M1 Physique Fondamentale & M1 General Physics — Université Paris-Saclay
Année 2016-2017

Final exam : Physics of complex systems

Vendredi 17 Mars 2017 – 3h

Take you time and read carefully, there are many questions you can address even if you haven't done the intermediate steps. The first exercise requires careful thinking but not complicated algebra.

Coupled order parameters (Y. Imry 1975)

We consider two Ising-like scalar order parameters M and ϕ, satisfying $\mathbb{Z}_2$ symmetry in the absence of an external field. They are coupled to each other and have different critical temperatures. The free energy density reads, in the absence of an external field :

$$F(M,\phi) = at\frac{M^2}{2} + d\frac{M^4}{4} + \alpha\tau\frac{\phi^2}{2} + \delta\frac{\phi^4}{4} + \frac{1}{2}gM^2\phi^2 \tag{1}$$

with $t = T - T_1$ and $\tau = T - T_2$ where $T_{1,2}$ are two temperatures such that $T_2 < T_1$ (T is the control temperature), a, d, α and δ are positive parameters and g a coupling constant which can be either positive or negative.

1. Explain why the free energy expansion has the form of Eq. (1).
2. When $g = 0$, describe, *without doing calculations*, the phase diagram as a function of temperature T.
3. Explain qualitatively the effect of g, depending on its sign. In particular, for which sign would we expect a phase not realized when $g = 0$?
4. We make the following change of variables : $M = \sqrt{\frac{a}{d}}\tilde{m}$ and $\phi = \sqrt{\frac{\alpha}{\delta}}\tilde{\phi}$. Show that the free energy density takes the form

$$F(\tilde{m},\tilde{\phi}) = \varepsilon\left[r\left(t\frac{\tilde{m}^2}{2} + \frac{\tilde{m}^4}{4}\right) + r^{-1}\left(\tau\frac{\tilde{\phi}^2}{2} + \frac{\tilde{\phi}^4}{4}\right) + \frac{1}{2}\lambda\tilde{m}^2\tilde{\phi}^2\right] \tag{2}$$

   and give the expression of r, ε and λ as a function of the parameters.
5. Finally, show that $f = F/\varepsilon$ can be put into the following form, after rescaling the order parameters $\tilde{m} \to m$ and $\tilde{\phi} \to \varphi$ by an appropriate factor that you will find :

$$\boxed{f(m,\varphi) = qt\frac{m^2}{2} + \frac{m^4}{4} + q^{-1}\tau\frac{\varphi^2}{2} + \frac{\varphi^4}{4} + \frac{1}{2}\lambda m^2\varphi^2} \tag{3}$$

   and give the expression of q.
6. By writing the quartic terms as a quadratic form (ie. $f = \vec{v}^{\,\mathsf{T}}A\vec{v} + \vec{B}\cdot\vec{v}$ with A a 2×2 matrix and $\vec{v}^{\mathsf{T}} = \begin{pmatrix} m^2 & \varphi^2 \end{pmatrix}$), show the following condition : $\lambda > -1$.
7. Write down the equations which solutions are the equilibrium order parameters.

8. We use the following labelling for the phases :

$$\begin{array}{ll} \text{phase I} & (m = 0, \varphi = 0) \\ \text{phase II} & (m \neq 0, \varphi = 0) \\ \text{phase III} & (m = 0, \varphi \neq 0) \\ \text{phase IV} & (m \neq 0, \varphi \neq 0) \end{array}$$

Find the expression of the order parameters in each phase and the corresponding free energy densities $f_{\rm I}$, $f_{\rm II}$, $f_{\rm III}$ and $f_{\rm IV}$ as a function of q, t, τ and λ. If you can't prove it, we give the result for phase IV :

$$f_{\rm IV} = \frac{1}{4(1-\lambda^2)}\left(2\lambda t\tau - q^2t^2 - q^{-2}\tau^2\right) \quad \text{and} \quad m^2 = \frac{\lambda\tau/q - qt}{1-\lambda^2}, \varphi^2 = \frac{\lambda qt - \tau/q}{1-\lambda^2} \tag{4}$$

9. First consider the case $\tau > t > 0$.
   a) which phases are easily excluded ?
   b) first look at the $\lambda > 1$ situation. Are all phases possible ?
   c) then look at the other case and conclude.
10. In the case where $\tau > 0 > t$:
    a) show that only phases II and IV are competing and compute their free energy difference. Sort out the case $\lambda > 1$.
    b) we introduce $\Delta T_c = T_1 - T_2$. Give a first interval for t.
    c) give the boundaries of phase IV as an interval for possible values of λ in terms of functions $\lambda_\pm(t)$ parametrized by ΔT_c and q : $\lambda_-(t) < \lambda < \lambda_+(t)$.
    d) Draw qualitatively the curves of $\lambda_-(t)$ and $\lambda_+(t)$ for $t < 0$ and find the condition of existence of their crossing points and their respective location. These points are denoted by G for the one at $\lambda = -1$ and K for the other.
    e) Infer a simpler condition for the region of phase IV when $\tau > 0 > t$.
11. Last, in the case where $0 > \tau > t$:
    a) is phase I possible ?
    b) compute the expression of the free energy differences $f_{\rm II} - f_{\rm III}$ and $f_{\rm IV} - f_{\rm III}$ (the expression of $f_{\rm IV} - f_{\rm II}$ remains unchanged). Sort out the case for $\lambda > 1$.
    c) For $\lambda^2 < 1$, reuse the results of question 10 to obtain the phase diagram.
12. Reproduce the phase diagrams corresponding to $q > 1$ and $q < 1$ of Fig. 1 on your paper. Can you locate each phase of the diagrams ? Place points G and K.

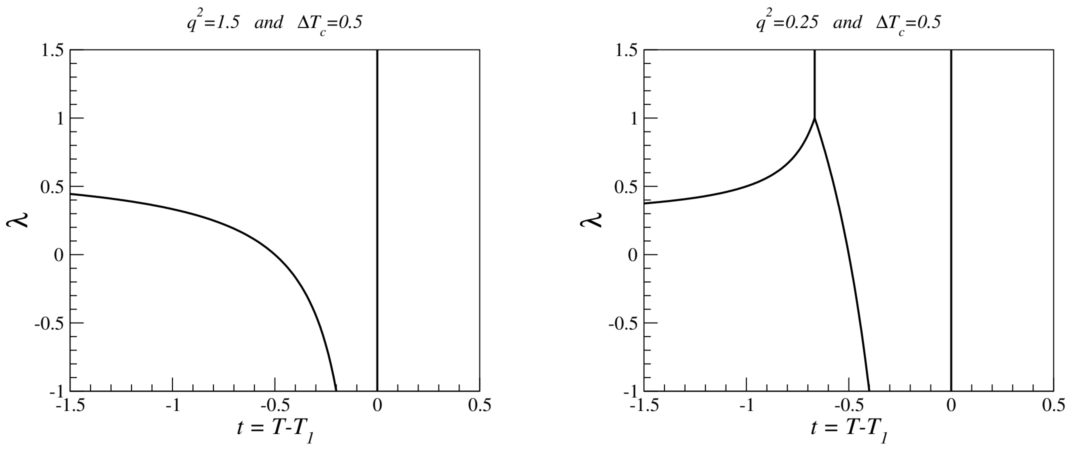

FIGURE 1 – Typical phase diagrams for $q > 1$ (left) and $q < 1$ (right).

Cahn-Hilliard equation

We consider the equation of motion governing a conserved scalar order parameter $\phi(\vec{r},t)$ the free energy of which is given by :

$$F = \int_V d\vec{r} \left\{ g(\vec{\nabla}\phi)^2 + f_L(\phi) \right\} \text{ with } f_L(\phi) = \tilde{a}\phi^2 + d\phi^4 \tag{5}$$

and $\tilde{a} = a(T - T_c)$, $a, d > 0$, T_c the critical temperature and V the volume of a *closed* system. The conservation of the total order parameter means that

$$\overline{\phi} = \frac{1}{V} \int_V \phi(\vec{r},t)\, d\vec{r} = \text{const.} \tag{6}$$

The Cahn-Hilliard equation identifies $\frac{\delta F}{\delta \phi}$ with a generalized chemical potential $\mu = \frac{\delta F}{\delta \phi}$. An inhomogeneity of this chemical potential induces a response through a current $\vec{J}$, which, to the lowest order in μ behaves according to a generalized Fick's law :

$$\vec{J} = -\Gamma \vec{\nabla}\mu \; . \tag{7}$$

Then, the conservation law translates in the usual continuity equation :

$$\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad \text{(Cahn-Hilliard)} \tag{8}$$

that governs the time evolution of the order parameter. It is relevant for the so-called spinodal decomposition, when a gas is suddenly quench below the critical temperature so that the system tries to reach the equilibrium phase such that gas and liquid coexist.

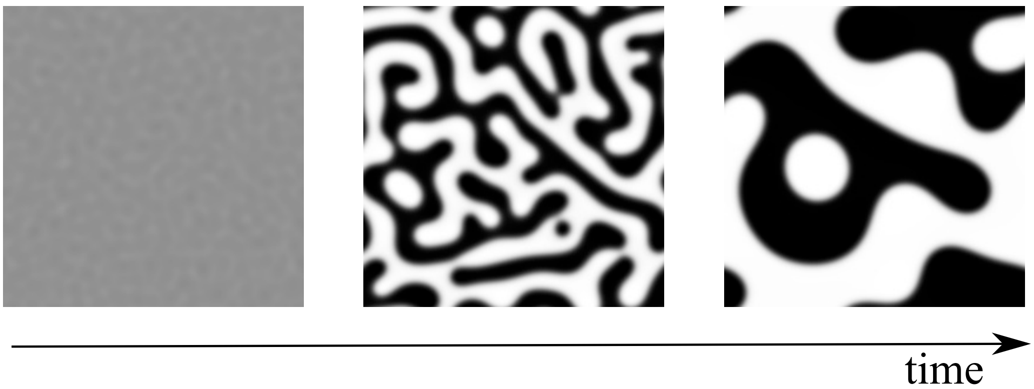

FIGURE 2 – Spinodal decomposition with coexistence of two phases and conserved order parameter.

Statics

1. What is the meaning of the g term in (5) ? Draw qualitatively $f_L(\phi)$ for $T > T_c$ and $T < T_c$. Discuss the positions of the extrema ϕ_m and the corresponding $f_L''(\phi_m)$.
2. What is the equation satisfied by an equilibrium configuration $\phi_0(\vec{r})$?
3. Linearize the equation around an equilibrium solution $\phi_0(\vec{r})$ by taking $\phi(\vec{r}) = \phi_0(\vec{r}) + \delta\phi(\vec{r})$ with $\delta\phi \ll 1$. Express the result as a function of the Laplacian operator $\vec{\nabla}^2 \equiv \Delta$, f_L'' and other parameters.
4. Discuss what happens around uniform phases corresponding to each extrema ϕ_m of f_L, above and below T_c. Identify in each case a characteristic length scale ξ as a function of $f_L''(\phi_m)$ first and then as a function of the parameters of f_L.

Dynamics

5. Show that the Cahn-Hilliard equation is compatible with the conservation of $\overline{\phi}$. What is the sign of Γ ?
6. Recall the equation for a *non-conserved* order parameter. What is the difference if one starts from a uniform phase ?
7. Give the explicit form of the partial differential equation governing the evolution of ϕ.
8. Perform a linear stability analysis of the equation around an extrema ϕ_m of f_L, assuming that $\phi(\vec{r},t) = \phi_m + \delta\phi(\vec{r},t)$. Express the equation as a function of Γ, g and $f_L''(\phi_m)$.
9. Consider the situation where ϕ_m is a minimum :
   a) rewrite the equation by introducing the length scale ξ.
   b) study this linearized equation using the Fourier transform :
   $$\delta\phi(\vec{r},t) = \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^d} \delta\hat{\phi}(\vec{k},\omega) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{9}$$
   Find the dispersion relation $\omega(k)$ of the modes, with $k = \|\vec{k}\|$.
   c) we first neglect the Δ^2 term. What is this kind of equation ? Find the Green's function $\chi(\vec{r},t)$ of this equation.
   d) by injecting the Green's function back in the full equation, for which times is the Δ^2 term negligible ?
10. Consider the situation where ϕ_m is a local maximum :
   a) rewrite the equation by introducing the length scale ξ.
   b) find the new dispersion relation $\omega(k)$ of the modes and draw it.
   c) show that some modes are unstable and give the typical length-scale of the instability. Interpret physically the short time behavior in relation with Fig. 2.