Compton effect as a double Doppler shift

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The Compton equation is normally developed using the photon model, but it is also capable of derivation based on a classical wave model of light, and there is a considerable history of such derivations, including one by Compton himself. A simple derivation in the laboratory frame based on waves scattered by a statistical aggregate of recoil electrons is developed in this paper, providing a bridge between microscopic events and macroscopic observation, and implications of coexisting wave and particle model derivations of the Compton effect are discussed. The parallel derivations based on contrasting models can also furnish an instructive exercise for students.

I. INTRODUCTION

The Compton effect, given by Eq. (1) below, is well known to every student of physics and is frequently cited as a pillar of the quantum theory. When they were first formulated in 1923, Compton[1] and Debye's[2] simultaneous derivations appeared to be a startling confirmation[3] of the then widely doubted photon model.[4–6] Yet Compton was fundamentally a classical physicist[7] looking for concrete answers,[8] and in a sense, the Compton model is a conservative one, treating radiation as being composed of a swarm of corpuscles and harking back to a pre-Maxwellian view. Following a brief review of the Compton theory, an alternative derivation will be developed based on the wave model.

II. THE STANDARD MODEL

$$\Delta\lambda = \lambda_c(1 - \cos\theta) \tag{1}$$

is the Compton equation (Compton shift), where $\lambda_c = h/m_e c$ is the so-called Compton wavelength. Since the radiation is considered as a corpuscle, it may also be shown by conservation of momentum that

$$\tan\phi = -1/(1+\alpha)\tan\frac{\theta}{2} = -\left(\frac{\lambda_0}{\lambda_0+\lambda_c}\right)\cot\frac{\theta}{2}, \tag{2}$$

where ϕ is the corresponding angle of the recoil electron (see Fig. 1), and $\alpha = h\nu_0/m_e c^2 = \lambda_c/\lambda_0$. Equation (2) together with conservation laws implies that $\phi < \pi/2$ for nonzero recoil velocity. Furthermore, the kinetic energy of the electron is given by

$$T_e = \frac{2m_e c^2\alpha^2\cos^2\phi}{(1+\alpha)^2 - \alpha^2\cos^2\phi}, \tag{3}$$

which is equivalent to

$$\beta = \frac{2(\alpha+\alpha^2)\cos\phi}{(1+\alpha)^2 + \alpha^2\cos^2\phi}. \tag{4}$$

Finally, the probability of the relative intensity of the scat-

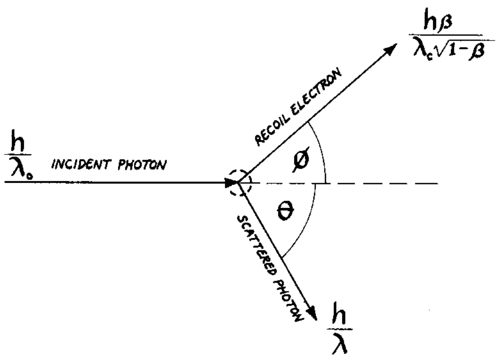

Fig. 1. Momentum diagram of photon collision with stationary electron, showing typical values of $\alpha = 1$. The de Broglie wavelength of the recoil electron is the ratio of the Lorentz-contracted Compton wavelength to β.

tered radiation per unit solid angle about θ is given by the Klein–Nishina equation,

$$\frac{I_\theta}{I_0} = \frac{e^4}{2m_e^2c^4} \frac{(1+\cos^2\theta)}{[1+\alpha(1-\cos\theta)]^3} \times\left(1 + \frac{\alpha^2(1-\cos\theta)^2}{(1+\cos^2\theta)[1+\alpha(1-\cos\theta)]}\right). \quad (5)$$

III. A WAVE MODEL ALTERNATIVE

Contrary to the popular impression, the photon model does not appear to be essential in explaining the Compton effect. In fact, it is noteworthy that in the standard theory the recoil electron is unbound and therefore not quantized, that only a fraction of the supposed photon's energy is transferred as a continuous function of recoil angle and radiation frequency, that the Compton shift is a continuous function of the observer angle, and that indeed the sole necessary reference to quantum theory in Eq. (1) is the presence of Planck's constant. (Of course it may simply be supposed that the electron first absorbs the whole incident quantum and then subsequently emits the reduced quantum, but this appears to beg the question, since there is no time delay, momentary equivalence of electron energy or momentum to those of the incident quantum, or other external evidence whatsoever for this interpretation.) Compton's own experiments relied on Bragg diffraction,[9] a wave effect,[10] to measure the actual wavelengths involved, and in his classic papers and text on x rays, Compton himself carefully discussed and rejected[11] (apparently on erroneous grounds)[9,11–13] a Doppler-shift wave derivation[14] similar to that in this paper, as well as noting the existence of Schrödinger's nonrelativistic wave derivation[15,16] of Compton scattering. Further Doppler derivations of the Compton effect have been made over the years by Halpern,[17] Breit,[18] Synge,[19] Olsen,[20,21] and Mellen,[22] among others (cf. *The Handbook of Physics*[23]), in addition to the unsuccessful but fruitful[5,24] time-averaged statistical wave approach promulgated by Bohr *et al.*[25] The present paper was largely stimulated by the elegant center-of-momentum approach of Mellen, despite our differing viewpoints, but for purposes of physical interpretation it has been found useful to rederive the Compton equation in the laboratory frame.

The incident radiation is taken to be moving in the zero direction. We shall tentatively assume that the recoil electron also goes off in (approximately) the zero direction. The point will be returned to later, but $\phi = 0$ is at least a kind of average value as well as constituting the limiting case when $\alpha \gg 1$ (and when incident radiation might be expected to be most corpuscle-like). It is necessary to make repeated use of the generalized Doppler shift of light given by

$$\lambda' = \lambda(1-\beta\cos A)/(1-\beta^2)^{1/2}, \quad (6)$$

where λ is the emitted wavelength, λ^1 is the observed wavelength, A is the angle between the direction of the light toward the observer and the motion of the emitter, and β has the usual significance of the ratio v/c.

λ_1, the wavelength received by the recoil electron, is related to λ_0, the incident wavelength in the laboratory frame, by

$$\lambda_1 = \lambda_0(1+\beta_0)/(1-\beta_0^2)^{1/2}, \quad (7)$$

where the zero subscript is used to distinguish beta in the forward direction from the more general beta of Eq. (4). The wavelength reradiated toward the observer, λ_2, undergoes an additional Doppler shift so that

$$\lambda_2 = \lambda_0\{[(1+\beta_0)(1-\beta_0\cos\theta)]/(1-\beta_0^2)\}, \quad (8)$$

which leads to

$$\Delta\lambda = \lambda_2 - \lambda_0 = \lambda_0[\beta_0/(1-\beta_0)](1-\cos\theta), \quad (9)$$

which is Eq. (1) iff

$$\lambda_0\beta_0/(1-\beta_0) = \lambda_c. \quad (10)$$

This condition, in turn, is equivalent to

$$\beta_0 = \alpha/(1+\alpha), \quad (11)$$

which Compton remarked "is the velocity which the electron must have in order to give the observed change of wavelength according to the Doppler principle."[12]

It seems likely that the success of the Doppler-shift derivation of the Compton effect must represent something more than a coincidence. In the view of the writers, individual recoil electrons can take on the range of values implied by Eq. (2), but the electromagnetic energy is reradiated in all directions with a continuous distribution of Compton shift (except along the direction of motion, where the double Doppler shifts exactly cancel out), and the observed shift of Eq. (1) is true for the statistical aggregate at each particular moment. (This may in part account for the broadening of the reradiated line, as remarked by Compton,[12,26–28] which has usually been assumed to be due purely to the spread of initial velocities of the electrons prior to ejection.[29,30] Equation (2) indicates that ϕ goes to zero as α goes to high values for incident radiation, and this is consistent with the possibility that large angles of recoil may be due to random deflections of the ejected electron by adjacent atoms in the parent material, which would become decreasingly significant with more energetic radiation. Since the angles of the recoil electrons form a cone about the x axis with no preferred direction, all but the longitudinal components cancel out on the average, and

$$\beta_0 = \beta_{\langle\phi\rangle}\cos\langle\phi\rangle \quad (12)$$

for some average value of ϕ. [Compton himself developed Eq. (11) which, on the basis of measured $\beta > \beta_0$, he termed an "effective" velocity.[11]] It is clear from the equations that $\langle\phi\rangle \to 0$ and $\beta_{\langle\phi\rangle} \to \beta_0$ as $\alpha \to \infty$. Comparison of Eqs. (4) and (11) indicates that Eq. (12) is exact for

$$\langle\phi\rangle = \text{arcos}\left(\frac{1+\alpha}{(2+4\alpha+\alpha^2)^{1/2}}\right) \quad (13)$$

and, despite the differences in the radiation model, Eq. (11) is reasonably approximated by Eq. (12) using Eq. (4) and ϕ

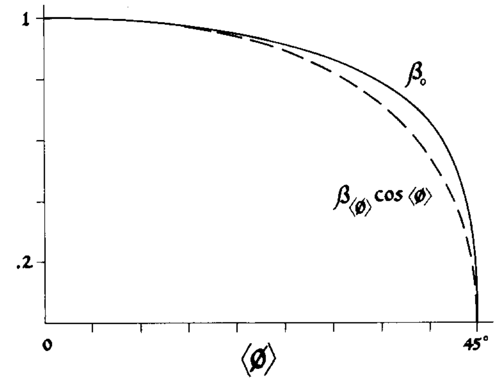

Fig. 2. $\beta\cos\langle\phi\rangle$ vs $\langle\phi\rangle$ (broken line) from computer-averaged Klein–Nishina $\langle\theta\rangle$ and Eqs. (2) and (4), compared with β_0 versus $\langle\phi\rangle$ (solid line) from Eqs. (11) and (13), showing general similarity and convergence at endpoints.

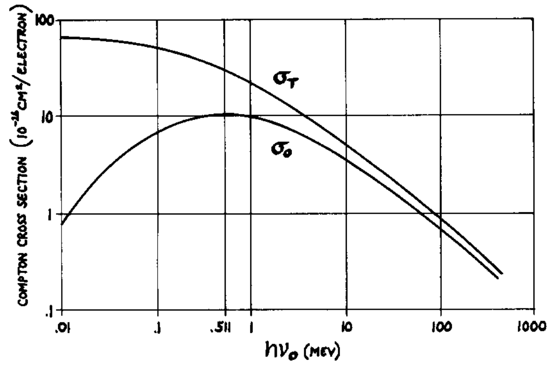

Fig. 3. Total scattering cross section σ_T (upper curve), showing decrease with increasing α; and energy-absorbing cross section σ_a (lower curve), showing maximum at $\alpha = 1(h\nu_0 = 0.511$ MeV). [Courtesy U. S. National Bureau of Standards.[41]]

derived from Eq. (2) in conjunction with computer-generated expectation values of θ from Eq. (5) (see Fig. 2). It might be remarked that the Klein–Nishina equation itself is somewhat idealized and can be modified by various factors,[31–33] including the increasing polarization of incident radiation at higher energies.[34,35]

It appears reasonable that the Compton shift can be derived from a wave model since, as is well known, if there is insufficient energy available to free the electron, m_e in Eq. (1) must be redefined as being the entire mass of the atom, λ_c becomes immeasurably small and the difference between wavelengths vanishes, and Compton scattering goes to the limiting case of Thomson scattering, which in turn was derived from a classical wave model. [It is interesting that in the limit $\alpha \to 0$, Eqs. (2) and (4) predict motion at $\pi/2$ for $\theta = 0$ and a vanishing recoil velocity, which correctly describe the transverse oscillations of the (bound) electron in Thomson scattering.] The fraction of the incident energy carried off by the supposedly scattered photon, $f = 1/[1 + (1 - \cos\theta)]$ (see Ref. 31) is readily seen to be just the Doppler-shifted ratio ν_2/ν_0, using Eqs. (8) and (11), and the increasing asymmetry in the forward direction of the Klein–Nishina distribution at higher recoil velocities[36] becomes intuitively reasonable when it is considered as a Doppler-modified Thomson scattering corrected for decreasing cross section at higher frequencies. Furthermore, it is consistent with the modern emphasis on quantized fields to consider the electromagnetic radiation holistically.

Finally, it is not individual interactions but an ensemble of events which is normally viewed macroscopically,[37] and hence perforce a statistical picture[38] (cf. Ref. 28), and it is claimed that the wave model successfully accounts for observations in the average, at least in the low-energy limit.[38,40] Indeed, some writers who accept the photon model also insist that the Doppler effect accounts for the Compton shift in individual events along the direction of observation, a view which differs in only the last-mentioned respect from a wave model.

IV. WAVE AND/OR PARTICLE?

It is interesting that if a kind of relativistic mass is defined for the postulated photon by setting $m_{ph}c^2 = h\nu_0$, maximum energy transfer between radiation and electrons occurs when the mass of the photon matches m_e, in analogy to the classic mechanics of particles (see Fig. 3).[41] However, if equivalently we set $m_{ph}c = h/\lambda_0$, then equating m_{ph} to the mass of the electron gives the alternate result that

$$\lambda_0 = h/m_e c. \tag{14}$$

That is, for maximum energy transfer the incident wavelength must be equal to the Compton wavelength of the electron, which suggests a classical resonance effect.

The relativistic Doppler shift can be derived using a photon model if one starts from conservation of energy and momentum,[42–45] and we have seen in this paper that Eq. (1) can be derived using either a wave model or the photon hypothesis (cf. Ref. 42). In accordance with Bohr's insistence that derivations in themselves can never constitute proof of a theory,[46] it is clear that despite the historic impact of Compton's paper, derivation of the Compton equation alone cannot properly be taken as verification of either the photon or the wave model (cf. Compton[47]), which indeed must ultimately be based on experimental evidence.

Experiments with modern equipment have verified simultaneity of appearance of the recoil electron and scattered radiation to within 5×10^{-10} s (see Ref. 48) (thus disposing of the historic Bohr–Kramers–Slater time-delay statistical theory) and indicate a broad maximum of wavelength around the predicted value[29,49] and a rather low coincidence rate.[50–52] While this may be consistent with the standard model, it does not clearly differentiate between implications of the two approaches, which can only be done by a careful study of isolated events and of phenomena outside the standard angles.[53] Indeed, without such investigations, there was an element of circularity in the historic citing of the Compton effect as definitive evidence for the photon nature of light. As Compton remarked some 60 years ago, "The close overlapping of the classical and quantum principles as applied to this problem...suggests that here may be a most profitable field for studying the connection between these two points of view."[14] The low-energy transition where the Klein–Nishina equation smoothly merges into the Thomson formula apparently embodies a discontinuity between the symmetric wave and the asymmetric photon scattering models for individual events which might repay examination.

Experimental confirmation apart, may the existence of both wave- and photon-based derivations of the Compton effect be taken as simply another instance of duality? Even this interpretation is not necessarily adequate, in the view of the writers. For, even without admitting the photon model, it is well established that Planck's constant serves as a proportionality factor in the transfer of energy and momentum between radiation and matter.[54] On the other hand, classical wave theory was extrapolated from macroscopic observations and was deficient in its consideration of directed momentum on the atomic level, despite its general validity. One may speculate that any model whatsoever which takes account of the above plus the conservation laws and is not inconsistent with previous laws of physics might be successful as a starting point in deriving equations dealing with the net interaction of radiation and matter, and that, consequently, agreement with the standard equations is not to be considered a confirmation of fundamental validity of the model but only as fulfillment of a necessary but not sufficient condition.

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