exp\Big(-\frac{1}{R}\Big)\frac{\partial v}{\partial r}(1^+)\bar{v}(1)\Big).
\end{aligned}$$

Using the transmission condition (4), we obtain

$$2\,\mathrm{Im}\Big(\exp\Big(-\frac{1}{R}\Big)[v_r]_1\overline{v}(1)\Big) = 0.$$

We then infer

$$\frac{\partial q_R}{\partial t} = -4\,\mathrm{Im}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\overline{v}\,dr + 2\,\mathrm{Im}\int_0^{+\infty}\frac{r^3}{R}\exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\overline{v}\,dr. \tag{13}$$

We now use the Cauchy–Schwarz inequality to obtain

$$\Big|-4\,\mathrm{Im}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\overline{v}\,dr\Big| \le 4\sqrt{q_R}\Big(\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big|\frac{\partial v}{\partial r}\Big|^2\,dr\Big)^{\frac{1}{2}} \le 4\sqrt{q_R}\Big\|\frac{\partial v}{\partial r}\Big\|_{L^2_{\mathrm{rad}}}.$$

Using once more the Cauchy–Schwarz inequality, we have

$$\Big|2\,\mathrm{Im}\int_0^{+\infty}\frac{r^3}{R}\exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}\overline{v}\,dr\Big| \le \sqrt{q_R}\Big(\int_0^{+\infty}\frac{r^3}{R^2}\exp\Big(-\frac{r}{R}\Big)\Big|\frac{\partial v}{\partial r}\Big|^2\,dr\Big)^{\frac{1}{2}} \le 2C\sqrt{q_R}\Big\|\frac{\partial v}{\partial r}\Big\|_{L^2_{\mathrm{rad}}}.$$

Therefore

$$\frac{\partial q_R}{\partial t} \le (4+2C)\sqrt{q_R}\Big\|\frac{\partial v}{\partial r}\Big\|_{L^2_{\mathrm{rad}}}.$$

We then have that for all $t < T$,

$$\sqrt{q_R(t)} \le \sqrt{q_R(0)} + (4+2C)\int_0^t\Big\|\frac{\partial v}{\partial r}\Big\|_{L^2_{\mathrm{rad}}}\,dt.$$

Letting $R \to +\infty$ provides that for all $v \in D(A_1)$, $q(t) < \infty$ since $q(0) < \infty$. We conclude by the density of $D(A_1)$ in H^1. □

Corollary 3.2. *Setting $q = \lim_{R\to+\infty} q_R$, we have*

$$\frac{\partial q}{\partial t} = -4\,\mathrm{Im}\int_0^{+\infty} r^2\frac{\partial v}{\partial r}\overline{v}\,dr.$$

Proof. We go back to (13):

$$\frac{\partial q_R}{\partial t} = -4\,\mathrm{Im}\int_0^{+\infty}\Big(1-\frac{r}{2R}\Big)\exp\Big(-\frac{r}{R}\Big)r^2\overline{v}\frac{\partial v}{\partial r}\,dr.$$

The function $r \to r^2\overline{v}\frac{\partial v}{\partial r}dr$ is integrable since $\frac{\partial v}{\partial r}, rv \in L^2_{\mathrm{rad}}$. We conclude by the Lebesgue dominated convergence theorem. □

3.2 The virial identity

To begin with, we recall that the energy $E(t) = E_0$ does not depend on t.

Proposition 3.3. *For any initial data v_0 in H^1 such that $q(0) < +\infty$ we have*

$$\frac{\partial V}{\partial t}(t) = -2E_0 - \Big(\Big|\frac{\partial v}{\partial r}(1^+)\Big|^2 - \Big|\frac{\partial v}{\partial r}(1^-)\Big|^2\Big).$$

Proof. We proceed as above, performing the computations for v_0 in $D(A_1)$ and then passing to the limit due to a density argument. We introduce

$$V_R = \mathrm{Im}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial v}{\partial r}(t,r)\overline{v}(t,r)\,dr.$$

We first compute $\frac{\partial V_R}{\partial t}(t)$, and then let $R \to +\infty$. To begin with, we have

$$\begin{aligned}\frac{\partial V_R}{\partial t}(t) &= -2\,\mathrm{Im}\int_0^{+\infty} r^2 \exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}\frac{\partial v}{\partial t}\,dr + \mathrm{Im}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big(\frac{\partial v}{\partial t}\bar v\Big)\,dr \\ &= -2\,\mathrm{Im}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}\frac{\partial v}{\partial t}\,dr - 2\,\mathrm{Im}\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\frac{\partial v}{\partial t}\bar v\,dr. \qquad (14)\end{aligned}$$

We now estimate the first term in the right-hand side of (14),

$$\begin{aligned}-2\,\mathrm{Im}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}\frac{\partial v}{\partial t}\,dr &= 2\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\,dr \\ &\quad + 2\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r} v|v|^2\,dr. \qquad (15)\end{aligned}$$

We then have

$$2\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\,dr = \mathrm{Re}\int_0^{+\infty}\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big|r\frac{\partial v}{\partial r}\Big|^2\,dr.$$

Integrating by parts, we infer

$$\begin{aligned}\mathrm{Re}\int_0^{+\infty}\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big|r\frac{\partial v}{\partial r}\Big|^2\,dr &= \mathrm{Re}\int_0^{1}\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big|r\frac{\partial v}{\partial r}\Big|^2\,dr + \mathrm{Re}\int_1^{+\infty}\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}\Big|r\frac{\partial v}{\partial r}\Big|^2\,dr \\ &= \exp\Big(-\frac{1}{R}\Big)\Big(\Big|\frac{\partial v}{\partial r}(1^-)\Big|^2 - \Big|\frac{\partial v}{\partial r}(1^+)\Big|^2\Big) + \mathrm{Re}\int_0^{+\infty}\frac{\exp(-\frac{r}{R})}{R}\Big|r\frac{\partial v}{\partial r}\Big|^2\,dr.\end{aligned}$$

On one hand, by the Lebesgue dominated convergence theorem, since $\frac{1}{r}|\frac{\partial v}{\partial r}|^2$ belongs to L^1,

$$\lim_{R\to+\infty}\mathrm{Re}\int_0^{+\infty}\frac{\exp(-\frac{r}{R})}{R}\Big|r\frac{\partial v}{\partial r}\Big|^2\,dr = 0.$$

On the other hand

$$\lim_{R\to+\infty}\exp\Big(-\frac{1}{R}\Big)\Big(\Big|\frac{\partial v}{\partial r}(1^-)\Big|^2 - \Big|\frac{\partial v}{\partial r}(1^+)\Big|^2\Big) = \Big|\frac{\partial v}{\partial r}(1^-)\Big|^2 - \Big|\frac{\partial v}{\partial r}(1^+)\Big|^2.$$

Therefore

$$\lim_{R\to+\infty} 2\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\,dr = \Big(\Big|\frac{\partial v}{\partial r}(1^-)\Big|^2 - \Big|\frac{\partial v}{\partial r}(1^+)\Big|^2\Big).$$

We now compute the second term in (15) as follows:

$$\begin{aligned}2\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}v|v|^2\,dr &= \frac{1}{2}\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}|v|^4\,dr \\ &= \frac{1}{2}\,\mathrm{Re}\int_0^{1} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}|v|^4\,dr + \frac{1}{2}\,\mathrm{Re}\int_1^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial}{\partial r}|v|^4\,dr.\end{aligned}$$

Since the function $r \to |v(r)|^4$ is continuous at 1, integrating by parts we have

$$2\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial \bar v}{\partial r}v|v|^2\,dr = -\,\mathrm{Re}\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)|v|^4\,dr. \qquad (16)$$

Then, using the Lebesgue dominated convergence theorem in (16), we have

$$\lim_{R\to+\infty} 2\,\mathrm{Re}\int_0^{+\infty} r^2\exp\Big(-\frac{r}{R}\Big)\frac{\partial\bar v}{\partial r}v|v|^2\,dr = -\|v\|^4_{L^4_{\mathrm{rad}}}.$$

We now pass to the limit in the second term in the right-hand side of (14). We first have

$$\begin{aligned}-2\,\mathrm{Im}\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\frac{\partial v}{\partial t}\bar v\,dr &= 2\,\mathrm{Re}\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)|v|^4\,dr\\ &+2\,\mathrm{Re}\int_0^{+\infty}\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\Big(\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\bar v\,dr.\end{aligned}$$

On one hand, using the Lebesgue dominated convergence theorem, we have

$$\lim_{R\to+\infty} 2\,\mathrm{Re}\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)|v|^4\,dr = 2\|v\|^4_{L^4_{\mathrm{rad}}}.$$

On the other hand, the second term reads also

$$\begin{aligned}2\,\mathrm{Re}\int_0^{+\infty}\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\Big(\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\bar v\,dr &= 2\,\mathrm{Re}\int_0^{1}\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\Big(\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\bar v\,dr\\ &+2\,\mathrm{Re}\int_1^{+\infty}\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\Big(\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\bar v\,dr.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}2\,\mathrm{Re}\int_0^{+\infty}\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\Big(\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\,dr &= -2\exp\Big(-\frac{1}{R}\Big)\Big(1-\frac{1}{2R}\Big)\mathrm{Re}\Big(\bar v(1)\Big[\frac{\partial v}{\partial r}\Big]_1\Big)\\ &-2\,\mathrm{Re}\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\Big|\frac{\partial v}{\partial r}\Big|^2\,dr\\ &+\mathrm{Re}\int_0^{+\infty} r\exp\Big(-\frac{r}{R}\Big)\Big(\frac{3}{2R}-\frac{r}{2R^2}\Big)\bar v\frac{\partial v}{\partial r}\,dr.\end{aligned}$$

Using once again the Lebesgue dominated convergence theorem, we obtain

$$\lim_{R\to+\infty} 2\,\mathrm{Re}\int_0^{+\infty}\exp\Big(-\frac{r}{R}\Big)\Big(1-\frac{r}{2R}\Big)\Big(\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big)\Big)\,dr = 2Z|v(1)|^2-2\Big\|\frac{\partial v}{\partial r}\Big\|^2_{L^2_{\mathrm{rad}}}.$$

Gathering these computations we conclude

$$\lim_{R\to+\infty}\frac{\partial V_R}{\partial t}(t) = \Big(\Big|\frac{\partial v}{\partial r}(1^-)\Big|^2-\Big|\frac{\partial v}{\partial r}(1^+)\Big|^2\Big)+\|v\|^4_{L^4_{\mathrm{rad}}}+2Z|v(1)|^2-2\Big\|\frac{\partial v}{\partial r}\Big\|^2_{L^2_{\mathrm{rad}}} = -2E_0-\Big[\Big|\frac{\partial v}{\partial r}\Big|^2\Big]_1.$$

This completes the proof of the proposition. □

3.3 Conclusion

In the previous subsection we have proved that

$$\frac{\partial V}{\partial t}(t) = -2E_0-\Big[\Big|\frac{\partial v}{\partial r}\Big|^2\Big]_1.$$

If $Z = 0$ and $E_0 < 0$, then the solution blows up in finite time. We assume below that we have a solution with negative energy. Here we are interested in the case $Z \neq 0$. For general solutions, we do not know the sign of $[|\frac{\partial v}{\partial r}|^2]_1$. In the next section we will investigate this issue using numerics. We shall observe that for solutions moving from the right to the left (going to the zero) the sign of $[|\frac{\partial v}{\partial r}|^2]_1$ is positive and then balance the negative energy.

4 Numerics

We solve our problem using second-order finite differences in r and the second-order implicit Crank–Nicolson scheme in time. We start this section by describing briefly the numerical method, next we discuss the numerical results. We refer to [10] for details.

4.1 The numerical method

We discuss here first the discretization of the delta function. Consider a solution to (2). For the discretization of the transmission condition, we write respectively to the right of r_0

$$\frac{\partial v}{\partial r}(t, r_0^+) = \frac{4v(t, r_0 + \Delta r) - v(t, r_0 + 2\Delta r) - 3v(t, r_0)}{2\Delta r},$$

and respectively to the left of r_0

$$\frac{\partial v}{\partial r}(t, r_0^-) = \frac{v(t, r_0 - 2\Delta r) - 4v(t, r_0 - \Delta r) + 3v(t, r_0)}{2\Delta r}.$$

Indeed, this approximation is a second order approximation in space. We have

$$4v(t, r_0 + \Delta r) - v(t, r_0 + 2\Delta r) - 2(3 - 2Z\Delta r)v(t, r_0) - v(t, r_0 - 2\Delta r) + 4v(t, r_0 - \Delta r) = 0.$$

Usual second-order scheme with finite differences is used inside the computational domain, except at the defect, and Crank–Nicolson scheme in time is performed. This reads for $r \neq r_0$:

$$i\frac{v_j^{n+1} - v_j^n}{\Delta t} - \frac{v_{j+1}^{n+\frac{1}{2}} - 2v_j^{n+\frac{1}{2}} + v_{j-1}^{n+\frac{1}{2}}}{\Delta r^2} - \frac{1}{r_j}\frac{v_{j+\frac{1}{2}}^{n+1} - v_{j-1}^{n+\frac{1}{2}}}{2\Delta r} - \frac{1}{4}(|v_j^{n+1}|^2 + |v_j^n|^2)(v_j^{n+1} + v_j^n) = 0,$$

where $v_j^{n+\frac{1}{2}} = \frac{v_j^{n+1} + v_j^n}{2}$. Our nonlinear problem is solved using a fixed point method at each time step.

For the boundary conditions, we use a PML method far away to the right of the defect to avoid spurious reflections (see [10] and the references therein). At the left boundary $r = 0$, we solve

$$i\frac{v_0^{n+1} - v_0^n}{\Delta t} - \frac{2v_1^{n+1} - 2v_0^{n+1}}{\Delta r^2} = \frac{2v_1^n - 2v_0^n}{\Delta r^2} + \frac{1}{4}(|v_0^{n+1}|^2 + |v_0^n|^2)(v_0^{n+1} + v_0^n).$$

4.2 The numerical results

In this subsection, we investigate the influence of the defect on the dynamics of traveling Gaussian solution that blows up in the case without defect $Z = 0$. We consider the following initial data (see Figure 1):

$$v_i(r) = 3\exp(i10r)\exp(-(r - 15)^2),$$

defined on the numerical domain $\Omega = (0, 20)$, that contains a PML band of width $L = 2$. The parameters of the band PML are chosen to absorb the reflected waves at the boundary of the computational domain (see [10, 18]). In our simulation the parameters are $\Delta r = 5 \times 10^{-3}$ and $\Delta t = 2.5 \times 10^{-5}$ for a computation performed with final time $T = 1$. Here we perform some numerical simulations for Z=0, i.e. without defect.

The blow-up structure shows in Figure 2 by the mass concentration of the solution around $r = 1$. To confirm this, we compute in Figure 3 the variation of the $L^2_{\text{rad}}(\mathbb{R}^2)$-norm of the gradient of the solution over time. We note that the solution blows up at $T^* = 0.6512$ and we observe that the norm $\|v_r\|^2_{L^2_{\text{rad}}}$ tends to ∞ when $t \to T^*$.

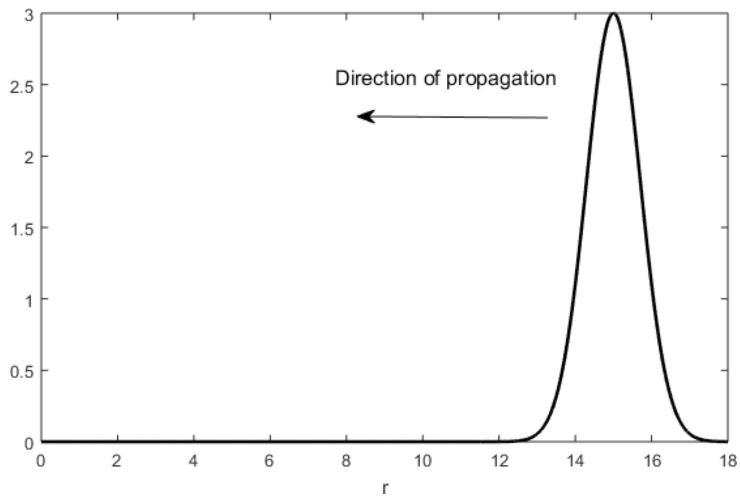

Figure 1. Spatial profile of the initial data.

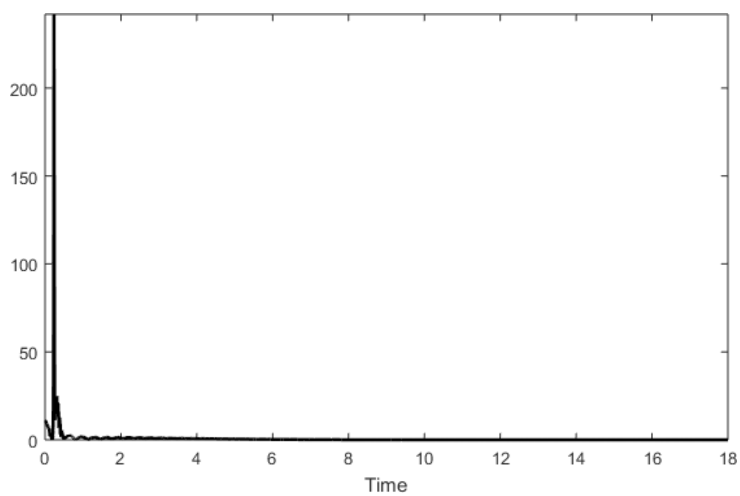

Figure 2. Formation of the singularity at $T = 0.6512$.

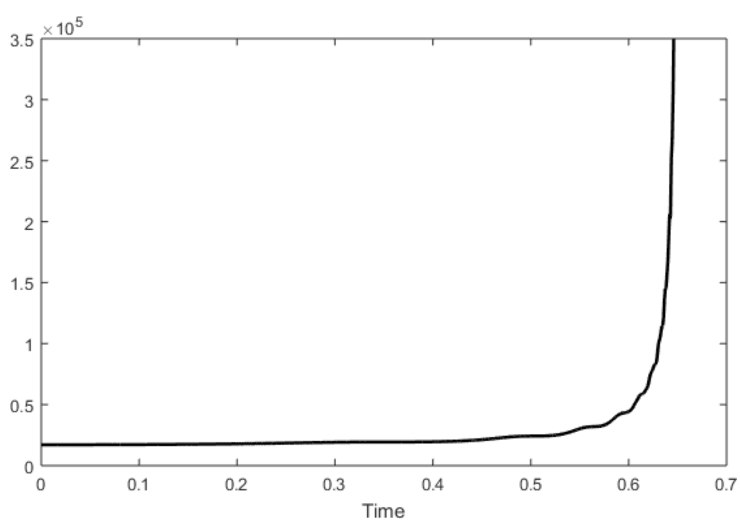

Figure 3. Variation of $\|v_r\|^2_{L^2_{\mathrm{rad}}}$ versus time for $Z = 0$.

Let M_n and E_n denote respectively the discrete mass and the energy at $t = t^n$. In Figures 4 and 5 we show the order of magnitude of the relative errors made for M_n and E_n versus time. We observe the conservation of mass and energy over time, and that a singularity appears for $t = T^*$.

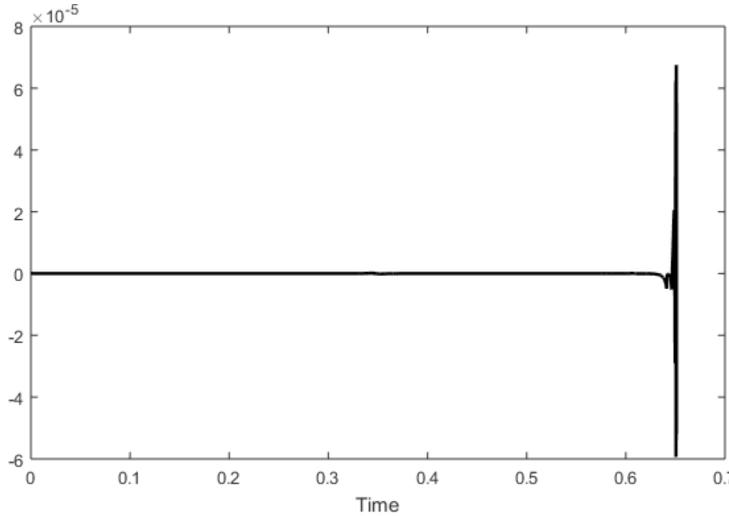

Figure 4. Plot of $\frac{M^{n+1}-M^n}{M^n}$ versus time.

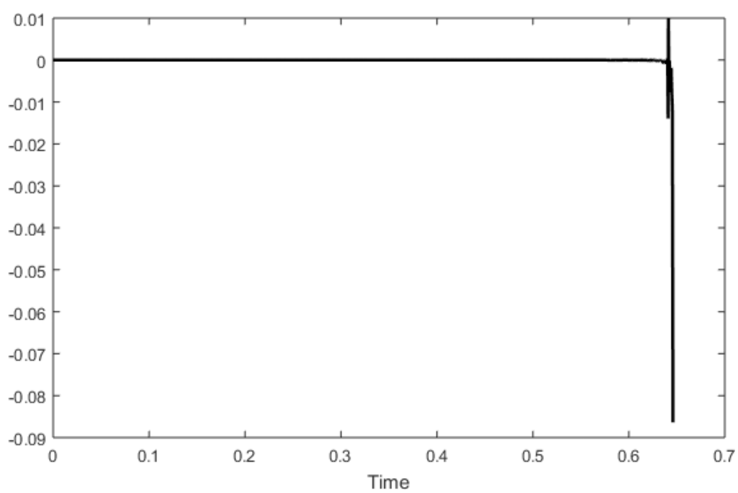

Figure 5. Plot of $\frac{E^{n+1}-E^n}{E^n}$ versus time.

Now, we consider a defect at $r = 10$ and we set $Z = 200$. Is this defect prevent or alter blow-up? After a phase of interaction with the defect Figure 6, we see in Figure 7 that the solution splits into two parts: a transmitted wave v_t and reflected one v_r. In our test case, we numerically have

$$\|v_t\|^2_{L^2(\mathbb{R}^2)} = 9.5828 < \|Q\|^2_{L^2(\mathbb{R}^2)} = 11.7009,$$

while the reflected part v_r comes out of the computational domain over time (it is absorbed by the PML band). We show in Figure 8 the variation of $(\|v_r\|_{L^2_{\text{rad}}})^2$ versus time. We observe that for $Z = 200$ the L^2_{rad}-norm of the gradient remains bounded along the flow. So, in this case test the defect prevents the blow-up. For this case, we numerically verify the sign of jump $[|\frac{\partial v}{\partial r}|^2]$ at $r_0 = 10$ (see the discussion in Section 3.3 above). We observe in Figure 9 that for $Z = 200$ the sign of the jump remains positive. We conclude that the defect splits the incident wave in one reflected part and one transmitted part. It can prevent blow-up if the mass of each part is smaller than the one of the ground state Q.

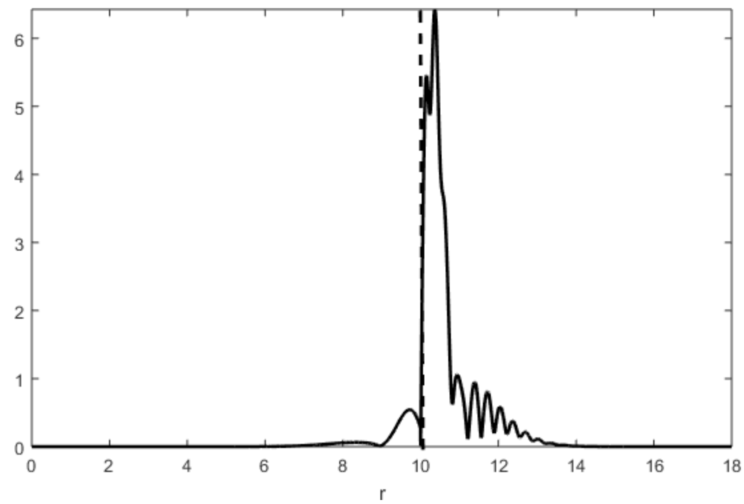

Figure 6. Solution profile then interacting with the defect for $Z = 200$ at $t = 0.2645$.

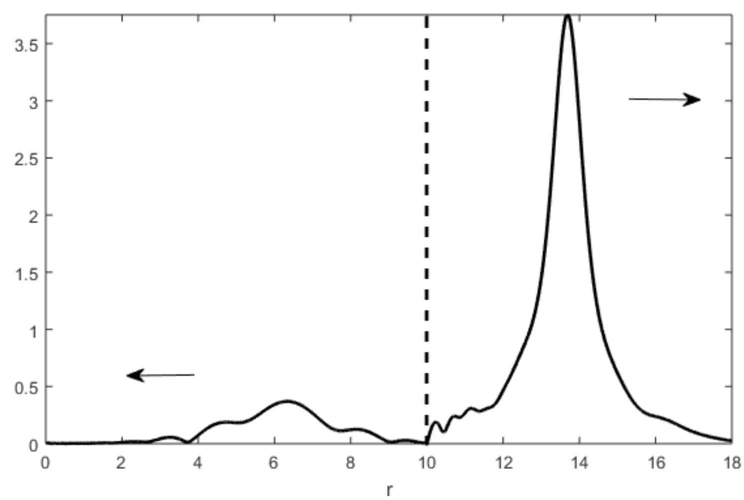

Figure 7. Solution profile after defect interaction for $Z = 200$ and $t = 0.4320$.

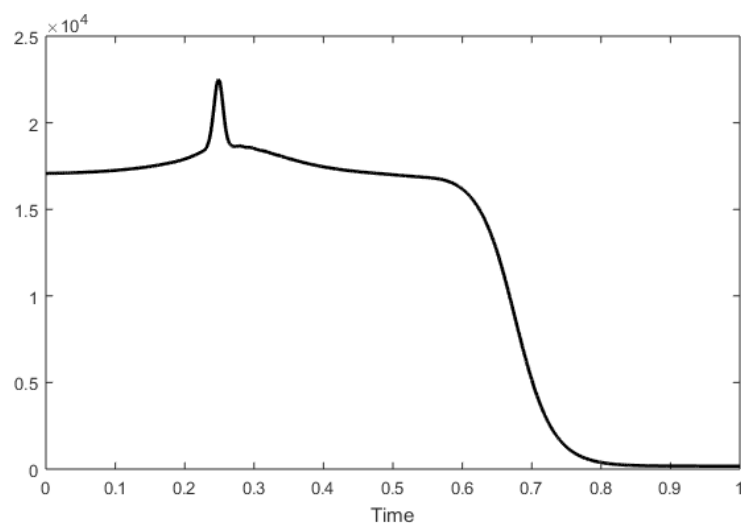

Figure 8. Variation of $\|v_r\|_{L^2}^2$ versus time for $Z = 200$.

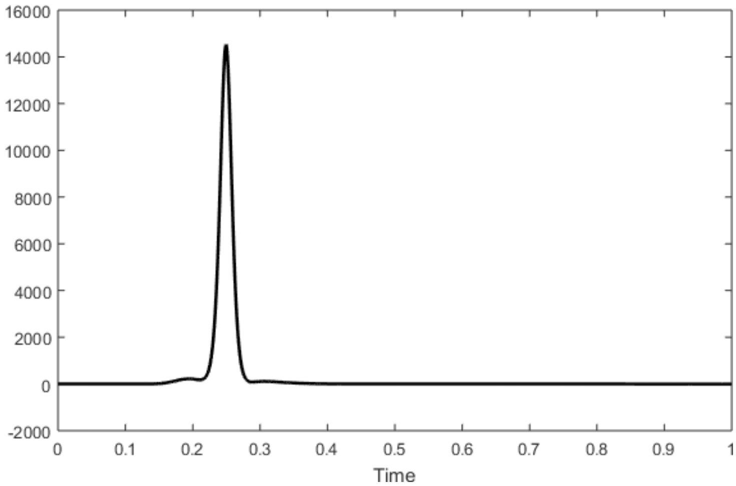

Figure 9. Evolution of $[|\frac{\partial v}{\partial r}|^2]_{r_0}$ versus time for $Z = 200$.

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