Lecture Notes
Distributions and Partial Differential Equations

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Contents

Foreword

These notes are based on a course I gave several times in Orsay, and once in Ritsumeikan University. As a matter of fact, they rely mostly on two excellent textbooks (in french), that I strongly recommend to give a look at.

[Bo] Jean-Michel Bony, Cours d'analyse - Théorie des distributions et analyse de Fourier, Editions de l'Ecole Polytechnique, Palaiseau, Ellipses Diffusion, 2010, ISBN: 2-7302-0775-9.

[Zu] Claude Zuily, Introduction aux distributions et équations aux dérivées partielles, Collection Sciences Sup, Dunod, Paris, 2002, ISBN: 2-10-005735-9.

Introduction

A distribution is a linear form on the space of smooth functions with compact support, satisfying some continuity property. It is not difficult to write down such a linear form, as for example the mapping

$$T_f : \varphi \mapsto \int f(x)\varphi(x)dx,$$

where f is a locally integrable function.

As a matter of fact, the form T_f is a distribution for any locally integrable function f, and each concept in distributions theory extends to distributions the corresponding notion for functions when it exists. In that sense, distributions are ”generalized functions”.

Of course, there is also a lot of distributions that are not of this form, which makes the theory worthwhile!

Chapter 1

Distributions in 1d

This chapter is devoted to basic calculus of distributions. We have chosen to begin with distribution of one variable to explore the basic ideas of the theory. That way, the reader has not to cope at the same time with several variables calculus.

1.1 Test functions

We recall here elementary facts about smooth functions of one real variable.

A smooth function on an open interval $I \subset \mathbb{R}$ is a function $\varphi : I \to \mathbb{C}$ whose derivatives of any order $\varphi', \varphi'', \ldots, \varphi^{(k)}, \ldots$ exist and is continuous on I. A linear combination of smooth functions is a smooth function, and we denote $\mathcal{C}^\infty(I)$ the vector space of smooth functions.

If $\varphi \in \mathcal{C}^\infty(I)$, and J is an open subset of I, the function defined on J by $x \mapsto \varphi(x)$ is a smooth function on J, that we denote $\varphi_{|_J}$. This function is called the restriction of φ to J.

When $F \subset \mathbb{R}$ is a closed interval, the assertion $\varphi \in \mathcal{C}^\infty(F)$ means that there exists an open interval $I \subset \mathbb{R}$ such that $F \subset I$, and a smooth function $\psi \in \mathcal{C}^\infty(J)$ such that $\psi_{|_F} = \varphi$.

It is true that the product of two smooth functions is smooth, and we have the so-called Leibniz formula

$$(\varphi_1\varphi_2)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} \varphi_1^{(j)} \varphi_2^{(k-j)}.$$

A smooth function $\varphi : \mathbb{R} \to \mathbb{C}$ satisfies the Taylor formula

$$\varphi(x) = \sum_{k=0}^{m} \frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^{m+1}}{m!}\int_0^1 (1-s)^m \varphi^{(m+1)}(sx)ds,$$

that one can prove integrating by parts the last term on the right.

Exercise 1.1.1 Prove Hadamard's lemma: if $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $\varphi(0) = 0$, there exists a function $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\varphi(x) = x\psi(x)$ for any $x \in \mathbb{R}$.

1.1.1 Support of a smooth function

If $\varphi \in \mathcal{C}^\infty(I)$, and J is an open subset of I, we say that φ vanishes on J if it vanishes at any point of J, or, equivalently, if $\varphi_{|_J}$ is the null function.

Definition 1.1.2 Let $\varphi : I \to \mathbb{C}$ a smooth function. The support of φ is the complementary of the union of all the open sets in I where φ vanishes. This set is denoted by $\operatorname{supp}\varphi$.

Note that the support of a function φ is a closed set. It is also the closure of the set of $x \in I$ such that $\varphi(x) \neq 0$. The following characterization is often useful:

$$x_0 \notin \operatorname{supp}\varphi \iff \exists V \text{ neighborhood of } x_0 \text{ such that } \varphi_{|_V} = 0.$$

Exercise 1.1.3 Show

$$\operatorname{supp}\varphi_1\varphi_2 \subset \operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2.$$

Are these two sets equal?

Of course, if $\varphi \in \mathcal{C}^\infty(I)$ vanishes on an open set $J \subset I$, all its derivatives vanishes also on J, and, therefore, for all integer k,

$$\operatorname{supp}\varphi^{(k)} \subset \operatorname{supp}\varphi.$$

Definition 1.1.4 We denote $\mathcal{D}(I) = \mathcal{C}_0^\infty(I)$ the vector space of functions which are $\mathcal{C}^\infty$ on I, and whose support is a compact subset of I.

If V is an open subset of I, one may identify a function $\varphi \in \mathcal{D}(V)$ with the function $\tilde{\varphi} \in \mathcal{D}(I)$, where the function $\tilde{\varphi}$ is defined as

$$\tilde{\varphi}(x) = \varphi(x) \text{ for } x \in V, \quad \tilde{\varphi}(x) = 0 \text{ for } x \in I \setminus V.$$

Indeed, $\tilde{\varphi}$ is smooth and with compact support on I for $\varphi \in \mathcal{D}(V)$. On the other hand, a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, can be identified with its restriction $\varphi_{|_I} \in \mathcal{D}(I)$ for any open set I that contains its support.

Definition 1.1.5 If $K \subset I$ is a compact subset, we denote $\mathcal{C}_K^\infty(I)$ the space of smooth functions on I with support in K.

Exercise 1.1.6 Let f and φ be two functions in $\in L^1(\mathbb{R})$. Show that the convolution $f * \varphi$ of f and φ, given by

$$f * \varphi(x) = \int f(x-y)\varphi(y)dy,$$

is defined almost everywhere, and is an L^1 function.

Suppose moreover that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, show that $f * \varphi$ is smooth. At last if f is also continuous, show that

$$\text{supp}\, f * \varphi \subset \text{supp}\, f + \text{supp}\, \varphi.$$

Answer: First of all, $f * \varphi$ is an almost everywhere defined and L^1 function. Indeed, using Fubini's theorem for non-negative functions,

$$\begin{aligned}\int |f * \varphi(x)|dx &\leq \iint |f(x-y)\varphi(y)|dydx \\ &\leq \int |\varphi(y)|\big(\int |f(x-y)|dx\big)dy \leq \|\varphi\|_{L^1}\|f\|_{L^1} < +\infty\end{aligned}$$

Therefore $f * \varphi$ is finite almost everywhere, and belongs to L^1.

Suppose now that φ is smooth with compact support. Changing variable we have

$$f * \varphi(x) = \int f(y)\varphi(x-y)dy.$$

Thus $f * \varphi$ is also smooth thanks to Lebsegue's theorem, since

- the function $x \mapsto f(y)\varphi(x-y)$ is smooth for all y, and

$$\partial_x^k(f(y)\varphi(x-y)) = f(y)\varphi^{(k)}(x-y),$$

- we have the domination

$$|f(y)\varphi^{(k)}(x-y)| \leq |f(y)| \sup |\varphi^{(k)}| \in L^1.$$

Eventually suppose that f is continuous. If $x \notin \text{supp}\, f + \text{supp}\, \varphi$, then for any $y \in \text{supp}\, f$, $x - y$ does not belong to $\text{supp}\, \varphi$, thus

$$f * \varphi(x) = \int f(y)\varphi(x-y)dy = 0.$$

1.1.2 Plateau functions

It is not immediately clear that $\mathcal{D}(I)$ is not reduced to the null function. One knows for example that the only compactly supported analytic function on $\mathbb{R}^2$ is this null function: indeed, such a function is bounded and analytic, therefore vanishes thanks to Picard's Theorem. However

Proposition 1.1.7 The set $\mathcal{D}(I)$ is not trivial.

Proof.— First of all, the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(t) = \begin{cases} e^{-1/t} & \text{as } t > 0, \\ 0 & \text{as } t \leq 0, \end{cases}$$

is smooth on $\mathbb{R}$. Indeed, it is easily seen to be $\mathcal{C}^\infty$ on $\mathbb{R}^*$, with $\varphi^{(k)}(t) = 0$ for $t < 0$. On the other hand, for $t > 0$, one can prove by induction that

$$\forall k \in \mathbb{N},\ \varphi^{(k)}(t) = P_k(\frac{1}{t})e^{-1/t}$$

where P_k is a polynomial of degree $2k$. Therefore, for any $k \geq 1$, $\varphi^{(k)}(t) \to 0$ as $t \to 0^+$, so that $\varphi^{(k-1)}(t)$ is $\mathcal{C}^1$ on $\mathbb{R}$. Now let $x_0 \in I$, and $r > 0$ such that $\overline{B(x_0, r)} = [x_0 - r, x_0 + r] \subset I$. We denote $\varphi_{x_0,r} : I \to \mathbb{R}^+$ the function given by

$$\varphi_{x_0,r}(x) = \varphi(r^2 - |x - x_0|^2).$$

This function is smooth, with $\operatorname{supp} \varphi_{x_0,r} = \overline{B(x_0, r)}$, thus $\varphi_{x_0,r} \in \mathcal{D}(I)$. □

As a matter of fact, we can even prove the

Proposition 1.1.8 Let $I \subset \mathbb{R}$ an open set, and K a compact subset of I. There exists a function $\psi \in \mathcal{D}(I)$ such that

i) $\psi = 1$ in a neighborhood of K,

ii) $\psi \in [0, 1]$.

Such a function is called a "plateau function" - from the french word plateau, which means a flat, high region. The usual english name for such functions is "cut-off" functions.

Proof.— Let $\varphi_{0,1} \in \mathcal{C}^\infty(\mathbb{R})$ be the function defined above. We denote by φ the function defined by

$$\varphi(x) = (\int \varphi_{0,1})^{-1}\varphi_{0,1}(x)$$

We also set, for any $\varepsilon > 0$,

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon}).$$

The function φ_ε is smooth, and its support is $\overline{B(0, \varepsilon)}$. Moreover

$$\int \varphi_\varepsilon(x)dx = \int \varphi(x)dx = 1.$$

We denote $K_\varepsilon = \{x \in \mathbb{R}, d(x, K) \leq \varepsilon\}$, and $\mathbf{1}_{K_\varepsilon}$ its characteristic function. We set $\chi = \mathbf{1}_{K_\varepsilon} * \varphi_\varepsilon$. We have $\operatorname{supp} \chi \subset K_\varepsilon + \overline{B(0, \varepsilon)} \subset K_{2\varepsilon}$, and, for $x \in K$,

$$\chi(x) = \mathbf{1}_{K_\varepsilon} * \varphi_\varepsilon(x) = \int \mathbf{1}_{K_\varepsilon}(x - y)\varphi_\varepsilon(y)dy = \int \mathbf{1}_{K_\varepsilon}(x - \varepsilon z)\varphi(z)dz = \int \varphi(z)dz = 1,$$

since $x - \varepsilon z \in K_\varepsilon$ for any $z \in \operatorname{supp} \varphi = \overline{B(0, 1)}$. □

Notice that the support of the function χ in the previous proposition can be as close to K as needed.

1.1.3 Convergence in $\mathcal{D}(I)$

The natural notion of convergence for continuous functions is that of uniform convergence, since it is the simplest one for which the limit of a sequence of continuous functions is continuous. For smooth, compactly supported function, it is clear what the correct notion is:

Definition 1.1.9 Let (φ_j) be a sequence of functions in $\mathcal{D}(I)$, and $\varphi \in \mathcal{D}(I)$. We say that (φ_j) tends to φ in $\mathcal{D}(I)$ (or in the $\mathcal{D}(I)$-sense), when

i) There exits a compact $K \subset I$ such that $\operatorname{supp} \varphi_j \subset K$ for all j, and $\operatorname{supp} \varphi \subset K$.

ii) For all $k \in \mathbb{N}$, $\|\varphi_j^{(k)} - \varphi^{(k)}\|_\infty := \sup |\varphi_j^{(k)} - \varphi^{(k)}| \to 0$ as $j \to +\infty$.

In that case we may write

$$\varphi = \mathcal{D} - \lim_{j \to +\infty} \varphi_j.$$

Exercise 1.1.10 Let $\varphi \in \mathcal{D}(\mathbb{R})$, and, for $t \neq 1$, denote $\psi_t \in \mathcal{D}(\mathbb{R})$ the function given by

$$\psi_t(x) = \frac{\varphi(tx) - \varphi(x)}{t-1}.$$

Show that the family (ψ_t) converges in $\mathcal{D}(\mathbb{R})$.

1.2 Definitions and examples

1.2.1 Definitions

Definition 1.2.1 Let $I \subset \mathbb{R}$ an open subset, and T a complex valued linear form on $\mathcal{D}(I)$. One says that T is a distribution on I when

$$\forall K \subset\subset I, \exists C > 0, \exists m \in \mathbb{N}, \forall \varphi \in \mathcal{C}_K^\infty(I),\ |T(\varphi)| \leq C \sum_{\alpha \leq m} \sup |\varphi^{(\alpha)}|.$$

We denote $\mathcal{D}'(I)$ the set of distributions on I, and for $T \in \mathcal{D}'(I)$, $\varphi \in \mathcal{D}(I)$, we denote $\langle T, \varphi \rangle := T(\varphi)$.

Proposition 1.2.2 A linear form T on $\mathcal{D}(I)$ is a distribution on I if and only if $T(\varphi_j) \to T(\varphi)$ for any sequence (φ_j) of functions in $\mathcal{D}(I)$ that converges to φ in the $\mathcal{D}(I)$-sense.

Proof.— Let T be a distribution on I, and (φ_j) a sequence in $\mathcal{D}(I)$ which converges to φ in $\mathcal{D}(I)$. There is a compact $K \subset I$ such that supp $\varphi_j \subset K$ for all $j \in \mathbb{N}$, and supp $\varphi \subset K$. There exist $C = C_K > 0$ and $m = m_K \in \mathbb{N}$ such that

$$\forall \psi \in \mathcal{C}_K^\infty(I),\ |T(\psi)| \leq C \sum_{\alpha \leq m} \sup |\psi^{(\alpha)}|.$$

In particular, for any $j \in \mathbb{N}$,

$$|T(\varphi_j) - T(\varphi)| = |T(\varphi_j - \varphi)| \leq C \sum_{\alpha \leq m} \sup |\varphi_j^{(\alpha)} - \varphi^{(\alpha)}|.$$

Therefore $T(\varphi_j) \to T(\varphi)$ as $j \to +\infty$, and we have proved the only if part of the proposition.

Suppose now that for any sequence (φ_j) of functions which converges in $\mathcal{D}(I)$, we have $T(\varphi_j) \to T(\varphi)$, where $\varphi = \mathcal{D} - \lim \varphi_j$. Suppose that the linear form T is *not* a distribution, that is

$$\exists K \subset \Omega, \forall C > 0, \forall m \in \mathbb{N}, \exists \varphi \in \mathcal{C}_K^\infty(\Omega) \text{ such that } |T(\varphi)| > C \sum_{\alpha \leq m} \sup |\varphi^{(\alpha)}|.$$

Then for any $j \in \mathbb{N}$, choosing $C = m = j$, there is a function $\varphi_j \in \mathcal{C}_K^\infty(\Omega)$ such that

$$|T(\varphi_j)| > j \sum_{\alpha \leq j} \sup |\varphi_j^{(\alpha)}|.$$

Let $\psi_j \in \mathcal{C}_K^\infty(I)$ given by $\psi_j = \varphi_j / |T(\varphi_j)|$. One has $|T(\psi_j)| = 1$, and

$$\mathcal{D} - \lim \psi_j = 0,$$

since for all $j \geq \alpha$,

$$\sup |\psi_j^{(\alpha)}| \leq \sum_{\alpha \leq j} \sup |\psi_j^{(\alpha)}| \leq \frac{1}{j} |T(\psi_j)| \leq \frac{1}{j}.$$

This is a contradiction, since one should have $T(\psi_j) \to 0$. □

As a set of continuous linear forms, $\mathcal{D}'(I)$ is of course a vector space on $\mathbb{C}$: if $T_1, T_2 \in \mathcal{D}'(I)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1 T_1 + \lambda_2 T_2$ is the distribution given by

$$\langle \lambda_1 T_1 + \lambda_2 T_2, \varphi \rangle := \lambda_1 \langle T_1, \varphi \rangle + \lambda_2 \langle T_2, \varphi \rangle.$$

For $T \in \mathcal{D}'(I)$, we may also denote $\bar{T}$ or T^* the distribution given by

$$\langle \bar{T}, \varphi \rangle = \overline{\langle T, \overline{\varphi} \rangle}.$$

Then, any distribution T can be written $T = T_1 + iT_2$ where T_1 and T_2 are real distributions, that is such that $\langle T, \varphi \rangle \in \mathbb{R}$ for any real-valued function φ. Indeed, this relation holds with

$$T_1 = \frac{1}{2}(T + \bar{T}) \text{ and } T_2 = \frac{1}{2i}(T - \bar{T}).$$

1.2.2 Regular distributions

If $f \in L^1_{loc}(I)$, the function $f\varphi$ is integrable for any $\varphi \in \mathcal{D}(I)$, and $T_f : \varphi \to \int f\varphi$ is a linear form. Moreover, if $K \subset \Omega$ is a compact subset of I, for any $\varphi \in \mathcal{C}^\infty_K(I)$, one has

$$|T_f(\varphi)| \leq \|f\|_{L^1(K)} \sup |\varphi|,$$

which shows that T_f is a distribution on I. Distributions of that form are called regular distributions.

As a matter of fact, one can identify $L^1_{loc}(I)$ to a part of $\mathcal{D}'(I)$, that is identify f with T_f, since the map $f \mapsto T_f$ is 1 to 1. The proof of this fact is the subject of the

Exercise 1.2.3 Let f and g be two functions in $L^1_{loc}(I)$.

i) Let $K_j = \{x \in I, d(x, \partial I) \geq 1/j, |x| \leq j\}$. Show that, for any $j \in \mathbb{N}^*$, $K_j \subset \mathring{K}_{j+1}$, and that $I = \bigcup_{j>0} K_j$.

ii) For any $j \in \mathbb{N}^*$, we denote ψ_j a plateau function above K_j, and (φ_ε) the family of functions defined in the proof of Proposition 1.1.8. show that $\psi_j f * \varphi_\varepsilon$ tends to $\psi_j f$ in $L^1(I)$.

iii) Show that if $T_f = T_g$, then $f = g$.

Answer: (i) is obvious.

(ii) We prove that if $f \in L^1(I)$, then $f_\varepsilon = f * \varphi_\varepsilon \in \mathcal{C}^\infty(I)$ tends to f in $L^1(I)$. We have

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &\leq |\int f(y)\varphi_\varepsilon(x-y)dy - f(x)| \\ &\leq |\int f(x - \varepsilon z)\varphi(z)dz - f(x)| \leq \int |f(x-\varepsilon z) - f(x)|\varphi(z)dz \end{aligned}$$

Then by Fubini-Tonelli,

$$\|f_\varepsilon - f\|_{L^1} \leq \int \big(\int |f(x-\varepsilon z) - f(x)|\varphi(z)dz\big)dx \leq \int \varphi(z)\|\tau_{\varepsilon z} f - f\|_{L^1} dz,$$

where $\tau_a f$ denotes the translation of the function f given by $\tau_a f(x) = f(x-a)$. The result is then a direct consequence of the dominated convergence theorem, taking into account the continuity of the translations in the L^p spaces, that is

$$\|\tau_{\varepsilon z} f - f\|_{L^p} \to 0 \text{ as } \varepsilon \to 0.$$

Indeed we have the domination

$$\|\tau_{\varepsilon z} f - f\|_{L^1}\varphi(z) \leq 2\|f\|_{L^1}\varphi(z).$$

(iii) Let $f \in L^1_{loc}(I)$. Suppose that $\langle T_f, \varphi\rangle = \int f\varphi = 0$ for any function $\varphi \in \mathcal{C}^\infty_0(I)$. We have

$$\psi_j f * \chi_\varepsilon(x) = \int f(y)\psi_j(y)\chi_\varepsilon(x-y)dy = 0,$$

since $y \mapsto \psi_j(y)\chi_\varepsilon(x-y)$ belongs to $\mathcal{C}^\infty_0(I)$. Thus $\|\psi_j f\|_{L^1} = 0$. Since this holds for any j, we have $f = 0$ in $L^1(I)$.

1.2.3 The Dirac mass

For $x_0 \in \mathbb{R}$, we denote $\delta_{x_0} : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ the linear form given by

$$\delta_{x_0}(\varphi) = \varphi(x_0).$$

For $K \subset \mathbb{R}$ a compact set, and any function $\varphi \in \mathcal{C}_K^\infty$, one has

$$|\delta_{x_0}(\varphi)| \leq \sup |\varphi|,$$

so that δ_{x_0} is a distribution on $\mathbb{R}$. It is called the Dirac mass at x_0.

That distribution is not a regular one. Indeed, otherwise we would have, for any function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $x_0 \notin \operatorname{supp} \varphi$,

$$\varphi(x_0) = 0 = \int f(x)\varphi(x)dx.$$

so that $f = 0$ a.e.. In particular, for a plateau function ψ above $K = \{x_0\}$, we would have

$$1 = \psi(x_0) = \langle T_f, \psi \rangle = \int f\psi = 0,$$

which is absurd.

1.2.4 Hadamard principal value of $1/x$

Let us consider the linear form $T : \mathcal{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ given by

$$T(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx.$$

This limit exists for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Indeed, for such a function, there exists a real number $A > 0$ such that $\operatorname{supp} \varphi \subset [-A, A]$. Then, there exists $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\varphi(x) = \varphi(0) + x\psi(x),$$

and we have

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx = \int_{\varepsilon<|x|<A} \frac{\varphi(x)}{x} dx = \int_{\varepsilon<|x|<A} \frac{\varphi(0)}{x} dx + \int_{\varepsilon<|x|<A} \psi(x)dx.$$

Since $x \mapsto \varphi(0)/x$ is an odd function, the first integral vanishes. Moreover, using for example Lebesgue's dominated convergence theorem, we have

$$\int_{\varepsilon<|x|<A} \psi(x)dx \to \int_{-A}^{A} \psi(x)dx \text{ as } \varepsilon \to 0^+.$$

Now we show that the well-defined linear form T is a distribution on $\mathbb{R}$. Let $K \subset \mathbb{R}$ be a compact set, and $A > 0$ a real number such that $K \subset [-A, A]$. For $\varphi \in \mathcal{C}_{[-A,A]}^\infty(\mathbb{R})$, we have seen

$$T(\varphi) = \int_{[-A,A]} \psi(x)dx, \text{ with } \psi(x) = \int_0^1 \varphi'(tx)dt.$$

Therefore

$$|T(\varphi)| \leq 2A \sup_{x\in[-A,A]} |\psi(x)| \leq 2A \sup_{x\in[-A,A]} |\varphi'(x)| \leq 2A \sum_{0\leq j\leq 1} \sup |\varphi^{(j)}(x)|,$$

and this shows that T satisfies the estimate in Definition 1.2.1. It is important to notice that the constant C in the estimate is here $2A$, that is indeed depend of the compact K.

This distribution is called (Hadamard's) principal value of $1/x$, and we denote it by

$$\langle \text{pv}(\frac{1}{x}), \varphi\rangle = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx.$$

1.3 The order of a distribution

In the three examples above, the number m asked for in the definition does not depend on the compact K. It is not the case generally, and this fact is useful enough so that it has been decided to give it a specific name.

1.3.1 Definition, examples

Definition 1.3.1 Let $T \in \mathcal{D}'(I)$. We say that T is of finite order $m \in \mathbb{N}$ when

$$\forall K \subset\subset \Omega, \exists C > 0, \forall \varphi \in \mathcal{C}_K^\infty(\Omega), |\langle T, \varphi\rangle| \leq C \sum_{|\alpha|\leq m} \sup |\partial^\alpha \varphi|.$$

The space of distributions of order m on I is denoted $\mathcal{D}'_{(m)}(I)$.

Notice that if a distribution is of order m, it is also of order m' for any $m' \geq m$. We shall say that $T \in \mathcal{D}'(I)$ is of exact order $m \geq 1$ when $T \in \mathcal{D}'_{(m)}(I) \setminus \mathcal{D}'_{(m-1)}(I)$.

We have seen that if T is a regular distribution, that is $T = T_f$ for a function $f \in L^1_{loc}(I)$, then T is of order 0. The Dirac mass δ_{x_0} is not a regular distribution, but is also of order 0. At last, we have shown that the distribution pv$(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$ is of order 1.

We ask now the question if pv$(\frac{1}{x})$is of order 0. Suppose that it is. For any $A > 0$, there would exist $C_A > 0$ such that for any $\varphi \in \mathcal{C}^\infty_{[-A,A]}$,

$$|\langle T, \varphi\rangle| \leq C_A \sup |\varphi|.$$

Now recall that the distribution pv$(\frac{1}{x})$ is given by a L^1_{loc} function out of 0. By this, we mean that if $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfies supp $\varphi \subset \mathbb{R}^*$, we have

$$\langle \text{pv}(\frac{1}{x}), \varphi\rangle = \int \frac{\varphi(x)}{x} dx.$$

Therefore, the only possible obstruction to the fact that pv$(1/x)$ is of order 0 would concern functions that are supported at the origin. Let us choose a sequence (φ_n) which supports get closer and closer to 0. We fix $\varphi_n \in \mathcal{C}^\infty_{[-2,2]}$ such that $\varphi_n \geq 0$ and

$$\varphi_n(x) = \begin{cases} 1 \text{ for } x \in [\frac{1}{n}, 1], \\ 0 \text{ for } x \notin [\frac{1}{2n}, 2]. \end{cases}$$

We know that there exists $C = C_2 > 0$ such that $|\langle T, \varphi_n \rangle| \leq C \sup \varphi_n \leq C$. On the other hand, we have the lower bound

$$\langle T, \varphi_n \rangle = \int_{\frac{1}{2n}}^{2} \frac{\varphi_n(x)}{x} dx \geq \int_{\frac{1}{n}}^{1} \frac{1}{x} dx \geq \ln n.$$

Therefore we should have $\ln n \leq C$ for all n, a contradiction. We have proved that pv$(1/x)$ is a distribution of exact order 1.

Exercise 1.3.2 Let $T : \mathcal{C}^\infty_0(\mathbb{R}) \to \mathbb{C}$ be the linear form

$$T(\varphi) = \sum_{j \geq 0} \varphi^{(j)}(j).$$

Show that T is a distribution, and that T is not of finite order.

1.3.2 Non-negative distributions

Definition 1.3.3 We say that $T \in \mathcal{D}'(I)$ is a non-negative distribution when $\langle T, \varphi \rangle \in \mathbb{R}^+$ for any function $\varphi \in \mathcal{C}^\infty_0(I)$ with values in $\mathbb{R}^+$.

Proposition 1.3.4 If $T \in \mathcal{D}'(I)$ is non-negative, then it is of order 0.

Proof.— Let $K \subset I$ be a compact set, and $\chi \in \mathcal{C}^\infty_0(I)$ a plateau function above K. For $\varphi \in \mathcal{C}^\infty_K(I)$ with real values, one has

$$\forall x \in I,\ -\chi \sup |\varphi| \leq \varphi(x) \leq \chi \sup |\varphi|,$$

thus

$$\langle T, \varphi + \chi \sup |\varphi| \rangle \geq 0 \text{ and } \langle T, \chi \sup |\varphi| - \varphi \rangle \geq 0.$$

This gives

$$|\langle T, \varphi \rangle| \leq |\langle T, \chi \rangle| \sup |\varphi|.$$

If $\varphi \in \mathcal{C}^\infty_K(I)$ is complex valued, we write $\varphi = \varphi_1 + i\varphi_2$ with φ_1, φ_2 real-valued, and, with what we have seen before,

$$|\langle T, \varphi \rangle| = |\langle T, \varphi_1 + i\varphi_2 \rangle| \leq |\langle T, \varphi_1 \rangle| + |\langle T, \varphi_2 \rangle| \leq C \sup |\varphi_1| + C \sup |\varphi_2| \leq C \sup |\varphi|,$$

and this concludes the proof of the proposition. □

Notice that non-negative distributions are therefore continuous linear forms on $\mathcal{C}^0(I)$, that is non-negative Radon measures.

1.4 Derivatives of distributions

Proposition 1.4.1 Let $T \in \mathcal{D}'(I)$. The linear form on $\mathcal{D}(I)$ defined by

$$\varphi \mapsto \langle T, -\varphi' \rangle$$

is a distribution, that we call the derivative of T, and that we denote by T'.

Proof.— Let $K \subset I$ be a compact set, and $C = C_K > 0$, $m = m_K \in \mathbb{N}$ the constants given by the fact that $T \in \mathcal{D}'(I)$. For $\varphi \in \mathcal{C}_K^\infty(I)$, we have

$$|\langle T', \varphi \rangle| = |\langle T, \varphi' | \leq C_K \sum_{\alpha \leq m} \sup |\varphi^{(1+\alpha)}| \leq C \sum_{\alpha \leq m+1} \sup |\varphi^{(\alpha)}|,$$

which shows that T' is a distribution. □

Notice that if $T \in \mathcal{D}'(I)$ is of order m, then T' is of order $m+1$ at most. However, to differentiate a distribution do not always increase the order, as in the case of regular distribution, for example:

Proposition 1.4.2 If $T = T_f$ with $f \in \mathcal{C}^1(\overline{I})$, then $T' = T_{f'}$.

Proof.— For $\varphi \in \mathcal{C}_0^\infty(I)$, integrating by parts we have

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = -\int_I f(x)\varphi'(x)dx = \int_I f'(x)\varphi(x)dx = \langle T_{f'}, \varphi \rangle.$$

□

Since T' is a distribution, it can be derivated. Iterating this idea, one can define the successive derivatives $T^{(k)}$ of T for all $k \in \mathbb{N}$. This means that distributions can be deviated to all order, with the formula

$$\langle T^{(k)}, \varphi \rangle = (-1)^k \langle T, \varphi^{(k)} \rangle.$$

Example 1.4.3 Let $H : \mathbb{R} \to \mathbb{C}$ denote the Heaviside function, namely $H(x) = \mathbf{1}_{\mathbb{R}^+}(x)$. The function H belongs to $L^1_{loc}(\mathbb{R})$, and we can denote $T = T_H$ the associated regular distribution. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = -\int_0^{+\infty} \varphi'(x)dx = \varphi(0),$$

so that $T' = \delta_0$.

Example 1.4.4 Let $f \in L^1_{loc}(\mathbb{R})$ be the function given by $f(x) = \ln(|x|)$, and $T = T_f \in \mathcal{D}'(\mathbb{R})$ the associated distribution. We want to compute T'; it seems reasonable that T' is related to the function $x \mapsto 1/x$, but this one is not in $L^1_{loc}(\mathbb{R})$.

However, let $A > 0$, and $\varphi \in \mathcal{C}_0^\infty([-A, A])$. We compute

$$\langle T', \varphi \rangle = -\int_{-A}^{A} \ln|x| \; \varphi'(x)dx = -\int_{-A}^{0} \ln(-x)\varphi'(x)dx - \int_0^A \ln(x)\varphi'(x)dx.$$

One can not integrate by parts ($x \mapsto \ln(x)$ is not in $\mathcal{C}^1([0,A])$), but it is an easy consequence of the dominated convergence theorem that

$$\langle T', \varphi\rangle = -\lim_{\varepsilon\to 0}\int_{-A}^{-\varepsilon} \ln(-x)\varphi'(x)dx - \lim_{\varepsilon\to 0}\int_{\varepsilon}^{A} \ln(x)\varphi'(x)dx$$

Now we integrate by parts, and we get

$$\langle T', \varphi\rangle = -\lim_{\varepsilon\to 0}\left(\ln(\varepsilon)(\varphi(-\varepsilon)-\varphi(\varepsilon)) - \int_{\varepsilon<|x|<A}\frac{\varphi(x)}{x}dx\right).$$

Eventually, Hadamard's lemma gives $\ln(\varepsilon)(\varphi(-\varepsilon)-\varphi(\varepsilon)) = \varepsilon\ln(\varepsilon)(\psi(-\varepsilon)+\psi(\varepsilon))$, where ψ is the smooth function such that $\varphi(x)=\varphi(0)+x\psi(x)$. Therefore

$$\langle T', \varphi\rangle = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<A}\frac{\varphi(x)}{x}dx = \langle \mathrm{vp}(1/x), \varphi\rangle,$$

so that $T' = \mathrm{pv}(1/x)$.

The only functions that have the null function as derivative are constant functions. For distributions in 1d, the same result holds true:

> **Proposition 1.4.5** If $T\in\mathcal{D}'(\mathbb{R})$ satisfies $T'=0$, then T is a regular distribution associated with a constant function.

Proof.— We start the proof by a remark that may be useful in other contexts. A function $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$ is the derivative of a function $\psi\in\mathcal{C}_0^\infty(\mathbb{R})$ if and only if $\int\varphi=0$. Indeed, if $\varphi=\psi'$ for a compactly supported ψ, then $\int_{-\infty}^{+\infty}\varphi(x)dx=\int_{-\infty}^{+\infty}\psi'(x)dx=0$. Conversely, if $\int\varphi=0$, the function $\psi : x\mapsto \int_{-\infty}^{x}\varphi(t)dt$ is compactly supported, with support included in that of φ, and satisfies $\psi'=\varphi$.

Now let $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$ a function such that $\int\chi=1$. For $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$, we have

$$\varphi=\varphi_1+\varphi_2 \text{ with } \varphi_1=\varphi-(\int\varphi)\chi, \text{ and } \varphi_2=(\int\varphi)\chi,$$

and $\langle T,\varphi\rangle=\langle T,\varphi_1\rangle+\langle T,\varphi_2\rangle$. Since $\int\varphi_1=0$, there exists $\psi\in\mathcal{C}_0^\infty(\mathbb{R})$ such that $\varphi_1=\psi'$. Thus

$$\langle T,\varphi\rangle=\langle T,\psi'\rangle+\big(\int\varphi\big)\langle T,\chi\rangle=-\langle T',\psi\rangle+\big(\int\varphi\big)\langle T,\chi\rangle=C\int\varphi=\langle T_C,\varphi\rangle,$$

where $C=\langle T,\chi\rangle$ is a constant, independant of φ. □

Note that Proposition 1.4.2 says in particular that the derivative of a regular distribution associated to a constant function is null. Therefore, the above statement is an equivalence.

1.5 Product by a smooth function

Proposition 1.5.1 Let $T \in \mathcal{D}'(I)$, and $f \in \mathcal{C}^\infty(I)$. The linear form

$$\varphi \mapsto \langle T, f\varphi \rangle$$

is a distribution, and we denote it fT.

Proof.— Suppose (φ_j) is a sequence of functions in $\mathcal{C}_0^\infty(I)$, that converges to 0 in the $\mathcal{D}(I)$-sense. There is a compact $K \subset\subset I$ such that supp $\varphi_j \subset K$ for all j, which implies supp $f\varphi_j \subset K$ for all j. Moreover, for any $\alpha \in \mathbb{N} \cup \{0\}$,

$$(f\varphi)^{(\alpha)} = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f^{(\beta)}\, \varphi^{(\alpha-\beta)}.$$

so that if we denote

$$M = \max_{\beta \leq \alpha} \sup_K |f^{(\beta)}|,$$

we see that

$$\sup |(f\varphi_j)^{(\alpha)}| \leq M \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sup |\varphi_j^{(\alpha-\beta)}|.$$

Since each of the terms in the sum tends to 0 as $j \to +\infty$, we have $(f\varphi_j)^{(\alpha)} \to 0$ in $\mathcal{D}(I)$. Therefore, since T is a distribution, $\langle T, f\varphi_j \rangle \to 0$. Proposition 1.2.2 shows that fT is a distribution on I. □

Exercise 1.5.2 For $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, show $f\delta_0 = f(0)\delta_0$.

Exercise 1.5.3 For $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$, show $fT_g = T_{fg}$.

Exercise 1.5.4 Show that $x\,\mathrm{pv}(1/x) = 1$.

As for the product of two smooth functions, there is a Leibniz formula for the product of a distribution by a function. We leave the proof of the following result to the reader.

Proposition 1.5.5 For $f \in \mathcal{C}^\infty(I)$, and $T \in \mathcal{D}'(I)$, we have

$$\forall \alpha \in \mathbb{N},\ (fT)^{(\alpha)} = \sum_{\beta \leq\, \alpha} \binom{\alpha}{\beta} f^{(\beta)}\, T^{(\alpha-\beta)}.$$

To finish with, we solve a (very simple) differential equation in $\mathcal{D}'(\mathbb{R})$.

Proposition 1.5.6 Let $I \subset \mathbb{R}$ be an open interval, and $a \in \mathcal{C}^\infty(I)$. The distributions in $\mathcal{D}'(I)$ that satisfy the differential equation

$$T' + aT = 0$$

are exactly the $\mathcal{C}^1$ solutions, that is the regular distributions T_f with $f : x \mapsto Ce^{-A(x)}$, for some constant $C \in \mathbb{C}$, where A is primitive of a in I.

Proof.— Let A be a primitive of a on I. For $T \in \mathcal{D}'(I)$, we have, using Leibniz formula,

$$(e^A T)' = ae^A T + e^A T' = e^A(T' + aT).$$

Thus

$$T' + aT = 0 \iff (e^A T)' = 0 \iff e^A T = T_C \iff T = e^{-A} T_C = T_{Ce^{-A}}.$$

□

Exercise 1.5.7 Solve in $\mathcal{D}'(I)$ the inhomogeneous equation $T' + aT = f$, for $f \in \mathcal{C}^\infty(I)$.

1.6 Support of a distribution

1.6.1 Smooth partition of unity

Proposition 1.6.1 Let $K \subset\subset \mathbb{R}$ be a compact set. Suppose that $K \subset \bigcup_{j=1}^N V_j$, where the V_j's are open subsets. Then there exist functions χ_j, $j = 1, \dots N$, such that

i) $\chi_j \in \mathcal{C}_0^\infty(V_j)$,

ii) $\sum_{j=1}^N \chi_j = 1$ in a neighborhood of K.

Proof.— First of all, there exist compact sets $K_j \subset \mathbb{R}$ such that $K_j \subset V_j$ and $K \subset \bigcup_{j=1}^N K_j$. Indeed, for $x \in K$, there exists a $j \in \{1, \dots, N\}$ and $r > 0$ such that $B(x, 2r) \subset V_j$. Denote then $B_x = B(x, r)$, so that $\overline{B_x} \subset V_j$. We have $K \subset \bigcup_{x \in K} B_x$, thus there exist B_{x_1}, ...,B_{x_p} such that

$$K \subset \bigcup_{j=1}^p B_{x_j}.$$

Then, for $j \in \{1, \dots, N\}$, we denote

$$A_j = \{\ell \in \{0, \dots, p\}, \overline{B_{x_\ell}} \subset V_j\}, \quad K_j = \bigcup_{\ell \in A_j} \overline{B_{x_\ell}} \cap K,$$

and the K_j satisfy the required properties.

Now for $j \in \{1, \dots, N\}$, we choose a plateau function $\psi_j \in \mathcal{C}_0^\infty(V_j)$ above K_j. In a neighborhood V of K, we have $\sum_{j=1}^N \psi_j \geq 1$. Then if $\theta \in \mathcal{C}_0^\infty(V)$ is a plateau function above K, and $\psi_0 = 1 - \theta$, we have

$$\Psi = \sum_{j=0}^N \psi_j \neq 0 \text{ on } \mathbb{R}.$$

Thus the functions $\chi_j = \psi_j / \Psi$ are $\mathcal{C}^\infty$ and satisfies the two properties of the proposition. □

1.6.2 Definitions

Definition 1.6.2 Let $T \in \mathcal{D}'(I)$, and $V \subset I$ an open set. We say that T vanishes in V when

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega),\ \operatorname{supp} \varphi \subset V \Rightarrow \langle T, \varphi \rangle = 0.$$

Proposition 1.6.3 Let (V_j) be a family of open subsets of I, and $V = \bigcup_j V_j$. If $T \in \mathcal{D}'(I)$ vanishes in each of the V_j's, then T vanishes in V.

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(I)$ such that $K = \operatorname{supp} \varphi \subset V$. Since K is compact, one can find $j_1, j_2, \dots, j_N$ such that $K \subset \bigcup_{k=1}^N V_{j_k}$. Then, let (χ_k) be an associated, smooth partition of unity. We have $\varphi = \sum_{k=1}^N \varphi \chi_k$ and

$$\langle T, \varphi \rangle = \sum_{k=1}^N \langle T, \varphi \chi_k \rangle = 0,$$

since $\operatorname{supp} \varphi \chi_k \subset V_k$. □

Definition 1.6.4 The support of a distribution $T \in \mathcal{D}'(I)$ is the complement of the largest open set (that is: the union of all the open sets) where T vanishes. We denote it $\operatorname{supp} T$.

Notice that $\operatorname{supp} T$ is closed, and the following characterizations are convenient:

- $x_0 \notin \operatorname{supp} T$ if and only if there is a neighborhood V of x_0 such that $\langle T, \varphi \rangle = 0$ for any $\varphi \in \mathcal{C}_0^\infty(V)$.
- $x_0 \in \operatorname{supp} T$ if and only if for any neighborhood V of x_0, one can find $\varphi \in \mathcal{C}_0^\infty(V)$ such that $\langle T, \varphi \rangle \neq 0$.

Example 1.6.5 *i)* Let $T = \delta_0$. If V is an open set that does not contain $\{0\}$, then $\langle T, \varphi \rangle = \varphi(0) = 0$ for any $\mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \varphi \subset V$. Thus $\operatorname{supp} T \subset \{0\}$. On the other hand, if v is an open set that contains 0, it contains an open ball of the form $B(0, 2r)$. We can then find a plateau function ψ over $\overline{B(0,r)}$ in $\mathcal{C}_0^\infty(V)$, and for this function we have $\langle T, \psi \rangle = \psi(0) = 1$. Therefore we have $0 \in \operatorname{supp} T$ and finally $\operatorname{supp} T = \{0\}$.

ii) If $T = T_f$ for some $f \in \mathcal{C}^0(I)$, with I an open subset of $\mathbb{R}$, we have $\operatorname{supp} T = \operatorname{supp} f$. Indeed, suppose that $x_0 \notin \operatorname{supp} f$. There is a neighborhood V of x_0 such that $f_{|V} = 0$. For $\varphi \in \mathcal{C}_0^\infty(V)$, we have thus $\langle T_f, \varphi \rangle = 0$, so that T_f vanishes on V, and $x_0 \notin \operatorname{supp} T_f$. Reciprocally, if $x_0 \notin \operatorname{supp} T_f$, there is a neighborhood V of x_0 such that, for all $\varphi \in \mathcal{C}_0^\infty(V)$, we have $\int f \varphi dx = \langle T_f, \varphi \rangle = 0$. We have seen (in Exercise 1.2.3) that this implies $f = 0$ in V, thus $x_0 \notin \operatorname{supp} f$.

1.6.3 Some properties

Lemma 1.6.6 Let $I \subset \mathbb{R}$ be an open set, and $T \in \mathcal{D}'(I)$.

i) For $k \in \mathbb{N}$, supp $T^{(k)} \subset$ supp T.

ii) For $f \in \mathcal{C}^\infty(I)$, supp $fT \subset$ supp $f \cap$ supp T.

Proof.— Let $x_0 \notin$ supp T. There exists a neighborhood V of x_0 such that for all $\psi \in \mathcal{C}_0^\infty(V)$, $\langle T, \psi\rangle = 0$. But if $\varphi \in \mathcal{C}_0^\infty(V)$, $\psi = \varphi^{(k)} \in \mathcal{C}_0^\infty(V)$, thus

$$\langle T^{(k)}, \varphi\rangle = (-1)^k \langle T, \varphi^{(k)}\rangle = 0.$$

Therefore $x_0 \notin$ supp $T^{(k)}$, which proves *i)*. Now we prove the second point. If $x_0 \notin$ supp $f \bigcup$ supp T, x_0 either belongs to (supp $f)^c$ or to (supp $T)^c$. In the first case, there exists a neighborhood V of x_0 such that $f_{|V} = 0$. For $\varphi \in \mathcal{C}_0^\infty(V)$, we have $\langle fT, \varphi\rangle = \langle T, f\varphi\rangle = 0$, thus $x_0 \notin$ supp(fT). In the latter case, there exists a neighborhood V of x_0 such that, for all $\psi \in \mathcal{C}_0^\infty(V)$, $\langle T, \psi\rangle = 0$. For $\varphi \in \mathcal{C}_0^\infty(V)$, $f\varphi$ ias a smooth function, which vanishes out of a compact set included in V. Thus $\langle fT, \varphi\rangle = \langle T, f\varphi\rangle = 0$, and $x_0 \notin$ supp fT. □

The following result is very useful.

Proposition 1.6.7 Let $\varphi \in \mathcal{C}_0^\infty(I)$ and $T \in \mathcal{D}'(I)$. If supp $\varphi \cap$ supp $T = \emptyset$, then $\langle T, \varphi\rangle = 0$.

Proof.— Let $x \in$ supp φ. We have, by assumption, $x \notin$ supp T, thus there is a neighborhood V_x of x on which T vanishes. From the covering of the compact supp φ with the open sets V_x, one can extract a finite covering

$$\text{supp}\,\varphi \subset \bigcup_{j=1}^{N} V_{x_j}.$$

Now let $\chi_1, \chi_2, \dots, \chi_N$ be an associated smooth partition of unity. We have

$$\langle T, \varphi\rangle = \sum_{j=1}^{N} \langle T, \chi_j \varphi\rangle = 0,$$

since$\chi_j \varphi \in \mathcal{C}_0^\infty(V_{x_j})$. □

Be careful: one may have $\varphi = 0$ on supp T and $\langle T, \varphi\rangle \neq 0$. For example, this is the case for $T = \delta_0'$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\varphi(0) = 0$, $\varphi'(0) = 1$.

1.6.4 Compactly supported distributions

For $I \subset \mathbb{R}$ an open set, we denote $\mathcal{E}'(\Omega)$ the vector space of distributions on I with compact support.

Proposition 1.6.8 Any compactly supported distribution has finite order. If $T \in \mathcal{E}'(I)$, with order m, then for any compact neighborhood $\tilde{K}$ of supp T, there is a constant $C > 0$ such that

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega),\ |\langle T, \varphi\rangle| \leq C \sum_{|\alpha|\leq m} \sup_{\tilde{K}} |\partial^\alpha \varphi|.$$

Proof.— Let $\tilde{K}$ be a compact neighborhood of supp T, ie. a compact set such that supp $T \subset \mathring{\tilde{K}}$, and $\chi \in \mathcal{C}_0^\infty(\mathring{\tilde{K}})$ a plateau function above supp T.

Since T is a distribution, there exist $C = C(\tilde{K}) > 0$, $m = m(\tilde{K}) \in \mathbb{N}$ such that

$$\forall \psi \in \mathcal{C}_0^\infty(\Omega),\ \text{supp}\, \psi \subset \tilde{K} \Rightarrow |\langle T, \psi\rangle| \leq\ C \sum_{|\alpha|\leq m} \sup |\partial^\alpha \psi|,$$

and from now on we denote m the smallest integer for which this property holds.

For any function $\varphi \in \mathcal{C}_0^\infty(\Omega)$, since supp $\varphi - \chi\varphi \cap$ supp $T = 0$, we have

$$\langle T, \varphi\rangle = \langle T, \chi\varphi\rangle + \langle T, \varphi - \chi\varphi\rangle = \langle T, \chi\varphi\rangle.$$

Since supp$(\chi\varphi) \subset \tilde{K}$, we have

$$|\langle T, \varphi\rangle| \leq |\langle T, \chi\varphi\rangle| \leq C \sum_{|\alpha|\leq m} \sup |\partial^\alpha(\chi\varphi)|.$$

At last, Leibniz's formula gives that

$$\sup |\partial^\alpha(\chi\varphi)| = \sup_{\tilde{K}} |\partial^\alpha(\chi\varphi)| \leq C \sum_{\beta\leq\alpha} \sup_{\tilde{K}} |\partial^\beta \varphi|,$$

which ends the proof of the proposition. □

Be careful: as it follows from the exercise below, one can not in general replace the compact set $\tilde{K}$ by supp T in the estimate of Proposition 1.6.8.

Exercise 1.6.9 Show that the linear form T on $\mathcal{C}_0^\infty(\mathbb{R})$ given by

$$T(\varphi) = \sum_{n\geq 0} \frac{1}{n}(\varphi(\frac{1}{n}) - \varphi(0))$$

is a distribution. Give its support K, and show that T is of order 1.

Show that there is no constant $C > 0$ such that, for any function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, it holds that

$$|\langle T, \varphi\rangle| \leq C\|\varphi\|_{\mathcal{C}^1(K)}.$$

Hint: Test this inequality on suitable plateau functions.

The next result permits us to identify $\mathcal{E}'(I)$ with the space of continuous linear forms on $\mathcal{C}^\infty(I)$.

Proposition 1.6.10 Let $T \in \mathcal{E}'(I)$, and $m \in \mathbb{N}$ its order. Let $\chi \in \mathcal{C}_0^\infty(I)$ be a plateau function above supp T. We denote $\tilde{T} : \mathcal{C}^\infty(\Omega) \to \mathbb{C}$ the linear form given by

$$\tilde{T}(f) = \langle T, \chi f \rangle.$$

Then

i) $\tilde{T}$ does not depend on the choice of χ.

ii) For $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\tilde{T}(\varphi) = \langle T, \varphi \rangle$.

iii) $\tilde{T}$ is continuous on $\mathcal{C}^\infty(\Omega)$, in the sense that

$$\exists \tilde{K} \subset \Omega, \exists C > 0, \forall f \in \mathcal{C}^\infty(\Omega), |\tilde{T}(f)| \leq C \sum_{\alpha \leq m} \sup_{\tilde{K}} |\partial^\alpha f|.$$

Last, if $L : \mathcal{C}^\infty(\Omega) \to \mathbb{C}$ is a linear form satisfying *ii)* and *iii)*, then $L = \tilde{T}$.

Notice that (iii) implies that, if $(\varphi_j) \in \mathcal{C}^\infty(\Omega)$ converges uniformly to $\varphi \in \mathcal{C}^\infty(\Omega)$ on every compact subset of Ω, and this is also true for their derivatives at all order, then $\langle T, \varphi_j \rangle \to \langle T, \varphi \rangle$.

Proof.— Let χ_1, χ_2 be two plateau functions above supp T. For $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\langle T, \chi_1 \varphi \rangle - \langle T, \chi_2 \varphi \rangle = \langle T, (\chi_1 - \chi_2)\varphi \rangle = 0,$$

since $\text{supp}(\chi_1 - \chi_2)\varphi \cap \text{supp}\, T = \emptyset$. The same way, $\varphi \in \mathcal{C}_0^\infty(\Omega)$, since $\text{supp}(1 - \chi)\varphi \cap \text{supp}\, T = \emptyset$,

$$\tilde{T}(\varphi) = \langle T, \chi \varphi \rangle = \langle T, \chi \varphi \rangle + \langle T, (1 - \chi)\varphi \rangle = \langle T, \varphi \rangle.$$

Thus (i) and (ii) are proved. Now let $\tilde{K}$ be a compact neighborhood of supp T. Since T is a distribution of order m, there exists $C = C(\tilde{K}) > 0$ such that

$$\forall \psi \in \mathcal{C}_0^\infty(\Omega),\ \text{supp}\, \psi \subset \tilde{K} \Rightarrow |\langle T, \psi \rangle| \leq\ C \sum_{|\alpha| \leq m} \sup |\partial^\alpha \psi|.$$

For $f \in \mathcal{C}^\infty(\Omega)$, supp $\chi f \subset \tilde{K}$, and

$$|\tilde{T}(f)| = |\langle T, \chi f \rangle| \leq C \sum_{|\alpha| \leq m} \sup |\partial^\alpha (\chi f)| \leq C' \sum_{|\alpha| \leq m} \sup_{\tilde{K}} |\partial^\alpha f|,$$

which proves iii).

Eventually, suppose L satisfies ii) and iii). Thanks to iii), if $f \in \mathcal{C}^\infty(I)$ satisfies supp $f \cap \tilde{K} = \emptyset$, we have $L(f) = 0$. Thus let $\chi \in \mathcal{C}_0^\infty(I)$ be a plateau function above $\tilde{K}$. We have

$$L(f) = L(\chi f) + L((1 - \chi)f) = L(\chi f) = \langle T, \chi f \rangle,$$

using ii). □

Eventually, we notice that a distribution T with compact support in $\Omega \subset \mathbb{R}^n$ can be extended to a distribution (with compact support) $\tilde{T}$ on $\mathbb{R}^n$. For $T \in \mathcal{E}'(\Omega)$, it suffices to set, for $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$,

$$\langle \tilde{T}, \varphi \rangle = \langle T, \varphi_{|\Omega} \rangle.$$

This remark may be of importance in particular if one wants to convolve or Fourier transform a distribution in $\mathcal{E}'(\Omega)$.

1.6.5 Distributions supported at one point only

Let $x_0 \in \mathbb{R}$. We are interested in distributions $T \in \mathcal{D}'(\mathbb{R})$ such that supp $T \subset \{x_0\}$. Without loss of generality, we suppose that $x_0 = 0$.

We need a technical result, which has an interest on his own.

Proposition 1.6.11 Let $T \in \mathcal{D}'(I)$ be a distribution of order m, with support in a compact set K. If $\varphi \in \mathcal{C}^\infty(I)$ satisfies

$$\forall k \in \mathbb{N}, \forall x \in K,\ k \leq m \Rightarrow \varphi^{(k)}(x) = 0,$$

then $\langle T, \varphi \rangle = 0$.

Proof.— We proceed in two steps.

– Let $\varphi \in \mathcal{C}_0^\infty(I)$, and $K \subset I$ a compact set such that

$$\forall \alpha \in \mathbb{N},\ \alpha \leq m \Rightarrow (\varphi^{(\alpha)})_{|K} = 0.$$

For all $\varepsilon > 0$, and all $\alpha \leq m$, setting $K_\varepsilon = K + \overline{B(0, \varepsilon)}$, we have

$$\sup_{K_\varepsilon} |\varphi^{(\alpha)}| = \mathcal{O}(\varepsilon^{m-\alpha+1}).$$

Indeed, for a given $\alpha \in \mathbb{N}$, there exists $x_\varepsilon \in K_\varepsilon$ such that

$$\sup_{K_\varepsilon} |\varphi^{(\alpha)}| = |\varphi^{(\alpha)}(x_\varepsilon)|.$$

Then one can find $x_0 \in K$ such that $|x_\varepsilon - x_0| \leq \varepsilon$, and Taylor's formula for $\varphi^{(\alpha)}$ at order $m - \alpha$ between x_0 and x_ε gives

$$\begin{aligned}\varphi^{(\alpha)}(x_\varepsilon) = &\sum_{\beta \leq m-\alpha} \frac{(x_\varepsilon - x_0)^\beta}{\beta!} \varphi^{(\alpha+\beta)}(x_0) \\ &+ \frac{(x_\varepsilon - x_0)^{(m-\alpha+1)}}{(m-\alpha)!} \int_0^1 (1-t)^{(m-\alpha)} \varphi^{(m+1)}(tx_\varepsilon + (1-t)x_0) dt.\end{aligned}$$

Since $x_0 \in K$, all the terms in the sum vanish, and, as stated,

$$|\varphi^{(\alpha)}(x_\varepsilon)| \leq \frac{\varepsilon^{(m-\alpha+1)}}{(m-\alpha+1)!} \sup |\varphi^{(m+1)}| = \mathcal{O}(\varepsilon^{m-\alpha+1}).$$

– Now let $\chi \in \mathcal{C}_0^\infty(]-1,1[)$ be a plateau function above $[-1/2,1/2]$ and, for any $\varepsilon > 0$, set $\chi_\varepsilon(x) = \chi(\frac{x}{\varepsilon})$. Since $\text{supp}(1-\chi_\varepsilon) \cap \text{supp}\, T = \emptyset$, we have, for any $\varepsilon > 0$,

$$\langle T, \varphi \rangle = \langle T, \chi_\varepsilon \varphi \rangle.$$

Since T is a distribution of order m, and because $\text{supp}\, \chi_\varepsilon \varphi \subset [-1,1]$ for any $\epsilon < 1$, there exists $C > 0$ such that, for any $\varepsilon > 0$,

$$|\langle T, \chi_\varepsilon \varphi \rangle| \leq C \sum_{k \leq m} \sup_{[-\varepsilon,\varepsilon]} |(\chi_\varepsilon \varphi)^{(k)})|.$$

Then, using Leinitz formula, we see that there exists a constant $M > 0$, such that

$$|\langle T, \varphi \rangle| \leq M \sum_{k \leq m} \sup_{[-\varepsilon,\varepsilon]} |\varphi^{(k)}| \leq M\varepsilon,$$

thanks to the above estimates. Since this holds for any $\varepsilon > 0$, we have $\langle T, \varphi \rangle = 0$ as claimed. □

The main result of this section is the

Proposition 1.6.12 Let $T \in \mathcal{D}'(I)$, with $T \neq 0$, and $x_0 \in I$. If $\text{supp}\, T \subset \{x_0\}$, there exists $N \in \mathbb{N}$ and $N+1$ complex numbers a_k for $0 \leq k \leq N$ such that $a_N \neq 0$ and

$$T = \sum_{|\alpha| \leq N} a_k \delta_{x_0}^{(k)}.$$

Proof.— We write the proof for $x_0 = 0$. Such a distribution T has compact support, thus is of finite order, and we denote N its exact order. Let $\chi \in \mathcal{C}_0^\infty(I)$ be a plateau function above $\{0\}$. For $\varphi \in \mathcal{C}_0^\infty(I)$ we have

$$\varphi(x) = \sum_{k \leq N} \frac{x^k}{k!} \varphi^{(k)}(0) \chi(x) + r(x),$$

where

$$r(x) = (1-\chi(x)) \sum_{k \leq N} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^{N+1}}{N!} \int_0^1 (1-t)^N \varphi^{(N+1)}(tx) dt.$$

Since the function r vanishes at order N at 0, the previous proposition asserts that

$$\langle T, \varphi \rangle = \sum_{k \leq N} \varphi^{(k)}(0) \langle T, \frac{x^k}{k!} \chi(x) \rangle.$$

This is precisely what we have claimed, if we set $a_k = \langle T, \frac{x^k}{k!} \chi(x) \rangle$. At last, if $a_N = 0$ then T would be of order $\leq N-1$. □

The coefficients a_k in this decomposition are unique. Indeed

Proposition 1.6.13 Let $x_0 \in I$, and $N \in \mathbb{N}$. The distributions $(\delta_{x_0}^{(k)})_{k \leq N}$ are linearly independent in $\mathcal{D}'(I)$.

Proof.— Again, we write the proof for $x_0 = 0$. Suppose that $\sum_{k\leq N} a_k\delta_0^{(k)} = 0$ for some $a_k \in \mathbb{C}$. Let $\chi \in \mathcal{C}_0^\infty(I)$ a plateau function above $\{0\}$, and $j \in \mathbb{N}$. We compute

$$\langle \delta_0^{(k)}, x^j\chi\rangle = (-1)^k[(x^j\chi)^{(k)}]_{|x=0}$$

But $[(x^\beta\chi)^{(k)}]_{|x=0}$ vanishes if $k \neq j$, and amounts to $k!$ otherwise. Thus, for any $k \leq N$,

$$0 = \langle \sum_{k\leq N} a_k\delta_0^{(k)}, x^j\chi\rangle = (-1)^j j! a_j,$$

which shows that all the a_k's are zero. □

1.7 Sequences of distributions

1.7.1 Convergence in $\mathcal{D}'$

Definition 1.7.1 Let (T_j) be a sequence of distributions in $\mathcal{D}'(I)$. We say that (T_j) converges to $T \in \mathcal{D}'(I)$ when, for any function $\varphi \in \mathcal{C}_0^\infty(\Omega)$, the sequence of complex numbers $(\langle T_j, \varphi\rangle)$ converges to$\langle T, \varphi\rangle$.

Example 1.7.2 Let $T_k \in \mathcal{D}'(\mathbb{R})$ be the distribution associated to the function $x \mapsto e^{ikx}$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and $A > 0$ a real number such that $\operatorname{supp}\varphi \subset [-A, A]$, we have

$$\langle T_k, \varphi\rangle = \int_{-A}^{A} e^{ikx}\varphi(x)dx \to 0$$

by Riemann-Lebesgue's Lemma (or more simply, here, integrating by parts). Thus the sequence (T_k) converges to 0 in $\mathcal{D}'(\mathbb{R})$.

Example 1.7.3 Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\int \chi = 1$, and (χ_ε) the sequence of functions defined by

$$\chi_\varepsilon(x) = \varepsilon^{-1}\chi(\frac{x}{\varepsilon}).$$

We also denote $(T_\varepsilon) = (T_{\chi_\varepsilon})$ the associated family of distributions. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, with $A > 0$ a real number such that $\operatorname{supp}\varphi \subset [-A, A]$, we have

$$\langle T_\varepsilon, \varphi\rangle = \int_{[-A,A]^n} \chi_\varepsilon(x)\varphi(x)dx = \int_{[-A/\varepsilon, A/\varepsilon]^n} \chi(y)\varphi(\varepsilon y)dy.$$

By the Dominated Convergence Theorem, we see that $\langle T_\varepsilon, \varphi\rangle \to \varphi(0)$ as $\varepsilon \to 0$. Therefore the sequence (χ_ε) converges to δ_0 in $\mathcal{D}'(\mathbb{R})$.

Exercise 1.7.4 (see also Exercise 1.A.8.) Show that the sequence $(\frac{1}{x \pm i\varepsilon})$ converges to $\text{vp}(\frac{1}{x}) \mp i\pi\delta_0$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0^+$. We shall denote

$$\frac{1}{x - i0} = \text{vp}(\frac{1}{x}) + i\pi\delta_0 \text{ and } \frac{1}{x + i0} = \text{vp}(\frac{1}{x}) - i\pi\delta_0.$$

The operations that we have defined on distributions are continuous with respect to the notion of convergence. More precisely

Proposition 1.7.5 If (T_j) converges to T in $\mathcal{D}'(I)$, then

i) For any $k \in \mathbb{N}$, $(T_j^{(k)})$ converges to $T^{(k)}$.

ii) For any $f \in \mathcal{C}^\infty(I)$, (fT_j) converges to fT.

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(I)$. We have clearly

$$\langle \partial^\alpha T_j, \varphi \rangle = (-1)^{|\alpha|} \langle T_j, \partial^\alpha \varphi \rangle \to (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle = \langle \partial^\alpha T, \varphi \rangle,$$

and

$$\langle fT_j, \varphi \rangle = \langle T_j, f\varphi \rangle \to \langle T, f\varphi \rangle = \langle fT, \varphi \rangle.$$

□

Exercise 1.7.6 Show that if (f_j) converges to f in $\mathcal{C}^\infty(I)$, then $(f_j T)$ converges to fT in $\mathcal{D}'(I)$.

1.7.2 Uniform Boundedness Principle

We state below, without proof, an important theoretical result, which is a "distribution version" of the well-known Banach-Steinhaus theorem.

Proposition 1.7.7 Let (T_j) be a sequence in $\mathcal{D}'(I)$, and $K \subset\subset I$ a compact subset. If, for any function $\varphi \in \mathcal{C}_0^\infty(I)$ with support in K, the sequence $(\langle T_j, \varphi \rangle)$ converges, then there exists $C > 0$ and $m \in \mathbb{N}$, independent of j, such that

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega), \ \text{supp}\, \varphi \subset K \Rightarrow |\langle T_j, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi|.$$

As a matter of fact, we will only use the following

Corollary 1.7.8 Let (T_j) be a sequence of distributions on I. If, for all functions $\varphi \in \mathcal{C}_0^\infty(I)$, the sequence $(\langle T_j, \varphi \rangle)$ converges in $\mathbb{C}$, then there exists a distribution $T \in \mathcal{D}'(I)$ such that $(T_j) \to T$ in $\mathcal{D}'(I)$.

Proof.— Let $T : \mathcal{C}_0^\infty(\Omega) \to \mathbb{C}$ be the linear form given by

$$T(\varphi) = \lim_{j\to+\infty} \langle T_j, \varphi\rangle.$$

We want to show that T is a distribution. So take $K \subset I$ a compact subset. Proposition 1.7.7 ensures that there is a constant $C > 0$ and a natural number m such that, for any $\varphi \in \mathcal{C}_0^\infty(I)$ with $\operatorname{supp}\varphi \subset K$, we have

$$|\langle T_j, \varphi\rangle| \leq C \sum_{|\alpha|\leq m} \sup |\partial^\alpha \varphi|.$$

Then we can pass to the limit $j \to +\infty$, and we get the required estimate. □

1.A Exercises

Exercice 1.A.1 Show that the following expressions, where $\varphi \in C_0^\infty(\mathbb{R})$, define distributions on $\mathbb{R}$:

$$\int_{\mathbb{R}} \varphi(x^2)dx, \quad \int_{\mathbb{R}} \varphi'(x)e^{x^2}dx, \quad \int_0^\pi \varphi'(x)\cos x\, dx, \quad \int_0^\infty \varphi'(x)\ln(x)dx.$$

Are these distributions regular ones? If not, give their order and their support.

Exercice 1.A.2 Show that the following linear form is a distribution on $\mathbb{R}$, that we denote $\mathrm{Fp}(\frac{1}{x^2})$:

$$\varphi \mapsto \lim_{\varepsilon\to 0+} \Big(\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}dx - 2\frac{\varphi(0)}{\varepsilon}\Big)$$

Find a link between $\mathrm{pv}(\frac{1}{x})$ and $\mathrm{Fp}(\frac{1}{x^2})$.

Exercice 1.A.3 For $n \in \mathbb{N}$, compute the successive derivatives of $\dfrac{x^n}{n!}H(x)$ in $\mathcal{D}'(\mathbb{R})$, where H is the Heaviside function. For $\alpha \in]0,1[$, compute the derivatives of the distribution associated to the function $x \mapsto |x|^\alpha H(x)$.

Exercice 1.A.4 1. Let $p \in \mathbb{N}$. Solve in $\mathcal{D}'(\mathbb{R})$ the equation $x^p T = 0$.

2. Solve in $\mathcal{D}'(\mathbb{R})$ the equations

a) $xT = 1$.

b) $x^2 T = 1$.

Exercice 1.A.5 We want to solve in $\mathcal{D}'(\mathbb{R})$ the differential equation

$$(E) \qquad x^2 T' + T = 0.$$

1. What are the solutions of (E) in $\mathcal{D}'(]0,+\infty[)$? In $\mathcal{D}'(\mathbb{R}^*)$?

2. Show that there is no distribution $T \in \mathcal{D}'(\mathbb{R})$ such that, for all $\varphi \in C_0^\infty(]0,+\infty[)$, we have

$$\langle T, \varphi\rangle = \int_0^{+\infty} \mathrm{e}^{1/x}\varphi(x)\, dx\ .$$

Hint. Use a sequence of functions (φ_n) of the form $\varphi_n(x) = \varphi(nx)$, where φ is a C^∞ function with suitably chosen support.

3. What are the restrictions to $\mathbb{R}^*$ of the solutions of (E) on $\mathbb{R}$?

4. Give all the solutions to (E) with support included in $\{0\}$.

5. Find all the solutions of (E) in $\mathcal{D}'(\mathbb{R})$.

Exercice 1.A.6 Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}$ by

$$f_n(x) = \frac{nx}{1+nx^2}.$$

Show that (f_n) converges in $\mathcal{D}'(\mathbb{R})$ to a distribution. Which one is it?

Exercice 1.A.7 1. Compute the limit in $\mathcal{D}'(\mathbb{R})$ of the following sequences of distributions :

$$A_n = n^{100}e^{inx}, B_n = \cos^2(nx), C_n = n\sin(nx)H(x), D_n = \frac{1}{n}\sum_{p=0}^{n-1}\delta_{\frac{p}{n}}, E_n = e^{inx}\ \mathrm{pv}\,\frac{1}{x}.$$

2. Show that the sequence (T_n) given by $T_n = n(\delta_{1/n} - \delta_{-1/n})$ converges in $\mathcal{D}'(\mathbb{R})$. Compare the order of the T_n the order of the limit of (T_n).

Exercice 1.A.8 Show that, for any $\varphi \in \mathcal{D}(\mathbb{R})$, the limit $\lim_{\varepsilon\to 0+}\int_{\mathbb{R}}\frac{\varphi(x)}{x-i\varepsilon}dx$ exists, and defines a distribution.

Exercice 1.A.9 Consider the linear form T on $\mathcal{C}_0^\infty(\mathbb{R})$ given by

$$T : \varphi \mapsto \sum_{k=1}^{+\infty}\frac{1}{\sqrt{k}}(\varphi(\frac{1}{k}) - \varphi(-\frac{1}{k})).$$

1. Show that T is a distribution, of order 1.

2. Show that T is not of order 0.

3. Show that the support of T is $S = \{0\} \cup \{\frac{1}{k},\ k \in \mathbb{Z}^*\}$.

4. Find a sequence (φ_j) of functions in $\mathcal{C}_0^\infty(\mathbb{R})$ such that, as $j \to +\infty$,

- for all $p \in \mathbb{N}^*$, $\varphi_j^{(p)} \to 0$ uniformly on S,
- $\langle T, \varphi_j\rangle \not\to 0$.

Compare with Definition 1.2.1.

Exercice 1.A.10 For any open interval I of $\mathbb{R}$, we denote P_I the linear, differential operator

$$P_I : T \in \mathcal{D}'(I)\ \mapsto x^2T'' + xT' + \left(x^2 - \frac{1}{4}\right)T \in \mathcal{D}'(I)\ .$$

We also denote $P = P_{\mathbb{R}}$, and we are looking for distributions $T \in \mathcal{D}'(\mathbb{R})$ such that $PT = 0$.

1) For $T \in \mathcal{D}'(\mathbb{R})$, we denote T^* the distribution defined by $\langle T^*, \varphi\rangle = \overline{\langle T, \overline{\varphi}\rangle}$. We shall say that T is real when $T = T^*$.

   (a) iF $T = T_f$ with $f \in L^1_{loc}(\mathbb{R})$, what does it mean for f that T is real?

   (b) Show $(T^*)^* = T$. What is $(fT)^*$ for $f \in \mathcal{C}^\infty(\mathbb{R})$?

   (c) Show that any distribution $T \in \mathcal{D}'(\mathbb{R})$ can be written $T = T_1 + iT_2$ where T_1 and T_2 are real. Show that $T = 0$ if and only if $T_1 = T_2 = 0$.

   (d) Explain why it is only necessary to cope with real distributions that satisfy $P_I T = 0$.

We denote Ker(P_I) the space of real distributions $T \in \mathcal{D}'(I)$ such that $P_I T = 0$.

2) (a) For $p, q \in \mathbb{N}$, compute $x^p \delta_0^{(q)}$.

   (b) For $k \in \mathbb{N}$, compute $P(\delta_0^{(k)})$.

   (c) Deduce that there is no non-trivial distribution in Ker(P) with support in $\{0\}$.

3) (a) Let $I \subset \mathbb{R}$ be an interval. Show that the solutions in $\mathcal{D}'(I)$ of the differential equation $U'' + U = 0$ are exactly the $\mathcal{C}^2$ solutions. *Hint:* set $V = e^{ix}U$. What about real solutions?

   (b) For $T \in \mathcal{D}'(]0, +\infty[)$, compute $(x^{1/2}T)''$. Deduce Ker$(P_{]0,+\infty[})$.

   (c) Give Ker$(P_{]-\infty,0[})$.

4) Let $T_+ \in$ Ker$(P_{]0,+\infty[)}$ and $T_- \in$ Ker$(P_{]-\infty,0[})$.

   (a) Show that the following distribution f belongs to $L^1_{loc}(\mathbb{R})$:

$$f(x) = T_+ \quad \text{as} \quad x > 0 \quad ; \quad f(x) = T_- \quad \text{as} \quad x < 0 \ .$$

   (b) Let then $S \in \mathcal{D}'(\mathbb{R})$ defined by $S = PT_f$. What can you sy of the support of S ?

   (c) Let $\chi \in C_0^\infty(\mathbb{R})$ a plateau function above $\{0\}$, with support in $[-1, 1]$. For $\varepsilon > 0$, we set $\chi_\varepsilon(x) = \chi(x/\varepsilon)$. Show that, for any $\varphi \in C_0^\infty(\mathbb{R})$, and all $\varepsilon > 0$, we have

$$\langle S, \chi_\varepsilon \varphi\rangle = \int_{\mathbb{R}} f(x)\psi_\varepsilon(x)\,dx$$

   where ψ_ε is uniformly bounded and supported in $[-\varepsilon, \varepsilon]$. Deduce that $\langle S, \chi_\varepsilon\varphi\rangle \to 0$ as $\varepsilon \to 0$, then deduce that $T_f \in$ Ker(P).

5) (a) Show that Ker(P) is exactly the space of functions f defined at question **(4a)**.

   (b) What is the dimension of Ker(P)? The dimension of the subspaces Ker$(P) \cap C^0(\mathbb{R})$ and Ker$(P) \cap C^1(\mathbb{R})$?

Exercice 1.A.11 We recall that for a function f of one real variable, and $a \in \mathbb{R}$, we denote $\tau_a f$ its translation by a defined as

$$\tau_a f(x) = f(x - a).$$

If $(T_n)_{n\in\mathbb{Z}}$ is a sequence of distributions, one says that the series $\sum_{n\in\mathbb{Z}} T_n$ converges when the sequence (S_N), given by $S_N = \sum_{n=-N}^{N} T_n$, converges. The limit of the sequence (S_N) is then called the sum of the series $\sum_{n\in\mathbb{Z}} T_n$.

Part A.– Let $T \in \mathcal{D}'(\mathbb{R})$. We denote $\tau_a T$ lthe linear form on $\mathcal{C}_0^\infty(\mathbb{R})$ given by

$$\tau_a T(\varphi) = \langle T, \tau_{-a}\varphi\rangle.$$

1. Show that $\tau_a T$ is a distribution. What is $\tau_a T$ when $T = T_f$ for some $f \in L^1_{loc}(\mathbb{R})$?

We shall say that $T \in \mathcal{D}'(\mathbb{R})$ is a-periodic when $\tau_a(T) = T$.

2. Show that the series $\sum_{n\in\mathbb{Z}} \delta_{2\pi n}$ converges in $\mathcal{D}'(\mathbb{R})$, and that its sum W is 2π-periodic.

3. Let us consider a sequence (c_n) of complexe numbers such that

$$\exists C > 0,\ \exists p \in \mathbb{N},\ |c_n| \leq C(1 + |n|)^p. \tag{1.A.1}$$

a) Show that, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have, for $n \in \mathbb{Z}^*$,

$$\int e^{-inx}\varphi(x)dx = \mathcal{O}(n^{-p-2}).$$

b) Deduce that $\sum\limits_{n\in\mathbb{Z}} c_n e^{inx}$ converges in $\mathcal{D}'(\mathbb{R})$, and show that its sum is a 2π-periodic distribution.

Part B.– Recall that if $f \in \mathcal{C}^\infty(\mathbb{R})$ is 2π-periodic, then

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f)e^{inx}, \quad c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx,$$

where the $c_n(f)$ are the Fourier coefficients of f, and where the series converges normally.

1. Let $\omega \in \mathcal{C}_0^\infty(\mathbb{R})$ a non-trivial, non-negative function. Show that $\sum_{k\in\mathbb{Z}} \tau_{-2k\pi}\omega$ converges and define a non-negative smooth function on $\mathbb{R}$.

2. Deduce the existence of a function $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\forall x \in \mathbb{R},\ \sum_{k\in\mathbb{Z}} \tau_{-2k\pi}\psi(x) = \sum_{k\in\mathbb{Z}} \psi(x + 2k\pi) = 1$$

3. Let T be a 2π-periodic distribution. We call Fourier coefficients of T the numbers

$$c_n(T) = \frac{1}{2\pi}\langle T, e^{-inx}\psi\rangle,\ n \in \mathbb{Z}, \tag{1.A.2}$$

a) Show that the sequence $(c_n(T))$ satisfies the property (1.A.1). Therefore, the series $\sum_{n\in\mathbb{Z}} c_n(T)e^{inx}$ converges in $\mathcal{D}'(\mathbb{R})$.

b) For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we denote $\widetilde{\varphi}$ the smooth, 2π-periodic function given by

$$\widetilde{\varphi}(x) = \sum_{k\in\mathbb{Z}} \tau_{2k\pi}\varphi(x) = \sum_{k\in\mathbb{Z}} \varphi(x - 2k\pi).$$

Show

$$\langle T, \varphi\rangle = \langle T, \psi\widetilde{\varphi}\rangle = 2\pi \sum_{n\in\mathbb{Z}} c_{-n}(T)\; c_n(\widetilde{\varphi}).$$

c) Deduce finally that, for any periodic distribution $T \in \mathcal{D}'(\mathbb{R})$, we have, with the $c_n(T)$ given by (1.A.2),

$$T = \sum_{n\in\mathbb{Z}} c_n(T)e^{inx}.$$

Part C.–

1. Compute the Fourier coefficients of the distribution $W = \sum_{n\in\mathbb{Z}} \delta_{2\pi n}$.

2. Deduce Poisson's formula:

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}), \quad \sum_{n\in\mathbb{Z}} \varphi(2n\pi) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \hat{\varphi}(n),$$

where $\hat{\varphi}$ is the Fourier transform of φ:

$$\hat{\varphi}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}\varphi(x)dx.$$

1.B Answers

Answer 1.B.1 1. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and $A > 0$ such that $\operatorname{supp}\varphi \subset [-A, A]$. Since $x \mapsto \varphi(x^2)$ is an even function,

$$T(\varphi) = \int_{\mathbb{R}} \varphi(x^2)dx = 2\int_0^{\sqrt{A}} \varphi(x^2)dx.$$

Then we can make the change of variable $y = x^2$ and we obtain

$$T(\varphi) := \int_0^A \varphi(y)\frac{1}{\sqrt{y}}dy.$$

Thus T is the regular distribution associated with the L^1_{loc} function $f : x \mapsto \dfrac{1}{\sqrt{x}}H(x)$. It is therefore of order 0 and its support is $\operatorname{supp} f = \mathbb{R}^+$.

2. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and $A > 0$ such that $\operatorname{supp}\varphi \subset [-A, A]$. Integrating by parts, we have

$$T(\varphi) := \int_{\mathbb{R}} \varphi'(x)e^{x^2}dx = \int_{-A}^{A} \varphi'(x)e^{x^2}dx = \cancel{\big[\varphi(x)e^{x^2}\big]_{-A}^{A}} - \int_{-A}^{A} 2x\varphi(x)e^{x^2}dx.$$

Therefore T is the regular distribution associated to the smooth function $f : x \mapsto -2xe^{x^2}$.

3. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. We integrate by parts and we get

$$T(\varphi) := \int_0^\pi \varphi'(x)\cos x\, dx = \big[\varphi(x)\cos x\big]_0^\pi + \int_0^\pi \varphi(x)\sin x dx = -\varphi(\pi) - \varphi(0) + \int_0^\pi \varphi(x)\sin x dx.$$

So $T = -\delta_\pi - \delta_0 + T_f$, with $f : x \mapsto \sin x\; \mathbf{1}_{[0,\pi]}(x)$. Thus T is of order 0 (even if it is not a regular distribution), and one can easily show that its support is $[0, \pi]$.

4. Let $A > 0$. For all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that supp $\varphi \subset [-A, A]$, we have

$$|T(\varphi)| \leq \int_0^A |\varphi'(x)|\, |\ln x| dx \leq \sup |\varphi'| \int_0^A |\ln x| dx,$$

so that, for $C_A = \int_0^A |\ln x| dx$,

$$|T(\varphi)| \leq C_A \sum_{k=0}^{1} \sup |\varphi^{(k)}|.$$

Therefore T is a distribution of order 1. Now we show that T is not of order 0 (as we did for pv $1/x$), so that, in particular, it can not be a regular distribution. For each n, we choose $\varphi_n \in \mathcal{C}^\infty_{[0,2]}$ such that $-1 \leq \varphi_n \leq 0$ and

$$\varphi_n(x) = \begin{cases} -1 \text{ for } x \in [\frac{1}{n}, 1], \\ 0 \text{ for } x \notin [\frac{1}{2n}, 2]. \end{cases}$$

Suppose that T is of order 0. Then there would exist $C = C_{[0,2]} > 0$ such that $|\langle T, \varphi_n\rangle| \leq C \sup |\varphi_n| \leq C$. On the other hand, we have, since φ_n and ln are of class $\mathcal{C}^1$ on $[1/2n, 2]$,

$$\langle T, \varphi_n\rangle = \int_{\frac{1}{2n}}^2 \varphi_n'(x) \ln x dx = -\int_{\frac{1}{2n}}^2 \frac{\varphi_n(x)}{x} dx \geq \int_{\frac{1}{n}}^1 \frac{1}{x} dx \geq \ln n.$$

Therefore we should have $\ln n \leq C$ for all n, a contradiction.

Concerning the support of T, it is clear that supp $T \subset \mathbb{R}^+$. Moreover if $x_0 > 0$, one can find $\delta > 0$ such that $]x_0 - \delta, x_0 + \delta[\subset]0, +\infty[$. Then if φ is a plateau function above $\{x_0\}$ in $\mathcal{C}_0^\infty(]x_0 - \delta, x_0 + \delta[)$, we can compute $\langle T, \varphi\rangle$ integrating by parts, and we see that $x_0 \in$ supp T. Thus $]0, +\infty[\subset$ supp T, and since supp T is closed, we obtain supp $T = \mathbb{R}^+$.

Answer 1.B.2 We denote T this linear form. Let A be a positive real number. For any $\varphi \in \mathcal{C}^\infty_{[-A,A]}$, we know, by Taylor's formula, that there exists $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x),$$

so that

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}\, dx = \int_{A>|x|>\varepsilon} \frac{\varphi(0)}{x^2}\, dx + \int_{A>|x|>\varepsilon} \frac{\varphi'(0)}{x} dx + \int_{A>|x|>\varepsilon} \psi(x))\, dx.$$

The second integral vanishes since we integrate an odd function on a symmetric interval, and we obtain

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2} dx = 2\frac{\varphi(0)}{\varepsilon} + \int_{A>|x|>\varepsilon} \psi(x)\, dx$$

Therefore

$$\lim_{\varepsilon \to 0+} \Big(\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2} dx - 2\frac{\varphi(0)}{\varepsilon}\Big) = \int_{|x|<A} \psi(x) dx.$$

Moreover, since

$$\psi(x) = \int_0^1 (1-t)\varphi''(tx)dt,$$

we see that

$$|\langle T, \varphi\rangle| \leq 2A \sup|\varphi''|,$$

so that T is a distribution, of order 2.

Now we compute the derivative of pv$(1/x)$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have, integrating by parts,

$$\langle \mathsf{pv}(\frac{1}{x})', \varphi\rangle = -\langle \mathsf{pv}(\frac{1}{x}), \varphi'\rangle = -\lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon}\frac{\varphi'(x)}{x}dx = -\lim_{\varepsilon\to 0^+}\big[\frac{\varphi(x)}{x}\big]_\varepsilon^{-\varepsilon} + \int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx.$$

But, using Hadamard's lemma, as $\varepsilon \to 0$,

$$\big[\frac{\varphi(x)}{x}\big]_\varepsilon^{-\varepsilon} = -2\frac{\varphi(0)}{\varepsilon} + o(1),$$

so that

$$\langle \mathsf{pv}(\frac{1}{x})', \varphi\rangle = -\langle \mathsf{Fp}(\frac{1}{x^2}), \varphi\rangle.$$

Notice that this computation gives another proof of the fact that Fp$(\frac{1}{x^2})$ is a distribution.

Answer 1.B.3

Answer 1.B.4 1. If $T \in \mathcal{D}'(\mathbb{R})$ satisfies $x^pT = 0$, then supp $T \subset \{0\}$. So either $T = 0$, or there exist complex numbers $a_0, a_1, \dots, a_N$ with $a_N \neq 0$ such that $T = \sum_{j=0}^N a_j\delta^{(j)}$. In the latter case we have $x^pT = \sum_{j=0}^N a_jx^p\delta^{(j)} = \sum_{j=p}^N a_j(-1)^p\frac{p!}{(p-j)!}\delta^{(j-p)}$, since for $p > j$, $x^p\delta^{(j)} = 0$, and for $0 \leq p \leq j$,

$$x^p\delta^{(j)} = (-1)^p\frac{p!}{(p-j)!}\delta^{(j-p)}.$$

Derivatives of Dirac masses at 0 are linearly independent, thus $a_j = 0$ for all $j \geq p$. Therefore we should have

$$T = \sum_{j=0}^{p-1} a_j\delta^{(j)}.$$

Conversely, the above computation shows that all the distributions of this type are solutions of the equation.

2. a) Let $T_0 \in \mathcal{D}'(\mathbb{R})$ be a solution. T_1 is another solution if and only if $T_0 - T_1$ solves $xT = 0$, i.e. $T_1 = T_0 + a\delta$ for some $a \in \mathbb{C}$. On the other hand, $T_0 = \mathsf{vp}(\frac{1}{x})$ is a solution: for $\varphi \in \mathcal{C}^\infty(\mathbb{R})$,

$$\langle xT, \varphi\rangle = \langle T, x\varphi\rangle = \lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon}\varphi(x)dx = \langle 1, \varphi\rangle.$$

Therefore, the set of solutions is $\mathcal{S} = \{\mathsf{vp}(\frac{1}{x}) + a\delta,\ a \in \mathbb{C}\}$.

b) The same way, we see that solutions can be written $T_0 + a_0\delta + a_1\delta'$ where T_0 is any solution, and $a_0, a_1 \in \mathbb{C}$. Therefore it is sufficient to find T_0 such that $x^2T_0 = 1$. But for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, with supp $\varphi \subset [-A, A]$,

$$\langle -x^2\,\mathsf{vp}(\frac{1}{x})', \varphi\rangle = \langle \mathsf{vp}(\frac{1}{x}), (x^2\varphi)'\rangle = \langle \mathsf{vp}(\frac{1}{x}), 2x\varphi + x^2\varphi'\rangle = \langle 2, \varphi\rangle + \int_{-A}^A x\varphi'(x)dx = \langle 1, \varphi\rangle.$$

Thus the set of solutions is $\mathcal{S} = \{\mathsf{vp}(\frac{1}{x})' + a_0\delta + a_1\delta',\ a_0, a_1 \in \mathbb{C}\}$. Notice that we have defined $\mathsf{Fp}(\frac{1}{x^2}) = -\,\mathsf{vp}(\frac{1}{x})'$.

Answer 1.B.5

Answer 1.B.6 Denote $T_n = T_{f_n}$. Let $A > 0$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset [-A, A]$, we have $\varphi(x) = \varphi(0) + x\psi(x)$ for some $\psi \in CC^\infty(\mathbb{R})$. Then

$$\langle T_n, \varphi\rangle = \int_{-A}^{A} \frac{nx}{1+nx^2}\varphi(x)dx = \varphi(0)\int_{-A}^{A} \frac{nx}{1+nx^2}dx + \int_{-A}^{A} \frac{nx^2}{1+nx^2}\psi(x)dx$$

The first integral vanishes since f_n is odd, thus

$$\langle T_n, \varphi\rangle = \int_{-A}^{A} \frac{nx^2}{1+nx^2}\psi(x)dx$$

But $\frac{nx^2}{1+nx^2}\psi(x) \to \psi(x)$ as $n \to +\infty$, and $|\frac{nx^2}{1+nx^2}\psi(x)| \leq |\psi(x)| \in L^1(\mathbb{R})$. By Lebesgue's Dominated Convergence Theorem, we obtain

$$\langle T_n, \varphi\rangle \ \to \int_{-A}^{A} \psi(x)dx = \langle \mathsf{pv}(\frac{1}{x}), \varphi\rangle.$$

Answer 1.B.7 1. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and let $A > 0$ be such that $\operatorname{supp}\varphi \subset [-A, A]$. Integrating by parts we obtain

$$\langle A_n, \varphi\rangle = \int_{-A}^{A} n^{100}e^{inx}\varphi(x)dx = -\frac{1}{in}\int_{-A}^{A} n^{100}e^{inx}\varphi(x)dx - \frac{1}{in}\langle A_n, \varphi\rangle.$$

Therefore by induction

$$\langle A_n, \varphi\rangle = (-\frac{1}{in})^{101}\int_{-A}^{A} n^{100}e^{inx}\varphi(x)dx = \mathcal{O}(\frac{1}{n}).$$

Thus $\langle A_n, \varphi\rangle \to 0$ as $n \to +\infty$ for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and $(A_n) \to 0$ in $\mathcal{D}'(\mathbb{R})$.

2. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and let $A > 0$ be such that $\operatorname{supp}\varphi \subset [-A, A]$. We have

$$\langle B_n, \varphi\rangle = \int_{-A}^{A} \varphi(x)\cos^2(nx)dx = \int_{-A}^{A} \varphi(x)\frac{1+\cos(2nx)}{2}dx = \int_{-A}^{A} \frac{\varphi(x)}{2} + o(1),$$

as $n \to +\infty$ thanks to Riemman-Lebesgue's Lemma. Thus $(B_n) \to T_{\frac{1}{2}}$ in $\mathcal{D}'(\mathbb{R})$.

3. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and let $A > 0$ be such that $\operatorname{supp}\varphi \subset [-A, A]$. Integrating by parts we get

$$\langle C_n, \varphi\rangle = \int_0^A n\sin(nx)\varphi(x)dx = \big[n\frac{\cos(nx)}{n}\varphi(x)\big]_0^A - \int_0^A \cos(nx)\varphi'(x)dx = -\varphi(0) + o(1)$$

as $n \to +\infty$ thanks to Riemman-Lebesgue's Lemma. Thus $(C_n) \to -\delta_0$ in $\mathcal{D}'(\mathbb{R})$.

4. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. We have, as $n \to +\infty$,

$$\langle D_n, \varphi\rangle = \sum_{p=0}^{n} \frac{1}{n}\varphi(\frac{p}{n}) \to \int_0^1 \varphi(x)dx,$$

since $\langle D_n, \varphi\rangle$ is the n-th Darboux sum for φ on the interval $[0,1]$. Thus (D_n) to $\mathbf{1}_{[0,1]}$ in $\mathcal{D}'(\mathbb{R})$.

5. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and let $A > 0$ be such that $\operatorname{supp}\varphi \subset [-A, A]$. By Hadamard's lemma, we know that there exists $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\varphi(x) = \varphi(0) + x\psi(x)$. Then

$$\begin{aligned}\langle E_n, \varphi\rangle = \langle \mathsf{pv}(\frac{1}{x}), e^{inx}\varphi(x)\rangle &= \lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<A} e^{inx}\frac{\varphi(x)}{x}dx\\ &= \varphi(0)\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<A}\frac{e^{inx}}{x}dx + \lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<A} e^{inx}\psi(x)dx\\ &= \varphi(0)\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<A}\frac{e^{inx}}{x}dx + \int_{-A}^{A} e^{inx}\psi(x)dx.\end{aligned}$$

The last term of this sum is $o(1)$ as $n \to +\infty$ by Riemman-Lebesgue's lemma. On the other hand, we have

$$\int_{\varepsilon<|x|<A}\frac{e^{inx}}{x}dx = \cancel{\int_{\varepsilon<|x|<A}\frac{\cos(nx)}{x}dx} + i\int_{\varepsilon<|x|<A}\frac{\sin(nx)}{x}dx,$$

since $x \mapsto \dfrac{\cos(nx)}{x}$ is an odd function. Now $x \mapsto \dfrac{\sin(nx)}{x}$ is continuous at 0, so that

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<A}\frac{e^{inx}}{x}dx = \int_{-A}^{A}\frac{\sin(nx)}{x}dx = \int_{-nA}^{nA}\frac{\sin(y)}{y}dy.$$

Summing up, we have obtained

$$\langle E_n, \varphi\rangle = i\varphi(0)\int_{-\infty}^{+\infty}\frac{\sin(y)}{y}dy + o(1),$$

so that $(E_n) \to i\pi\delta_0$ in $\mathcal{D}'(\mathbb{R})$.

6. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. We have, as $n \to +\infty$,

$$\langle\frac{1}{n}(\delta_{1/n} - \delta_{-1/n}), \varphi\rangle = \frac{1}{n}(\varphi(\frac{1}{n}) - \varphi(-\frac{1}{n})) = \frac{\varphi(\frac{1}{n}) - \varphi(0)}{1/n} - \frac{\varphi(-\frac{1}{n}) - \varphi(0)}{1/n} \to 2\varphi'(0).$$

Thus $\frac{1}{n}(\delta_{1/n} - \delta_{-1/n}) \to 2\delta_0'$ in $\mathcal{D}'(\mathbb{R})$.

Answer 1.B.8 Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and let $A > 0$ be such that $\operatorname{supp}\varphi \subset [-A, A]$. For $\varepsilon > 0$ we have

$$\langle\frac{1}{x - i\varepsilon}, \varphi\rangle = \int_{-A}^{A}\frac{\varphi(x)}{x - i\varepsilon}dx = \int_{-A}^{A} x\frac{\varphi(x)}{x^2 + \varepsilon^2}dx + i\varepsilon\int_{-A}^{A}\frac{\varphi(x)}{x^2 + \varepsilon^2}dx,$$

and we study each term of the sum separately.

– By Hadamard's lemma, we can write $\varphi(x) = \varphi(0) + x\psi(x)$ for some smooth function ψ. Then

$$\int_{-A}^{A} x\frac{\varphi(x)}{x^2 + \varepsilon^2}dx = \cancel{\varphi(0)\int_{-A}^{A}\frac{x}{x^2 + \varepsilon^2}dx} + \int_{-A}^{A} x^2\frac{\psi(x)}{x^2 + \varepsilon^2}dx,$$

since $x \mapsto \frac{x}{x^2+\varepsilon^2}$ is an odd function. Then it follows easily, for example using the dominated convergence theorem, that

$$\lim_{\varepsilon\to 0^+} \int_{-A}^{A} x\frac{\varphi(x)}{x^2+\varepsilon^2}dx = \int_{-A}^{A} \varphi(x)dx = \langle \mathsf{pv}(\frac{1}{x}), \varphi\rangle.$$

– We perform the change of variable $x \leftarrow\!\shortmid y = x/\varepsilon$, and we get, as $\varepsilon \to 0^+$,

$$\varepsilon \int_{-A}^{A} \frac{\varphi(x)}{x^2+\varepsilon^2}dx = \int_{-A/\varepsilon}^{A/\varepsilon} \frac{\varphi(\varepsilon y)}{y^2+1}dy \to \varphi(0)\int_{-\infty}^{+\infty} \frac{dy}{y^2+1} = \pi\varphi(0).$$

Summing up, we have obtained that, as $\varepsilon \to 0^+$,

$$\frac{1}{x-i\varepsilon} \to \mathsf{pv}(\frac{1}{x}) + i\pi\delta_0.$$

Answer 1.B.9 1. Let $A > 0$, and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ with support included in $[-A, A]$. By Taylor's formula, we have $\varphi(x) = \varphi(0) + x\psi(x)$, for $\psi \in \mathcal{C}^\infty(\mathbb{R})$ given by

$$\psi(x) = \int_0^1 \varphi'(tx)dt.$$

Then, for all $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} |\frac{1}{\sqrt{k}}(\varphi(\frac{1}{k}) - \varphi(-\frac{1}{k}))| \leq 2\sup_{[0,1]}|\psi| \sum_{k=1}^{N}\frac{1}{k^{3/2}} \leq 2\sup_{[0,1]}|\varphi'|\sum_{k=1}^{N}\frac{1}{k^{3/2}}.$$

Passing to the limit $N \to +\infty$, we see that the linear form T is well defined, and that

$$|\langle T, \varphi\rangle| \leq \big(2\sum_{k=1}^{+\infty}\frac{1}{k^{3/2}}\big)\sum_{j=0}^{1}\sup|\varphi^{(j)}|,$$

so that it is indeed a distribution of order 1.

2. For $j \in \mathbb{N}$, we denote φ_j a function in $\mathcal{C}_0^\infty(\mathbb{R})$, non-negative, such that $\varphi_j(x) = 1$ for $x \in [1/j, 1]$ and $\varphi_j(x) = 0$ for $x \leq 1/(j+1)$ and for $x \geq 2$. All the φ_j are supported in the compact $K = [0, 2]$, and $\sup|\varphi_j| = 1$. If T has order 0, there exists a constant$C = C_K > 0$ such that, for all $j \in \mathbb{N}^*$, $|\langle T, \varphi_j\rangle| \leq C$. But

$$\langle T, \varphi_j\rangle = \sum_{k=1}^{j}\frac{1}{\sqrt{k}} \to +\infty \text{ as } j \to +\infty,$$

which is a contradiction. Thus T is of exact order 1.

3. Let $k \in \mathbb{N}$. For any open set V which contains the compact $\{1/k\}$, there is a plateau function$\psi_k \in \mathcal{C}_0^\infty(V)$ such that $\mathsf{supp}\,\psi_k \subset]1/(k-1), 1/(k+1)[\cap V$, $\psi_k(1/k) = 1$ and $0 \leq \psi_k \leq 1$. For this ψ_k, we have

$$\langle T, \psi_k\rangle = \frac{1}{\sqrt{k}} \neq 0,$$

so that $1/k \in \operatorname{supp} T$ for all $k \in \mathbb{N}$. The same way, we have also $1/k \in \operatorname{supp} T$ for all $k \in -\mathbb{N}^*$. Sincesupp T is closed, we even have $S = \{0\} \cup \{\frac{1}{k},\ k \in \mathbb{Z}^*\} \subset \operatorname{supp} T$.

Conversely, if $x_0 \notin S$, there is a neighborhood V of x_0 that does not intersect with S, and for any $\varphi \in \mathcal{C}_0^\infty(V)$, we have $\langle T, \varphi \rangle = 0$, so that $x_0 \notin \operatorname{supp} T$.

4. We consider her again the functions φ_j of question (2), and we set $\tilde{\varphi}_j = j^{-3/2}\varphi_j$. We get

$$\langle T, \tilde{\varphi}_j \rangle \to \sum_{j \geq 1} \frac{1}{j^2} = \frac{\pi^2}{6} \neq 0 \text{ as } j \to +\infty,$$

and $\tilde{\varphi}_j^{(p)} = 0$ on S for $p \geq 1$, thus the sequence $(\tilde{\varphi}_j)$ answers to the question.

We also have $\sup |\tilde{\varphi}_j| = 1/j^{3/2} \to 0$ as $j \to +\infty$. Therefore, there is no constant $C > 0$ and no natural number k such that, for all $j \in \mathbb{N}$,

$$|\langle T, \tilde{\varphi}_j \rangle| \leq C \sum_{p \leq k} \sup_{x \in \operatorname{supp} T} |\tilde{\varphi}_j^{(p)}(x)|.$$

This shows that in general, one can't replace $\sup |\partial^\alpha \varphi|$ by $\sup\limits_{x \in \operatorname{supp} T} |\partial^\alpha \varphi|$ in the definition of a distribution.

Answer 1.B.10 1. **a.** For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\langle (T_f)^*, \varphi \rangle = \overline{\int f(x)\overline{\varphi(x)}dx} = \int \overline{f(x)}\varphi(x)dx$. Thus the distribution T_f is real if and only if f is real-valued.

b. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have $\langle (T^*)^*, \varphi \rangle = \overline{\langle T^*, \overline{\varphi} \rangle}$ et $\langle T^*, \overline{\varphi} \rangle = \overline{\langle T, \overline{\overline{\varphi}} \rangle} = \overline{\langle T, \varphi \rangle}$, so that $(T^*)^* = T$. Also, $\langle (fT)^*, \varphi \rangle = \overline{\langle fT, \overline{\varphi} \rangle} = \overline{\langle T, \overline{\overline{f}\varphi} \rangle} = \langle T^*, \overline{f}\varphi \rangle = \langle \overline{f}T^*, \varphi \rangle$, so that $(fT)^* = \overline{f}T^*$.

c. Let $T_1 = \frac{1}{2}(T + T^*)$ and let $T_2 = \frac{1}{2i}(T - T^*)$. We have $T = T_1 + iT_2$, and $T_1^* = \frac{1}{2}(T^* + T) = T_1$, $T_2^* = -\frac{1}{2i}(T^* - T) = T_2$. Moreover if $T = T_1 + iT_2 = 0$, with real T_1, T_2, we also have $0 = T^* = T_1 - iT_2$, thus $T_1 = T_2 = 0$.

d. Since P_I has real coefficient, we see that $(P_I T)^* = P_I(T^*)$ using question (b). In particular $P_I T$ is real when T is real. Thus $T = T_1 + iT_2$ satisfies $P_I T = 0$, or $P_I T_1 + i P_I T_2 = 0$ if and only if $P_I T_1 = P_I T_2 = 0$ (cf. question (c)). Therefore, it suffices to know real solutions to get all the solutions.

2.a. By Leibniz formula, we obtain $x^p \delta_0^{(q)} = 0$ for $q < p$ and $x^p \delta_0^{(q)} = (-1)^p \frac{q!}{(q-p)!} \delta_0^{(q-p)}$ for $q \geq p$.

b. For $k \geq 2$, we have $P(\delta_0^{(k)}) = ((k+1)^2 - \frac{1}{4})\delta_0^{(k)} + k(k-1)\delta_0^{(k-2)}$ and, simply, $P(\delta_0^{(k)}) = ((k+1)^2 - \frac{1}{4})\delta_0^{(k)}$ for $k \in \{0, 1\}$.

c. Suppose that $T \neq 0$ in $\operatorname{Ker}(P)$ is supported in $\{0\}$. Then there exists $N \in \mathbb{N}$ eand $a_0, \ldots, a_N \in \mathbb{R}$, with $a_N \neq 0$ such that $T = \sum_{k=0}^N a_k \delta_0^{(k)}$. Thus

$$0 = PT = \sum_{k=0}^N a_k P(\delta_0^{(k)}) = a_N((N+1)^2 - \frac{1}{4})\delta_0^{(N)} + T_{N-1},$$

where T_{N-1} is a linear combination of derivatives of order at most $N-1$ of δ_0. This $a_N = 0$, which is a contradiction. Therefore, the null distribution is the only distribution in $\operatorname{Ker}(P)$ with support in $\{0\}$.

3.a. Let $U \in \mathcal{D}'(I)$, and $V = e^{ix}U$. We have $U = e^{-ix}V$ so that $U'' = -e^{-ix}V - 2ie^{-ix}V' + e^{-ix}V''$. Thus $U'' + U = 0$ if and only if $V'' - 2iV' = 0$, that is $V' = Ce^{2ix}$ for a certain constant $C \in \mathbb{C}$. This last equation can also be written $(V - \frac{C}{2i}e^{2ix})' = 0$, thus $V = \frac{C}{2i}e^{2ix} + B$, and $U = Ae^{ix} + Be^{-ix}$ for some $A, B \in \mathbb{C}$. The real solutions are therefore $U = a\cos x + b\sin x$ with $a, b \in \mathbb{R}$.

b. Let $T \in \mathcal{D}'(]0, +\infty[)$. The function $x \mapsto x^{1/2}$ is $\mathcal{C}^\infty$ on $]0, +\infty[$, so that Leibniz formula gives

$$(x^{1/2}T)'' = x^{1/2}T'' + x^{-1/2}T' - \frac{1}{4}x^{-3/2}T = x^{-3/2}P_{]0,+\infty[}T - x^{1/2}T$$

Therefore $T \in \mathsf{Ker}(P_{]0,+\infty[})$ if and only if $(x^{1/2}T)'' + (x^{1/2}T) = 0$ and $x^{-1/2}\,T = a_+\cos x + b_+\sin x$ for two constants $a_+, b_+ \in \mathbb{R}$. So we have

$$\mathsf{Ker}(P_{]0,+\infty[}) = \big\{x \mapsto a_+\frac{\cos x}{\sqrt{x}} + b_+\frac{\sin x}{\sqrt{x}}, a_+, b_+ \in \mathbb{R}\big\}$$

c. We proceed the same way with $(-x)^{1/2}T$. We get

$$\mathsf{Ker}(P_{]-\infty,0[}) = \big\{x \mapsto a_-\frac{\cos x}{\sqrt{-x}} + b_-\frac{\sin x}{\sqrt{-x}}, a_-, b_- \in \mathbb{R}\big\}$$

4. a. The function f belongs to $L^1_{loc}(\mathbb{R})$ since $|\frac{\cos x}{\sqrt{\pm x}}H(\pm x)| \leq \frac{1}{\sqrt{\pm x}}H(\pm x) \in L^1_{loc}(\mathbb{R})$ and $|\frac{\sin x}{\sqrt{\pm x}}H(\pm x)| \leq \frac{1}{\sqrt{\pm x}}H(\pm x) \in L^1_{loc}(\mathbb{R})$.

b. Since $T_f|_{]0,+\infty[} = T_+$ belongs to $\mathsf{Ker}(P_{]0,+\infty[})$, $S|_{]0,+\infty[} = 0$. The same way, $S|_{]-\infty,0[} = 0$, and $\mathsf{supp}\, S \subset \{0\}$.

c. We have

$$\begin{aligned}\langle S, \chi_\varepsilon\varphi\rangle =& \langle PT_f, \chi_\varepsilon\varphi\rangle = \langle x^2(T_f)'' + x(T_f)' + (x^2 - \frac{1}{4})T_f, \chi_\varepsilon\varphi\rangle \\ =& \langle T_f, (x^2\chi_\varepsilon\varphi)'' - (x\chi_\varepsilon\varphi)' + (x^2 - \frac{1}{4})\chi_\varepsilon\varphi = \int f(x)\psi_\varepsilon(x)dx,\end{aligned}$$

with

$$\begin{aligned}\psi_\varepsilon(x) &= (x^2\chi_\varepsilon\varphi)'' - (x\chi_\varepsilon\varphi)' + (x^2 - \frac{1}{4})\chi_\varepsilon\varphi \\ &= \frac{x^2}{\varepsilon^2}\chi''(\frac{x}{\varepsilon})\varphi + \frac{x}{\varepsilon}\chi'(\frac{x}{\varepsilon})(3\varphi + 2x\varphi') + \chi(\frac{x}{\varepsilon})\big((x^2\varphi)'' - (x\varphi)' + (x^2 - \frac{1}{4})\varphi\big).\end{aligned}$$

Since $\mathsf{supp}\,\chi_\varepsilon \subset [-\varepsilon, \varepsilon]$, we clearly have $\mathsf{supp}\,\psi_\varepsilon \subset [-\varepsilon, \varepsilon]$. But for $x \in [-\varepsilon, \varepsilon]$ the above expression gives

$$|\psi_\varepsilon(x)| \leq \|\chi''\|_\infty\|\varphi\|_\infty + \|\chi'\|_\infty\|3\varphi + 2x\varphi'\|_\infty + \|\chi\|_\infty\|(x^2\varphi)'' - (x\varphi)' + (x^2 - \frac{1}{4})\varphi\|_\infty.$$

This shows that the functions ψ_ϵ are bounded on $\mathbb{R}$ uniformly with respect to ε. In particular, the dominated convergence theorem gives

$$\int f(x)\psi_\varepsilon(x)dx \to 0 \text{ as } \quad x \to 0,$$

since $f(x)\psi_\varepsilon(x) \to 0$ for $x \neq 0$, thus a.e..

At last, since supp $S \subset \{0\}$, and $\varphi - \chi_\varepsilon\varphi$ vanishes in a neighborhood of 0, we have $\langle S, \chi_\varepsilon\varphi\rangle = \langle S, \varphi\rangle$, so that $0 = S = PT_f$.

5. a. Let $T \in \mathrm{Ker}(P)$. There exist $T_+ \in \mathrm{Ker}(P_{]0,+\infty[})$ and $T_- \in \mathrm{Ker}(P_{]-\infty,0[})$ such that $T|_{]0,+\infty[} = T_+$ and $T|_{]-\infty,0[} = T_-$. Then, let f be the function we have built in question **(4a)** for this T_+ and this T_-. We know that supp$(T - T_f) \subset \{0\}$, and, thanks to question (4), that $T - T_f \in \mathrm{Ker}(P)$. It follows from question (2) that $T = T_f$.

5. b. Any element $T \in \mathrm{Ker}(P)$ can be written

$$T = (a_+\frac{\cos x}{\sqrt{x}} + b_+\frac{\sin x}{\sqrt{x}})H(x) + (a_-\frac{\cos x}{\sqrt{-x}} + b_-\frac{\sin x}{\sqrt{-x}})H(-x).$$

Thus $\dim \mathrm{Ker}(P) = 4$. However this element of $\mathrm{Ker}(P)$ is continuous at 0 if and only if $a_+ = a_- = 0$, so that $\mathrm{Ker}(P) \cap C^0(\mathbb{R})$ has dimension 2. At last $T = b_+\frac{\sin x}{\sqrt{x}}H(x) + b_-\frac{\sin x}{\sqrt{-x}}H(-x)$ is differentiable at 0 if and only if $b_- = b_+ = 0$, so that $\mathrm{Ker}(P) \cap C^1(\mathbb{R})$ only contains the null function.

Answer 1.B.11 Part A.– 1. Let $A > 0$. For any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that supp $\varphi \subset [-A, A]$, we have supp $\tau_{-a}\varphi \subset [-B, B]$, taking for example $B = A + a$. Since T is a distribution, there is $C = C_B > 0$ and $k = k_B \in \mathbb{N}$ such that

$$|\langle \tau_a T, \varphi\rangle| = |\langle T, \tau_{-a}\varphi\rangle| \leq C \sum_{j\leq k} \sup |\partial_x^j(\tau_{-a}\varphi)| \leq C \sum_{j\leq k} \sup |\partial_x^j\varphi|.$$

This shows that the linear form $\tau_a T$ is a distribution. Moreover, for $f \in L^1_{loc}(\mathbb{R})$,

$$\langle \tau_a T_f, \varphi\rangle = \int f(x)\varphi(x + a)dx = \int \tau_a f(y)\varphi(y)dy,$$

so that $\tau_a T_f = T_{\tau_a f}$.

2. Let $S_N = \sum_{n=-N}^{N} \delta_{2\pi n}$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, there is $N_0 \in \mathbb{N}$ such that supp $\varphi \subset [-2\pi N_0, 2\pi N_0]$. For all $N \geq N_0$, we thus have

$$\langle \sum_{n=-N}^{N} \delta_{2\pi n}, \varphi\rangle = \sum_{n=-N_0}^{N_0} \varphi(2\pi n).$$

Otherwise stated, the sequence $(\langle S_N, \varphi\rangle)$ is constant starting from N_0, thus converges, which also proves that the serie $\sum_{n\in\mathbb{Z}} \delta_{2\pi n}$ converges in $\mathcal{D}'(\mathbb{R})$. For any N large enough we also have

$$\langle \tau_{2\pi} W, \varphi\rangle = \langle W, \tau_{-2\pi}\varphi\rangle = \langle S_N, \tau_{-2\pi}\varphi\rangle = \sum_{n=-N}^{N} \varphi(2\pi(n - 1)) = \sum_{n=-N_0}^{N_0} \varphi(2\pi n) = \langle W, \varphi\rangle,$$

so that W is 2π-periodic.

3. a) Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and $A > 0$ such that supp $\varphi \subset [-A, A]$. Integrating by parts, for $n \neq 0$,

$$\begin{aligned}\int e^{-inx}\varphi(x)dx &= -\int_{-A}^{A} \frac{1}{-in} e^{-inx}\varphi'(x)dx \\ &= \cdots = (-1)^{p+2} \int_{-A}^{A} \frac{1}{(-in)^{p+2}} e^{-inx}\varphi^{(p+2)}(x)dx = \mathcal{O}(n^{-p-2}).\end{aligned}$$

b) Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, and $A > 0$ such that $\operatorname{supp}\varphi \subset [-A, A]$. For $N \in \mathbb{N}$ we have

$$|\langle \sum_{n=-N}^{N} c_n e^{inx}, \varphi\rangle| \leq \sum_{n=-N}^{N} |c_n|\, |\int e^{inx}\varphi(x)dx| \lesssim |c_0| \,+ \sum_{\substack{n=-N\\ n\neq 0}}^{N} \frac{(1+|n|)^p}{n^{p+2}} \lesssim \sum_{n=-N}^{N} \frac{1}{n^2},$$

so that the serie $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ converges in $\mathcal{D}'(\mathbb{R})$. Moreover, using question 1) ($\tau_a T_f = T_{\tau_a f}$), we get

$$\tau_{2\pi}\big(\sum_{n=-N}^{N} c_n e^{inx}\big) = \sum_{n=-N}^{N} c_n \tau_{2\pi}(e^{inx}) = \sum_{n=-N}^{N} c_n e^{inx}.$$

Passing to the limit $N \to +\infty$, we obtain that the sum is indeed 2π-periodic.

Part B.– 1. Let $A > 0$ such that $\operatorname{supp}\omega \subset [-A, A]$. For $x \in \mathbb{R}$ fixed, $\omega(x + 2\pi n) = 0$ for all $n \notin [\frac{-A-x}{\pi}, \frac{A-x}{\pi}]$. Thus for $x \in [-B, B]$, $\omega(x + 2\pi n) = 0$ for any $n \in \mathbb{N}$ such that $n \notin [\frac{-A-B}{\pi}, \frac{A+B}{\pi}]$. The serie

$$\sum_{n\in\mathbb{Z}} \tau_{-2\pi n}\omega(x)$$

is thus localy finite, therefore converges (for example uniformly on every compact), and its sum is $\mathcal{C}^\infty$.

2. We suppose furthermore that $\omega = 1$ on $[0, 2\pi]$, so that the function $x \mapsto \sum_{n\in\mathbb{Z}} \tau_{-2\pi n}\omega(x)$ is positive. It follows from the previous questions that the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(x) = \frac{\omega(x)}{\sum_{n\in\mathbb{Z}} \omega(x + 2\pi n)}.$$

is $\mathcal{C}_0^\infty(\mathbb{R})$ and satisfies $\sum_{k\in\mathbb{Z}} \tau_{-2k\pi}\psi = 1$.

3. a) First of all, we note that $\operatorname{supp}(e^{inx}\psi) \subset \operatorname{supp}\omega = K$ for any n. Since T is a distribution, there exist $C = C_K > 0$ and $m = m_K \in \mathbb{N}$ such that, for all $n \in Z$,

$$|c_n| \leq C \sum_{j\leq m} \sup |\partial_x^j(e^{-inx}\psi)|,$$

so that $|c_n| = \mathcal{O}((1 + |n|)^m)$.

b) Since $\sum_{k\in\mathbb{Z}} \tau_{-2k\pi}\psi = 1$ et $\tau_{2\pi k}T = T$ for all k, we have

$$\begin{aligned}\langle T, \varphi\rangle &= \langle T, (\sum_{k\in Z} \tau_{-2\pi k}\psi)\varphi\rangle = \sum_{k\in Z} \langle T,\ \tau_{-2\pi k}\psi\varphi\rangle\\ &= \sum_{k\in Z} \langle \tau_{2\pi k}T,\ \tau_{-2\pi k}\psi\varphi\rangle = \sum_{k\in Z} \langle T, \psi\ \tau_{2\pi k}\varphi\rangle = \langle T, \psi\widetilde{\varphi}\rangle.\end{aligned}$$

c) The function $\widetilde{\varphi}$ is $\mathcal{C}^\infty$ and 2π-periodic, thus it is equal to the sum of its Fourier serie. Therefore

$$\langle T, \varphi\rangle = \langle T, \psi\widetilde{\varphi}\rangle = \sum_{n\in\mathbb{Z}} c_n(\widetilde{\varphi})\langle T, e^{inx}\psi\rangle = 2\pi \sum_{n\in\mathbb{Z}} c_{-n}(T)\ c_n(\widetilde{\varphi}).$$

Going back to the definition of the coefficients $c_n(\widetilde{\varphi})$, we get

$$\begin{aligned}\langle T, \psi\widetilde{\varphi}\rangle &= \sum_{n\in\mathbb{Z}} c_{-n}(T) \int_0^{2\pi} \widetilde{\varphi}(x) e^{-inx} dx = \sum_{n\in\mathbb{Z}} c_{-n}(T) \int_0^{2\pi} \sum_{k\in\mathbb{Z}} \varphi(x-2\pi k) e^{-inx} dx \\ &= \sum_{n\in\mathbb{Z}} c_{-n}(T) \int \varphi(x) e^{-inx} dx = \int \varphi(x) \sum_{n\in\mathbb{Z}} c_{-n}(T) e^{-inx} dx = \langle \sum_{n\in\mathbb{Z}} c_{-n}(T) e^{-inx}, \varphi\rangle,\end{aligned}$$

so that $T = \sum_{n\in\mathbb{Z}} c_n(T) e^{inx}$.

Part C.– 1. Let ψ be as above. By definition,

$$c_n(W) = \frac{1}{2\pi}\langle W, e^{-inx}\psi\rangle = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\langle \delta_{2\pi k}, e^{-inx}\psi\rangle = \frac{1}{2\pi}.$$

2. It suffices to write, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\sum_{n\in\mathbb{Z}} \varphi(2n\pi) = \langle W, \varphi\rangle = \langle \sum_{n\in\mathbb{Z}} c_n(W) e^{inx}, \varphi\rangle = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \int e^{inx}\varphi(x) dx = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \hat{\varphi}(n).$$

Chapter 2

Several variables calculus

2.1 Partial derivatives of functions

2.1.1 Definitions

Definition 2.1.1 Let $\Omega \in \mathbb{R}^n$ an open set, and $f : \Omega \to \mathbb{C}$ a function. The fonction f is differentiable at $x \in \Omega$ if there is a linear form $L_x : \mathbb{R}^n \to \mathbb{C}$ such that

$$f(x+h) = f(x) + L_x(h) + o(\|h\|) \text{ quand } h \to 0.$$

We denote $L_x = d_x f = df(x)$ the differential of f at x.

For a differentiable function $f : \Omega \to \mathbb{C}$, we denote $\partial_j f : \Omega \to \mathbb{C}$ the map defined by

$$\partial_j f(x) = d_x f(e_j),$$

where $(e_1, e_2, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. The function $\partial_j f$ is called the j-th partial derivative of f. One can easily see that

$$\partial_j f(x) = \lim_{t \to 0} \frac{f(x+te_j) - f(x)}{t}\cdot \tag{2.1.1}$$

Therefore, $\partial_j f(x)$ is the directional derivative of f at the point x in the direction of the vector e_j, or the derivative of the j-th partial map associated to f, that one gets fixing all the coordinates of x but the j-th one.

There are functions whose partial derivatives all exist, but that are not differentiables. However, the mean value theorem gives the following

Proposition 2.1.2 A function $f : \Omega \to \mathbb{C}$ is differentiable on Ω, and the function $x \mapsto d_x f$ is continuous on Ω if and only if the functions given by (2.1.1) exist and are continuous on Ω.

When this proposition applies, one says that f is of $\mathcal{C}^1$ class on Ω, or that it belongs to $\mathcal{C}^1(\Omega)$. More generally, for $k \in \mathbb{N}$, we denote $\mathcal{C}^k(\Omega)$ the set (vector space) of functions $f : \Omega \to \mathbb{C}$ whose partial derivatives $\partial_1 f, \partial_2 f, \dots, \partial_n f$ belong to $\mathcal{C}^{k-1}(\Omega)$.

In general, the order in which one computes repeated partial derivatives matters, but this is not the case for $\mathcal{C}^2$ functions:

Proposition 2.1.3 If $f \in \mathcal{C}^2(\Omega)$, then, for any $j, k \in \{0, \dots, n\}$,

$$\partial_j(\partial_k f) = \partial_k(\partial_j f)$$

In particular for $\mathcal{C}^\infty$ functions, one can compute partial derivatives of f in any order. It is therefore very convenient to use multiindices.

2.1.2 Multiindices

Let $f \in \mathcal{C}^\infty(\Omega)$, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ a multiindex. We denote $\partial^\alpha f$ the function

$$\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f.$$

The number

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

is the order of the partial derivative, and it is called the length of α. We also denote

$$\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$$

and, for $\beta \in \mathbb{N}^n$ such that $\beta_j \leq \alpha_j$ for all j, which we will write $\beta \leq \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha - \beta)!} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \dots \binom{\alpha_n}{\beta_n}.$$

With these notations, Leibniz's formula for the derivatives of a product of functions generalize easily to the case of partial derivatives of functions of many variables. Its proof is exactly the same.

Proposition 2.1.4 Let f and g be two functions in $\mathcal{C}^\infty(\Omega)$, and $\alpha \in \mathbb{N}^n$ a multiindex. We have

$$\partial^\alpha (fg) = \sum_{\beta \in \mathbb{N}^n,\ \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\alpha f\ \partial^{\alpha - \beta} g.$$

Proof.— We prove the result by induction over $|\alpha|$. If $|\alpha| = 1$, $\partial^\alpha = \partial_j$ for some $j \in \{1, \dots, n\}$, and

$$\partial_j(fg) = (\partial_j f)g + f(\partial_j g),$$

which is the above formula. Suppose then that the formula is true for all multiindices of length $\leq m$. Let $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m + 1$. There exists $j \in \{1, \dots, n\}$ et $\beta \in \mathbb{N}^n$ of length m such that

$$\alpha = \beta + 1_j,$$

where $1_j = (0, \dots, 0, 1, 0, \dots, 0)$ with a 1 as j-th coordinate. With these notations

$$\partial^\alpha(fg) = \partial^{\beta+1_j}(fg) = \partial^\beta(\partial_j(fg)) = \partial^\beta((\partial_j f)g) + \partial^\beta(f(\partial_j g)).$$

Since β is of length m, the induction assumption gives

$$\begin{aligned}
\partial^\alpha(fg) &= \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma(\partial_j f)\ \partial^{\beta-\gamma} g + \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma f\ \partial^{\beta-\gamma}(\partial_j g) \\
&= \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^{\gamma+1_j} f\ \partial^{\beta-\gamma} g + \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma f\ \partial^{\beta+1_j-\gamma} g \\
&= \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^{\gamma+1_j} f\ \partial^{\alpha-(\gamma+1_j)} g + \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma f\ \partial^{\alpha-\gamma} g
\end{aligned}$$

We change the multiindex in the first sum: $\gamma \leftarrow \gamma + 1_j$, and we get

$$\begin{aligned}
\partial^\alpha(fg) &= \sum_{1_j\leq\gamma\leq\alpha} \binom{\beta}{\gamma - 1_j} \partial^\gamma f\ \partial^{\alpha-\gamma} g + \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma f\ \partial^{\alpha-\gamma} g \\
&= \partial^\alpha f\ g + \sum_{1_j\leq\gamma\leq\beta} (\binom{\beta}{\gamma - 1_j} + \binom{\beta}{\gamma}) \partial^\gamma f\ \partial^{\alpha-\gamma} g + f\partial^\alpha g \\
&= \partial^\alpha f\ g + \sum_{1_j\leq\gamma\leq\beta} \binom{\beta + 1_j}{\gamma} \partial^\gamma f\ \partial^{\alpha-\gamma} g + f\partial^\alpha g,
\end{aligned}$$

which is the required result. □

We can continue the analogy with the 1 variable case: if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we denote x^α the real number

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Then we can show, the same way as for Leibniz's formula, that for $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$,

$$(x + y)^\alpha = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha-\beta}.$$

With these notations, Taylor's formula can be written

Proposition 2.1.5 Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{C}$ a function of class $\mathcal{C}^{m+1}$. Let $a, b \in \Omega$, such that the segment $[a, b]$ is included in Ω. We have

$$f(b) = \sum_{|\alpha|\leq m} \frac{(b-a)^\alpha}{\alpha!} \partial^\alpha f(a) + (m + 1) \sum_{|\alpha|=m+1} \frac{(b-a)^\alpha}{\alpha!} \int_0^1 (1-t)^m \partial^\alpha f(a + t(b-a))dt.$$

Proof.— We have already seen that if $\varphi : \mathbb{R} \to \mathbb{C}$ is smooth, we have

$$\varphi(1) = \sum_{k=0}^{m} \frac{1}{k!}\varphi^{(k)}(0) + \frac{1}{m!}\int_0^1 (1-s)^m \varphi^{(m+1)}(sx)ds.$$

We shall use this result for the function $\varphi : t \mapsto f(a + t(b-a))$. Notice that

$$\varphi'(t) = \sum_{j=1}^{n} (b-a)_j (\partial_j f)(a + t(b-a))$$

and more generally that

$$\varphi^{(k)}(t) = \sum_{j_1, j_2, \dots, j_k = 1}^{n} (b-a)_{j_1} \dots (b-a)_{j_k} (\partial_{j_1} \dots \partial_{j_k} f)((a + t(b-a)).$$

This sum only contains terms of the form $(b-a)^\alpha (\partial^\alpha f)(a+t(b-a))$ with $\alpha \in \mathbb{N}^n$ of length $|\alpha| = k$. Therefor we can write, for some coefficients $c_\alpha \in \mathbb{R}$,

$$\sum_{j_1, j_2, \dots, j_k = 1}^{n} (b-a)_{j_1} \dots (b-a)_{j_k} (\partial_{j_1} \dots \partial_{j_k} f)((a+t(b-a)) = \sum_{|\alpha|=k} c_\alpha (b-a)^\alpha (\partial^\alpha f)(a+t(b-a)).$$

Denoting $x = b - a$, this equality between two polynomials in x implies that the c_α are given by

$$c_\alpha = \#\{(j_1, j_2, \dots, j_k) \in \{1, \dots, n\}^k,\ x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} = x_{j_1} \dots x_{j_k}\}.$$

thus

$$c_\alpha = \binom{k}{\alpha_1}\binom{k-\alpha_1}{\alpha_2} \dots \binom{k - \alpha_1 - \dots - \alpha_{n-1}}{\alpha_n} = \frac{k!}{\alpha!}.$$

Indeed, one has to choose first α_1 numbers among $j_1, \dots j_k$ that should be 1, then α_2 among the $k - \alpha_1$ numbers left that should be 2, …

Since $a + t(b-a)_{|t=0} = a$, and $a + t(b-a)_{|t=1} = b$, we obtain the stated formula. □

Exercise 2.1.6 Show that, for $k \in \mathbb{N}$ and $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we have

$$(x_1 + x_2 + \dots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\, x^\alpha.$$

Corollary 2.1.7 (Hadamard's formula) Let $f \in \mathcal{C}^{m+1}(\mathbb{R}^n)$. If $f(0) = 0$, there exist n functions g_1, g_2,…g_n of class $\mathcal{C}^m(\mathbb{R}^n)$ such that

$$f(x) = \sum_{j=1}^{n} x_j g_j(x).$$

2.1.3 The Chain rule

Proposition 2.1.8 Let $\Omega \subset \mathbb{R}^n$ an open set, and $\psi : \Omega \to \mathbb{R}^p$, $\psi = (\psi_1, \psi_2, \dots, \psi_p)$, a function of class $\mathcal{C}^k$. Let $\Omega' \subset \mathbb{R}^p$ an open set that contains $\psi(\Omega)$, and $f : \Omega' \to \mathbb{C}$ a $\mathcal{C}^k$ function.

Then $f \circ \psi$ is $\mathcal{C}^k$ on Ω. Moreover

$$d_x(f \circ \psi) = d_{\psi(x)} f . d_x \psi = \nabla \psi(x) \cdot \nabla f(\psi(x)),$$

or, for any $i \in \{1, \dots, n\}$,

$$\partial_i (f \circ \psi)(x) = \sum_{j=1}^{p} \partial_j f(\psi(x)) \partial_i \psi_j(x).$$

2.2 Stokes formula

Here, we recall, without proof, some notions and results pertaining to the field of elementary differential geometry. We will provide in Section 2.4 another presentation of the same material - with slightly less general assumptions - in the context of distribution theory, with proofs. Of course it will be the occasion of generalizing our presentation of distributions to the multidimensional case.

2.2.1 Surface measure

Definition 2.2.1 Let $F : \mathbb{R}^{n-1} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. We denote

$$S = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n,\ x_n = F(x_1, x_2, \dots, x_{n-1}\}.$$

We call surface measure on S the measure $\sigma \in \mathcal{D}'(\mathbb{R}^n)$ given by

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n),\ \langle \sigma, \varphi \rangle = \int_{\mathbb{R}^{n-1}} \varphi(x', F(x')) \sqrt{1 + \|\nabla F(x')\|^2} dx'.$$

Example 2.2.2 Let $\gamma : [0,1] \to \mathbb{R}^2$ be a parametrized curve on the plane, with $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. We suppose that $\gamma \in \mathcal{C}^1$, and that $\gamma'(t) \neq 0$ for all $t \in [0,1]$. The length of the curve γ is

$$L(\gamma) = \int_0^1 \|\gamma'(t)\| dt.$$

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function,the integral of f along γ is

$$\int_\gamma f(x) dx = \int_0^1 f(\gamma(t)) \|\gamma'(t)\| dt.$$

Suppose now that $\gamma_1'(t) \neq 0$ for all $t \in [0,1]$, i.e. that the tangent vector to γ is never vertical. Then γ_1 is an invertible function, and the curve γ is the graph of the function $F : \mathbb{R} \to \mathbb{R}$ given by $F(x) = \gamma_2 \circ (\gamma_1)^{-1}(x)$:

$$\gamma = \{(x_1, x_2) \in \mathbb{R}^2,\ x_2 = F(x_1)\}.$$

By the change of variable $x = \gamma_1(t)$, with $a = \gamma_1(0)$ and $b = \gamma_1(1)$, we get

$$\int_\gamma f(x)dx = \int_0^1 f(\gamma_1(t), \gamma_2(t))\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}dt = \int_a^b f(x, F(x))\sqrt{1 + F'(x)^2}dx.$$

One may think that this definition only concerns surfaces in $\mathbb{R}^n$ that are the graph of a function. However a surface is a subset S of $\mathbb{R}^n$ given by an equation of the form

$$G(x_1, x_2, \dots, x_n) = 0,$$

where G is a $\mathcal{C}^1$ function such that $\nabla G(x) \neq 0$ for all $x \in S$. The hyperplane tangent to the surface S at the point $x \in S$ is then by definition the hyperplane normal to the vector $\nabla G(x)$ and containing x, that is

$$T_x S = \{z \in \mathbb{R}^n,\ \nabla G(x) \cdot (z - x) = 0\}.$$

For a point $x_0 \in S$, there is thus at least one index j such that $\partial_j G(x_0) \neq 0$, and one can apply the local inversion theorem: there is a neighborhood U of x_0 and a function $F : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$x \in V \cap S \Longleftrightarrow x_j = F(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

Therefore, localy, any surface is a graph, and one can apply the above definition up to a permutation of the coordinates. Of course, one should then ”glue” the results obtained locally, for example using a suitable partition of unity.

In any case, the definition of the surface measure seems to depend on a choice of a function F to describe the surface. As a matter of fact, one can show that this is not the case: any choice would lead to the same measure.

2.2.2 Stokes Formula

Let us now state the famous Stokes formula. First, we need some vocabulary.

Definition 2.2.3 Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that Ω is an open set of class $\mathcal{C}^k$ when there exists a function $\rho \in \mathcal{C}^k(\mathbb{R}^n)$ such that

i) $\Omega = \{x \in \mathbb{R}^n,\ \rho(x) < 0\}$,

ii) For all $x \in \partial\Omega = \{x \in \mathbb{R}^n,\ \rho(x) = 0\}$, we have $\nabla\rho(x) \neq 0$.

Then we say that ρ defines Ω, and the set $\partial\Omega$ is called the boundary of Ω.

Of course, for a given regular open set, there are a lot of possible choices for the function ρ. Here follows some canonical examples:

Examples 2.2.4 – For the ball of radius $R > 0$, $B(0, R) = \{x \in \mathbb{R}^n,\ \|x\| < R\}$, one can take $\rho(x) = \|x\|^2 - R^2$, where $\|x\| = \sqrt{\sum_{j=1}^n x_{j^2}}$ is the euclidian norm. Notice that the choice $x \mapsto \|x\| - R$ is not allowed, since it is not a smooth function.

– For $\Omega =]0, 1[$, we can take $\rho(x) = x(x - 1)$.

– When Ω is a sub-graph, i.e. $\Omega = \{x \in \mathbb{R}^n, x_n < F(x')\}$, one can take $\rho(x) = x_n - F(x')$, which has the same regularity as F.

Definition 2.2.5 Let Ω be a $\mathcal{C}^1$ open set, and $\rho \in \mathcal{C}^1(\mathbb{R}^n)$ a function that defines Ω. For $x \in \partial\Omega$, the vector

$$\nu(x) = \frac{\nabla\rho(x)}{\|\nabla\rho(x)\|}$$

is called the outgoing unitary normal vector to Ω at point x.

This definition would not make sense if $\nu(x)$ depends on the choice of the function ρ. We can prove, and we admit, that it is not the case.

Examples 2.2.6 – For the ball $B(0, R)$, with $\rho(x) = \|x\|^2 - R^2$, we get $\nu(x) = \frac{x}{R}$, which corresponds to what we know from classical geometry: the outgoing unitary normal vector at a point x of the sphere has the direction of the radius of the sphere corresponding to x.

– For $\Omega =]0, 1[$, with $\rho(x) = x(x - 1)$, we get $\nu(0) = -1$ and $\nu(1) = 1$.

– For a sub-graph, with $\rho(x) = x_n - F(x')$ we have

$$\nu(x) = \frac{1}{\sqrt{1 + \|\nabla F(x')\|^2}} \begin{pmatrix} -\nabla F(x') \\ 1 \end{pmatrix}$$

Proposition 2.2.7 (Stokes formula) Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a compactly supported vector field. Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^1$ open set, σ its surface measure and ν the outgoing unitary normal vector field to Ω. If $X \in \mathcal{C}^1(\overline{\Omega})$, denoting $\operatorname{div} X$ its divergence:

$$\operatorname{div} X = \sum_{j=1}^n \partial_j X_j,$$

we have

$$\int_\Omega \operatorname{div} X(x)dx = \langle \sigma, \nu \cdot X \rangle = \int_{\partial\Omega} X(x) \cdot \nu(x) d\sigma(x).$$

Corollary 2.2.8 (Stokes formula (2)) Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^1$ open set, σ its surface measure and ν the outgoing unitary normal vector field to Ω. Let also $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field on $\overline{\Omega}$, and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We have

$$\int_\Omega \varphi(x) \operatorname{div} F(x) dx = -\int_\Omega \nabla\varphi(x) \cdot F(x) dx + \int_{\partial\Omega} \varphi(x) F(x) \cdot \nu(x) d\sigma(x).$$

Proof.— We only have to apply Proposition 2.2.7 to the vector field $X = \varphi F$, noticing that

$$\operatorname{div}(\varphi F) = \sum_{j=1}^n (\partial_j \varphi) F_j + \varphi \sum_{j=1}^n \partial_j F_j = \nabla\varphi \cdot F + \varphi \operatorname{div} F.$$

□

Example 2.2.9 Take $\Omega =]0,1[\subset \mathbb{R}$. The surface measure on $\partial\Omega$ is $\sigma = \delta_0 + \delta_1$. For $\varphi \in \mathcal{C}_0^1(\mathbb{R})$ and $F \in \mathcal{C}^1(\mathbb{R})$, Stokes Formula (2) is

$$\begin{aligned} \int_0^1 \varphi(x) F'(x) dx &= -\int_0^1 \varphi'(x) F(x) dx + \langle \delta_0 + \delta_1, \varphi(x) F(x) \nu(x) \rangle \\ &= -\int_0^1 \varphi'(x) F(x) dx + \varphi(1) F(1) - \varphi(0) F(0). \end{aligned}$$

In the previous example, in dimension 1, we have seen that Stokes formula reduces to the usual integration by parts formula. This aspect is even better seen in the following case: take $\varphi \in \mathcal{C}_0^1(\Omega)$ and $f \in \mathcal{C}^\infty(\Omega)$. If we set $F : \Omega \to \mathbb{R}^n$ to be the function given by $F(x) = (0, \dots, 0, f(x), 0, \dots, 0)$ with $f(x)$ at the j-th place, we have $\operatorname{div} F = \partial_j f$ and

$$\begin{aligned} \int_\Omega \varphi(x) \partial_j f(x) dx = \int_\Omega \varphi(x) \operatorname{div} F(x) dx &= -\int_\Omega \nabla\varphi(x) \cdot F(x) dx + \int_{\partial\Omega} \varphi(x) F(x) \cdot \nu(x) d\sigma(x) \\ &= -\int_\Omega \partial_j \varphi(x) f(x) dx + \int_{\partial\Omega} \varphi(x) f(x) \nu_j(x) d\sigma(x). \end{aligned}$$

From Stokes formula follows the also well-known

Corollary 2.2.10 (Green-Riemann formula) Let $\Omega \subset \mathbb{R}^2$ a $\mathcal{C}^1$ open set whose boundary $\partial\Omega$ is given by the parametrized curve $\gamma : [0,1] \to \mathbb{R}^2$, with $\gamma'(t) \neq 0$ for all $t \in [0,1]$, which is positively oriented. For P and Q two $\mathcal{C}^1$ function on $\mathbb{R}^2$, we have

$$\iint_\Omega (\partial_x Q(x,y) - \partial_y P(x,y)) dx dy = \int_\gamma P dx + Q dy.$$

Proof.— First we recall that the RHS of this equation means

$$\int_\gamma P dx + Q dy = \int_0^1 \big(P(\gamma(t)) \gamma_1'(t) + Q(\gamma(t)) \gamma_2'(t) \big) dt.$$

Applying Stokes formula to the vector field $X = \begin{pmatrix} -Q \\ P \end{pmatrix}$, we get

$$\iint_\Omega (\partial_x Q(x,y) - \partial_y P(x,y))dxdy = -\int_\Omega \text{div}\, X dx = -\int_{\partial\Omega} X \cdot \nu d\sigma.$$

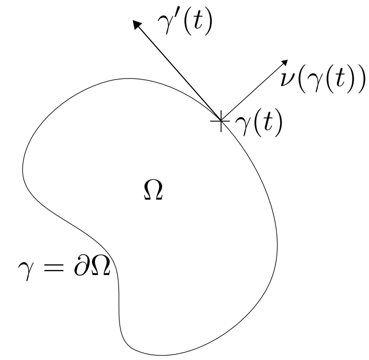


Figure 2.1: A positively oriented boundary in $\mathbb{R}^2$

The assumption that γ is positively oriented means that Ω lies at the left of the curve γ, or equivalently that $\gamma(t)$ goes counterclockwise. Then we only have to remark that $\nu(\gamma(t)))$ is the image of the normed vector $\gamma'(t)/\|\gamma'(t)\|$ by the rotation of angle $-\pi/2$, that is

$$\nu(\gamma(t))) = \frac{1}{\|\gamma'(t)\|} \begin{pmatrix} \gamma_2'(t) \\ -\gamma_1'(t) \end{pmatrix},$$

and that $d\sigma(\gamma(t)) = \|\gamma'(t)\|dt$. □

2.3 Distributions on $\mathbb{R}^n$

In this section, we review very briefly the definitions for distributions in an open subset Ω of $\mathbb{R}^n$. As a matter of fact, almost everything we have seen about distributions in 1d extends without difficulty to the present multi-dimensional case.

First of all, the support of a smooth function φ on Ω is the closure of the complement of the set of points where φ vanishes. The space $\mathcal{C}_0^\infty(\Omega)$ of compactly supported smooth function is not reduced to the null function, since for example, it contains the functions (see Proposition 1.1.7)

$$\varphi_{x_0,r}(x) = \begin{cases} 0 & \text{for } x \notin B(x_0,r), \\ \exp\left(\dfrac{-1}{r^2 - \|x - x_0\|^2}\right) & \text{for } x \in B(x_0,r), \end{cases}$$

where $x_0 \in \Omega$ and $r > 0$ is such that $B(x_0,r) \subset \Omega$. The space $\mathcal{C}_0^\infty(\Omega)$ contains also plateau functions: as a matter of fact, the proof of Proposition 1.1.8 holds replacing $\mathbb{R}$ by $\mathbb{R}^n$. This is also true for the proof of Proposition 1.6.1. Thus there exists also smooth partitions of unity associated to a finite covering of a compact subset K of $\Omega \subset \mathbb{R}^n$.

The notion of convergence in $\mathcal{C}_0^\infty(\Omega)$ is the same as in dimension 1, replacing successive derivatives by partial derivatives at all order:

Definition 2.3.1 A sequence (φ_j) of $\mathcal{C}_0^\infty(\Omega)$ converges to φ in $\mathcal{C}_0^\infty(\Omega)$ when

i) There exists a compact $K \subset \Omega$ that contains the support of all the φ_j and the support of φ.

ii) For any $\alpha \in \mathbb{N}^n$, $\partial^\alpha(\varphi_j - \varphi)$ converges uniformly to 0 on K.

At last, a distribution on $\Omega \subset \mathbb{R}^n$ is a continuous linear form on $\mathcal{C}_0^\infty(\Omega)$:

Definition 2.3.2 A distribution on Ω is a linear form on $\mathcal{C}_0^\infty(\Omega)$ such that if (φ_j) converges to φ in $\mathcal{C}_0^\infty(\Omega)$, then $T(\varphi_j) \to T(\varphi)$. The vector space of distributions on Ω is denoted $\mathcal{D}'(\Omega)$.

As in dimension 1 this definition is equivalent to the following property.

Proposition 2.3.3 Let $\Omega \subset \mathbb{R}^n$ be open, and T be a complex valued linear form on $\mathcal{C}_0^\infty(\Omega)$. T is a distribution if and only if

$$\forall K \subset \Omega, \exists C > 0, \exists k \in \mathbb{N}, \forall \varphi \in \mathcal{C}_K^\infty(\Omega), |T(\varphi)| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi|.$$

The space of distributions on Ω is denoted $\mathcal{D}'(\Omega)$. The notion of support and that of the order of a distribution are exactly the same as in dimension 1.

One can of course multiply a distribution in $\mathcal{D}'(\Omega)$ by a smooth function $f \in \mathcal{C}^\infty(\Omega)$, and define partial derivatives $\partial^\alpha T$ of a distribution T by the formulas

$$\langle fT, \varphi \rangle = \langle T, f\varphi \rangle \quad \text{and} \quad \langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle, \alpha \in \mathbb{N}^n.$$

2.4 Surface measure: a distribution point of view

When the surface S is not globally a graph, as for example the sphere $S_R^{n-1} = \{x \in \mathbb{R}^n, \|x\| = R\}$, the Definition 2.2.1 does not apply directly. Of course we can use a smooth partition of unity to compute the measure surface on S, but we propose here another way, using distribution theory. To start with, we notice that

Proposition 2.4.1 Let Ω be a $\mathcal{C}^1$ open subset of $\mathbb{R}^n$, and $1_\Omega \in L^1_{loc}(\mathbb{R}^n)$ be its characteristic function. Then, for $j = 1 \dots n$, the support of the distribution $\partial_j 1_\Omega$ is included in $\partial\Omega$.

Proof.— Let $x_0 \in \mathbb{R}^n \setminus \partial\Omega$. There exists $\delta > 0$ such that $B_\infty(x_0, \delta) = \{x \in \mathbb{R}^n,\ \max_j |x_j| < \delta\} \subset \mathbb{R}^n \setminus \partial\Omega$. For any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B_\infty(x_0, \delta)$, we have

$$\langle \partial_j 1_\Omega, \varphi\rangle = -\langle 1_\Omega, \partial_j\varphi\rangle = -\int_\Omega \partial_j\varphi(x)dx.$$

If $x_0 \notin \Omega$, then $B_\infty(x_0, \delta) \cap \Omega = \emptyset$, and this integral vanishes. If $x_0 \in \Omega$, then $B_\infty(x_0, \delta) \subset \Omega$ and we get, by Fubini-Tonelli,

$$\begin{aligned}\langle \partial_j 1_\Omega, \varphi\rangle &= -\int_{B_\infty(x_0,\delta)} \partial_j\varphi(x)dx \\ &= -\int \dots \int \Big(\int_{x_{0,j}-\delta}^{x_{0,j}+\delta} \partial_j\varphi(x)dx_j\Big)dx' = 0.\end{aligned}$$

In both cases, we see that $x_0 \notin \operatorname{supp} T$. □

Proposition 2.4.2 Let Ω be a $\mathcal{C}^2$ open subset of $\mathbb{R}^n$, and ν its unitary outgoing normal vector. The surface measure on $\partial\Omega$ is the distribution

$$-\nu \cdot \nabla 1_\Omega = -\sum_{j=1}^n \nu_j \partial_j 1_\Omega.$$

Remark 2.4.3 Notice that the function ν is not smooth. As a matter of fact, it is not even defined everywhere on $\mathbb{R}^n$ since $\nabla\rho$ may vanish. Thus the above definition uses two generalizations of the product of a distribution by a function:

- First of all, the function f needs only to be defined in a neighborhood of the support of the distribution. Indeed if χ is any plateau function above $\operatorname{supp} T$, then, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\langle (\chi f)T, \varphi\rangle = \langle T, \chi f\varphi\rangle = \langle T, f\varphi\rangle$, and we can consider the left hand side as the generalization of the product of T by functions that are only defined, or smooth, near the support of T.
- Second, the function only needs to be $\mathcal{C}^k$ when the distribution is of order k. We postponed this slightly technical generalization to an appendix at the end of this chapter.

Proof.— We write the proof in dimension 2 to simplify a bit the notations. The computations in the general case are exactly the same. Let us consider the surface S in $\mathbb{R}^2$ given as the graph $x_2 = F(x_1)$, where F is a $\mathcal{C}^2$ function. We denote Ω the open subset of $x = (x_1, x_2) \in \mathbb{R}^2$ such that $\rho(x_1, x_2) = x_2 - F(x_1) < 0$, so that $S = \partial\Omega$. We have

$$\nabla\rho(x) = \begin{pmatrix} -F'(x_1) \\ 1\end{pmatrix} \qquad \text{and} \qquad \|\nabla\rho(x)\| = \sqrt{1 + F'(x_1)^2}.$$

For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we compute

$$I = \langle -\nu \cdot \nabla 1_\Omega, \varphi\rangle = -\langle \sum_{j=1}^2 \rho_j \partial_j 1_\Omega, \frac{\varphi}{\|\nabla\rho\|}\rangle = \sum_{j=1}^2 \int_\Omega \partial_j\big(\rho_j(x)\frac{\varphi(x)}{\|\nabla\rho(x)\|}\big)dx.$$

Note that, thanks to the above remark and Proposition 2.4.1, we can suppose that φ vanishes out of a small neighborhood of S. For $A > 0$ such that supp$(\varphi) \subset [-A, A]^2$, we have

$$I = \int_{-A}^A \int_{-A}^{F(x_1)} \partial_1\big(\frac{\partial_1\rho(x)}{\|\nabla\rho(x)\|}\varphi(x)\big)dx_2dx_1 + \int_{-A}^A \int_{-A}^{F(x_1)} \partial_2\big(\frac{\partial_2\rho(x)}{\|\nabla\rho(x)\|}\varphi(x)\big)dx_2dx_1 = I_1 + I_2.$$

One can compute directly

$$I_2 = \int_{-A}^A \frac{\partial_2\rho(x_1, F(x_1))}{\|\nabla\rho(x_1, F(x_1))\|}\varphi(x_1, f(x_1))dx_1 = \int_{-A}^A \frac{1}{\|\nabla\rho(x_1, F(x_1))\|}\varphi(x_1, F(x_1))dx_1.$$

For the computation of I_1, we notice that

$$\partial_1\big(\int_{-A}^{F(x_1)} \frac{\partial_1\rho(x_1, x_2)}{\|\nabla\rho(x_1, x_2)\|}\varphi(x_1, x_2)dx_2\big) = \int_{-A}^{F(x_1)} \partial_1\big(\frac{\partial_1\rho(x)}{\|\nabla\rho(x)\|}\varphi(x)\big)dx_2 + F'(x_1)\frac{\partial_1\rho(x_1, F(x_1))}{\|\nabla\rho(x_1, F(x_1))\|}\varphi(x_1, F(x_1)),$$

and thus,

$$I_1 = \int_{-A}^A \partial_1\big(\int_{-A}^{F(x_1)} \frac{\partial_1\rho(x_1, x_2)}{\|\nabla\rho(x_1, x_2)\|}\varphi(x_1, x_2)dx_2\big)dx_1 - \int_{-A}^A f'(x_1)\frac{\partial_1\rho(x_1, f(x_1))}{\|\nabla\rho(x_1, f(x_1))\|}\varphi(x_1, f(x_1))dx_1.$$

The first of these two integrals vanishes since $\varphi(A, \cdot) = \varphi(-A, \cdot) = 0$, and we obtain

$$\begin{aligned} I &= -\int_{-A}^A F'(x_1)\frac{\partial_1\rho(x_1, F(x_1))}{\|\nabla\rho(x_1, F(x_1))\|}\varphi(x_1, F(x_1))dx_1 + \int_{-A}^A \frac{1}{\|\nabla\rho(x_1, F(x_1))\|}\varphi(x_1, F(x_1))dx_1 \\ &= \int_{-A}^A \frac{F'(x_1)^2 + 1}{\|\nabla\rho(x_1, F(x_1))\|}\varphi(x_1, F(x_1))dx_1 = \int \varphi(x_1, F(x_1))\ \sqrt{F'(x_1)^2 + 1}\ dx_1. \end{aligned}$$

This finishes the proof of the proposition. □

Example 2.4.4 Let $\Omega =]0, 1[\subset \mathbb{R}$, and $\rho : x \mapsto x(x-1)$. Near $\partial\Omega = \{0, 1\}$, the unitary outgoing normal vector to Ω is

$$\nu(x) = \frac{2x - 1}{|2x - 1|},$$

so that $\nu(0) = -1$ and $\nu(1) = 1$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have

$$\langle \sigma, \varphi\rangle = -\langle \nu\partial_x 1_\Omega, \varphi\rangle = \int_0^1 (\nu(x)\varphi(x))'dx = \nu(1)\varphi(1) - \nu(0)\varphi(0) = \varphi(1) + \varphi(0).$$

Therefore, we have $\sigma = \delta_0 + \delta_1$.

2.4.1 The surface measure on the sphere

As a fundamental example, we compute now the surface measure on the sphere. We recall first what are called spherical coordinates.

Proposition 2.4.5 Let $n \geq 2$, and $\varphi :]0, +\infty[\times]0, \pi[\times \cdots \times]0, \pi[\times[0, 2\pi[\to \mathbb{R}^n$ the function given by

$$\varphi(r, \theta_1, \dots, \theta_{n-2}, \theta_{n-1}) = \begin{pmatrix} r \cos\theta_1 \\ r \sin\theta_1 \cos\theta_2 \\ \vdots \\ r \sin\theta_1 \dots \sin\theta_{n-2} \cos\theta_{n-1} \\ r \sin\theta_1 \dots \sin\theta_{n-2} \sin\theta_{n-1}. \end{pmatrix}$$

The jacobian J_φ of φ is

$$J_\varphi(r, \theta_1, \dots, \theta_{n-2}, \theta_{n-1}) = |\det \nabla\varphi| = r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3} \dots \sin\theta_{n-2},$$

and φ is a $\mathcal{C}^\infty$-diffeomorphism onto $\mathbb{R}^n \setminus \{x_n = x_{n-1} = 0\}$.

Proposition 2.4.6 The surface measure on the sphere S_R^{n-1} is the distribution given by

$$\langle \sigma_R, \varphi \rangle = \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} \varphi(R, \theta_1, \dots, \theta_{n-1}) R^{n-1} (\sin\theta_1)^{n-2} (\sin\theta_2)^{n-3} \dots \sin\theta_{n-2} d\theta_1 \dots d\theta_{n-1}.$$

for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

In particular for $R = 1$, $\omega = \sigma_1$ frequently denotes the surface measure on $S^{n-1} = S_1^{n-1}$, so that

$$\langle \sigma_R, \varphi \rangle = \int_{S_1^{n-1}} \varphi(R\omega) R^{n-1} d\omega = R^{n-1} \langle \omega, \varphi(R\cdot) \rangle.$$

Proof.– Let $\Omega = B(0, R)$ and $\rho(x) = |x|^2 - R^2$. We have $\nabla\rho(x) = 2x$, and $\nu(x) = \frac{x}{|x|}$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, that we can suppose to be supported out of a neighborhood of 0, we get

$$\langle \sigma_R, \varphi \rangle = \langle -\nu \cdot \nabla 1_\Omega, \varphi \rangle = \int_\Omega \sum_{j=1}^n \partial_j ((\frac{x_j}{|x|} \varphi(x)) dx$$
$$= \int_\Omega \sum_{j=1}^n \partial_j (\frac{x_j}{|x|}) \varphi(x) dx + \int_\Omega \sum_{j=1}^n \frac{x_j}{|x|} \partial_j \varphi(x) dx.$$

Now

$$\sum_{j=1}^n \partial_j (\frac{x_j}{|x|}) = \sum_{j=1}^n \frac{|x| - x_j^2/|x|}{|x|^2} = \frac{n-1}{|x|},$$

and, on the other hand, if we denote $x = r\omega$ with $r \in]0, +\infty[$ and $\omega \in S_1^{n-1}$, we notice that

$$\partial_r(\varphi(r\omega)) = \omega \cdot \nabla\varphi(r\omega).$$

Thus,

$$\langle \sigma_R, \varphi\rangle = (n-1)\int_{S_1^{n-1}}\int_0^R \frac{\varphi(r\omega)}{r} r^{n-1} dr d\omega + \int_{S_1^{n-1}}[\int_0^R \partial_r(\varphi(r\omega))r^{n-1}dr]d\omega.$$

At last, integrating by parts, and since φ is suppose to vanish near 0, we get

$$\langle \sigma_R, \varphi\rangle = (n-1)\int_{S_1^{n-1}}\int_0^R \frac{\varphi(r\omega)}{r} r^{n-1} dr d\omega \\ + \int_{S_1^{n-1}} \varphi(R\omega)R^{n-1}d\omega - (n-1)\int_{S_1^{n-1}}\int_0^R (\varphi(r\omega))r^{n-2}drd\omega,$$

which is the statement we have announced. □

2.4.2 A proof of Stokes' formula

We show here how to deduce Stokes formula (Proposition 2.2.7) starting from the distributional definition of the surface measure.

Proposition 2.4.7 Let Ω be a $\mathcal{C}^2$ open set in $\mathbb{R}^n$, and ν be the unit normal vector field outgoing from Ω. If σ is the surface measure on $\partial\Omega$, then for all $j \in \{1, \dots, n\}$,

$$\nu_j \sigma = -\partial_j 1_\Omega.$$

Proof.— Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(x) = 1$ if $x \le -1$ and $\chi(x) = 0$ for $x \ge 0$. Let also $\rho \in \mathcal{C}^2(\mathbb{R}^n)$ be a defining function for the regular open set Ω. For $\alpha > 0$, we set

$$\chi_\alpha(x) = \chi(\alpha\rho(x)).$$

If $x \notin \Omega$, then $\chi_\alpha(x) = 0$ since $\rho(x) \ge 0$, and if $x \in \Omega$, then $\chi_\alpha(x) = 1$ for any large enough α, since then $\alpha\rho(x) \le -1$. Moreover $\chi_\alpha \le 1$, thus, by the dominated convergence theorem, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have

$$\int \chi_\alpha(x)\varphi(x)dx \to \int_\Omega \varphi(x)dx,$$

i.e. the family (χ_α) converges to 1_Ω in $\mathcal{D}'(\mathbb{R}^n)$ as $\alpha \to +\infty$.

Since derivation and multiplication by a function are continuous operations in $\mathcal{D}'(\mathbb{R}^n)$, we also have, for any $j \in \{1, \dots, n\}$,

$$-\nu_j \sum_{k=1}^n \nu_k \partial_k \chi_\alpha \to \nu_j \sigma,$$

in $\mathcal{D}'(\mathbb{R}^n)$ as $\alpha \to +\infty$. But for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, that we can suppose to be supported near $\partial\Omega$, we have

$$\langle -\nu_j \sum_{k=1}^n \nu_k \partial_k \chi_\alpha, \varphi\rangle = \langle -\sum_{k=1}^n \partial_k\chi_\alpha, \nu_k\nu_j\varphi\rangle \\ = -\alpha\int \chi'(\alpha\rho(x))\sum_{k=1}^n \partial_k\rho(x)\nu_k(x)\nu_j(x)\varphi(x)dx.$$

Since $\sum_{k=1}^n \partial_k\rho(x)\nu_k(x) = \|\nabla\rho(x)\|$, we get

$$\langle -\nu_j \sum_{k=1}^n \nu_k\partial_k\chi_\alpha, \varphi\rangle = -\alpha\int \chi'(\alpha\rho(x))\partial_j\rho(x)\varphi(x)dx = -\int \partial_j(\chi_\alpha(x))\varphi(x)dx.$$

Passing to the limit $\alpha \to +\infty$, we eventually obtain $\nu_j\sigma = -\partial_j 1_\Omega$. □

Stokes formula is a very simple corollary of this proposition. Indeed

$$\langle\sigma, \nu\cdot X\rangle = \sum_{j=1}^n \langle\sigma, \nu_j X_j\rangle = \sum_{j=1}^n \langle\nu_j\sigma, X_j\rangle = -\sum_{j=1}^n \langle\partial_j 1_\Omega, X_j\rangle = \langle 1_\Omega, \sum_{j=1}^n \partial_j X_j\rangle.$$

2.A Exercises

Exercice 2.A.1 (A proof of Stone-Weierstrass' theorem) Let $\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)$, such that $\varphi(0) > \varphi(x) \geq 0$ for all $x > 0$. For $k \in \mathbb{N}^*$ we set

$$\varphi_k(x) = \left(\int \varphi(x)^k dx\right)^{-1} \varphi(x)^k.$$

1. Show that φ_k is well-defined, and that $\displaystyle\int \varphi_k(x)dx = 1$. Explain why we can suppose $\varphi(0) = 1$ without changing the φ_k. We do so in what follows.

2. Let $\alpha > 0$. Show that there exists $p \in \mathbb{N}$ such that $A_p = \emptyset$, where

$$A_j = \{x \in \mathbb{R}^n,\ |x| \geq \alpha \text{ and } \varphi(x) \geq 1 - \frac{1}{j}\}.$$

One may consider $\displaystyle\bigcap_j A_j$. Deduce that there is a constant $C_1 > 0$ such that

$$\int_{|x|>\alpha} (\varphi(x))^n dx \leq C_1(1 - \frac{1}{k})^n.$$

3. Show that there is a constant $C_2 > 0$ such that $\displaystyle\int \varphi(x)^k dx \geq C_2(1 - \frac{1}{2p})^k$.

4. Deduce that for all $\alpha > 0$, $\displaystyle\int_{|x|>\alpha} \varphi_k(x)dx \to 0$ when $k \to +\infty$.

5. Show that if $f \in \mathcal{C}(K)$ where K is a compact subset of $\mathbb{R}^n$, $f * \varphi_k \to f$ uniformly on K. One may start with the case where f vanishes at the boundary of K.

6. We denote $K = \overline{B(0,1)}$, and $\varphi(x) = (1 - \frac{|x|^2}{4})_+ = \max(0, 1 - \frac{|x|^2}{4})$. Show that for any function $f \in \mathcal{C}^\infty(K)$, there is a sequence (P_k) of polynomials such that, for all $\alpha \in \mathbb{N}^n$, $\|\partial^\alpha(P_k - f)\|_{L^\infty(K)} \to 0$ as $k \to +\infty$.

Exercice 2.A.2 1. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $f(x,0) = f(0,y) = 0$ for any $(x,y) \in \mathbb{R}^2$. Show that there is $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $f(x,y) = xy\psi(x,y)$.

2. Show that the following expression defines a distribution of $\mathcal{D}'(\mathbb{R}^2)$:

$$\varphi \mapsto \lim_{\epsilon\to 0^+} \int_{|x|,|y|>\epsilon} \frac{\varphi(x,y)}{xy}\, dxdy.$$

Hint. Consider the function $f : (x,y) \mapsto \varphi(x,y) - \varphi(x,0) - \varphi(0,y) - \varphi(0,0)$.

Exercise 2.A.3 1. For $\varphi \in C_0^\infty(\mathbb{R}^2)$, we denote $\langle T, \varphi\rangle = \int_{\mathbb{R}} \varphi(x,x)dx$ et $\langle T^+, \varphi\rangle = \int_0^{+\infty} \varphi(x,x)dx$.

Show that T and $T^+ \in \mathcal{D}'(\mathbb{R}^2)$. Compute $\partial_x T + \partial_t T$ and $\partial_x T^+ + \partial_t T^+$.

2. Let $D = \{(x,t) \in \mathbb{R}^2 \mid t \geq |x|\}$. We denote E the regular distribution associated to $\mathbf{1}_D$, the characteristic function of D. Compute $\partial_t E - \partial_x E$, then compute $\partial_t^2 E - \partial_x^2 E$.

Exercice 2.A.4 1. Let $u \in \mathcal{C}_0^1(\mathbb{R},\mathbb{C})$ such that $u(0) = 0$.

a) Show that for $x \neq 0$, $\dfrac{u(x)}{x} = \displaystyle\int_0^1 u'(tx)dt = \int_0^1 t^{-1/4}(t^{1/4}u'(tx))dt.$

b) Deduce that $\|\dfrac{u(x)}{x}\|_{L^2(\mathbb{R})} \leq 2\|u'\|_{L^2(\mathbb{R})}$.

c) Is the condition $u(0) = 0$ necessary for the above inequality to hold?

2. Now let $n \geq 3$ and $\epsilon > 0$. We set

$$\Omega_\epsilon = \{x \in \mathbb{R}^n,\ |x| > \epsilon\}, \quad \partial\Omega_\epsilon = \{x \in \mathbb{R}^n,\ |x| = \epsilon\}.$$

a) Let $u \in \mathcal{C}_0^1(\mathbb{R}^n,\mathbb{C})$, and $F = (F_1,\dots,F_n)$ with $F_j(x) = \dfrac{x_j}{|x|^2}|u(x)|^2$. Compute $\operatorname{div} F$ and show that, for ϵ small enough,

$$(n-2)\int_{\Omega_\epsilon} \frac{|u(x)|^2}{|x|^2}dx + 2\,\mathrm{Re}\int_{\Omega_\epsilon} \frac{1}{|x|^2}\ u(x)x\cdot\overline{\nabla u(x)}dx = -\int_{\partial\Omega_\epsilon} \frac{|u(x)|^2}{|x|}d\sigma(x).$$

b) Then, show that

$$(n-2)\int_{\Omega_\epsilon} \frac{|u(x)|^2}{|x|^2}dx \leq 2\left(\int_{\Omega_\epsilon} \frac{|u(x)|^2}{|x|^2}dx\right)^{1/2}\left(\int_{\Omega_\epsilon} |\nabla u(x)|^2dx\right)^{1/2} + \epsilon^{n-2}\int_{S^{n-1}} |u(\epsilon\omega)|^2d\omega.$$

c) Deduce that there is a constant $C_n > 0$, independent of u and of ϵ such that

$$\int_{\Omega_\epsilon} \frac{|u(x)|^2}{|x|^2}dx \leq C(n)\left(\int_{\Omega_\epsilon} |\nabla u(x)|^2dx + \epsilon^{n-2}\int_{S^{n-1}} |u(\epsilon\omega)|^2d\omega\right).$$

Hint: use the identity $2ab \leq \frac{1}{2}a^2 + 2b^2$.

d) Using Fatou's Lemma, show that there is a constant $C'_n > 0$ such that, for any function $u \in \mathcal{C}_0^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2}dx \leq C'_n\int_{\mathbb{R}^n} |\nabla u(x)|^2dx.$$

2.B Answers

Solution 2.B.1

Solution 2.B.2

Solution 2.B.3 1. Let $A > 0$. For $\varphi \in \mathcal{C}^\infty_{[-A,A]^2}$ we have

$$|\langle T, \varphi\rangle| \leq 2A \sup|\varphi| \text{ and } |\langle T^+, \varphi\rangle| \leq A \sup|\varphi|,$$

which shows that T and T^+ are distributions of order 0 on $\mathbb{R}^2$. Still for $\varphi \in \mathcal{C}^\infty_{[-A,A]^2}$, we have

$$\begin{aligned}\langle \partial_x T, \varphi\rangle &= -\langle T, \partial_x\varphi\rangle = -\int_{-A}^{A} (\partial_x\varphi)(x,x)dx = -\int_{-A}^{A} \partial_x(\varphi(x,x))dx + \int_{-A}^{A} \partial_t\varphi(x,x)dx \\ &= 0 + \langle T, \partial_t\varphi\rangle = -\langle \partial_t T, \varphi\rangle.\end{aligned}$$

Thus $\partial_x T + \partial_t T = 0$. The same way, we have $\partial_x T^+ + \partial_t T^+ = \delta_0$ since

$$\begin{aligned}\langle \partial_x T^+, \varphi\rangle &= -\langle T^+, \partial_x\varphi\rangle = -\int_{0}^{A} (\partial_x\varphi)(x,x)dx = -\int_{0}^{A} \partial_x(\varphi(x,x))dx + \int_{0}^{A} \partial_t\varphi(x,x)dx \\ &= \varphi(0,0) + \langle T, \partial_t\varphi\rangle = \varphi(0,0) - \langle \partial_t T, \varphi\rangle.\end{aligned}$$

2. Let $A > 0$ and $\varphi \in \mathcal{C}^\infty_{[-A,A]^2}$. We have

$$\begin{aligned}\langle \partial_t E - \partial_x E, \varphi\rangle &= -\langle E, \partial_t\varphi - \partial_x\varphi\rangle = -\iint_{t\geq|x|} \partial_t\varphi(t,x) - \partial_x\varphi(t,x)dtdx \\ &= -\int_{x\in\mathbb{R}} \big(\int_{|x|}^{+\infty} \partial_t\varphi(t,x)dt\big)dx - \int_{t\geq 0} \big(\int_{-t}^{t} \partial_x\varphi(t,x)dx\big)dt \\ &= -\int_{\mathbb{R}} \varphi(|x|,x)dx - \int_0^{+\infty} \varphi(t,t) - \varphi(t,-t)dt = 2\langle T^+, \varphi\rangle.\end{aligned}$$

Therefore

$$\langle \partial_t^2 E - \partial_x^2 E, \varphi\rangle = -\langle (\partial_t - \partial_x)E, (\partial_t + \partial_x)\varphi\rangle = -2\langle T^+, (\partial_t + \partial_x)\varphi\rangle = 2\langle (\partial_t + \partial_x)T^+, \varphi\rangle = \langle 2\delta_0, \varphi\rangle,$$

and $\partial_t^2 E - \partial_x^2 E = 2\delta_0$.

Solution 2.B.4

Chapter 3

The Laplacean

3.1 A Fundamental solution of the Laplacean

The Laplace operator, or Laplacean, is the differential operator Δ defined on $\mathcal{D}'(\mathbb{R}^n)$ by

$$\Delta T = \sum_{j=1}^{n} \partial_j^2 T = \mathrm{div}(\nabla T).$$

This operator appears everywhere in physics, and more generally in the modelisation of a lot of problems of the real world. It is also a natural object in geometry. Notice that it is sometimes the operator $-\Delta$ which is called the Laplace operator, since $-\Delta$ is a positive operator: for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$(-\Delta\varphi, \varphi)_{L^2} = \int \|\nabla\varphi\|^2 dx \geq 0.$$

3.1.1 Green's formula for the Laplacean

Proposition 3.1.1 For $u \in \mathcal{C}^2(\overline{\Omega})$ and $\varphi \in \mathcal{C}_0^2(\Omega)$, we have

$$\int_\Omega \big(\Delta u(x)\varphi(x) - u(x)\Delta\varphi(x)\big)dx = \int_{\partial\Omega} \big(\varphi(x)\nabla u(x)\cdot\nu(x) - u(x)\nabla\varphi(x)\cdot\nu(x)\big)d\sigma(x).$$

Proof.– Since $\Delta u = \mathrm{div}\,\nabla u$, Stokes formula (2) gives

$$\begin{aligned}\int_\Omega \Delta u(x)\varphi(x)dx &= \int_\Omega \varphi(x)\,\mathrm{div}\,\nabla u(x)dx\\ &= -\int_\Omega \nabla\varphi(x)\cdot\nabla u(x)dx + \int_{\partial\Omega}\varphi(x)\nabla u(x)\cdot\nu(x)d\sigma(x),\end{aligned}$$

and the same way,

$$\begin{aligned}\int_\Omega u(x)\Delta\varphi(x)dx &= \int_\Omega u(x)\,\mathrm{div}\,\nabla\varphi(x)dx\\ &= -\int_\Omega \nabla\varphi(x)\cdot\nabla u(x)dx + \int_{\partial\Omega}u(x)\nabla\varphi(x)\cdot\nu(x)d\sigma(x).\end{aligned}$$

The proposition follows subtracting these two equalities. □

3.1.2 The Laplacean of a radial function

Proposition 3.1.2 Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be a radial function, i.e. such that there exits $F :]0, +\infty[\to \mathbb{C}$ with

$$f(x) = F(|x|).$$

If F (thus f) is $\mathcal{C}^2$, we have

$$\Delta f(x) = F''(|x|) + \frac{n-1}{|x|} F'(|x|).$$

Proof.— This is a simple computation: for $x \neq 0$,

$$\partial_j f(x) = \partial_j (F(|x|)) = F'(|x|) \frac{x_j}{|x|},$$

and

$$\partial_j^2 f(x) = F''(|x|) \frac{x_j^2}{|x|^2} + \frac{|x| - x_j^2/|x|}{|x|^2} F'(|x|).$$

Therefore

$$\Delta f(x) = \sum_{j=1}^{n} F''(|x|) \frac{x_j^2}{|x|^2} + (\frac{1}{|x|} - \frac{x_j^2}{|x|^3}) F'(|x|) = F''(|x|) + \frac{n-1}{|x|} F'(|x|).$$

□

3.1.3 Computation of $\Delta(\frac{1}{\|x\|^{n-2}})$

The function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \dfrac{1}{\|x\|^{n-2}}$ is in $L^1_{loc}(\mathbb{R}^n)$, thus defines a distribution $T = T_f$. We want to compute ΔT_f in $\mathcal{D}'(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and $R > 0$ be such that $\operatorname{supp} \varphi \subset B(0, R)$. First of all, we have

$$\langle \Delta T_f, \varphi \rangle = \langle T_f, \Delta \varphi \rangle = \int_\Omega f(x) \Delta \varphi(x) dx.$$

Since the function f is not smooth on Ω, one can not use directly Stokes' formula, but by the dominated convergence theorem, we have,

$$\int_\Omega f(x) \Delta \varphi(x) dx = \lim_{\varepsilon \to 0^+} \int_{\Omega_{R,\varepsilon}} f(x) \Delta \varphi(x) dx,$$

where $\Omega_{R,\varepsilon} = B(0,R) \setminus B(0,\varepsilon)$. Notice that $\partial\Omega_{R,\varepsilon} = S_R^{n-1} \bigcup S_\varepsilon^{n-1}$, where $\nu(x) = x/R$ for $x \in S_R^{n-1}$ and $\nu(x) = -x/\varepsilon$ for $x \in S_\varepsilon^{n-1}$. Thus

$$\begin{aligned}\int_{\Omega_{R,\varepsilon}} & f(x)\Delta\varphi(x)dx \\ &= \int_{\Omega_{R,\varepsilon}} \Delta f(x)\varphi(x)dx + \int_{\partial\Omega_{R,\varepsilon}} f(x)\nabla\varphi(x)\cdot\nu(x) - \varphi(x)\nabla f(x)\cdot\nu(x)d\sigma(x).\end{aligned}$$

Now in $\Omega_{R,\varepsilon}$, we have, writing $F(r) = 1/r^{n-2}$,

$$\begin{cases} \nabla(|x|^{n-2}) = F'(|x|)\dfrac{x}{|x|} = -\dfrac{(n-2)x}{|x|^n}, \\ \Delta f(x) = F''(|x|) + \dfrac{n-1}{|x|}F'(x) = \dfrac{(n-2)(n-1)}{|x|^n} - \dfrac{(n-2)}{|x|^{n-1}}\dfrac{(n-1)}{|x|} = 0. \end{cases}$$

Since moreover φ vanishes on S_R^{n-1}, we get

$$\begin{aligned}\int_{\Omega_{R,\varepsilon}} & f(x)\Delta\varphi(x)dx \\ &= \int_{S_\varepsilon^{n-1}} f(x)\nabla\varphi(x)\cdot\nu(x) - \varphi(x)\nabla f(x)\cdot\nu(x)d\sigma(x) \\ &= \int_{S_1^{n-1}} \frac{1}{\varepsilon^{n-2}}\nabla\varphi(\varepsilon\omega)\cdot(-\omega)\varepsilon^{n-1}d\omega + \int_{S_1^{n-1}} \varphi(\varepsilon\omega)\frac{(n-2)\varepsilon\omega}{\varepsilon^n}\cdot(-\omega)\varepsilon^{n-1}d\omega.\end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain

$$\int_\Omega f(x)\Delta\varphi(x)dx = -(n-2)\varphi(0)|S_1^{n-1}|.$$

We have proved the

Proposition 3.1.3 Let $E_n \in \mathcal{D}'(\mathbb{R}^n)$ be the distribution associated to the function $f \in L^1_{loc}(\mathbb{R}^n)$ given by

$$f(x) = \frac{c_n}{|x|^{n-2}}, \qquad \text{with } c_n = \frac{1}{(2-n)|S_1^{n-1}|}.$$

Then

$$\Delta E_n = \delta_0,$$

Anticipating a bit, we introduce right now a very important notion.

Definition 3.1.4 Let $P = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$ be a differential operator with constant coefficients on $\mathbb{R}^n$. A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of P when $PE = \delta_0$.

Example 3.1.5 We have seen that distribution $E_n \in \mathcal{D}'(\mathbb{R}^n)$ defined as

$$E_n = \begin{cases} \dfrac{c_n}{|x|^{n-2}}, \quad c_n = -\dfrac{1}{(n-2)|\mathbb{S}^{n-1}|} & \text{for } n \geq 3, \\ \dfrac{1}{2\pi} \ln |x| & \text{for } n = 2, \\ xH(x) & \text{for } n = 1, \end{cases}$$

is a fundamental solution of the Laplacean $P = \Delta$ in $\mathcal{D}'(\mathbb{R}^n)$.

Exercise 3.1.6 Show that $\partial_{\bar{z}}(\frac{1}{\pi z}) = \delta_0$, i.e. $\frac{1}{\pi z}$ is a fundamental solution of the operator $\partial_{\bar{z}}$ (read "d-bar") in $\mathcal{D}'(\mathbb{R}^2)$.

B. Malgrange and L. Ehrenpreis have shown independently in 1954/1955, that any (non-trivial) differential operator with constant coefficients has a fundamental solution. There even exists different more or less explicit formulas for theses distributions, that are however very difficult to use. On the other hand one knows that this is not true for differential operators with non-constant coefficients, yet relatively simple.

3.2 Harmonic Functions

A function $\varphi \in \mathcal{C}^2(\overline{\Omega})$ is harmonic in Ω when $\Delta\varphi(x) = 0$ for any $x \in \Omega$. Harmonic functions have remarkable properties, as for example the so-called mean value formula.

Proposition 3.2.1 Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\varphi \in \mathcal{C}^2(\overline{\Omega})$ a harmonic function on Ω. For any $x_0 \in \Omega$, we have

$$\varphi(x_0) = \frac{1}{vol(B(x_0, R))} \int_{B(x_0,R)} \varphi(x)dx = \frac{1}{|S^{n-1}(x_0, R)|} \int_{S^{n-1}(x_0,R)} \varphi(x)d\sigma(x),$$

where $R > 0$ is such that $B(x_0, R) \subset \Omega$.

Proof.— Let E_n be the fundamental solution of the Laplacean defined above. For any $\varepsilon > 0$ less than R, we denote

$$\Omega_{R,\varepsilon} = B(x_0, R) \setminus B(x_0, \varepsilon).$$

We have $\partial\Omega_{R,\varepsilon} = S_R^{n-1} \bigcup S_\varepsilon^{n-1}$, where $\nu(x) = x/R$ for $x \in S_R^{n-1}$, and $\nu(x) = -x/\varepsilon$ for $x \in S_\varepsilon^{n-1}$.

Since φ is harmonic and $\Delta E_n = 0$ on $\Omega_{R,\varepsilon}$, we have

$$\int_{\Omega_{R,\varepsilon}} (\varphi\Delta E_n - E_n\Delta\varphi)dx = 0.$$

Green's formula gives

$$\int_{\partial\Omega_{R,\varepsilon}} (\varphi\nabla E_n \cdot \nu - E_n\nabla\varphi \cdot \nu)dx = 0.$$

Since φ is harmonic in Ω, we have, by Stokes' formula

$$0 = \int_{\Omega_{R,\varepsilon}} 1 \times \Delta\varphi \, dx = -\int_{\Omega_{R,\varepsilon}} \Delta(1)\varphi \, dx + \int_{\partial\Omega_{R,\varepsilon}} 1 \times \nabla\varphi \cdot \nu \, d\sigma = \int_{\partial\Omega_{R,\varepsilon}} \nabla\varphi \cdot \nu \, d\sigma.$$

Since E_n is a constant function on $\partial\Omega_{R,\varepsilon}$, we get

$$\int_{\partial\Omega_{R,\varepsilon}} E_n \nabla\varphi \cdot \nu \, d\sigma = 0,$$

and finaly

$$\int_{\partial\Omega_{R,\varepsilon}} \varphi \nabla E_n \cdot \nu \, d\sigma = 0.$$

Moreover

$$\nabla E_n = c_n \nabla\big(\frac{1}{\|x\|^{n-2}}\big) = \frac{1}{|S_1^{n-1}|} \frac{x}{\|x\|^n},$$

thus, for any $\varepsilon > 0$ small enough,

$$0 = \frac{1}{|S_1^{n-1}|} \int_{S_R^{n-1}} \varphi(x) \frac{1}{R^{n-1}} d\sigma(x) - \frac{1}{|S_1^{n-1}|} \int_{S_\varepsilon^{n-1}} \varphi(x) \frac{1}{\varepsilon^{n-1}} d\sigma(x).$$

Last, wheb $\varepsilon \to 0^+$,

$$\int_{S_\varepsilon^{n-1}} \varphi(x) \frac{1}{\varepsilon^{n-1}} d\sigma(x) = \int_{S_1^{n-1}} \varphi(\varepsilon\omega) d\omega \to |S_1^{n-1}| \varphi(0),$$

and we have proved that

$$\varphi(0) = \frac{1}{|S_1^{n-1}| R^{n-1}} \int_{S_R^{n-1}} \varphi(x) d\sigma(x) = \frac{1}{|S_R^{n-1}|} \int_{S_R^{n-1}} \varphi(x) d\sigma(x).$$

To finish the proof, we notice that $\nabla(|x|^2) = 2x$, $\Delta(|x|^2) = 2n$, and thus

$$\begin{aligned}
\int_{B(0,R)} \varphi(x) dx &= \frac{1}{2n} \int_{B(0,R)} (\varphi(x)\Delta(|x|^2) - \Delta\varphi(x)|x|^2) dx \\
&= \frac{1}{2n} \int_{S_R^{n-1}} [\varphi(x)\nabla(|x|^2) \cdot \nu(x) - |x|^2 \nabla\varphi(x) \cdot \nu(x)] d\sigma(x) \\
&= \frac{R}{n} \int_{S_R^{n-1}} \varphi(x) d\sigma(x) - \frac{R^2}{2n} \int_{S_R^{n-1}} \nabla\varphi(x) \cdot \nu(x) d\sigma(x) = \frac{R}{n} \int_{S_R^{n-1}} \varphi(x) d\sigma(x).
\end{aligned}$$

We have in particular

$$vol(B(0,R)) = \int_{B(0,R)} 1 \, dx = \frac{R}{n} \int_{S_R^{n-1}} d\sigma(x) = \frac{R}{n} |S_R^{n-1}|,$$

so that

$$\frac{1}{vol(B(0,R))} \int_{B(0,R)} \varphi(x) dx = \frac{1}{|S_R^{n-1}|} \int_{S_R^{n-1}} \varphi(x) d\sigma(x),$$

as stated. □

Corollary 3.2.2 (The maximum Principle) Let $\Omega \subset \mathbb{R}^n$ be a connected open set, and $\varphi \in \mathcal{C}^2(\overline{\Omega})$ a harmonic function on Ω. If φ has a local maximum in Ω, then φ is a constant function on Ω.

Proof.— Suppose that there exists $x_0 \in \Omega$ and an open set $V \subset \Omega$ that contains x_0, and such that

$$\forall x \in V,\ \varphi(x) \leq \varphi(x_0).$$

There exists $R > 0$ such that $B(x_0, R) \subset V$, and the mean value formula gives

$$\varphi(x_0) = \frac{1}{|B(x_0, R)|} \int_{B(x_0,R)} \varphi(x)dx,$$

which we can also write

$$\int_{B(x_0,R)} \varphi(x) - \varphi(x_0)dx = 0.$$

Since the function $\varphi(x) - \varphi(x_0)$ is continuous and non-positive on V, we get that $\varphi(x) = \varphi(x_0)$ for all $x \in B(x_0, R)$.

Thus, the set $A = \{x \in \Omega,\ \varphi(x) = \varphi(x_0)\}$ is open. It is also closed by definition, and it is not empty since x_0 is in A. As Ω is connected, we obtain that $A = \Omega$. □

3.A Exercises

Exercise 3.A.1 1. Let $f \in \mathcal{C}^1(\mathbb{R}^n)$, $n \geq 2$, and $R > 0$. Show that for all $k \in \{1, \dots, n\}$,

$$\int_{|x|\leq R} \partial_k f(x)dx = \frac{1}{R} \int_{|x|=R} x_k f(x)d\sigma(x),$$

where σ is the surface measure on the sphere of radius R.

2. Suppose that $f \in \mathcal{C}^3(\mathbb{R}^n)$ is harmonic. Why do we have, for any $R > 0$ and any $k \in \{1, \dots, n\}$,

$$\partial_k f(0) = \frac{1}{vol(B(0, R))} \int_{|x|\leq R} \partial_k f(x)dx \quad ?$$

3. Deduce that if, moreover, f is bounded on $\mathbb{R}^n$, then f is a constant function.

Exercise 3.A.2 For $R > 0$, σ_R is the surface measure on the sphere $S_R \subset \mathbb{R}^n$, and we denote $\mu_R = \dfrac{1}{\sigma_R(S_R)}\sigma_R$.

1. Using Stokes formula, compute $\int x_i d\mu_R$ et $\int x_i x_j d\mu_R$.

2. Give the limit as $R \to 0$ of the family of distributions $T_R = \dfrac{1}{R^2}(\mu_R - \delta_0)$.

Hint. Use Taylor's expansion of φ at 0.

3. Let $f \in C(\mathbb{R}^n, \mathbb{R})$ be a f subharmonic function, that is for all $R > 0$ and all $x \in \mathbb{R}^n$,

$$f(x) \leq (f * \mu_R)(x) = \int_{S_R} f(x-y) d\mu_R(y).$$

Show that Δf is a non-negative distribution.

4. Deduce that if f is continuous and satisfies the mean value property, then f is harmonic.

Chapter 4

Convolution of distributions

4.1 Differentiation and integration inside the bracket

Proposition 4.1.1 Let $\Omega \subset \mathbb{R}^p$ be an open set, and $T \in \mathcal{D}'(\Omega)$. Let also $\varphi \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^q)$. If there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi \subset K \times \mathbb{R}^q$, then the function

$$G : y \in \mathbb{R}^q \mapsto \langle T, \varphi(\cdot, y) \rangle$$

is $\mathcal{C}^\infty$, and, for $\alpha \in \mathbb{N}^q$,

$$\partial^\alpha G(y) = \langle T, \partial_y^\alpha \varphi(\cdot, y) \rangle$$

Remark 4.1.2 *i)* We have written $\langle T, \varphi(\cdot, y) \rangle$ in place of $\langle T, \varphi_y \rangle$, where $\varphi_y \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is the function given by $\varphi_y(x) = \varphi(x, y)$.

ii) The assumption $\operatorname{supp} \varphi \subset K \times \mathbb{R}^q$ means that for any $y \in \mathbb{R}^q$, the support of φ_y is included in K. It holds in particular when $\varphi \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^q)$.

iii) For a regular distribution $T = T_f$, with $f \in L^1_{loc}(\Omega)$, we have $G(y) = \int f(x)\varphi(x, y)dy$, so that, under the above assumptions, we get $G \in \mathcal{C}^\infty(\mathbb{R}^q)$ and

$$\partial^\alpha G(y) = \int f(x) \partial_y^\alpha \varphi(x, y) dy.$$

Proof.— Let $y_0 \in \mathbb{R}^q$ and $x \in \Omega$. For $h \in \mathbb{R}^q$, Taylor's formula at order 1 gives

$$\varphi(x, y_0 + h) = \varphi(x, y_0) + \sum_{j=1}^{q} \partial_{y_j} \varphi(x, y_0) h_j + r(x, y_0, h),$$

$$\text{with } r(x, y_0, h) = 2 \sum_{|\alpha| \le 2} \frac{h^\alpha}{\alpha!} \int_0^1 (1 - t) \partial_y^\alpha \varphi(x, y_0 + th) dt.$$

Since $x \mapsto r(x, y_0, h))$ is $\mathcal{C}^\infty$ with support in K,

$$|\langle T, r(x, y_0, h)\rangle| \leq C \sum_{|\beta|\leq k} \sup |\partial_x^\beta r(x, y_0, h)|$$

for a constant $C > 0$ and an integer $k \in \mathbb{N}$ independent of y_0 and h. But for $|h| \leq 1$,

$$|\partial_x^\beta r(x, y_0, h)| \leq 2 \sum_{|\alpha|\leq 2} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t)\partial_x^\beta \partial_y^\alpha \varphi(x, y_0 + th)dt \leq C|h|^2 \sum_{|\alpha|\leq 2} \sup_{K\times\overline{B}(0,1)} |\partial_x^\beta \partial_y^\alpha \varphi(x, y)|,$$

Thus

$$|\langle T, r(x, y_0, h)\rangle| = \mathcal{O}(|h|^2).$$

and,

$$G(y + h) = G(y) + \sum_{j=1}^q \langle T, \partial_{y_j} \varphi(x, y_0)\rangle h_j + O(|h|^2),$$

which shows that G is differentiable at y - in particular G is continuous, and that

$$\partial_j G(y) = \langle T, \partial_{y_j} \varphi(x, y)\rangle.$$

Then one can replace φ by $\partial_{y_j}\varphi(x, y)$ in the above discussion. We see that for all j, $\partial_j \varphi$ is differentiable, thus in particular continuous. So G is $\mathcal{C}^1$, and the statement of the proposition is true for any $|\alpha| = 1$. One can easily get the general case by induction. □

Proposition 4.1.3 Let $\Omega \subset \mathbb{R}^p$ be an open set,and $T \in \mathcal{D}'(\Omega)$. Let also $\varphi \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^q)$. Then

$$\int_{\mathbb{R}^q} \langle T, \varphi(\cdot, y)\rangle dy = \langle T, \int_{\mathbb{R}^q} \varphi(\cdot, y)dy\rangle$$

Proof.— We start with the case $q = 1$. Let $\varphi \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R})$. We choose $A > 0$ and a compact set $K \subset \Omega$ such that supp $\varphi \subset K \times [-A, A]$. We denote $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$ the function given by

$$\psi(x, y) = \int_{t<y} \varphi(x, t)dt.$$

The function ψ belongs to $\mathcal{C}^\infty(\Omega \times \mathbb{R})$, and for any y, supp$(x \mapsto \psi(x, y))$ is included in K. Therefore the above proposition applies: the function

$$G(y) = \langle T, \psi(x, y)\rangle = \langle T, \int_{t<y} \varphi(x, t)dt\rangle$$

is smooth, and

$$G'(y) = \langle T, \partial_y \psi(x, y)\rangle = \langle T, \varphi(x, y)\rangle.$$

Integrating, we get

$$\langle T, \int_{t<y} \varphi(x, t)dt\rangle = G(y) = \int_{t<y} G'(t)dt = \int_{t<y} \langle T, \varphi(x, t)\rangle dt.$$

This gives the proposition, taking $y = A$ for example.

For $q > 1$, we proceed by repeated integrations. Let $\varphi \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^q)$. We can suppose that $\operatorname{supp}\varphi \subset K \times [-A, A]^q$ for a compact subset $K \subset \Omega$, and $A > 0$. We denote $\psi_q : \Omega \times \mathbb{R}^q \to \mathbb{C}$ the function given by

$$\psi_q(x, y', y_q) = \int_{t<y_q} \varphi(x, y', t)dt,$$

where we have denoted $y = (y', y_q) \in \mathbb{R}^{q-1} \times \mathbb{R}$. Using the result in the case $q = 1$, we get

$$\langle T, \int_{\mathbb{R}} \varphi(x, y', t)dt\rangle = \int_{\mathbb{R}} \langle T, \varphi(x, y', t)\rangle dt.$$

Then we denote $\psi_{q-1} \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^{q-1})$ the function given by (where now $y' \in \mathbb{R}^{q-2}$),

$$\psi_{q-1}(x, y', y_{q-1}) = \int_{t_2<y_{q-1}} (\int_{\mathbb{R}} \varphi(x, y', t_2, t_1)dt_1)dt_2.$$

We obtain

$$\begin{aligned}\langle T, \psi_{q-1}(x, y', y_{q-1})\rangle = G(y_{q-1}) &= \int_{t_2<y_{q-1}} G'(t)dt = \int_{t<y_{q-1}} \langle T, \int_{\mathbb{R}} \varphi(x, y', t, t_1)dt_1\rangle dt \\ &= \int_{t_2<y_{q-1}} \int_{\mathbb{R}} \langle T, \varphi(x, y', t, t_1)\rangle dt_1 dt.\end{aligned}$$

The proposition then follows by induction. □

4.2 Tensor products

4.2.1 Tensor products of functions

Definition 4.2.1 Let $f : \mathbb{R}^p \to \mathbb{C}$ and $g : \mathbb{R}^q \to \mathbb{C}$ be two functions. The function $f \otimes g$ is defined on $\mathbb{R}^{p+q}$ by

$$f \otimes g(x) = f(x_1)g(x_2), \text{ where } x = (x_1, x_2) \in \mathbb{R}^{p+q}, x_1 \in \mathbb{R}^p, x_2 \in \mathbb{R}^q.$$

This function is called the tensor product of f and g.

For example, the monomials $\mathbb{R}^n \ni x \mapsto x^\alpha$, $\alpha \in \mathbb{N}^n$, are tensor products: one may write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2},$$

for any decomposition $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^q$, $p + q = n$. As a matter of fact, one can define inductively the tensor product of many functions, and one easily sees that

$$x^\alpha = x_1^{\alpha_1} \otimes x_2^{\alpha_2} \otimes \cdots \otimes x_n^{\alpha_n}.$$

When a function f of n variables can be written as a tensor product of n functions of a single variable, one says that f is a function with separate variables.

Proposition 4.2.2 Let $\mathcal{T} \subset \mathcal{C}_0^\infty(\mathbb{R}^p \times \mathbb{R}^q)$ be the space of finite linear combinations of tensor products of functions in $\mathcal{C}_0^\infty(\mathbb{R}^p)$ and $\mathcal{C}_0^\infty(\mathbb{R}^q)$. For any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{p+q})$, one can find a sequence (φ_j) in $\mathcal{T}$ such that

$$\varphi_j \to \varphi \text{ in } \mathcal{C}_0^\infty(\mathbb{R}^{p+q}).$$

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{p+q})$, and $K = K_1 \times K_2 \subset \mathbb{R}^p \times \mathbb{R}^q$ be a compact that contains the support of φ. Then, let $(\chi_1, \chi_2) \in \mathcal{C}_0^\infty(\mathbb{R}^p) \times \mathcal{C}_0^\infty(\mathbb{R}^q)$ such that χ_j is a plateau function above K_j, and set $\tilde{K} = \operatorname{supp} \chi_1 \otimes \chi_2$.

By Stone's theorem, we know that there is a sequence P_j of polynomials such that, for all $\alpha \in \mathbb{N}^n$, $\partial^\alpha P_j \to \partial^\alpha \varphi$ uniformly on $\tilde{K}$. We set

$$\varphi_j(x_1, x_2) = \chi_1(x_1)\chi_2(x_2)P_j(x_1, x_2) = \sum_{|\beta| \leq m_j} a_{j,\beta}\chi_1(x_1)x_1^{\beta_1}\chi_2(x_2)x_2^{\beta_2},$$

where m_j is the degree of P_j. The function φ_j belongs to $\mathcal{T}$,and one can easily see that $\varphi_j \to \varphi$ in $\mathcal{C}_0^\infty(\mathbb{R}^{p+q})$. □

Proposition 4.2.3 Let $\Omega \subset \mathbb{R}^p$ be an open set, and let $T \in \mathcal{D}'(\Omega)$.

$$\partial_j T = 0 \text{ for all } j \in \{1, \dots, p\} \iff T = T_C \text{ for some constant } C \in \mathbb{C}.$$

Proof.— Let $\varphi_1, \varphi_2 \dots, \varphi_n$ be functions in $\mathcal{C}_0^\infty(\mathbb{R})$, and $\varphi = \varphi_1 \otimes \ \dots \ \otimes \varphi_n$. Let also $\chi_1, \chi_2, \dots, \chi_n$ be functions in $\mathcal{C}_0^\infty(\mathbb{R})$ such that $\int \chi_j = 1$ for all j. We have

$$\begin{aligned}\langle T, \varphi \rangle =& \langle T, \varphi_1 \otimes \ \dots \ \otimes \varphi_n \rangle \\ =& \langle T, \big(\varphi_1 - (\int \varphi_1)\chi_1\big) \otimes \ \dots \ \otimes \varphi_n \rangle + \langle T, (\int \varphi_1)\chi_1 \otimes \ \dots \ \otimes \varphi_n \rangle\end{aligned}$$

But $\varphi_1 - (\int \varphi_1)\chi_1$ belongs to $\mathcal{C}_0^\infty(\mathbb{R})$ an has a vanishing integral, thus it can be written ψ' for some $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$. Thus

$$\langle T, \big(\varphi_1 - (\int \varphi_1)\chi_1\big) \otimes \ \dots \ \otimes \varphi_n \rangle = \langle T, \psi' \otimes \dots \ \otimes \varphi_n \rangle = -\langle \partial_1 T, \psi' \otimes \dots \ \otimes \varphi_n \rangle = 0,$$

and

$$\langle T, \varphi \rangle = \big(\int \varphi_1\big)\langle T, \chi_1 \otimes \ \dots \ \otimes \varphi_n \rangle.$$

By induction, we get

$$\langle T, \varphi \rangle = \big(\int \varphi_1\big)\big(\int \varphi_2\big) \dots \big(\int \varphi_n\big)\langle T, \chi_1 \otimes \ \dots \ \otimes \chi_n \rangle = \big(\int \varphi\big)\langle T, \chi \rangle = \langle T_C, \varphi \rangle,$$

where $C = \langle T, \chi \rangle$. □

4.2.2 Tensor products of distributions

When $(f_1, f_2) \in \mathcal{C}^0(\Omega_1)\times\mathcal{C}^0(\Omega_2)$, where $\Omega_1 \subset \mathbb{R}^p$ and $\Omega_2 \subset \mathbb{R}^q$ are open sets, for $\varphi \in \mathcal{C}_0^\infty(\Omega_1\times\Omega_2)$ we have,

$$\langle T_{f_1\otimes f_2}, \varphi\rangle = \langle T_{f_1,x_1}, \langle T_{f_2,x_2}, \varphi(x_1, x_2)\rangle\rangle.$$

In particular if $\varphi = \varphi_1 \otimes \varphi_2$,

$$\langle T_{f_1\otimes f_2}, \varphi\rangle = \langle T_{f_1}, \varphi_1\rangle\langle T_{f_2}, \varphi_2\rangle.$$

Proposition 4.2.4 Let $(T_1, T_2) \in \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2)$.

i) For any $\varphi \in \mathcal{C}_0^\infty(\Omega_1 \times \Omega_2)$, the function $\psi : x \mapsto \langle T_{2,y}, \varphi(x, y)\rangle$ belongs to $\mathcal{C}_0^\infty(\Omega_1)$.

ii) The linear form $T : \varphi \in \mathcal{C}_0^\infty(\Omega_1 \times \Omega_2) \mapsto \langle T_1, \psi\rangle$ is a distribution in $\mathcal{D}'(\Omega_1 \times \Omega_2)$. We denote it $T = T_1 \otimes T_2$, and it is called the tensor product of T_1 and T_2.

Proof.— Let $K = K_1 \times K_2$ be a compact subset of $\mathbb{R}^P \times \mathbb{R}^q$. Since T_1 is a distribution, there is a constant $C_1 > 0$ and an integer $k_1 \geq 0$ such that for all function $\psi \in \mathcal{C}^\infty_{K_1}$,

$$|\langle T_1, \psi\rangle| \leq C_1 \sum_{|\alpha|\leq k_1} \text{sup}\, |\partial^\alpha \psi|.$$

The same way, there is a constant $C_2 > 0$ and an integer $k_2 \geq 0$ such that for all function $\psi \in \mathcal{C}^\infty_{K_2}$,

$$|\langle T_2, \psi\rangle| \leq C_2 \sum_{|\beta|\leq k_2} \text{sup}\, |\partial^\beta \psi|.$$

Then let $\varphi \in \mathcal{C}_0^\infty(\Omega_1 \times \Omega_2)$ be such that supp $\varphi \subset K$. The assertion (i) follows easily from Proposition 4.1.1. Furthermore we have supp $\psi \subset K_1$ and $\partial^\alpha\psi(x) = \langle T_{2,y}, \partial_x^\alpha \varphi(x, y)\rangle$. Thus

$$|\langle T, \varphi\rangle| = |\langle T_1, \psi\rangle| \leq C_1 \sum_{|\alpha|\leq k_1} \text{sup}\, |\langle T_{2,y}, \partial_x^\alpha\varphi(x, y)\rangle| \leq C_1C_2 \sum_{|\alpha|\leq k_1, |\beta|\leq k_2} \text{sup}\, |\partial_y^\beta \partial_x^\alpha \varphi(x, y)|.$$

Notice that the constants C_2 et k_2 do not depend on x, since for all x, supp$(y \mapsto \varphi(x, y) \subset K_2$. □

Proposition 4.2.5 Let $(T_1, T_2) \in \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2)$. The distribution $T = T_1 \otimes T_2$ is the only element in $\mathcal{D}'(\Omega_1 \times \Omega_2)$ such that, for all $(\varphi_1, \varphi_2) \in \mathcal{C}_0^\infty(\Omega_1) \times \mathcal{C}_0^\infty(\Omega_2)$,

$$\langle T, \varphi_1 \otimes \varphi_2\rangle = \langle T_1, \varphi_1\rangle\langle T_2, \varphi_2\rangle.$$

Proof.— First of all, it is clear that $T_1 \otimes T_2$ has this property:

$$\langle T_1 \otimes T_2, \varphi_1 \otimes \varphi_2\rangle = \langle T_{1,x}, \langle T_{2,y}, \varphi_1(x)\varphi_2(y)\rangle\rangle = \langle T_2, \varphi_2\rangle\langle T_1, \varphi_1\rangle.$$

Now let T and $\widetilde{T}$ be two distributions in $\mathcal{D}'(\Omega_1 \times \Omega_2)$ satisftying this property. For $\varphi \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, there is a sequence (φ_j) of functions in $\mathcal{T}$ such that $\varphi_j \to \varphi$ in $\mathcal{D}'(\Omega_1 \times \Omega_2)$. But we obviously have $\langle T - \widetilde{T}, \varphi_j \rangle = 0$, and therefore

$$0 = \lim_{j \to +\infty} \langle T - \widetilde{T}, \varphi_j \rangle = \langle T - \widetilde{T}, \varphi \rangle.$$

Thus $T = \widetilde{T}$. □

Proposition 4.2.6 For $(T_1, T_2) \in \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2)$, it holds that

$$\operatorname{supp} T_1 \otimes T_2 = \operatorname{supp} T_1 \times \operatorname{supp} T_2.$$

Proof.— Let $y_0 \notin \operatorname{supp} T_2$. There exists a neighborhood V of y_0 such that for all $\psi \in \mathcal{C}_0^\infty(V)$, $\langle T_2, \psi \rangle = 0$. Thus, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^p \times V)$, we get

$$\langle T_1 \otimes T_2, \varphi \rangle = \langle T_1, \langle T_2, \varphi \rangle \rangle = \langle T_1, 0 \rangle = 0,$$

which shows that $\operatorname{supp} T_1 \otimes T_2 \subset \mathbb{R}^p \times \operatorname{supp} T_2$. The same way $\operatorname{supp} T_1 \otimes T_2 \subset \operatorname{supp} T_1 \times \mathbb{R}^q$, and thus

$$\operatorname{supp} T_1 \otimes T_2 \subset (\mathbb{R}^p \times \operatorname{supp} T_2) \cap (\operatorname{supp} T_1 \times \mathbb{R}^q) = \operatorname{supp} T_1 \times \operatorname{supp} T_2.$$

Conversely, suppose that $(x_0, y_0) \in \operatorname{supp} T_1 \times \operatorname{supp} T_2$. Let W be a neighborhood of (x_0, y_0). There are neighborhoods U of x_0 and V of y_0 such that $W \supset U \times V$. Since $x_0 \in \operatorname{supp} T_1$ and $y_0 \in \operatorname{supp} T_2$, there us $\varphi_1 \in \mathcal{C}_0^\infty(U)$ and $\varphi_2 \in \mathcal{C}_0^\infty(U)$ such that

$$\langle T_1 \otimes T_2, \varphi_1 \otimes \varphi_2 \rangle = \langle T_1, \varphi_1 \rangle \langle T_2, \varphi_2 \rangle \neq 0.$$

Therefore $(x_0, y_0) \in \operatorname{supp} T_1 \otimes T_2$. □

Proposition 4.2.7 Let $(T_1, T_2) \in \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2)$, where $\Omega_1 \subset \mathbb{R}^p$, $\Omega_2 \subset \mathbb{R}^q$ are two open sets. For $j \in \{1, \dots, p+q\}$, we have

$$\partial_j (T_1 \otimes T_2) = \begin{cases} (\partial_j T_1) \otimes T_2 \text{ pour } 1 \leq j \leq p, \\ T_1 \otimes (\partial_{j-p} T_2) \text{ pour } p+1 \leq j \leq p+q. \end{cases}$$

Proof.— It is sufficient to prove this for test functions $\varphi \in \mathcal{C}_0^\infty(\Omega_1 \times \Omega_2)$ of the form $\varphi = \varphi_1 \otimes \varphi_2$. But

$$\langle \partial_j (T_1 \otimes T_2), \varphi_1 \otimes \varphi_2 \rangle = -\langle T_1 \otimes T_2, \partial_j (\varphi_1 \otimes \varphi_2) \rangle,$$

and $\partial_j(\varphi_1 \otimes \varphi_2) = (\partial_j \varphi_1) \otimes \varphi_2$ for $j \leq p$, yet $\partial_j(\varphi_1 \otimes \varphi_2) = \varphi_1 \otimes (\partial_{j-p} \varphi_2)$ for $j \geq p+1$. □

Proposition 4.2.8 Let (T_j) be a sequence of distributions in $\mathcal{D}'(\Omega_1)$, and (S_j) a sequence of distributions in $\mathcal{D}'(\Omega_2)$, where $\Omega_1 \subset \mathbb{R}^p$, $\Omega_2 \subset \mathbb{R}^q$ are two open sets. If $(T_j) \to T$ in $\mathcal{D}'(\Omega_1)$ and $(S_j) \to S$ in $\mathcal{D}'(\Omega_2)$, then

$$T_j \otimes S_j \to T \otimes S \text{ in } \mathcal{D}'(\Omega_1 \times \Omega_2).$$

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(\Omega_1 \times \Omega_2)$, and $K = K_1 \times K_2$ be a compact set that contains the support of φ. Then, let $\psi_n : x \mapsto \langle S_n, \varphi(x, \cdot)\rangle$. Thanks to Proposition 4.1.1, we know that $\psi_n \in \mathcal{C}^\infty(\Omega_1)$, and that

$$\partial^\alpha \psi_n(x) = \langle S_n, \partial_x^\alpha \varphi(x, \cdot)\rangle.$$

Moreover we see that supp $\psi_n \subset K_1$ for any n.

Since $T_n \to T$ in $\mathcal{D}'(\Omega_1)$, it suffices to show $\psi_n \to \psi = \langle S, \varphi(x_1, .)\rangle$ in $\mathcal{C}_0^\infty(\Omega_1)$. The only remaining task is thus to prove uniform convergence of $\partial^\alpha \psi_n$ to $\partial^\alpha \psi$ on K_1.

Let (x_n) be a sequence in K_1 that converges to $x \in K_1$. We have

$$\partial^\alpha \psi_n(x_n) = \langle S_n, \partial_x^\alpha \varphi(x_n, \cdot)\rangle,$$

and $\partial_x^\alpha \varphi(x_n, \cdot) \to \partial_x^\alpha \varphi(x, \cdot)$ in $\mathcal{C}_0^\infty(\Omega_2)$. Indeed, supp$(y \mapsto \partial_x^\alpha \varphi(x_n, y)) \subset K_2$, and

$$|\partial_y^\beta \partial_x^\alpha \varphi(x_n, y) - \partial_y^\beta \partial_x^\alpha \varphi(x, y)| \leq C|x_n - x|,$$

because $\partial_y^\beta \partial_x^\alpha \varphi$ is $\mathcal{C}^1$ with compact support. Therefore, as $n \to +\infty$,

$$\partial^\alpha \psi_n(x_n) - \partial^\alpha \psi(x) = \langle S_n, \partial_x^\alpha \varphi(x_n, \cdot) - \partial_x^\alpha \varphi(x, \cdot)\rangle \to 0,$$

and this finishes the proof of the proposition. □

Exercice 4.2.9 (Distributions that are independant of one or many variables) Let $T \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, where $\Omega_1 \subset \mathbb{R}^p$ et $\Omega_2 \subset \mathbb{R}^q$ are open sets. Show that the following two assertions are equivalent:

i) $\partial_j T = 0$ for all $j \in \{p+1, \ldots, p+q\}$,

ii) There exists $S \in \mathcal{D}'(\Omega_1)$ such that $T = S \otimes 1$.

Application: Solve in $\mathcal{D}'(\mathbb{R}^2)$ the equation $\partial_1 \partial_2 T = 0$.

4.3 Convolution

For $f, g \in L^1(\mathbb{R}^n)$ we have defined $f * g \in L^1(\mathbb{R}^n)$ by

$$f * g(x) = \int f(x-y)g(y)dy.$$

Thus for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\langle T_{f*g}, \varphi\rangle &= \int f * g(x)\varphi(x)dx = \iint f(x-y)g(y)\varphi(x)dxdy = \iint f(z)g(y)\varphi(y+z)dzdy \\ &= \iint (f \otimes g)(z, y)\varphi(y+z)dzdy = \langle T_{f\otimes g}, \varphi^\Delta\rangle,\end{aligned}$$

where φ^Δ is the function on $\mathbb{R}^n \times \mathbb{R}^n$ given by $\varphi^\Delta(x, y) = \varphi(x+y)$. Therefore, the only reasonable choice for the definition of the convolution of two distributions $T, S \in \mathcal{D}'(\mathbb{R}^n)$ is

$$\langle T * S, \varphi\rangle = \langle T \otimes S, \varphi^\Delta\rangle. \tag{4.3.1}$$

However, we immediately see that it will not be possible to define in general the convolution of two distributions, since the function φ^Δ has not compact support for a general $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Nevertheless, as in the definition of the action of a distribution with compact support on general smooth functions, we will be able to give a meaning to the R. H. S. of (4.3.1) under some natural assumption on the distributions T and S.

4.3.1 The key lemma

Definition 4.3.1 Let $\Omega \in \mathbb{R}^n$ be an open set, and $T \in \mathcal{D}'(\Omega)$. We denote $\mathcal{E}(T) = \mathcal{E}(\operatorname{supp} T)$ the set of functions $\varphi \in \mathcal{C}^\infty(\Omega)$ such that

$$\operatorname{supp} T \cap \operatorname{supp} \varphi \text{ is compact.}$$

Of course, it holds that $\mathcal{C}_0^\infty(\Omega) \subset \mathcal{E}(T)$ for any distribution T.

Proposition 4.3.2 Let $T \in \mathcal{D}'(\Omega)$. There exists a unique linear form $\widetilde{T}$ on $\mathcal{E}(T)$ satisfying the two following properties:

i) $\widetilde{T}$ extends T: if $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\widetilde{T}(\varphi) = \langle T, \varphi\rangle$.

ii) $\widetilde{T}$ is continuous on $\mathcal{E}(T)$, in the following sense:
if (φ_j) is a sequence of $\mathcal{E}(T)$ such that $\operatorname{supp}\varphi_j \cap \operatorname{supp} T$ is included in a compact set $K \subset \mathbb{R}^n$ independent of j, and if for all $\alpha \in \mathbb{N}^n$, $(\partial^\alpha \varphi_j) \to \partial^\alpha\varphi$ uniformly on every compact subset of Ω, then $\widetilde{T}(\varphi_j) \to \widetilde{T}(\varphi)$

Proof.— We first exhibit such a linear form. Let $\varphi \in \mathcal{E}(T)$, and $\chi \in \mathcal{C}_0^\infty(\Omega)$, a plateau function above $\operatorname{supp}\varphi \cap \operatorname{supp} T$. We set

$$\widetilde{T}(\varphi) = \langle T, \chi\varphi\rangle.$$

If χ_1 and χ_2 are such plateau functions, we have

$$\operatorname{supp} T \cap \operatorname{supp}(\chi_1 - \chi_2)\varphi \subset \operatorname{supp} T \cap \operatorname{supp}\varphi \cap \operatorname{supp}(\chi_1 - \chi_2) = \emptyset,$$

thus $\widetilde{T}$ does not depend on the choice of χ. In particular $\widetilde{T}$ is linear: if φ_1 and φ_2 belongs to $\mathcal{E}(T)$, then

$$\widetilde{T}(\varphi_1 + \varphi_2) = \langle T, \chi(\varphi_1 + \varphi_2)\rangle$$

where we can take for χ a plateau function above $(\operatorname{supp}\varphi_1 \cup \operatorname{supp}\varphi_2) \cap \operatorname{supp} T$. Thus

$$\widetilde{T}(\varphi_1 + \varphi_2) = \langle T, \chi\varphi_1\rangle + \langle T, \chi\varphi_2\rangle = \widetilde{T}(\varphi_1) + \widetilde{T}(\varphi_2)$$

We also have $\langle T, (1-\chi)\varphi\rangle = 0$, so that $\widetilde{T}$ satisfies the property (i). Last, for a plateau function above K ,

$$\widetilde{T}(\varphi_j - \varphi) = \langle T, \chi(\varphi_j - \varphi)\rangle \to 0$$

since $(\chi(\varphi_j - \varphi))$ goes to 0 in $\mathcal{C}_0^\infty(\Omega)$, and $\widetilde{T}$ satisies property (ii).

Concerning uniqueness, suppose $\widetilde{\widetilde{T}}$ is a linear form on $\mathcal{E}(T)$ that satisfies (i) and (ii). Let (K_j) be an exhaustion sequence of Ω, that is K_j is a compact set, $K_j \subset \mathring{K}_{j+1}$, and $\Omega = \cup_j K_j$. Let also $\chi_j \in \mathcal{C}_0^\infty(\mathring{K}_{j+1})$ be a plateau function over K_j. For $\varphi \in \mathcal{E}(T)$, we have $\widetilde{\widetilde{T}}(\chi_j\varphi) \to \widetilde{\widetilde{T}}(\varphi)$ thanks to (ii). Indeed

- For all j, we have $\operatorname{supp} \chi_j\varphi \cap \operatorname{supp} T \subset \operatorname{supp} \varphi \cap \operatorname{supp} T = K$, that is a compact set independent of j.
- There exists $j_0 \in \mathbb{N}$ such that $K \subset K_{j_0}$. Thus, for all $j \geq j_0$,
$$\partial^\alpha(\chi_j\varphi) = \partial^\alpha\varphi,$$
which shows that $\partial^\alpha\chi_j\varphi \to \partial^\alpha\varphi$ uniformly on every compact subset of Ω.

Since $\widetilde{\widetilde{T}}(\chi_j\varphi) = \langle T, \chi_j\varphi\rangle$ thanks to (i), for $\varphi \in \mathcal{E}(T)$, we also have
$$\widetilde{\widetilde{T}}(\varphi) = \lim_{j\to+\infty} \langle T, \chi_j\varphi\rangle.$$

Finally, since χ_j is a plateau function over $\operatorname{supp}\varphi \cap \operatorname{supp} T$ for any $j \geq j_0$, we also have
$$\lim_{j\to+\infty} \langle T, \chi_j\varphi\rangle = \langle T, \chi_{j_0}\varphi\rangle = \widetilde{T}(\varphi),$$

and thus $\widetilde{\widetilde{T}}(\varphi) = \widetilde{T}(\varphi)$. □

In the particular case of a distribution $T \in \mathcal{D}'(\Omega)$ with compact support, it is obvious that $\mathcal{E}(T) = \mathcal{C}^\infty(\Omega)$. Thus the previous proposition states that one can extend in a unique way the distribution T to a continuous linear form on $\mathcal{C}^\infty(\Omega)$, as we already know (at least in dimension 1) from Section 1.6.4.

4.3.2 Convolvable Pairs

Let us recall that a map $s : X \to Y$ is said to be proper when $s^{-1}(K)$ is a compact set in X for any compact set $K \subset Y$.

Definition 4.3.3 Let F and G be to closed subset of $\mathbb{R}^n$. We say that the pair $\{F, G\}$ is convolvable when the map
$$\begin{array}{rccc} s: & F \times G & \to & \mathbb{R}^n \\ & (x, y) & \mapsto & x + y \end{array}$$
is proper.

For example, in $\mathbb{R}$ the pair $\{\mathbb{R}, \mathbb{R}\}$ is not convolvable, but the pair $\{\mathbb{R}^+, \mathbb{R}^+\}$ is. Indeed

- If $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $s(x, y) = x + y$, then $s^{-1}([-a, a]) = \{(x, y) \in \mathbb{R} \times \mathbb{R},\ -a \leq x + y \leq a\}$ is the set of points in the plane between the lines $y = -a - x$ and $y = a - x$, which is not compact.

- Now if $s : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is given by $s(x, y) = x + y$, then $s^{-1}([-a, a]) = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+,\ -a \leq x + y \leq a\} = \{(x, y),\ 0 \leq x,\ 0 \leq y \leq a - x\}$ which is a compact set.

Notice that a pair $\{F, G\}$ of subsets in $\mathbb{R}^n$ is convolvable if and only if, denoting S the map

$$S: \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (x, y) & \mapsto & x + y \end{array}$$

the set $(S_{|_{F \times G}})^{-1}(K)$ is a compact set for any compact $K \subset \mathbb{R}^n$. Indeed

$$(S_{|_{F \times G}})^{-1}(K) = S^{-1}(K) \cap F \times G = s^{-1}(K).$$

Proposition 4.3.4 If F or G is a compact set, then the pair $\{F, G\}$ is convolvable.

Proof.— Suppose that F is compact. We set $s : F \times G \to \mathbb{R}^n$ the map given by $s(x, y) = x + y$. Let $K \subset \mathbb{R}^n$ be a compact set, and $(x_j, y_j)_j$ a sequence of $s^{-1}(K)$. The sequence (z_j) given by $z_j = x_j + y_j$is a sequence of the compact set K, thus one can find a subsequence $(z_{j_k})_k$ which is convergent. Since F is compact, we can also extract from (x_{j_k}) a subsequence $(x_{j_{k_\ell}})$ which converges. Then, the sequence $y_{j_{k_\ell}} = z_{j_{k_\ell}} - x_{j_{k_\ell}}$ converges too, and we have found a a subsequence $(x_{j_{k_\ell}}, y_{j_{k_\ell}})_\ell$ from $(x_j, y_j)_j$ that is convergent. Therefore $s^{-1}(K)$ is a compact set, and $\{F, G\}$ is convolvable. □

Exercise 4.3.5 Let F and G be two closed subset of $\mathbb{R}^n$.

i) Give an example where $F + G$ is not closed.

ii) Show that if F is compact, $F + G$ is closed.

iii) Show that if $\{F, G\}$ is convolvable, then $F + G$ is closed.

Let us prove (iii), since we will need this property below. Let (z_j) be a sequence of $F + G$ that converges to some z in $\mathbb{R}^n$. There exists $(x_j) \subset F$ and $(y_j) \subset G$ such that $z_j = x_j + y_j$. The set $K = \{z\} \cup \{z_j, j \in \mathbb{N}\}$ is compact, thus $s^{-1}(K)$ is compact too. Since $(x_j, y_j)_j \subset s^{-1}(K)$, there exists a subsequence $(x_{j_k}, y_{j_k})_k$ that converges to some (x, y) in $F \times G$. Now $x_{j_k} + y_{j_k} \to z$, so that we have $z = x + y$, i.e. $z \in F + G$. Thus the limit of any convergent sequence of elements in $F + G$ belongs to $F + G$, and this proves that $F + G$ is closed.

4.3.3 Definition of the convolution of distributions

Two distributions $T, S \in \mathcal{D}'(\mathbb{R}^n)$ are called convolvables when the pair $\{\operatorname{supp} T, \operatorname{supp} S\}$ is convolvable.

Proposition 4.3.6 Let $T, S \in \mathcal{D}'(\mathbb{R}^n)$ be two convolvable distributions. For any function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, the function φ^Δ belongs to $\mathcal{E}(T \otimes S)$.

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We want to show that $A = \text{supp}\,\varphi^\Delta \cap \text{supp}\,T \otimes S = \text{supp}\,\varphi^\Delta \cap \text{supp}\,T \times \text{supp}\,S$ is a compact set. This is the case since $s : \text{supp}\,T \times \text{supp}\,S \to \mathbb{R}^n$, $s(x, y) = x + y$ is a proper map and $A = s^{-1}(\text{supp}\,\varphi)$. □

Proposition 4.3.7 Let $T, S \in \mathcal{D}'(\mathbb{R}^n)$ be two convolvable distributions. The linear form

$$\varphi \mapsto \langle \widetilde{T \otimes S}, \varphi^\Delta \rangle$$

is a distribution on $\mathbb{R}^n$. We denote it $T * S$, and it is called the convolution product of T and S.

Proof.— Let (φ_j) be a sequence of functions in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ which converges to φ in $\mathcal{C}_0^\infty(\mathbb{R}^n)$. We are going to show that $\langle T * S, \varphi_j \rangle \to \langle T * S, \varphi \rangle$ using property (ii).

- There is a compact set $K \subset \mathbb{R}^n$ that contains the support of φ and of all the functions φ_j. Thus $\text{supp}\,\varphi_j^\Delta \cap \text{supp}\,T \otimes \text{supp}\,S \subset s^{-1}(K) = \widetilde{K} \subset \mathbb{R}^{2n}$ which is a compact set independent of j .
- For all $\alpha \in \mathbb{N}^{2n}$, $\partial^\alpha \varphi_j^\Delta$ converges to $\partial^\alpha \varphi^\Delta$ uniformly on every compact of $\mathbb{R}^{2n}$. Indeed we notice that
$$\partial^\alpha \varphi_j^\Delta(x, y) = \partial_x^{\alpha_1} \partial_y^{\alpha_2}(\varphi(x + y)) = (\partial^{\alpha_1+\alpha_2}\varphi)(x + y).$$
Thus, let $C \subset \mathbb{R}^{2n}$ be a compact set. We can write $C = C_1 \cup C_2$, where $C_1 = (C \cap \widetilde{K})$ and $C_2 = (C \cap (\mathring{\widetilde{K}})^c)$ are two compact set, and
$$\sup_C |\partial^\alpha \varphi_j^\Delta - \partial^\alpha \varphi^\Delta| \leq \sup_{C_1} |[\partial^{\alpha_1+\alpha_2}(\varphi_j - \varphi)]^\Delta| + \sup_{C_2} |[\partial^{\alpha_1+\alpha_2}(\varphi_j - \varphi)]^\Delta|.$$
Then we notice that the second term vanishes (by continuity φ_j and φ vanishes at the boundary of K), so that
$$\sup_C |\partial^\alpha \varphi_j^\Delta - \partial^\alpha \varphi^\Delta| \leq \sup_{C_1} |[\partial^{\alpha_1+\alpha_2}(\varphi_j - \varphi)]^\Delta|.$$
Therefore, we only have to prove that $\partial^\alpha \varphi_j^\Delta$ converges to $\partial^\alpha \varphi^\Delta$ converges on any compact subset C_1 of $\tilde{K}$. But if $(x_j, y_j) \to (x, y) \in C_1$, we have
$$\partial^\alpha \varphi_j^\Delta(x_j, y_j) = (\partial^{\alpha_1+\alpha_2}\varphi)(x_j + y_j) \to (\partial^{\alpha_1+\alpha_2}\varphi)(x + y) = \partial^\alpha \varphi^\Delta(x, y),$$
since $\partial^{\alpha_1+\alpha_2}\varphi_j$ converges uniformly to $\partial^{\alpha_1+\alpha_2}\varphi$ on K.

Thus, we have $\langle T * S, \varphi_j \rangle = \langle \widetilde{T \otimes S}, \varphi_j \rangle \to \langle \widetilde{T \otimes S}, \varphi \rangle = \langle T * S, \varphi \rangle$. □

This definition of the convolution of distributions extends, as required, the convolution of functions. Indeed

Proposition 4.3.8 If $T = T_f$ and $S = T_g$ for $f, g \in L^1_{loc}(\mathbb{R}^n)$, and if the pair $\{\text{supp}\, f, \text{supp}\, g\}$ is convolvable, then

$$T_f * T_g = T_{f*g},$$

where $f * g$ is the function in $L^1_{loc}(\mathbb{R}^n)$ given by $f * g(x) = \displaystyle\int f(x-y)g(y)dy$.

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We have

$$\langle T * S, \varphi\rangle = \langle \widetilde{T \otimes S}, \varphi\rangle = \langle T \otimes S, \chi\varphi^\Delta\rangle = \iint f(x)g(y)\chi(x,y)\varphi(x+y)dxdy,$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is identically 1 near $\text{supp}\, T \otimes S \cap \text{supp}\, \varphi^\Delta$. It is easily checked that $(x,y) \mapsto f(x)g(y)\chi(x,y)\varphi(x+y)$ is a function in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$, and changing variables according to $(x,y) \mapsto (z,y) = (x+y,y)$ gives

$$\langle T * S, \varphi\rangle = \iint f(z-y)g(y)\varphi(z)dzdy = \langle T_{f*g}, \varphi\rangle.$$

□

4.3.4 Main properties of the convolution

Proposition 4.3.9 Let $T, S \in \mathcal{D}'(\mathbb{R}^n)$ be two convolvable distributions. Then

$$T * S = S * T.$$

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a plateau function over $\text{supp}\, T \times \text{supp}\, S \cap \text{supp}\, \varphi^\Delta$. We have

$$\langle T * S, \varphi\rangle = \langle T_x \otimes S_y, \chi(x,y)\varphi^\Delta(x,y)\rangle = \langle S_y, \langle T_x, \chi(x,y)\varphi^\Delta(x,y)\rangle\rangle.$$

We set $\widetilde{\chi}(y,x) = \chi(x,y)$, so that the function $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a plateau function above $\text{supp}\, S \times \text{supp}\, T \cap \text{supp}\, \varphi^\Delta$. Since $\varphi^\Delta(x,y) = \varphi^\Delta(y,x)$, we get, as stated

$$\langle T * S, \varphi\rangle = \langle S_y, \langle T_x, \widetilde{\chi}(y,x)\varphi^\Delta(y,x)\rangle\rangle = \langle \widetilde{S \otimes T}, \varphi^\Delta\rangle = \langle S * T, \varphi\rangle.$$

□

Proposition 4.3.10 For any $T \in \mathcal{D}'(\mathbb{R}^n)$, T and δ_0 are convolvable, and

$$T * \delta_0 = \delta_0 * T = T.$$

Proof.— The pair $\{\text{supp}\, T, \text{supp}\, \delta_0\}$ is always convolvable since $\text{supp}\, \delta_0 = \{0\}$ is a compact set. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ ta plateau function over $\text{supp}\, T \times \text{supp}\, \delta_0 \cap \text{supp}\, \varphi^\Delta$, we have

$$\langle T * \delta_0, \varphi\rangle = \langle \widetilde{T \otimes \delta_0}, \varphi^\Delta\rangle = \langle T_x, \langle \delta_{y=0}, \chi(x,y)\varphi(x+y)\rangle\rangle = \langle T_x, \chi(x,0)\varphi(x)\rangle = \langle T, \varphi\rangle.$$

□

Proposition 4.3.11 Let $T, S \in \mathcal{D}'(\mathbb{R}^n)$ be two convolvable distributions. For any $j \in \{1, \dots, n\}$, it holds that

$$\partial_j(T * S) = (\partial_j T) * S = T * (\partial_j S).$$

As an immediate consequence, we get $\partial^\alpha(T * S) = (\partial^\alpha T) * S = T * (\partial^\alpha S)$ for any $\alpha \in \mathbb{N}^n$.

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2n})$ a plateau function above $\text{supp}\, T \otimes S \cap \text{supp}\, \varphi^\Delta$. We have

$$\langle \partial_j(T * S), \varphi\rangle = -\langle T * S, \partial_j \varphi\rangle = -\langle T \otimes S, \chi(\partial_j\varphi)^\Delta\rangle.$$

But

$$(\partial_j\varphi)^\Delta(x,y) = (\partial_j\varphi)(x+y) = \partial_{x_j}(\varphi(x+y)) = \partial_{x_j}(\varphi^\Delta)(x,y),$$

so that

$$\begin{aligned}\langle \partial_j(T * S), \varphi\rangle &= -\langle T \otimes S, \chi\partial_{x_j}\varphi^\Delta\rangle \\ &= -\langle T \otimes S, \partial_{x_j}(\chi\varphi^\Delta)\rangle + \langle T \otimes S, \partial_{x_j}(\chi)\varphi^\Delta\rangle \\ &= \langle(\partial_j T) \otimes S, \chi\varphi^\Delta\rangle = \langle \partial_j T * S, \varphi\rangle.\end{aligned}$$

We have used the fact that $\langle T \otimes S, \partial_{x_j}(\chi)\varphi^\Delta\rangle = 0$, since $\partial_{x_j}\chi = 0$ near $\text{supp}\, T \otimes S \cap \text{supp}\, \varphi^\Delta$. □

Proposition 4.3.12 Let $T, S \in \mathcal{D}'(\mathbb{R}^n)$ be two convolvable distributions. We have

$$\text{supp}(T * S) \subset \text{supp}\, T + \text{supp}\, S.$$

Proof.— We denote again $s : \text{supp}\, T \times \text{supp}\, S \to \mathbb{R}^n$ the proper map given by $s(x,y) = x + y$. Let $x \notin \text{supp}\, T + \text{supp}\, S$. Since $\text{supp}\, T + \text{supp}\, S$ is closed, there exists $\delta > 0$ such that $B(x,\delta) \cap (\text{supp}\, T + \text{supp}\, S) = \emptyset$. For $\varphi \in \mathcal{C}_0^\infty(B(x,\delta))$, we have

$$\text{supp}\, \varphi^\Delta \cap (\text{supp}\, T \times \text{supp}\, S) = s^{-1}(\text{supp}\, \varphi) \subset s^{-1}(B(x,\delta)) = \emptyset.$$

Therefore $\langle T * S, \varphi\rangle = \langle \widetilde{T \otimes S}, \varphi^\Delta\rangle = 0$,and $x \notin \text{supp}(T * S)$. □

Proposition 4.3.13 Let (T_j) and (S_j) be two sequences of distributions, such that $T_j \to T$ and $S_j \to S$ in $\mathcal{D}'(\mathbb{R}^n)$. If there is a compact set $K \subset R^n$ such that $\text{supp}\, T_j \subset K$ for all j, then $T_j * S_j \to T * S$ to $\mathcal{D}'(\mathbb{R}^n)$.

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We have

$$\langle T_j * S_j, \varphi\rangle = \langle \widetilde{T_j \otimes S_j}, \varphi^\Delta\rangle = \langle T_j \otimes S_j, \chi_j \varphi^\Delta\rangle,$$

where $\chi_j \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is any function with value 1 in a neighborhood of supp $T_j \times$ supp $S_j \cap$ supp φ^Δ. But

$$\text{supp}\, T_j \times \text{supp}\, S_j \cap \text{supp}\, \varphi^\Delta \subset K \times \mathbb{R}^n \cap \text{supp}\, \varphi^\Delta \subset s^{-1}(\text{supp}\, \varphi)$$

is a compact set, independent of j, since the map $s : K \times \mathbb{R}^n \to \mathbb{R}^n$ given by $s(x, y) = x + y$ is proper.

Therefore one can take $\chi_j = \chi$, where $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is a plateau function above $s^{-1}(\text{supp}\, \varphi)$. Then we get

$$\langle T_j * S_j, \varphi\rangle = \langle T_j \otimes S_j, \chi\varphi^\Delta\rangle \to \langle T \otimes S, \chi\varphi^\Delta\rangle = \langle T * S, \varphi\rangle,$$

thanks to Proposition 4.2.8. □

4.3.5 Particular cases : $\mathcal{E}'(\mathbb{R}^n) * \mathcal{C}^\infty(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n) * \mathcal{C}_0^\infty(\mathbb{R}^n)$

For a function $f : \mathbb{R}^n \to \mathbb{C}$, we denote $\check{f}$ the function defined by $\check{f}(x) = f(-x)$. Of course $\check{f}$ is smooth when f is smooth, supp $\check{f} = -$ supp f, and $\check{\check{f}} = f$.

Proposition 4.3.14 Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that T or f is compactly supported. Then $g : x \mapsto \langle \widetilde{T}, f(x - \cdot)\rangle$ is a smooth function on $\mathbb{R}^n$, and $T * f = g$.

Proof.— For any fixed $x \in \mathbb{R}^n$, the set

$$\text{supp}\, T \cap \text{supp}\, f(x - \cdot) = \text{supp}\, T \cap (\text{supp}\, \check{f} + x),$$

is compact, thus g is well defined on $\mathbb{R}^n$.

Let us prove that g is $\mathcal{C}^\infty$ at each point $x_0 \in \mathbb{R}^n$, using Proposition 4.1.1. For $x \in B(x_0, r)$, we have

$$\text{supp}\, T \cap \text{supp}\, f(x - \cdot) = \text{supp}\, T \cap (\text{supp}\, \check{f} + x) \subset \text{supp}\, T \cap (\text{supp}\, \check{f} + \overline{B(x_0, r)}) = K,$$

which is a compact set. Then let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a plateau function above K. We have

$$g(x) = \langle \widetilde{T}, f(x - \cdot)\rangle = \langle T_y, \chi(y) f(x - y)\rangle,$$

and $(x, y) \mapsto \chi(y) f(x - y)$ is a function in $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, whose support is included in $\mathbb{R}^n \times$ supp χ. Since supp χ does not depend on $x \in B(x_0, r)$, Proposition 4.1.1 shows that g is a smooth function on $B(x_0, r)$.

Let then $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ a plateau function above supp $T \times$ supp $f \cap$ supp φ^Δ. We get

$$\begin{aligned}\langle T * f, \varphi\rangle &= \langle \widetilde{T \otimes f}, \varphi^\Delta\rangle = \langle T \otimes f, \psi\varphi^\Delta\rangle \\ &= \langle T_x, \langle f(y), \psi(x, y)\varphi(x + y)\rangle\rangle = \langle T_x, \int f(y)\psi(x, y)\varphi(x + y) dy\rangle \\ &= \langle T_x, \int f(z - x)\psi(x, z - x)\varphi(z) dz\rangle,\end{aligned}$$

after the change of variable $y \mapsto z = x+y$. Furthermore, the function $(x,z) \mapsto \psi(x, z-x)\varphi(z)f(z-x)$ belongs to $\mathcal{C}_0^\infty(\mathbb{R}^{2n})$, and exchanging the integral and the the bracket gives

$$\langle T * f, \varphi\rangle = \int \langle T_x, f(z-x)\psi(x,z-x)\varphi(z)\rangle dz = \int \langle \widetilde{T_x}, f(z-x)\rangle \varphi(z)dz,$$

since, for each fixed z, $\psi(x, z-x)$ is equal to 1 in a neighborhood of $\operatorname{supp} T \cap \operatorname{supp} f(z-\cdot)$. Thus, as claimed, we get $\langle T * f, \varphi\rangle = \langle g, \varphi\rangle$. □

Proposition 4.3.15 Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. If T or f is compactly supported, we have

$$T * f(0) = \langle T, \check{f}\rangle \text{ and } \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n),\ \langle T * f, \varphi\rangle = \langle T, \check{f} * \varphi\rangle.$$

Proof.— We know that $S * \check{h}(0) = \langle \tilde{S}, \check{h}(0-\cdot)\rangle = \langle S, h\rangle$. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we thus have

$$\langle T * f, \varphi\rangle = ((T * f) * \check{\varphi})(0).$$

Now we have $(T * f) * \check{\varphi} = T * (f * \check{\varphi})$. Indeed, for $x \in \mathbb{R}^n$, $(T * f) * \check{\varphi}$ is the convolution of two smooth functions, one of which has compact support. Thus

$$(T * f) * \check{\varphi}(x) = \int (T * f)(x-y)\check{\varphi}(y)dy = \int \langle T_z, \chi(z) f(x-y-z)\rangle \check{\varphi}(y)dy,$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is any plateau function above $\operatorname{supp} T \cap \operatorname{supp} f(x-\cdot)$, a compact set. Since

$$(y,z) \mapsto \chi(z) f(x-y-z)\varphi(y)$$

is a smooth and compactly supported function, one can apply Proposition 4.1.3 and we get

$$(T * f) * \check{\varphi}(x) = \langle T_z, \chi(z) \int f(x-y-z)\check{\varphi}(y)dy\rangle = \langle T_z, \chi(z)(f * \check{\varphi})(x-z)\rangle.$$

Since we can assume that χ is a plateau function above

$$\operatorname{supp} T \cap \operatorname{supp}(f * \check{\varphi})(x-\cdot) \subset \operatorname{supp} T \cap (\operatorname{supp} \check{f} + \operatorname{supp} \check{\varphi} + x),$$

which contains $\operatorname{supp} T \cap \operatorname{supp} f(x-\cdot) = \operatorname{supp} T \cap (\operatorname{supp} \check{f} + x)$, we get eventually

$$(T * f) * \check{\varphi}(x) = \langle \widetilde{T}_z, f * \check{\varphi})(x-z)\rangle = T * (f * \check{\varphi})(x).$$

Thus

$$\langle T * f, \varphi\rangle = T * (f * \check{\varphi})(0) = \langle T, \check{f} * \varphi\rangle,$$

where we have again used the identity $\overbrace{f * \check{\varphi}} = \check{f} * \varphi$. □

Exercise 4.3.16 Show that $\langle T * f, \varphi\rangle = \langle \widetilde{T}, \check{f} * \varphi\rangle$ when T and f are only supposed to be convolvable.

For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, the R.H.S. $\langle \widetilde{T}, \check{f} * \varphi \rangle$ is well defined. Indeed

$$\operatorname{supp} T \cap \operatorname{supp} \check{f} * \varphi \subset \operatorname{supp} T \cap \operatorname{supp} \check{f} + \operatorname{supp} \varphi,$$

and

$$\begin{aligned} x \in \operatorname{supp} T \cap \operatorname{supp} \check{f} + \operatorname{supp} \varphi &\Leftrightarrow \begin{cases} x \in \operatorname{supp} T, \\ \exists (y,z) \in \operatorname{supp} f \times \operatorname{supp} \varphi \text{ such that } x = -y + z, \end{cases} \\ &\Leftrightarrow \begin{cases} x \in \operatorname{supp} T, \\ \exists y \in \operatorname{supp} f \text{ such that } x + y \in \operatorname{supp} \varphi. \end{cases} \\ &\Leftrightarrow x \in \Pi_1(\operatorname{supp} T \times \operatorname{supp} f \cap \operatorname{supp} \varphi^\Delta). \end{aligned}$$

Since T and f are convolvable, this last set is compact, so that $\operatorname{supp} T \cap \operatorname{supp} \check{f} * \varphi$ is also compact.

Now let $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a plateau function near $\Pi_1(\operatorname{supp} T \times \operatorname{supp} f \cap \operatorname{supp} \varphi^\Delta)$. We have

$$\langle \widetilde{T}, \check{f} * \varphi \rangle = \langle T, \theta \check{f} * \varphi \rangle = \langle T_x, \int \theta(x) \check{f}(x-y)\varphi(y)dy \rangle = \langle T_x, \int \theta(x) f(z) \varphi(x+z) dz \rangle,$$

by a change of variable. Since the function $(x, y) \mapsto \theta(x)f(z)\varphi(x+z)$ is smooth with compact support (because the pair $\{\operatorname{supp} f, \operatorname{supp} \theta\}$ is convolvable), integrating in the bracket we obtain

$$\langle \widetilde{T}, \check{f} * \varphi \rangle = \int \langle T_x, \theta(x) f(z)\varphi(x+z) \rangle dz = \int f(z) \langle (\theta T)_x, \varphi(x+z) \rangle dz.$$

Using Proposition 4.3.14 we get

$$\langle \widetilde{T}, \check{f} * \varphi \rangle = \int f(z) \big(\widecheck{\theta T} * \varphi \big)(z) dz = \langle \widecheck{\theta T} * \varphi, f \rangle.$$

Finally, we use Proposition 4.3.15 in the case $\mathcal{E}'(\mathbb{R}^n) * \mathcal{C}^\infty(\mathbb{R}^n)$, and we get

$$\langle \widetilde{T}, \check{f} * \varphi \rangle = \langle \widecheck{\theta T}, \check{\varphi} * f \rangle = \langle T, \widecheck{\check{\varphi} * f} = \langle T, \check{f} * \varphi \rangle.$$

Proposition 4.3.17 Let $\Omega \in \mathbb{R}^n$ be an open set, and $T \in \mathcal{D}'(\Omega)$. There is a sequence (ψ_j) of functions in $\mathcal{C}_0^\infty(\Omega)$ such that $\psi_j \to T$ in $\mathcal{D}'(\Omega)$. Otherwise stated, $\mathcal{C}_0^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$.

Proof.— We proceed by troncation and regularization. Let (K_j) be an exhaustion sequence for Ω, and for all j, let $\chi_j \in \mathcal{C}_0^\infty(\Omega)$ be a plateau function over K_j. Let also $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, with support in $B(0,1)$ such that $\int \rho = 1$, and set $\rho_j(x) = j^n \rho(jx)$ so that (ρ_j) is an approximation of the identity. We notice that $(\check{\rho}_j)$ is also an approximation of the identity.

Let $T_j \in \mathcal{D}'(\mathbb{R}^n)$ be the distribution defined by

$$\langle T_j, \varphi \rangle = \langle T, \chi_j \varphi \rangle, \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

Since they have compact support, the distributions T_j and ρ_j are convolvable,and $T_j * \rho_j$ has compact support. Thanks to the previous proposition, $\psi_j = T_j * \rho_j$ belongs to $\mathcal{C}_0^\infty(\mathbb{R}^n)$, with support included in supp $T_j +$ supp $\rho_j \subset K_j + B(0, 1/j) \subset \Omega$ for j large enough. Thus we can consider it as a function in $\mathcal{C}_0^\infty(\Omega) \subset \mathcal{D}'(\Omega)$, and for $\varphi \in \mathcal{C}_0^\infty(\Omega)$ we have,

$$\langle \psi_j, \varphi \rangle = \langle T_j * \rho_j, \varphi \rangle = \langle T_j, \check{\rho}_j * \varphi \rangle.$$

Since $T_j \to T$ in $\mathcal{D}'(\Omega)$, and $\check{\rho}_j * \varphi \to \varphi$ in $\mathcal{C}_0^\infty(\Omega)$, we obtain that $\langle \psi_j, \varphi \rangle \to \langle T, \varphi \rangle$ and $\psi_j \to T$ in $\mathcal{D}'(\Omega)$. □

Proposition 4.3.18 Let T, S and R be three distributions in $\mathcal{D}'(\mathbb{R}^n)$. If two of them have compact support, then

$$(T * S) * R = T * (S * R).$$

Proof.— Suppose for example that T and S have compact support. First, notice that each of the convolution product in the above formula is well defined. Then, let (ρ_ε) be an approximation of the identity. We set $f_\varepsilon = T * \rho_\varepsilon$, $g_\varepsilon = S * \rho_\varepsilon$ and $h_\varepsilon = R * \rho_\varepsilon$. We know that $f_\varepsilon, g_\varepsilon$ and h_ε are smooth functions, and that, for all $\varepsilon \in]0, 1]$, f_ε has compact support, which is included in $K_T =$ supp $T + \overline{B(0, 1)}$. The same way, supp $g_\varepsilon \subset K_S =$ supp $S + \overline{B(0, 1)}$.

Thanks to Proposition 4.3.13, $f_\varepsilon \to T$, $g_\varepsilon \to S$ and $h_\varepsilon \to R$ in $\mathcal{D}'(\mathbb{R}^n)$, since supp $\rho_\varepsilon \subset \overline{B(0, 1)}$, which is a fixed compact set. Then again with Proposition 4.3.13, since for example the functions g_ε have their support in a fixed compact, we see that

$$f_\varepsilon * g_\varepsilon \to T * S \text{ and } g_\varepsilon * h_\varepsilon \to S * R \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Eventually, since the functions f_ε and $(f_\varepsilon * g_\varepsilon)$ are supported in a fixed compact set, we get

$$(f_\varepsilon * g_\varepsilon) * h_\varepsilon \to (T * S) * R \text{ and } f_\varepsilon * (g_\varepsilon * h_\varepsilon) \to T * (S * R) \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Therefore, we are left with the proof that, for three smooth functions with two fo them compactly supported, it holds true that

$$(f_\varepsilon * g_\varepsilon) * h_\varepsilon = f_\varepsilon * (g_\varepsilon * h_\varepsilon),$$

but this follows easily from Fubini's theorem. □

Exercise 4.3.19 Compute $(H * \delta') * 1$ and $H * (\delta' * 1)$ dans $\mathcal{D}'(\mathbb{R})$ and conclude.

Remark 4.3.20 The notion of convolvable pair can be generalized: one can also talk of a finite set of convolvable sets. Then, in particular, one can show that the above associativity property holds as soon as $\{\text{supp}\, T, \text{supp}\, S, \text{supp}\, R\}$ is convolvable, which of course holds when two of the supports are compact.

4.4 Application to constant coefficients PDE's

4.4.1 Notations

Let $P \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial of n variables with complex coefficients,

$$P(X) = \sum_{|\alpha| \leq m} a_\alpha X^\alpha, \quad X \in \mathbb{R}^n,$$

with $a_\alpha \in \mathbb{C}$. The integer $m \in \mathbb{N}$ is the degree of P. We denote $P(\partial)$ the operator on $\mathcal{D}'(\Omega)$ given by

$$\mathcal{D}'(\Omega) \ni T \mapsto P(\partial)T = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha T \in \mathcal{D}'(\Omega).$$

Those operators are called linear partial differential operators with constant coefficients. The equation

$$P(\partial)T = F,$$

where $F \in \mathcal{D}'(\Omega)$ is given, and $T \in \mathcal{D}'(\Omega)$ is the unknown, is called a linear Partial Differential Equation (PDE) of order m with constant coefficients. When $F \neq 0$, it is said to be inhomogeneous, or with source term F.

Remark 4.4.1 For $n = 1$, the equation $P(\partial)T = F$ is a linear differential equation of order m with constant coefficients, that can be explicitely solved. For $n > 1$, the situation is drastically different and it may even be very difficult to show existence of solutions. For equations of order $m = 1$ however, the so-called "method of characteristics" transforms the study of such a PDE to that of a system of differential equations, and the theory of PDE's really starts with equation of order 2.

Example 4.4.2 Here follows a list of linear PDE's of order 2 with constant coefficients, together with their associated polynomial. Each of these equations has specific properties différentes (that is: the solutions of these quations have different properties).

i) The Laplace (or Poisson) equation: $\Delta T = F$ dans $\mathcal{D}'(\mathbb{R}^n)$, $P(X) = \sum_{j=1}^n X_j^2$.

ii) The wave equation: $\partial_{tt}^2 T - \Delta T = F$ dans $\mathcal{D}'(\mathbb{R}^{1+n})$, $P(X) = X_0^2 - \sum_{j=1}^n X_j^2$.

iii) The heat equation: $\partial_t T - \Delta T = F$ dans $\mathcal{D}'(\mathbb{R}^{1+n})$, $P(X) = X_0 - \sum_{j=1}^n X_j^2$.

iv) The Schrödinger equation: $i\partial_t T - \Delta T = F$ dans $\mathcal{D}'(\mathbb{R}^{1+n})$, $P(X) = iX_0 - \sum_{j=1}^n X_j^2$.

4.4.2 Fundamental solutions

Definition 4.4.3 Let $P = P(\partial)$ be a linear partial differential operators with constant coefficients. One says that $E \in \mathcal{D}'(\Omega)$ is a fundamental solution of P when $PE = \delta_0$.

Notice that in physics, fundamental solutions are often called Green functions.

Example 4.4.4 We have seen that the distribution $E_n \in \mathcal{D}'(\mathbb{R}^n)$ given by

$$E_n = \begin{cases} \dfrac{c_n}{|x|^{n-2}}, \quad c_n = -\dfrac{1}{(n-2)|\mathbb{S}^{n-1}|} & \text{for } n \geq 3, \\ \dfrac{1}{2\pi} \ln|x| & \text{for } n = 2, \\ xH(x) & \text{for } n = 1, \end{cases}$$

is a fundamentental solution of the Laplacea $P = \Delta$ in $\mathcal{D}'(\mathbb{R}^n)$.

Example 4.4.5 We also know that $\partial_{\bar z}(\frac{1}{\pi z}) = \delta_0$. Thus $\frac{1}{\pi z}$ is a fundamentental solution of the operator $\partial_{\bar z}$ (read "d-bar") in $\mathcal{D}'(\mathbb{R}^2)$.

Proposition 4.4.6 If P has a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$, then for all $F \in \mathcal{E}'(\mathbb{R}^n)$, the equation $PT = F$ has a solution.

Proof.— Since F has compact support, E and F are convolvable, and

$$P(E * F) = (PE) * F = \delta * F = F,$$

so that $T = E * F$ is a solution. □

B. Malgrange and L. Ehrenpreis have proved, independently in 1954/1955, that any linear partial differential operators with constant coefficients has a fundamental solution. As a matter of fact, different mathematicians have provided more or less explicit expressions for these solutions, but they are not very easy to use. On the other hand, it has been proved that some linear partial differential operators with non-constant coefficients, even some simple ones, don't have a fundamental solution.

We will see in Chapter 5 below that the Fourier transform is a useful tool for computing fundamental solutions. But even if we have no explicit formula, any knowledge about a fundamental solution for P may give valuable informations on the solutions of $PT = F$ for general L.H.S. F. Indeed we have

$$T = \delta * T = (PE) * T = E * (PT),$$

for $T \in \mathcal{E}'(\mathbb{R}^n)$, say. In that direction, we have for example the following

Proposition 4.4.7 Let P be a linear partial differential operators with constant coefficients. Suppose that P has a fundamental solution E whose restriction to $\mathbb{R}^n \setminus \{0\}$ is a $\mathcal{C}^\infty$ function. Then, for any open set $\Omega \subset \mathbb{R}^n$, and any $F \in \mathcal{C}^\infty(\Omega)$, the solutions of the equation $PT = F$ in $\mathcal{D}'(\Omega)$ are $\mathcal{C}^\infty$ in Ω.

Proof.— Let $T \in \mathcal{D}'(\Omega)$ be such that $PT = F$, and $x_0 \in \Omega$. We want to show that T is $\mathcal{C}^\infty$ near x_0. Thus let $\chi \in \mathcal{C}_0^\infty(\Omega)$ be a plateau function above x_0. We have $\chi T \in \mathcal{E}'(\Omega)$, an we can consider χT as an element of $\mathcal{E}'(\mathbb{R}^n)$. Thus we have $\chi T = E * (P(\chi T))$. But since $\chi = 1$ near x_0, Leibniz formula gives

$$P(\chi T) = \chi P(T) + R = \chi F + R,$$

where $R \in \mathcal{E}'(\mathbb{R}^n)$ satisfies $x_0 \notin \operatorname{supp} R$. The fact that R is compactly supported stems form the fact that $R = P(\chi T) - \chi P(T) = 0$ out of the support of χ.

Therefore we have

$$\chi T = E * (P(\chi T)) = E * (\chi F) + E * R.$$

Since $E * (\chi F) \in \mathcal{C}^\infty(\mathbb{R}^n)$ (we have extended χF to a function in $\mathcal{C}_0^\infty(\mathbb{R}^n)$), we are left with the proof that $E * R$ is smooth near x_0.

Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a plateau function above 0. Since $x_0 \notin \operatorname{supp} R$, we can choose θ so that $x_0 \notin \operatorname{supp} T + \operatorname{supp} \theta$. Moreover

$$E * R = (\theta E) * R + ((1-\theta)E) * R = (\theta E) * R + \mathcal{C}^\infty(\mathbb{R}^n),$$

since $(1-\theta)E \in \mathcal{C}^\infty(\mathbb{R}^n)$ by assumption. Last, if the support of $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is close enough to x_0, we have

$$\operatorname{supp} \varphi \cap \operatorname{supp}(\theta E) * R \subset \operatorname{supp} \varphi \cap (\operatorname{supp} \theta + \operatorname{supp} R) = \emptyset,$$

so that $(\theta E) * R = 0$ near x_0. □

Corollary 4.4.8 Let $\Omega \subset \mathbb{R}^n$, with $n \geq 2$. If $T \in \mathcal{D}'(\Omega)$ satisfies $\Delta T = 0$, then $T \in \mathcal{C}^\infty(\Omega)$.

Corollary 4.4.9 Let T be a distribution which is holomorphic in $\Omega \subset \mathbb{R}^2$, i.e. satisfying $\partial_{\bar{z}} T = 0$. Then T is $\mathcal{C}^\infty$, therefore holomorphic in the usual sense.

4.4.3 Singular support

The following notion is very convenient when one discusses about the regularity of distributions that are solutions of PDE's.

Definition 4.4.10 Let $T \in \mathcal{D}'(\Omega)$. We say that $x \in \Omega$ is not in the singular support of T, and we denote $x \notin \operatorname{suppsing}(T)$, when there is a neighborhood V of x such that $T|_V$ is $\mathcal{C}^\infty$. Otherwise stated,

$$\operatorname{suppsing}(T) = \big(\{x \in \Omega,\ T \text{ is } \mathcal{C}^\infty \text{ near } x\}\big)^c.$$

The singular support of a distribution T is a closed set, included in $\operatorname{supp} T$. Differentiation and product by a smooth function do not increase the singular support:

$$\operatorname{suppsing}(fT) \subset \operatorname{suppsing}(T) \text{ for } f \in \mathcal{C}^\infty,$$
$$\operatorname{suppsing}(\partial^\alpha T) \subset \operatorname{suppsing}(T) \text{ for } \alpha \in \mathbb{N}^n.$$

It P is a differential operator with smooth coefficients, we have thus

$$\forall T \in \mathcal{D}'(\Omega), \quad \operatorname{suppsing}(PT) \subset \operatorname{suppsing} T.$$

The converse inclusion is false in general, and the following definition is worthwile.

Definition 4.4.11 A differential operator P is hypoelliptic on Ω when

$$\forall T \in \mathcal{D}'(\Omega), \quad \operatorname{suppsing} T \subset \operatorname{suppsing}(PT).$$

Exercise 4.4.12 Show that ∂_1 is not hypoelliptic on $\mathbb{R}^2$. Consider $T = 1 \otimes \delta \in \mathcal{D}'(\mathbb{R}^2)$.

If P is hypoelliptic and $E \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of P, then

$$\text{suppsing}\, E \subset \text{suppsing}(PE) = \text{suppsing}(\delta_0) = \{0\},$$

thus E is $\mathcal{C}^\infty$ on $\mathbb{R}^n \setminus \{0\}$. Proposition 4.4.7 is essentially the converse of this statement:

Proposition 4.4.13 Let P be a constant coefficients linear partial differential operator. If P has a fundamental solution E whose restriction to $\mathbb{R}^n \setminus \{0\}$ is $\mathcal{C}^\infty$, then P is hypoelliptic on $\mathbb{R}^n$.

Proof.— Let $T \in \mathcal{D}'(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$ be such that $x \notin \text{suppsing}\, PT$. There is a neighborhood Ω of x such that the restriction of PT to Ω is $\mathcal{C}^\infty$. Proposition 4.4.7 then states that $T \in \mathcal{C}^\infty(\Omega)$, that is $x \notin \text{suppsing}\, T$. Therefore $\text{suppsing}\, T \subset \text{suppsing}(PT)$. □

Proposition 4.4.14 Let $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n)$ be two convolvable distributions. We have

$$\text{suppsing}(T_1 * T_2) \subset \text{suppsing}(T_1) + \text{suppsing}(T_2).$$

Proof.— We suppose that T_1 and T_2 have compact support. Then, $\text{suppsing}\, T_1$ and $\text{suppsing}\, T_2$ are also compact, and there are two plateau functions $\chi_j = \chi_{j,\varepsilon} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ above $\text{suppsing}\, T_j + \bar{B}(0, \varepsilon/2)$ such that $\chi_{j,\varepsilon} = 0$ out of $\text{suppsing}\, T_j + B(0, \varepsilon)$. Then we get

$$T_1 * T_2 = [(\chi_1 + (1 - \chi_1))T_1] * [(\chi_2 + (1 - \chi_2))T_2] = (\chi_1 T_1) * (\chi_2 T_2) + R,$$

where $R \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus

$$\begin{aligned} \text{suppsing}\, T_1 * T_2 &= \text{suppsing}(\chi_1 T_1) * (\chi_2 T_2) \\ &\subset \text{supp}(\chi_1 T_1) * (\chi_2 T_2) \subset \text{supp}(\chi_{1,\varepsilon} T_1) + (\chi_{2,\varepsilon} T_2). \end{aligned}$$

The result follows letting $\varepsilon \to 0$. □

Chapter 5

The Fourier Transform

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is the function $\mathcal{F}(f) \in L^\infty(\mathbb{R}^n)$ given by

$$\mathcal{F}(f)(\xi) = \int e^{-ix\cdot\xi} f(x)dx, \text{ avec } \|\mathcal{F}(f)\|_{L^\infty} \leq \|f\|_{L^1}.$$

The important role played by the Fourier transform in PDE's theory is mainly due to the fact that, when these objects are well-defined,

$$\mathcal{F}(\partial_j f)(\xi) = i\xi_j \mathcal{F}(f)(\xi).$$

Otherwise stated, $\mathcal{F}$ transforms the action of a differential operator with constant coefficients to that of the product by a polynomial. This would be worthless without an inversion formula giving back the function f in terms of $\mathcal{F}(f)$, as

$$f(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \mathcal{F}(f)(\xi)d\xi,$$

Unfortunately, this formula only makes sense when $\mathcal{F}(f) \in L^1(\mathbb{R}^n)$, and this is not the case in general for $f \in L^1$. Thus we shall start below by introducing a large enough class of functions that is stable by $\mathcal{F}$, and for which the two formula above hold.

5.1 The space $\mathcal{S}(\mathbb{R}^n)$

5.1.1 Definitions and examples

Definition 5.1.1 We denote $\mathcal{S}(\mathbb{R}^n)$ the set of functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\forall(\alpha, \beta) \in \mathbb{N}^n, \exists C_{\alpha,\beta} > 0, \quad \sup |x^\alpha \partial^\beta \varphi(x)| \leq C_{\alpha,\beta}.$$

The set $\mathcal{S}(\mathbb{R}^n)$ is a vector space, and it is called the Schwartz space.

Example 5.1.2 *i)* $\mathcal{C}_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

ii) For $z \in \mathbb{C}$ such that Re $z > 0$, the function$\varphi(x) = e^{-z|x|^2}$ belongs to $\mathcal{S}(\mathbb{R}^n)$.

iii) If $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi_1\varphi_2 \in \mathcal{S}(\mathbb{R}^n)$.

iv) No rational function (even smooth ones) belongs to $\mathcal{S}(\mathbb{R}^n)$.

Often, one rephrases the définition of $\mathcal{S}(\mathbb{R}^n)$ saying that a smooth function belongs to $\mathcal{S}(\mathbb{R}^n)$ when φ and all is derivatives are rapidly decreasing. The topology on $\mathcal{S}(\mathbb{R}^n)$ we will work with is that given by the familly $(N_p)_{p\in\mathbb{N}}$ of semi-norms (that are norms, as a matter of fact) given by

$$N_p(\varphi) = \sum_{|\alpha|,|\beta|\ \leq p} \sup |x^\alpha \partial^\beta \varphi(x)|.$$

It is clear that for $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, we have the equivalence

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow \forall p \in \mathbb{N}, N_p(\varphi) < +\infty.$$

Proposition 5.1.3 If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $\partial^\beta\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $P(x)\varphi \in \mathcal{S}(\mathbb{R}^n)$ for any polynomial P.

Proof.— This follows immediately from the fact that

$$N_p(x^\alpha\partial^\beta\varphi) = \sum_{|\lambda|,|\mu|\leq p} \sup |x^\lambda \partial^\mu (x^\alpha\partial^\beta\varphi(x))| \leq N_{p+q}(\varphi) \tag{5.1.1}$$

when $|\alpha|, |\beta| \leq q$. □

5.1.2 Convergence in $\mathcal{S}(\mathbb{R}^n)$ and density results

Definition 5.1.4 Let (φ_j) be a sequence of functions in $\mathcal{S}(\mathbb{R}^n)$. One says that (φ_j) converges to φ in $\mathcal{S}(\mathbb{R}^n)$ when, for all $p \in \mathbb{N}$,

$$N_p(\varphi_j - \varphi) \to 0 \text{ as } j \to +\infty.$$

Remark 5.1.5 Let $\alpha \in \mathbb{N}^n$. It follows from (5.1.1) that, if (φ_j) converges to φ in $\mathcal{S}(\mathbb{R}^n)$, then $(x^\alpha\varphi_j)$ converges to $x^\alpha\varphi$ and $(\partial^\alpha\varphi_j)$ converges to $(\partial^\alpha\varphi)$ in $\mathcal{S}(\mathbb{R}^n)$. Otherwise stated, multiplication by a polynomial and derivation are continuous operations in $\mathcal{S}(\mathbb{R}^n)$.

Proposition 5.1.6 For any $q \in [1, +\infty]$, $\mathcal{S}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$. More precisely $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|x^\alpha\partial^\beta\varphi\|_{L^q} \leq C_q N_p(\varphi)^{1-1/q} N_{p+n+1}(\varphi)^{1/q} \text{ for } |\alpha|, |\beta| \leq p.$$

Proof.— When $\alpha = \beta = 0$, the above inequality gives $\mathcal{S} \subset L^q$, so that we only have to prove it. We set$g(x) = x^\alpha \partial^\beta \varphi(x)$. Then

$$\begin{aligned}\|x^\alpha \partial^\beta \varphi\|_{L^q}^q = \int |g(x)|^q dx \leq \sup |g(x)|^{q-1} \int |g(x)| dx \\ \leq \sup |g(x)|^{q-1} \sup(1+|x|)^{n+1} |g(x)| \int \frac{1}{(1+|x|)^{n+1}} dx \\ \leq C_n N_0(g)^{q-1} N_{n+1}(g) = C_n N_p(\varphi)^{q-1} N_{p+n+1}(\varphi).\end{aligned}$$

□

Notice that since $N_p(\varphi) \leq N_{p+n+1}(\varphi)$, we obviously have,

(5.1.2) $$\|x^\alpha \partial^\beta \varphi\|_{L^q} \leq C_q N_{p+n+1}(\varphi).$$

When $q = 1$, Proposition 5.1.6 et (5.1.2) give the same bound

(5.1.3) $$\|x^\alpha \partial^\beta \varphi\|_{L^1} \leq C N_{p+n+1}(\varphi),$$

but for $q = \infty$, Proposition 5.1.6 is sharper than (5.1.2) and gives

(5.1.4) $$\|x^\alpha \partial^\beta \varphi\|_{L^\infty} \leq C N_p(\varphi),$$

which, as a matter of fact, is a direct consequence of the definition of N_p.

Since $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in the $L^p(\mathbb{R}^n)$ spaces for $p \in [1, +\infty[$, we get the

Corollary 5.1.7 The set $\mathcal{S}(\mathbb{R}^n)$ is dense in all the $L^p(\mathbb{R}^n)$ for $p \in [1, +\infty[$.

We also have the important

Proposition 5.1.8 The space $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$: for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, there is a sequence (φ_j) of functions in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ which converges to φ in $\mathcal{S}(\mathbb{R}^n)$.

Proof.— Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a plateau function over $B(0,1)$. We set $\varphi_j(x) = \varphi(x)\chi(x/j)$. The functions φ_j are smooth with compact support, and equal φ in $B(0,j)$. By Leibniz's formula, we obtain

$$\partial^\beta(\varphi - \varphi_j)(x) = \partial^\beta \varphi(x)(1 - \chi(x/j)) + \sum_{|\gamma| \geq 1, \gamma \leq \beta} C_\beta^\gamma \frac{1}{j^{|\gamma|}} \partial^{\beta-\gamma} \varphi(x) (\partial^\gamma \chi)(\frac{x}{j}).$$

Thus, as $j \to +\infty$,

$$\|x^\alpha \partial^\beta (\varphi - \varphi_j)(x)\|_\infty \leq \sup_{|x| \geq j} |x^\alpha \partial^\beta \varphi(x)| + \frac{C}{j} \sum_{\gamma \leq \beta} \|x^\alpha \partial^{\beta-\gamma} \varphi\|_\infty \to 0.$$

Indeed

$$\sup_{|x| \geq j} |x^\alpha \partial^\beta \varphi(x)| \leq \frac{1}{j^2} \sup \big| |x|^2 x^\alpha \partial^\beta \varphi(x) \big| \leq \frac{1}{j^2} N_{p+2}(\varphi).$$

□

5.2 The Fourier transform in $\mathcal{S}(\mathbb{R}^n)$

5.2.1 Definition and first properties

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have $\varphi \in L^1(\mathbb{R}^n)$, so that $\mathcal{F}(\varphi)$ is well defined ans belongs to $L^\infty(\mathbb{R}^n)$.

Definition 5.2.1 For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we denote $\hat{\varphi}$, $\mathcal{F}(\varphi)$ or even $\mathcal{F}_{x\to\xi}(\varphi(x))$ the function in $L^\infty(\mathbb{R}^n)$ given by

$$\hat{\varphi}(\xi) = \mathcal{F}(\varphi)(\xi) = \mathcal{F}_{x\to\xi}(\varphi(x)) = \int e^{-ix\cdot\xi}\varphi(x)dx.$$

The linear map $\varphi \mapsto \mathcal{F}(\varphi)$ is called the Fourier transform.

Here follows some of the properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ that we have asked for in the introduction.

Proposition 5.2.2 Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

i) The function $\mathcal{F}(\varphi)$ is $\mathcal{C}^1$, and $\partial_j\mathcal{F}(\varphi)(\xi) = \mathcal{F}_{x\to\xi}(-ix_j\varphi(x))$.

ii) For all $j \in \{1, \dots, n\}$, we have $\mathcal{F}(\partial_j\varphi)(\xi) = i\xi_j\mathcal{F}(\varphi)(\xi)$.

iii) For $a \in \mathbb{R}^n$, $\mathcal{F}_{x\to\xi}(\varphi(x-a)) = e^{-ia\cdot\xi}\mathcal{F}(\varphi)(\xi)$.

iv) For $a \in \mathbb{R}^n$, $\mathcal{F}_{x\to\xi}(e^{ia\cdot x}\varphi(x)) = \mathcal{F}(\varphi)(\xi - a)$.

Proof.— i) The function $(x,\xi) \mapsto e^{-ix\cdot\xi}\varphi(x)$ is $\mathcal{C}^1$ on $\mathbb{R}^n$, et

$$|\partial_{\xi_j}(e^{-ix\cdot\xi}\varphi(x))| = |-ix_je^{-ix\cdot\xi}\varphi(x)| = |x_j\varphi(x)| \in L^1(\mathbb{R}^n).$$

By Lebesgue theorem, we see that $\mathcal{F}(\varphi)$ is $\mathcal{C}^1$ and

$$\partial_{\xi_j}\mathcal{F}(\varphi)(\xi) = \int -ix_je^{-ix\cdot\xi}\varphi(x) = \mathcal{F}(-ix_j\varphi(x)).$$

ii) We write the proof for $j = 1$. Integrating by parts, we get

$$\int \partial_1\varphi(x)e^{-ix\cdot\xi}dx_1 = i\xi_1\int \varphi(x)e^{-ix\cdot\xi}dx_1$$

Now we integrate with respect to the variable x'. By Fubini, since $\varphi, \partial_1\varphi \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, we have

$$\int \partial_1\varphi(x)e^{-ix\cdot\xi}dx = i\xi_1\int \varphi(x)e^{-ix\cdot\xi}dx.$$

To get (iv), we only have to write

$$\mathcal{F}_{x\to\xi}(e^{ia\cdot x}\varphi(x)) = \int e^{-ix\cdot\xi}e^{ia\cdot x}\varphi(x)dx = \int e^{-i(x-a)\cdot\xi}\varphi(x)dx = \mathcal{F}(\varphi)(\xi-a).$$

Eventually, performing a change of variable, we have

$$\mathcal{F}_{x\to\xi}(\varphi(x-a)) = \int e^{ix\cdot\xi}\varphi(x-a)dx = \int e^{i(x+a)\cdot\xi}\varphi(x)dx = e^{-ia\cdot\xi}\mathcal{F}(\varphi)(\xi),$$

and this is property (iii). □

Because of the presence of a factor $i=\sqrt{-1}$ in (i) and (ii), it is convenient to use the notation

$$D_j = \frac{1}{i}\partial_j.$$

Then, for example, (ii) becomes $\mathcal{F}(D_j\varphi) = \xi_j\mathcal{F}(\varphi)$, and (i) is $D_j\mathcal{F}(\varphi) = -\mathcal{F}(x_j\varphi)$. Summing up, we have

$$\begin{cases} \widehat{D_j\varphi} = \xi_j\widehat{\varphi}, \\ \widehat{x_j\varphi} = -D_j\widehat{\varphi}. \end{cases}$$

Notice also that, by (i), $\mathcal{F}(\varphi)$ is $\mathcal{C}^\infty$ since $x^\alpha\varphi\in\mathcal{S}$ for all $\alpha\in\mathbb{N}^n$.

5.2.2 Gaussians (1)

Proposition 5.2.3 For $z\in\mathbb{C}$ with $\operatorname{Re} z>0$, we have

$$\mathcal{F}_{x\to\xi}(e^{-z|x|^2}) = (\frac{\sqrt{\pi}}{\sqrt{z}})^n e^{-|\xi|^2/4z},$$

where $\sqrt{z} = e^{\frac{1}{2}\ln(z)}$, and $\ln z$ is the principal determination of the logarithm in $\mathbb{C}\setminus\mathbb{R}^-$.

Proof.— First of all, we notice that

$$\mathcal{F}_{x\to\xi}(e^{-z|x|^2}) = \int e^{-i(x_1\xi_1+x_2\xi_2+\dots+x_n\xi_n)}e^{-z(x_1^2+\dots x_n^2)}dx_1\dots dx_n = \prod_{j=1}^n\int e^{-ix_j\xi_j}e^{-zx_j^2}dx_j$$

Thus it is sufficient to prove the formula for functions of 1 variable. We suppose first that $z\in]0,+\infty[$, and we set $\varphi_z(x)=e^{-zx^2}$. We have seen that

$$\partial_\xi\hat{\varphi}(\xi) = \int e^{-ix\xi}(-ix)e^{-zx^2}dx$$

Integrating by parts we get,

$$\partial_\xi\hat{\varphi}(\xi) = \frac{i}{2z}\int e^{-ix\xi}\varphi'(x)dx = -\frac{i}{2z}\int(-i\xi)e^{-ix\xi}\varphi(x)dx = -\frac{\xi}{2z}\hat{\varphi}(\xi).$$

Thus $\hat{\varphi}(\xi) = e^{-\xi^2/4z}\hat{\varphi}(0)$. But

$$\hat{\varphi}(0) = \int e^{-zx^2}dx = \frac{\sqrt{\pi}}{\sqrt{z}},$$

which give the result in this case $z \in]0, +\infty[$.

Then, let $\Omega = \{z \in \mathbb{C}, \operatorname{Re} z > 0\}$. For a fixed $\xi \in \mathbb{R}^n$, we denote $\varphi : \Omega \to \mathbb{C}$ the function given by

$$\varphi(z) = \int e^{-ix\cdot\xi} e^{-z|x|^2} dx.$$

The function φ is holomorphic on Ω. Indeed

- $z \mapsto e^{-ix\cdot\xi} e^{-z|x|^2}$ is holomorphic for all $x \in \mathbb{R}^n$,
- If $K \subset \Omega$ is compact, there is $\varepsilon > 0$ such that $\operatorname{Re} z > \varepsilon$ for all $z \in K$, and

$$|e^{-ix\cdot\xi} e^{-z|x|^2}| \leq e^{-\varepsilon|x|^2} \in L^1(\mathbb{R}^n).$$

But we know that for $z \in]0, +\infty[$,

$$\varphi(z) = (\frac{\sqrt{\pi}}{\sqrt{z}})^n e^{-|\xi|^2/4z}.$$

Since the R.H.S. is also holomorphic in Ω, this equality is still true for all z in Ω. □

Exercise 5.2.4 Compute $\mathcal{F}_{x\to\xi}(e^{-zx^2})$ by calculating $\int_{\operatorname{Im}\zeta=\xi} e^{-z\zeta^2} d\zeta$ using the residue theorem.

5.2.3 The Inversion formula

Proposition 5.2.5 The Fourier transform is an isomorphism on the vector space $\mathcal{S}(\mathbb{R}^n)$. Its inverse $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}(\varphi)(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \varphi(\xi) d\xi = \frac{1}{(2\pi)^n} \widetilde{\mathcal{F}(\varphi)}(x).$$

Moreover $\mathcal{F}$ is bicontinuous on $\mathcal{S}(\mathbb{R}^n)$, in the following sense : for all $p \in \mathbb{N}$, there is $C_p > 0$ such that

$$N_p(\mathcal{F}(\varphi)), N_p(\mathcal{F}^{-1}(\varphi)) \leq C_p N_{p+n+1}(\varphi).$$

Proof.— a) First of all, we prove that $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$. Let $p \in \mathbb{N}$, and $|\alpha|, |\beta| \leq p$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have seen that

$$\xi^\alpha \partial_\xi^\beta \mathcal{F}(\varphi)(\xi) = (-i)^{|\alpha|+|\beta|} \mathcal{F}_{x\to\xi}(\partial^\alpha(x^\beta \varphi(x))).$$

Then, Leibniz's formula gives

$$\partial^\alpha(x^\beta \varphi) = \sum_{\gamma\leq\alpha} C_\alpha^\gamma \partial^\gamma(x^\beta) \partial^{\alpha-\gamma}\varphi,$$

so that $\partial^\alpha(x^\beta\varphi) \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Thus $\mathcal{F}(\partial^\alpha(x^\beta\varphi)) \in L^\infty(\mathbb{R}^n)$, and $\mathcal{F}(\varphi) \in \mathcal{S}(\mathbb{R}^n)$.

Moreover

$$\begin{aligned}|\xi^\alpha \partial_\xi^\beta \mathcal{F}(\varphi)(\xi)| \leq & \|\mathcal{F}_{x\to\xi}(\partial^\alpha(x^\beta\varphi(x)))\|_{L^\infty} \\ \leq & \|\partial^\alpha(x^\beta\varphi(x))\|_{L^1} \\ \lesssim & N_{n+1}(\partial^\alpha(x^\beta\varphi)) \\ \lesssim & N_{n+1+p}(\varphi)\end{aligned}$$

thanks to Proposition 5.1.6. Thus $N_p(\mathcal{F}(\varphi)) \leq CN_{n+1+p}(\varphi)$.

b) Now we show that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it holds that

$$\varphi(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\hat{\varphi}(\xi)d\xi.$$

We start with

$$\int e^{ix\cdot\xi}\hat{\varphi}(\xi)d\xi = \int e^{ix\cdot\xi}\big(\int e^{-iy\cdot\xi}\varphi(y)dy\big)d\xi,$$

but the function $g : (y,\xi) \mapsto e^{ix\cdot\xi}e^{ix\cdot\xi}\varphi(y)$ is not in $L^1(\mathbb{R}_y^n \times \mathbb{R}_\xi^n)$, since $|g(y,\xi)| = |\varphi(y)|$, and one can not exchange the order in which we integrate. However, by the Dominated convergence Theorem,

$$\int e^{ix\cdot\xi}\hat{\varphi}(\xi)d\xi = \lim_{\varepsilon\to 0^+}\int e^{ix\cdot\xi}e^{-\varepsilon|\xi|^2}\hat{\varphi}(\xi)d\xi,$$

and

$$\int e^{ix\cdot\xi}e^{-\varepsilon|\xi|^2}\big(\int e^{-iy\cdot\xi}\varphi(y)dy\big)d\xi = \int\big(\int e^{i(x-y)\cdot\xi}e^{-\varepsilon|\xi|^2}d\xi\big)\varphi(y)dy.$$

Therefore, thanks to Proposition 5.2.3,

$$\int e^{ix\cdot\xi}\hat{\varphi}(\xi)d\xi = \lim_{\varepsilon\to 0^+}(\frac{\sqrt{\pi}}{\sqrt{\varepsilon}})^n\int e^{-|x-y|^2/4\varepsilon}\varphi(y)dy.$$

We change variable according to $u = (x-y)/2\sqrt{\varepsilon}$, and we get

$$\int e^{ix\cdot\xi}\hat{\varphi}(\xi)d\xi = \lim_{\varepsilon\to 0^+}(2\sqrt{\pi})^n\int e^{-|u|^2}\varphi(x-2\sqrt{\varepsilon}u)du.$$

Agin by the Dominated convergence Theorem, we finally obtain

$$\int e^{ix\cdot\xi}\hat{\varphi}(\xi)d\xi = (2\sqrt{\pi})^n\varphi(x)\int e^{-|u|^2}du = (2\pi)^n\varphi(x).$$

c) The remaining part of the proposition follows from the fact that $\mathcal{F}^{-1}(\varphi) = \frac{1}{(2\pi)^n}\widetilde{\mathcal{F}(\varphi)}$. □

5.2.4 Parseval and Plancherel

Proposition 5.2.6 (Parseval's formula, a.k.a. the "Lemme des chapeaux") Let φ and ψ be two functions in $\mathcal{S}(\mathbb{R}^n)$. Then

i) $\displaystyle\int \hat{\varphi}\psi = \int \varphi\hat{\psi}$.

ii) $\displaystyle\int \varphi\overline{\psi} = (2\pi)^{-n}\int \hat{\varphi}\overline{\hat{\psi}}$

Proof.— (i) Using Fubini's theorem, we have

$$\int \hat{\varphi}(x)\psi(x)dx = \int(\int e^{-ix\cdot y}\varphi(y)dy)\psi(x)dx = \int(\int e^{-ix\cdot y}\psi(x)dx)\varphi(y)dy = \int \varphi\hat{\psi}.$$

(ii) We apply (i) to φ and $\omega = (2\pi)^{-n}\overline{\hat{\psi}}$. We have

$$\widehat{\omega}(\xi) = (2\pi)^{-n}\int e^{-ix\cdot\xi}\overline{\hat{\psi}(x)}dx = \overline{\mathcal{F}^{-1}(\hat{\psi})(\xi)} = \overline{\psi(x)},$$

so that

$$\int \hat{\varphi}\omega = (2\pi)^{-n}\int \hat{\varphi}\overline{\hat{\psi}} = \int \varphi\overline{\psi}.$$

□

In terms of the hermitian scalar product in $L^2(\mathbb{R}^n, \mathbb{C})$, (ii) can be written

$$(\varphi, \psi)_{L^2} = \frac{1}{(2\pi)^n}(\hat{\varphi}, \hat{\psi})_{L^2}.$$

In particular, for $\psi = \varphi$, we obtain the famous Plancherel formula in $\mathcal{S}(\mathbb{R}^n)$:

Corollary 5.2.7 (Plancherel) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it holds that

$$\|\varphi\|_{L^2} = (2\pi)^{-n/2}\|\hat{\varphi}\|_{L^2}.$$

Exercise 5.2.8 Show that $\mathcal{F}(\bar{\varphi}) = \overline{\widecheck{\mathcal{F}(\varphi)}}$.

5.2.5 Convolution and the Fourier transform in $\mathcal{S}(\mathbb{R}^n)$

Proposition 5.2.9 Let φ and ψ be in $\mathcal{S}(\mathbb{R}^n)$. We have

i) $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$, and $\widehat{\varphi * \psi} = \hat{\varphi}\hat{\psi}$.

ii) $\widehat{\varphi\psi} = (2\pi)^{-n}\hat{\varphi} * \hat{\psi}$.

Proof.— (i) Note that it may not be true that φ and ψ are convolvable in the sense of distributions. However, $\varphi * \psi$ is well defined as an element in L^1 since $\varphi, \psi \in L^1$. Let $p \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha|, |\beta| \leq p$. For $x \in \mathbb{R}^n$, we have

$$x^\alpha\partial^\beta(\varphi * \psi)(x) = x^\alpha\partial_x^\beta \int \varphi(x-y)\psi(y)dy.$$

Since the function $y \mapsto \partial_x^\beta \varphi(x-y)\psi(y)$ is integrable for any x, and dominated by sup $|\partial^\beta \varphi|\ |\psi(y)| \in L^1$, we get

$$\begin{aligned} x^\alpha \partial^\beta(\varphi * \psi)(x) &= \int x^\alpha \partial_x^\beta \varphi(x-y)\psi(y)dy = \int (x-y+y)^\alpha \partial_x^\beta \varphi(x-y)\psi(y)dy \\ &= \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int (x-y)^\gamma y^{\alpha-\gamma} \partial_x^\beta \varphi(x-y)\psi(y)dy \\ &= \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int (x-y)^\gamma (\partial^\beta \varphi)(x-y) y^{\alpha-\gamma}\psi(y)dy \\ &= \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (y^\gamma \partial^\beta \varphi) * (y^{\alpha-\gamma}\psi)(x). \end{aligned}$$

Thus, by Young's inequality,

$$\|x^\alpha \partial^\beta(\varphi * \psi)\|_{L^\infty} \leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \|y^\gamma \partial^\beta \varphi\|_{L^1}\ \|y^{\alpha-\gamma}\psi\|_{L^\infty}.$$

Then Proposition 5.1.6, in particular (5.1.3) and (5.1.4), gives

(5.2.5) $$N_p(\varphi * \psi) \leq C N_{p+n+1}(\varphi) N_p(\psi),$$

and this shows that $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$. Eventually, Fubini's theorem implies that

$$\begin{aligned} \mathcal{F}(\varphi * \psi)(\xi) &= \int e^{-ix\cdot\xi} \varphi * \psi(x)dx = \int e^{-ix\cdot\xi}\big(\int \varphi(x-y)\psi(y)dy\big)dx \\ &= \int \psi(y)\big(\int e^{-ix\cdot\xi}\varphi(x-y)\psi(y)dx\big)dy \\ &= \int \psi(y)\big(\int e^{-i(y+z)\cdot\xi}\varphi(z)dz\big)dy \\ &= \int e^{-iy\cdot\xi}\psi(y)dy \int e^{-iz\cdot\xi}\varphi(z)dz = \hat{\varphi}(\xi)\hat{\psi}(\xi). \end{aligned}$$

We pass to the proof of (ii). For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $u = \mathcal{F}^{-1}(\varphi), v = \mathcal{F}^{-1}(\psi)$, we have

$$\begin{aligned} \widehat{\varphi\psi}(\xi) =& \mathcal{F}(\hat{u}\hat{v})(\xi) = \mathcal{F}(\mathcal{F}(u * v))(\xi) = (2\pi)^n \widecheck{u * v}(\xi) \\ =& (2\pi)^n \int u(-\xi-\eta)v(\eta)d\eta = (2\pi)^n \int \check{u}(\xi+\eta)v(\eta)d\eta \\ =& (2\pi)^{-n} \int \hat{\varphi}(\xi+\eta)\check{\hat{\psi}}(\eta)d\eta = (2\pi)^{-n} \int \hat{\varphi}(\xi-\eta)\hat{\psi}(\eta)d\eta \\ =& (2\pi)^{-n} \hat{\varphi} * \hat{\psi}(\xi). \end{aligned}$$

□

5.3 The space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions

5.3.1 Definition, examples

Definition 5.3.1 A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is said to be (a) tempered (distribution) when there exists $C > 0$ and $p \in \mathbb{N}$ such that, for all test function$\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$|\langle T, \varphi\rangle \leq CN_p(\varphi).$$

Proposition 5.3.2 If T is a tempered distribution, then T extends in a unique way as a linear form linéaire $\widetilde{T}$ on $\mathcal{S}(\mathbb{R}^n)$ continuous in the following sense: if $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$, then $\langle \widetilde{T}, \varphi_j\rangle \to \langle \widetilde{T}, \varphi\rangle$.

Proof.— Let $\varphi \in \mathcal{S}(\mathbb{R})$. There is a sequence (φ_j) in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$. The sequence $(\langle T, \varphi_j\rangle)_j$ is a Cauchy sequence in $\mathbb{C}$ since T is tempered:

(5.3.6)
$$|\langle T, \varphi_j - \varphi_k\rangle| \leq CN_p(\varphi_j - \varphi_k).$$

Its limit does not depend on the choice of the sequence (φ_j), since when $\varphi_j, \psi_j \to \varphi$, we have

$$|\langle T, \varphi_j - \psi_j\rangle| \leq CN_p(\varphi_j - \psi_j) \to 0.$$

Thus we can let $\tilde{T}$ be the linear form on $\mathcal{S}(\mathbb{R}^n)$ given by

$$\langle \tilde{T}, \varphi\rangle = \lim_{j\to+\infty} \langle T, \varphi_j\rangle,$$

where (φ_j) is any sequence in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ which converges to φ. Let us prove now that $\tilde{T}$ is continuous. For this, we will show that

(5.3.7)
$$|\langle \tilde{T}, \varphi\rangle| \leq CN_p(\varphi).$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Indeed let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be any function, and let (φ_j) be a sequence of smooth compactly supported functions that converges to φ in $\mathcal{S}$. We have, as in (5.3.6)

$$|\langle T, \varphi_j - \varphi_k\rangle| \leq CN_p(\varphi_j - \varphi_k).$$

Letting $k \to +\infty$ in both sides, we get

$$|\langle T, \varphi_j\rangle - \langle \tilde{T}, \varphi\rangle| \leq CN_p(\varphi_j - \varphi),$$

so that

$$|\langle \tilde{T}\varphi\rangle| \leq |\langle T, \varphi_j\rangle - \langle \tilde{T}, \varphi\rangle| + |\langle T, \varphi_j\rangle| \leq CN_p(\varphi_j - \varphi) + CN_p(\varphi_j).$$

Let $\varepsilon > 0$ be a real number. Since $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$, there exists $J_\varepsilon \in \mathbb{N}$ such that, for $j \geq J_\varepsilon$,

$$|N_p(\varphi_j) - N_p(\varphi)| \leq \frac{\varepsilon}{2C}.$$

Thus we have, for any $\varepsilon > 0$,

$$|\langle \tilde{T}\varphi \rangle| \leq \varepsilon + N_p(\varphi),$$

and this proves (5.3.7).

Finally, if T_1 is another continuous extension of T to $\mathcal{S}(\mathbb{R}^n)$, we should have

$$\langle T_1, \varphi \rangle = \langle T_1, \varphi - \varphi_j \rangle + \langle T_1, \varphi_j \rangle \to 0 + \langle \tilde{T}, \varphi \rangle,$$

so that $T_1 = \tilde{T}$. □

Example 5.3.3 If $T \in \mathcal{E}'(\mathbb{R}^n)$, there is $C > 0$, $m \in \mathbb{N}$, such that for any $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, and in particular in $\mathcal{S}(\mathbb{R}^n)$,

$$|\langle T, \varphi \rangle \leq C \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi| \leq C N_p(\varphi).$$

Thus $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, that is compactly supported distributions are tempered.

Example 5.3.4 For $p \in [1, +\infty]$, we have $L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Indeed, if $f \in L^p(\mathbb{R}^n)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and for $q \in [1, +\infty]$ such that $1/p + 1/q = 1$, we have

$$|\langle T_f, \varphi \rangle| \leq |\int f(x)\varphi(x) dx| \leq \|f\|_{L^p} \|\varphi\|_{L^q} \leq C \|f\|_{L^p} N_{n+1}(\varphi),$$

thanks to (5.1.2). Taking into account Corollary 5.1.7, we get in particular that $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

Example 5.3.5 Let $f \in L^1_{loc}(\mathbb{R}^n)$ be such that $|f(x)| \leq C(1 + |x|)^p$ for some constant $C > 0$ and a $p \in \mathbb{N}$. Then

$$\begin{aligned} |\langle T_f, \varphi \rangle| \leq & C \int (1+|x|)^p |\varphi(x)| dx \leq C \int (1+|x|)^{p+n+1} |\varphi(x)| \frac{1}{(1+|x|)^{n+1}} dx \\ \leq & C N_{p+n+1}(\varphi) \int \frac{1}{(1+|x|)^{n+1}} dx \leq C_n N_{p+n+1}(\varphi). \end{aligned}$$

Thus $f \in \mathcal{S}'(\mathbb{R}^n)$. In particular, polynomials define tempered distributions, and we also have $L^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

Proposition 5.3.6 If $T \in \mathcal{S}'(\mathbb{R}^n)$, then $x^\alpha \partial^\beta T \in \mathcal{S}'(\mathbb{R}^n)$ for all $\alpha, \beta \in \mathbb{N}^n$.

Proof.– It is sufficient to show that $x_j T$ and $\partial_j T$ are tempered distributions when T is. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle x_j T, \varphi \rangle = \langle T, x_j \varphi \rangle$ and $\langle \partial_j T, \varphi \rangle = -\langle T, \partial_j \varphi \rangle$, the proposition follows directly from (5.1.1). □

Exercise 5.3.7 Show that the function $x \mapsto e^x e^{ie^x}$ is not bounded by a polynomial, but belongs to $\mathcal{S}'(\mathbb{R})$. Hint: it is the derivative of a tempered distribution.

The multiplication of a tempered distribution by a smooth function not always yield a tempered distribution. However, it is the case when the function has moderate growth, in the following sense.

Definition 5.3.8 A function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ has moderate growth when for any $\beta \in \mathbb{N}^n$, there is $C_\beta > 0$ and $m_\beta \in \mathbb{N}$ such that

$$|\partial^\beta f(x)| \leq C_\beta (1+|x|)^{m_\beta}.$$

We denote $\mathcal{O}_M(\mathbb{R}^n)$ the set of such functions.

Proposition 5.3.9 If $T \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{O}_M(\mathbb{R}^n)$, then $fT \in \mathcal{S}'(\mathbb{R}^n)$.

Proof.— Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $\alpha, \beta \in \mathbb{N}^n$. Leibniz's formula gives

$$|x^\alpha \partial^\beta (f\varphi)| \leq \sum_{\gamma \leq \alpha} \binom{\beta}{\gamma} |x^\alpha \partial^\gamma f| \, |\partial^{\beta-\gamma}\varphi| \leq C_\gamma \sum_{\gamma \leq \alpha} \binom{\beta}{\gamma} (1+|x|)^{m_\gamma} |\partial^{\beta-\gamma}\varphi|.$$

Then, as in (5.1.1), we obtain

$$N_p(f\varphi) \leq C N_{p+M}(\varphi),$$

where $M = \max_{|\gamma|\leq p} m_\gamma$. Thus, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we obtain

$$|\langle fT, \varphi\rangle| = |\langle T, f\varphi\rangle| \leq C N_p(f\varphi) \leq C' N_{p+M}(\varphi),$$

and this means that fT is a tempered distribution. □

Exercise 5.3.10 Show that vp$(1/x) \in \mathcal{S}'(\mathbb{R})$.

Exercise 5.3.11 Show that for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and all $p \in \mathbb{N}$, there is a constant $C > 0$ such that, for all $a \in \mathbb{R}$, $N_p(\tau_a \varphi) \leq C(1+|a|)^p$. Then show $(x \mapsto e^x) \notin \mathcal{S}'(\mathbb{R})$.

5.3.2 Convergence in $\mathcal{S}'(\mathbb{R}^n)$

Definition 5.3.12 Let (T_j) be a sequence of tempered distributions. One says that (T_j) tends to T in $\mathcal{S}'(\mathbb{R}^n)$ when for any function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it holds that $\langle T_j, \varphi\rangle \to \langle T, \varphi\rangle$.

As it is the case in $\mathcal{D}'(\mathbb{R}^n)$, this notion convergence, a weak one, implies a stronger one. We admit the following result.

Proposition 5.3.13 If for any function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the sequence $(\langle T_j, \varphi\rangle)$ converges, then there exists a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ such that $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^n)$.

Remark 5.3.14 When $T_j \to T$ in $\mathcal{S}'(\mathbb{R})$, it is true that $T_j \to T$ in $\mathcal{D}'(\mathbb{R})$ since $\mathcal{C}^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. The converse is not true in general, as shown by the following example: for any sequence $(a_j)_j$ of complex numbers, the sequence $(a_j\delta_j)$ tends to 0 in $\mathcal{D}'(\mathbb{R})$. However it only converges (necessarily to 0) in $\mathcal{S}'(\mathbb{R}^n)$ if (a_j) has moderate growth, i.e. there is $C > 0$, $p \in \mathbb{N}$ such that

$$|a_j| \leq (1+j)^p.$$

Remark 5.3.15 If $(f_j) \to f$ in $L^p(\mathbb{R}^n)$, then $(f_j) \to f$ in $\mathcal{S}'(\mathbb{R}^n)$. If $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^n)$, then $fT_j \to fT$ in $\mathcal{S}'(\mathbb{R}^n)$ for all $f \in \mathcal{O}_M(\mathbb{R}^n)$.

5.4 The Fourier Transform in $\mathcal{S}'(\mathbb{R}^n)$

5.4.1 Definition

Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution. The linear form on $\mathcal{S}(\mathbb{R}^n)$ given by $\varphi \mapsto \langle T, \hat{\varphi}\rangle$ is a tempered distribution since there exist $C > 0$ and $p \in \mathbb{N}$ such that

$$|\langle T, \hat{\varphi}\rangle| \leq CN_p(\hat{\varphi}) \leq C'N_{p+n+1}(\varphi),$$

thanks to Proposition 5.2.5.

Definition 5.4.1 For $T \in \mathcal{S}'(\mathbb{R}^n)$, we denote $\hat{T} = \mathcal{F}(T)$ the tempered distribution given by

$$\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle,\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Example 5.4.2 *i)* For $f \in L^1$, we have by Parseval's lemma (Proposition 5.2.6),

$$\langle \widehat{T_f}, \varphi\rangle = \int f(x)\hat{\varphi}(x)dx = \int \hat{f}(x)\varphi(x)dx,$$

so that $\widehat{T_f} = T_{\hat{f}}$.

ii) For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \widehat{\delta_0}, \varphi\rangle = \int \varphi(x)dx$, thus $\widehat{\delta_0} = 1$.

Proposition 5.4.3 The Fourier Transform $\mathcal{F}$ is an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$. Its inverse is $\mathcal{F}^{-1} = (2\pi)^{-n}\check{\mathcal{F}}$. Moreover $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous on $\mathcal{S}'(\mathbb{R}^n)$, in the following sense: if $T_j \to T \in \mathcal{S}'(\mathbb{R}^n)$, then $\mathcal{F}(T_j) \to \mathcal{F}(T)$ in $\mathcal{S}'(\mathbb{R}^n)$.

These results follow immediatlely form the above definition and Proposition 5.2.5. The same way, transfering to $\mathcal{S}'(\mathbb{R}^n)$ the properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$, we obtain easily the following identities in $\mathcal{S}'(\mathbb{R}^n)$:

$$\mathcal{F}(D_jT) = \xi_j\hat{T},\ \mathcal{F}(x_jT) = -D_j\hat{T},\ \text{et}\ \ \mathcal{F}\circ\mathcal{F}(T) = (2\pi)^n\ \check{T}.$$

Example 5.4.4 $\hat{1} = \mathcal{F}\circ\mathcal{F}(\delta_0) = (2\pi)^n\ \check{\delta_0} = (2\pi)^n\delta_0$.

5.4.2 Gaussians (2)

For $z \in \mathbb{C}^*$ with $\mathrm{Re}\, z = 0$, the function $x \mapsto e^{-z|x|^2}$ is not in $\mathcal{S}(\mathbb{R}^n)$. However it is bounded, thus defines an element of $\mathcal{S}'(\mathbb{R}^n)$, and we want to compute its Fourier transform.

Proposition 5.4.5 Let $\lambda \in \mathbb{R}^*$ and $T = e^{-i\lambda|x|^2} \in \mathcal{S}'(\mathbb{R}^n)$. It holds that

$$\widehat{T} = \mathcal{F}_{x\to\xi}(e^{-i\lambda|x|^2}) = \Big(\frac{\sqrt{\pi}}{\sqrt{|\lambda|}}\Big)^n e^{-in\pi\,\mathrm{sign}(\lambda)/4}e^{i|\xi|^2/4\lambda}.$$

Proof.— Let (z_j) be a sequence of complex numbers in $\{z \in \mathbb{C}, \mathrm{Re}\, z > 0\}$ such that $z_j \to i\lambda$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the dominated convergence theorem gives that, as quand $j \to +\infty$,

$$\int e^{-z_j|x|^2}\varphi(x)dx \to \int e^{-i\lambda|x|^2}\varphi(x)dx,$$
$$\int e^{-|\xi|^2/4z_j}\varphi(\xi)d\xi \to \int e^{i\lambda|\xi|^2/4\lambda}\varphi(\xi)d\xi.$$

Thus, setting $T_j = e^{-z_j|x|^2}$, we have $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^n)$. By continuity of the Fourier transform, we also have $\mathcal{F}(T_j) \to \mathcal{F}(T)$. But we know that (see Proposition (5.2.3)),

$$\mathcal{F}(T_j) = \Big(\frac{\sqrt{\pi}}{\sqrt{z_j}}\Big)^n e^{-|\xi|^2/4z_j},$$

where $\sqrt{z_j} = \sqrt{|z_j|}\ e^{i\,\mathrm{arg}(z_j)/2} \to \sqrt{|\lambda|}\ e^{i\pi\,\mathrm{sign}(\lambda)/4}$. Thus we have, in $\mathcal{S}'(\mathbb{R}^n)$,

$$\mathcal{F}(T_j) \to \Big(\frac{\sqrt{\pi}}{\sqrt{|\lambda|}}\Big)^n e^{-in\pi\,\mathrm{sign}(\lambda)/4}e^{-i\lambda|\xi|^2/4\lambda} = \mathcal{F}(T),$$

as stated. □

5.4.3 Symmetries

We study here how certain symmetries of a tempered distribution T translates to its Fourier transform. A convenient way to formulate these properties of T consists in expressing them it terms of the action of an invertible matrix A on T.

Proposition 5.4.6 Let $A \in \mathsf{GL}_n(\mathbb{R})$ be an non singular matrix, and $T \in \mathcal{D}'(\mathbb{R}^n)$. The linear form $T \circ A$ defined on $\mathcal{C}^\infty(\mathbb{R}^n)$ by

$$\langle T \circ A, \varphi \rangle = \frac{1}{|\det A|} \langle T, \varphi \circ A^{-1} \rangle,$$

is a distribution. Moreover, $T \circ A \in \mathcal{S}'(\mathbb{R}^n)$ whenever $T \in \mathcal{S}'(\mathbb{R}^n)$.

Proof.— For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and $M = (m_{ij})$ a matrix with real coefficients, we have

$$\partial_j(\varphi \circ M)(x) = \nabla\varphi(Mx) \cdot \partial_j(Mx) = \sum_{k=1}^n m_{k,j}(\partial_k \varphi \circ M)(x),$$

and, iterating,

$$\partial_{j_1,j_2,\dots,j_s}(\varphi \circ M)(x) = \sum_{k_1,\dots,k_s=1}^n m_{k_1,j_1} \dots m_{k_s,j_s} (\partial_{k_1,k_2,\dots,k_s}\varphi)(Mx).$$

In particular, for any $\alpha \in \mathbb{N}^n$, there is a constant $C > 0$ such that

$$\|\partial^\beta(\varphi \circ M)\|_{L^\infty} \leq C \sum_{|\beta|=|\alpha|} \sup |\partial^\beta \varphi|,$$

and, for all $p \in \mathbb{N}$,

$$N_p(\varphi \circ M) \leq C N_p(\varphi).$$

Then let $T \in \mathcal{D}'(\mathbb{R}^n)$, and $K \subset \mathbb{R}^n$ be a compact set. There exist $C > 0$ and $k \in \mathbb{N}$ such that for all function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with support in K,

$$|\langle T \circ A, \varphi \rangle| \leq \frac{1}{|\det A|} |\langle T, \varphi \circ A^{-1} \rangle| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha(\varphi \circ A^{-1})| \leq C' \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi|,$$

and this shows that $T \circ A$ is a distribution.

Suppose moreover that $T \in \mathcal{S}'(\mathbb{R}^n)$. There exist $C > 0$ and $p \in \mathbb{N}$ such that, for any function $\varphi \in \mathbb{C}_0^\infty(\mathbb{R}^n)$,

$$|\langle T \circ A, \varphi \rangle| \leq \frac{1}{|\det A|} |\langle T, \varphi \circ A^{-1} \rangle| \leq C N_p(\varphi \circ A^{-1}) \leq C' N_p(\varphi),$$

thus $T \circ A$ is a tempered distribution. □

Proposition 5.4.7 For $A \in \mathsf{GL}_n(\mathbb{R})$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, it holds that

$$\mathcal{F}(T \circ A) = \frac{1}{|\det A|} \mathcal{F}(T) \circ {}^t A^{-1}.$$

Proof.— Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We have $\langle \widehat{T \circ A}, \varphi \rangle = \frac{1}{|\det A|}\langle T, \hat{\varphi} \circ A^{-1} \rangle$. But

$$\hat{\varphi} \circ A^{-1}(\xi) = \int e^{-ix\cdot A^{-1}\xi}\varphi(x)dx = \int e^{-i\,{}^tA^{-1}x\cdot\xi}\varphi(x)dx$$
$$= \int e^{-i\,y\cdot\xi}\varphi({}^tAy)|\det A|dy = |\det A|\,\widehat{\varphi \circ {}^tA}(\xi).$$

Thus

$$\langle \widehat{T \circ A}, \varphi \rangle = \langle \hat{T}, \varphi \circ {}^tA \rangle = |\det A^{-1}| \langle \hat{T} \circ {}^tA^{-1}, \varphi \rangle.$$

□

Definition 5.4.8 A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is even (resp. odd) when $\check{T} = T$ (resp. $\check{T} = -T$).

Corollary 5.4.9 Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution.

i) If T is even (resp. odd), then $\hat{T}$ is even (resp. odd).

ii) If T is homogeneous of degree $p \in \mathbb{R}$, then $\hat{T}$ is homogeneous of degree $-p-n$.

Proof.— We apply Proposition 5.4.7 with $A = A_\lambda = \lambda I$, $\lambda \in \mathbb{R}^*$. In that case, we get

(5.4.8) $$\widehat{T \circ A_\lambda} = |\lambda|^{-n}\,\hat{T} \circ A_{1/\lambda}.$$

Since $\check{T} = T \circ A_{-1}$, we obtain

$$T = \pm\check{T} \Longleftrightarrow \hat{T} = \pm\widehat{T \circ A_{-1}} = \pm\hat{T} \circ A_{-1} \Longleftrightarrow \hat{T} = \pm\check{\hat{T}},$$

and this finishes the proof of (i).

On the opter hand, for $\lambda > 0$, $\langle T \circ A_\lambda, \varphi \rangle = \lambda^{-n}\langle T, \varphi(x/\lambda)\rangle$. Thus T is homogeneous of degree p if and oly if $T \circ A_\lambda = \lambda^p T$. Taking(5.4.8) into account, this is equivalent to $\hat{T} \circ A_{1/\lambda} = \lambda^{n+p}\hat{T}$, that is $\hat{T}$ is homogeneous of degree $-p-n$. □

5.4.4 The Fourier Transform on L^1 and L^2

Here we sum up briefly the main properties of the Fourier transform of a tempered distribution given by an L^1 or an L^2 function.

Proposition 5.4.10 If $T = T_f \in L^1(\mathbb{R}^n)$, then $\mathcal{F}(T) = T_{\hat{f}}$. More precisely

i) $\mathcal{F}(T)$ is the continuous function given by $\mathcal{F}(T)(\xi) = \int e^{-ix\cdot\xi}f(x)dx$, and $\mathcal{F}(T)(\xi) \to 0$ when $\|\xi\| \to +\infty$.

ii) If moreover $\mathcal{F}(T)$ belongs to $L^1(\mathbb{R}^n)$, then $\mathcal{F}^{-1}(\mathcal{F}(T)) = T$ almost everywhere.

Proof.— The fact that $\hat{f}$ is a continuous function follows easily from Lebesgue theorem of continuity for integral with parameters,and the fact that $\hat{f}$ goes to 0 at infinity is called the Riemann-Lebesgue lemma. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle = \int f(\xi)\hat{\varphi}(\xi)d\xi = \int f(\xi)(\int e^{-ix\cdot\xi}\varphi(x)dx)d\xi.$$

Since the function $(x, \xi) \mapsto f(\xi)e^{-ix\cdot\xi}\varphi(x)$ belongs to $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, we have

$$\langle \hat{T}, \varphi \rangle = \int \varphi(x)(\int e^{-ix\cdot\xi} f(\xi)d\xi)dx = \langle \hat{f}, \varphi \rangle,$$

and this ends the proof of (i). We also know that $\mathcal{F}^{-1}(T_{\hat{f}}) = T_f$ in $\mathcal{S}'(\mathbb{R}^n)$. If $\hat{f} \in L^1(\mathbb{R}^n)$, we thus have $T_{\mathcal{F}^{-1}(\hat{f})} = T_f$ in $\mathcal{D}'(\mathbb{R}^n)$, and this implies that $\mathcal{F}^{-1}(\hat{f}) = f$ almost everywhere. □

Exercise 5.4.11 Let $T \in \mathcal{D}'(\mathbb{R}^n)$. Show that if $f \in L^1(\mathbb{R}^n)$ is such that, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\langle T, \varphi \rangle = \langle f, \hat{\varphi} \rangle$, then $T = \hat{f}$.

Proposition 5.4.12 The map $T \in \mathcal{S}'(\mathbb{R}^n) \mapsto (2\pi)^{-n/2}\mathcal{F}(T) \in \mathcal{S}'(\mathbb{R}^n)$ induces an isometry on $L^2(\mathbb{R}^n)$.

Proof.— Let $f \in L^2(\mathbb{R}^n)$. There is a sequence $(\varphi_j)_j$ of function in $\mathcal{S}(\mathbb{R}^n)$ that converges to f in $L^2(\mathbb{R}^n)$. Plancherel equality (Corollary 5.2.7) gives

$$\|\hat{\varphi}_p - \hat{\varphi}_q\|_{L^2} = (2\pi)^{n/2}\|\varphi_p - \varphi_q\|_{L^2}.$$

Thus $(\widehat{\varphi_j})$ is a Cauchy sequence of $L^2(\mathbb{R}^n)$, that converges to a certain function $g \in L^2(\mathbb{R}^n)$. But (φ_j) converges to f in $\mathcal{S}'(\mathbb{R}^n)$, so that $\hat{\varphi}_j$ converges to $\hat{f}$ in $\mathcal{S}'(\mathbb{R}^n)$. Thus $\hat{f} = g$, and $\hat{f} \in L^2(\mathbb{R}^n)$. Therefore one can pass to the limit in Plancherel's equality , and we get

$$\|\hat{\varphi}_j\|_{L^2} = (2\pi)^{n/2}\|\varphi_j\|_{L^2}.$$

This finishes the proof of the proposition. □

Remark 5.4.13 There are functions f in $L^2(\mathbb{R}^n)$ such that $x \mapsto e^{-ix\cdot\xi}f(x)$ is not integrable whatever the value of ξ is (for example, for $n = 1$, $f(x) = (1 + |x|)^{-3/4}$). Nevertheless, for $R > 0$, the function g_R given by

$$g_R(\xi) = \int_{|x|<R} e^{-ix\cdot\xi}f(x)dx$$

tends to $\hat{f}$ in $L^2(\mathbb{R}^n)$ thanks to the Proposition 5.4.12, since $f1_{|x|<R} \to f$ in $L^2(\mathbb{R}^n)$. Thus, for $f \in L^2(\mathbb{R}^n)$,

$$\hat{f}(\xi) = \lim_{R\to+\infty} \int_{|x|<R} e^{-ix\cdot\xi}f(x)dx.$$

5.5 The Fourier Transform of compactly supported distributions

The Fourier transform exchanges the speed of decay at infinity of a function with the regularity of its image, as shown by example by the following inequality

$$\|D^\alpha \hat{\varphi}\|_{L^\infty} = \|\mathcal{F}(x^\alpha \varphi)\|_{L^\infty} \leq \|x^\alpha \varphi\|_{L^1},\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The best speed of decay at infinity for a function is achieved for compactly supported ones, and one should not be surprised a lot by the results of this section.

5.5.1 Smoothness

Proposition 5.5.1 If $T \in \mathcal{E}'(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}(T)$ is the smooth function on $\mathbb{R}^n$ given by

$$\mathcal{F}(T)(\xi) = \langle T_x, e^{-ix\cdot\xi}\rangle.$$

Moreover, there is an integer $m \in \mathbb{N}$ such that, for all $\alpha \in \mathbb{N}^n$, there is a $C_\alpha > 0$ satisfying

$$|\partial^\alpha \mathcal{F}(T)(\xi)| \leq C_\alpha (1 + \|\xi\|)^m.$$

Proof.— Differantiating under the bracket, we see that the function $v(\xi)$ given by

$$v(\xi) = \langle T_x, e^{-ix\cdot\xi}\rangle$$

is a $\mathcal{C}^\infty$ function (as a matter of fact, one should write $v(\xi) = \langle T_x, \chi(x) e^{-ix\cdot\xi}\rangle$, for some plateau function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ above the support of T, and then apply the theorem). We also have

$$\partial^\alpha v(\xi) = \langle T_x, (-ix)^\alpha e^{-ix\cdot\xi}\rangle,$$

where, for some constant $C > 0$, an integer $m \in \mathbb{N}$ and a compact set K that depend only on T,

$$|\partial^\alpha v(\xi)| \leq C \sum_{|\beta| \leq m} \sup_{x \in K} |\partial_x^\beta \big((-ix)^\alpha e^{-ix\cdot\xi}\big)| \leq C_\alpha (1 + \|\xi\|)^m$$

Last, we have $\hat{T} = v$. Indeed, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, thanks to the result about integrating under the bracket,

$$\langle \hat{T}, \varphi\rangle = \langle T_x, \chi(x) \int e^{-ix\cdot\xi} \varphi(\xi) d\xi\rangle = \int \langle T_x, \chi(x) e^{-ix\cdot\xi}\rangle \varphi(\xi) d\xi,$$

where $\hat{T}(\xi) = \langle T_x, \chi(x) e^{-ix\cdot\xi}\rangle$. □

As an example, we compute now the Fourier transform of the surface measure σ_R of the sphere $\mathcal{S}_R^2$ with radius R in $\mathbb{R}^3$. This is an important result, that we shall use in the study of the wave equation.

Proposition 5.5.2 For $R > 0$, it holds that

$$\widehat{\sigma_R}(\xi) = 4\pi R \frac{\sin(R\|\xi\|)}{\|\xi\|}.$$

Proof.— First of all, we notice that σ_R is rotation-invariant: if $A \in \mathrm{GL}_n(\mathbb{R})$ is an orthogonal matrix, then

$$\langle \sigma_R \circ A, \varphi \rangle = \langle \sigma_R, \varphi \circ A^{-1} \rangle = \int \varphi(A^{-1}x) d\sigma_R(x).$$

But

$$\int_0^R \int_{|y|=r} \varphi(y) d\sigma_r(y) dr = \int_{|x|\leq R} \varphi(y) dy = \int_{|x|\leq R} \varphi(A^{-1}x) dx = \int_0^R \int_{|x|=r} \varphi(A^{-1}x) d\sigma_r(x) dr,$$

so that, derivating with respect to R,

$$\int_{|y|=R} \varphi(y) d\sigma_R(y) = \int_{|x|=R} \varphi(A^{-1}x) d\sigma_R(x),$$

and $\sigma_R \circ A = \sigma_R$.

Therefore, we also have $\widehat{\sigma_R} = \widehat{\sigma_R \circ A} = \frac{1}{|\det A|} \widehat{\sigma_R} \circ {}^t A^{-1} = \widehat{\sigma_R} \circ A$, and $\widehat{\sigma_R}$ is also rotation-invariant.

Now for $\xi \in \mathbb{R}^3 \setminus \{0\}$, there is a rotation which sends ξ on $(\|\xi\|, 0, 0)$, and

$$\begin{aligned} \widehat{\sigma_R}(\xi) =& \widehat{\sigma_R}(\|\xi\|, 0, 0) = \int e^{-ix_1\|\xi\|} d\sigma_R(\xi) \\ =& \int_0^{2\pi} \int_0^{\pi} e^{-iR\|\xi\| \cos\theta} R^2 \sin\theta d\theta d\varphi \\ =& 2\pi R^2 \int_0^{\pi} e^{-iR\|\xi\| \cos\theta} \sin\theta d\theta. \end{aligned}$$

We get the result setting $t = \cos\theta$. □

5.5.2 Analyticity (1): Paley-Wiener's theorem

Wa have proved that the Fourier transform of a compactly supported distribution is a function in $\mathcal{C}^\infty$. As a matter of fact, it is even an analytic function in $\mathbb{C}^n$, in the following sense.

Definition 5.5.3 Let $F : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ be a function defined on the open set Ω of $\mathbb{C}^n$. We suppose that F is a $\mathcal{C}^1$ function of the $2n$ real variables $(\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \dots, \mathrm{Re}\, z_n, \mathrm{Im}\, z_n)$. Then F is said to be holomorphic in $\mathbb{C}^n$ when

$$\forall z \in \Omega, \forall j \in \{1, \dots, n\},\ \partial_{\overline{z_j}} F(z) = 0,$$

where $\partial_{\overline{z_j}} F = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})F$.

Holomorphic functions of several variables are a vast subject of study by themselves. As a first step, one should notice that they are simply holomorphic functions of each of their variables, the other ones being fixed.

We start by the study of the Fourier transform of distribution given by compactly supported smooth functions.

Proposition 5.5.4 (Paley-Wiener's theorem) *i)* Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\varphi \subset B(0,r)$. Then $\hat{\varphi}$ extends as a holomorphic function F on $\mathbb{C}^n$, which satisfies

(5.5.9) $$\forall N \in \mathbb{N}, \exists C_N > 0, \forall z \in \mathbb{C}^n,\ |F(z)| \leq C_N(1+\|z\|)^{-N}e^{r\|\operatorname{Im} z\|}$$

ii) Conversely, if $F : \mathbb{C}^n \to \mathbb{C}$ is an holomorphic function which satisfies property (5.5.9), then there is a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B(0,r)$ and $\hat{\varphi}(\xi) = F(\xi)$ for all $\xi \in \mathbb{R}^n$.

Remark 5.5.5 *i)* If $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is not the null function, its Fourier transform is not compactly supported. Indeed, the only holomorphic function in $\mathbb{C}^n$ with compact support is the null function.

ii) For $z = \xi \in \mathbb{R}$, (5.5.9) gives

$$\forall N \in \mathbb{N},\ \hat{\varphi}(\xi) = \mathcal{O}((1+\|\xi\|)^{-N}).$$

Proof.– Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$. It is easily seen that

$$F(z) = \hat{\varphi}(z) = \int e^{-ix\cdot z}\varphi(x)dx$$

is holomorphic on $\mathbb{C}^n$ thanks to Lebesgue theorem of derivation under the integral sign. For $\alpha \in \mathbb{N}^n$, we also get

$$z^\alpha F(z) = \int z^\alpha e^{-ix\cdot z}\varphi(x)dx = \int (-D_x)^\alpha(e^{-ix\cdot z})\varphi(x)dx = \int e^{-ix\cdot z}D^\alpha\varphi(x)dx.$$

Thus we see that, for all $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$(1+\|z\|)^N|F(z)| \leq C_N e^{r\|\operatorname{Im} z\|}.$$

Notice that here, we have use the following notations for $z = (z_1, \dots, z_N) \in \mathbb{C}^N$:

$$\|z\| = \sqrt{\sum_{j=1}^N |z_j|^2}, \text{ and } \|\operatorname{Im} z\| = \sqrt{\sum_{j=1}^N (\operatorname{Im} z_j)^2}$$

Now we prove (ii). Suppose that F is holomorphic on $\mathbb{C}^n$ and satisfies (5.5.9). Then $F|_{\mathbb{R}^n} \in L^1$, and the function

$$\varphi(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}F(\xi)d\xi,$$

belongs to $\mathcal{C}^\infty$. Since $\hat{\varphi}(\xi) = F(\xi)$, it suffices to prove that $\operatorname{supp}\varphi \subset B(0,r)$.

We admit for a while that, for all $\eta \in \mathbb{R}^n$, it holds that

(5.5.10) $$\varphi(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot(\xi+i\eta)}F(\xi+i\eta)d\xi.$$

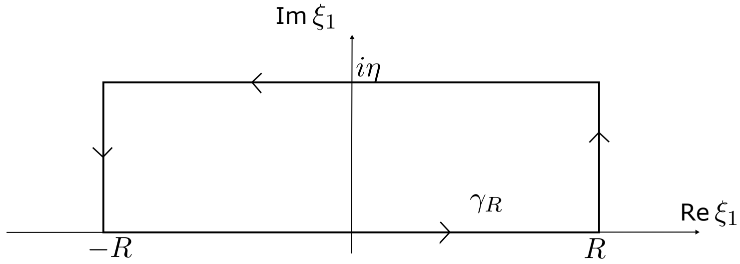


Figure 5.1: The change of contour

We set $\eta = \lambda\frac{x}{|x|}$, $\lambda > 0$, so that $x \cdot (\xi + i\eta) = x \cdot \xi + i\lambda|x|$, $|\eta| = \lambda$, and for $N = n + 1$, (5.5.9) gives

$$|e^{ix\cdot(\xi+i\eta)}F(\xi + i\eta)| \leq C_n e^{-\lambda|x|}(1 + |x|)^{-n-1}e^{r\lambda}.$$

Thus, for all $\lambda > 0$,

$$|\varphi(x)| \leq C_n e^{(r-|x|)\lambda} \int (1 + |x|)^{-n-1} dx,$$

and, passing to the limit $\lambda \to +\infty$, we get that $\varphi(x) = 0$ when $|x| > r$.

Finally, we go back to (5.5.10). The function $g : z_1 \mapsto e^{ix\cdot z}F(z_1, z')$ is holomorphic in $\mathbb{C}$, so that by Cauchy formula, for any $R > 0$ we have

$$\int_{\gamma_R} e^{ix\cdot z}F(z_1, z')dz_1 = 0,$$

where γ_R is the boundary of the rectangle drawn in Figure 5.1. Using (5.5.9), we easily see that the integrals on the left side and on the right side of the rectangle goes to 0 as $R \to +\infty$. Thus, for any $\eta_1 \in \mathbb{R}$, we have

$$\varphi(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot(\xi+i(\eta_1,0,\dots,0))}F(\xi + i(\eta_1, 0, \dots, 0))d\xi.$$

We can repeat this argument for the function $\varphi_1(x)$ given as the R.H.S. of this equality, and this finishes the proof. □

5.5.3 Analyticity (2): Paley-Wiener-Schwartz's theorem

The pressing result says that the Fourier transform of a compactly supported smooth function with support included in $B(0, r)$ increases "a little slower" than $e^{r\|\,\mathrm{Im}\, z\|}$ as z gets away from the real axis. In the case of general compactly supported distributions with support included in $B(0, r)$, we show now that they increase just "a little faster" than $e^{r\|\,\mathrm{Im}\, z\|}$.

Proposition 5.5.6 (Paley-Wiener-Schwartz's theorem) *i)* Let $T \in \mathcal{E}'(\mathbb{R}^n)$, be a compactly supported distribution, and let $m \in \mathbb{N}$ be is order. Let also $r > 0$ be such that $\mathrm{supp}\, T \subset B(0, r)$. Then $\xi \to \hat{T}(\xi)$ is a smooth function on $\mathbb{R}^n$ that extends has a holomorphic function F on $\mathbb{C}^n$, which satisfies

(5.5.11) $$\exists C > 0, \forall z \in \mathbb{C}^n,\ |F(z)| \leq C(1 + \|z\|)^m e^{r\|\,\mathrm{Im}\, z\|}$$

ii) Conversely, if $F : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function satisfying (5.5.11), then ther exists $T \in \mathcal{E}'(\mathbb{R}^n)$ such that $\operatorname{supp} T \subset B(0,r)$ and $\hat{T}(\xi) = F(\xi)$ for all $\xi \in \mathbb{R}^n$.

Proof.— For $z \in \mathbb{C}$, we set $F(z) = \langle T_x, e^{-iz\cdot x}\rangle$, where $z \cdot x = \sum_{j=1}^n z_j x_j$. Thanks to the theorem of differentiation under the bracket, we see easily that $\partial_{\overline{z_j}} F(z) = 0$ for all j. On the other hand, since $T \in \mathcal{E}'(\mathbb{R}^n)$, there is a $C > 0$ and a $m \in \mathbb{N}$ such that, for any compact $K \subset \mathbb{R}^n$ with $\operatorname{supp} T \subset \mathring{K}$,

$$|F(z)| \leq C \sum_{|\alpha|\leq m} \sup_K |\partial_x^\alpha (e^{-iz\cdot x})|.$$

For all $\varepsilon > 0$, we can choose $K = \overline{B(0, R+\varepsilon)}$, and we get

$$|F(z)| \leq C(1 + \|z\|)^m e^{(R+\varepsilon)\|\operatorname{Im} z\|},$$

that is (5.5.11).

Conversely, suppose that F is holomorphic on $\mathbb{C}^n$ and satisfies (5.5.11). For $z = \xi \in \mathbb{R}^n$, we get

$$|F(\xi)| \leq C(1 + |\xi|)^m,$$

so that $F|_{\mathbb{R}^n} \in \mathcal{S}'(\mathbb{R}^n)$. Then we set $T = \mathcal{F}^{-1}(F|_{\mathbb{R}^n}) \in \mathcal{S}'(\mathbb{R}^n)$, and we only have to show that $\operatorname{supp} T \subset B(0,r)$.

Let (ρ_ε) be an approximation of the identity. Since supp $\rho_\varepsilon \subset B(0,\varepsilon)$, it follows from the point (i) in Paley-Wiener's theorem that $\widehat{\rho_\varepsilon}$ extends as a holomorphic function on $\mathbb{C}^n$ such that, for all $N \in \mathbb{N}$, there is $C_N > 0$ with

$$|\widehat{\rho_\varepsilon}(z)| \leq C_N (1 + |z|)^{-N} e^{\varepsilon|\operatorname{Im} z|}.$$

We set $F_\varepsilon(z) = F(z)\widehat{\rho_\varepsilon}(z)$. By the previous inequality and (5.5.11), we know that for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that

$$|F_\varepsilon(z)| \leq C(1 + |z|)^{-N} e^{(r+\varepsilon)|\operatorname{Im} z|}.$$

The point (ii) in Paley-Wiener's theorem says that there is a function $\varphi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that supp $\varphi_\varepsilon \subset B(0, r+\varepsilon)$, and $\widehat{\varphi_\varepsilon} = F_\varepsilon$.

Now let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} \psi \cap B(0,r) = \emptyset$. There is an $\varepsilon_0 > 0$ such that $\operatorname{supp} \psi \subset B(0, R+\varepsilon_0)^c$. Thus we have

$$\begin{aligned}
\langle T, \psi\rangle =& \langle \mathcal{F}(T), \mathcal{F}^{-1}(\psi)\rangle = \int F(\xi)\mathcal{F}^{-1}(\psi)(\xi)d\xi \\
=& \lim_{\varepsilon\to 0^+} \int F_\varepsilon(\xi)\mathcal{F}^{-1}(\psi)(\xi)d\xi = \lim_{\varepsilon\to 0^+} \int \widehat{\varphi_\varepsilon}(\xi)\mathcal{F}^{-1}(\psi)(\xi)d\xi \\
=& \lim_{\varepsilon\to 0^+} \int \varphi_\varepsilon(x)\psi(x)dx = 0,
\end{aligned}$$

and this shows that $\operatorname{supp} T \subset B(0,r)$. □

5.6 Convolution and the Fourier transform in $\mathcal{S}'(\mathbb{R}^n)$

Another important feature of the Fourier transform is that it behaves nicely with respect to the convolution of distributions. Of course, we have first to examine whether this convolution is a tempered distribution. This happens in different circumstances that we explore along this section. In particular, we shall define the convolution of a tempered distribution with a function in $\mathcal{S}(\mathbb{R}^n)$ as an element of $\mathcal{S}'(\mathbb{R}^n)$, and show that the Fourier transform of this convolution is the product of the Fourier transform of each term.

5.6.1 $\mathcal{E}'(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n)$

Proposition 5.6.1 Let $T \in \mathcal{E}'(\mathbb{R}^n)$.

i) If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi * T \in \mathcal{S}(\mathbb{R}^n)$. Moreover there exists $m \in \mathbb{N}$ such that

$$\forall p \in \mathbb{N}, \exists C_p > 0, \forall f \in \mathcal{S}(\mathbb{R}^n),\ N_p(\varphi * T) \leq C_p N_{p+m}(\varphi)$$

ii) For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, $\widehat{\varphi * S}(\xi) = \hat{\varphi}(\xi)\hat{S}(\xi)$.

Proof.— Since φ belongs to $\mathcal{C}^\infty$, we know that $T * \varphi$ is a smooth fonction, and that

$$x^\alpha \partial^\beta (T * \varphi)(x) = \langle T_y, x^\alpha \partial_x^\beta \varphi(x-y)\rangle.$$

On the other hand, since T is compactly supported, T has finite order m, and there is a constant $C > 0$ such that, if $\operatorname{supp} T \subset B(0,R)$,

$$\begin{aligned}
|x^\alpha \partial^\beta (T * \varphi)(x)| &\leq C \sum_{|\mu|\leq m} \sup_{|y|\leq R} |\partial_y^\mu (x^\alpha \partial_x^\beta \varphi(x-y))| \\
&\leq C \sum_{|\mu|\leq m} \sup_{|y|\leq R} |x^\alpha \partial^{\beta+\mu} \varphi(x-y)| \\
&\leq C \sum_{|\mu|\leq m} \sup_{|y|\leq R} |((x-y)+y)^\alpha \partial^{\beta+\mu} \varphi(x-y)| \\
&\leq C N_{p+k}(\varphi),
\end{aligned}$$

for $|\alpha|, |\beta| \leq p$. The last inequality follows from the binomial formula. This proves (i). For (ii), we use the theorem of integration under the bracket:

$$\begin{aligned}
\widehat{T * \varphi}(\xi) &= \int e^{-ix\cdot\xi} \langle T_y, \varphi(x-y)\rangle dx \\
&= \langle T_y, \int e^{-ix\cdot\xi} \varphi(x-y) dx\rangle = \langle T_y, \int e^{-i(y+z)\cdot\xi} \varphi(z) dz\rangle \\
&= \langle T_y, e^{-iy\cdot\xi}\rangle \hat{\varphi}(\xi) = \hat{T}(\xi)\hat{\varphi}_j(\xi).
\end{aligned}$$

The only point here is to justify the second equality, though $(x,y)\mapsto\varphi(x-y)$ is not supposed to be compactly supported. Let then (φ_j)ne a sequence of functions in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ that tends to φ in $\mathcal{S}(\mathbb{R}^n)$. From the previous computation it follows that

$$\widehat{T*\varphi_j}(\xi)=\hat{T}(\xi)\hat{\varphi}_j(\xi).$$

But from (i), we know that $T*\varphi_j\to T*\varphi$ in $\mathcal{S}(\mathbb{R}^n)$. Since the Fourier transform is continuous on $\mathcal{S}(\mathbb{R}^n)$ (cf. Proposition 5.2.5), we thus have $\widehat{T*\varphi_j}\to\widehat{T*\varphi}$. Last, we see easily that $\hat{T}(\xi)\hat{\varphi}_j(\xi)\to\hat{T}(\xi)\hat{\varphi}(\xi)$ in $\mathcal{S}(\mathbb{R}^n)$. □

5.6.2 $\mathcal{E}'(\mathbb{R}^n)*\mathcal{S}'(\mathbb{R}^n)$

Proposition 5.6.2 Let $T\in\mathcal{S}'(\mathbb{R}^n)$, and $S\in\mathcal{E}'(\mathbb{R}^n)$. Then $T*S\in\mathcal{S}'(\mathbb{R}^n)$ and $\widehat{T*S}=\hat{S}(\xi)\hat{T}$.

Proof.— Let $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$. There is a sequence (ψ_j) of functions in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\psi_j\to S$ in $\mathcal{D}'(\mathbb{R}^n)$. Proposition 4.3.15 then gives $\langle T*\psi_j,\varphi\rangle=\langle T,\check{\psi}_j*\varphi\rangle$. But $T*\psi_j\to T*S$ in $\mathcal{D}'(\mathbb{R}^n)$ thanks to Proposition 4.3.13, and $\check{\psi}_j*\varphi\to\check{S}*\varphi$ dans $\mathcal{C}_0^\infty(\mathbb{R}^n)$. Thus

$$\langle T*S,\varphi\rangle=\lim_{j\to+\infty}\langle T,\check{\psi}_j*\varphi\rangle=\lim_{j\to+\infty}\langle T*\psi_j,\varphi\rangle=\langle T,\check{S}*\varphi\rangle.$$

But since $T\in\mathcal{S}'(\mathbb{R}^n)$, there are $C,C'>0$ and $p,m\in\mathbb{N}$ such that

$$|\langle T,\check{S}*\varphi\rangle|\le CN_p(\check{S}*\varphi)\le C'N_{p+m}(\varphi),$$

where the last inequality follows from Proposition 5.6.1. This show that $T*S\in\mathcal{S}'(\mathbb{R}^n)$.

Last, for $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\langle\widehat{T*S},\varphi\rangle=&\langle T*S,\hat{\varphi}\rangle=\langle T,\check{S}*\hat{\varphi}\rangle=(2\pi)^{-n}\langle\hat{\hat{\check{T}}},\check{S}*\hat{\varphi}\rangle\\=&(2\pi)^{-n}\langle\widehat{\check{T}},\widehat{\check{S}}\widehat{\hat{\varphi}}\rangle=\langle\widehat{\check{T}},\widehat{\check{S}}\check{\varphi}\rangle=\langle\widehat{T},\widehat{S}\varphi\rangle=\langle\hat{S}\hat{T},\varphi\rangle.\end{aligned}$$

Thus $\widehat{T*S}=\hat{S}\hat{T}$ as stated. □

5.6.3 $\mathcal{S}(\mathbb{R}^n)*\mathcal{S}'(\mathbb{R}^n)$

For $T\in\mathcal{E}'(\mathbb{R}^n)$ and $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n)$, we have seen that $T*\varphi$ is the smooth function given by

$$T*\varphi(x)=\langle T_y,\varphi(x-y)\rangle.$$

When $T\in\mathcal{S}'(\mathbb{R}^n)$, the R.H.S. of this equality still has a meaning as soon as $\varphi\in\mathcal{S}(\mathbb{R}^n)$. More precisely

Proposition 5.6.3 Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The function $x \mapsto \langle T_y, \varphi(x-y)\rangle$ is well defined and belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$.

Proof.— First we prove that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the function $\psi : y \mapsto \varphi(x-y)$ belongs to $\mathcal{S}(\mathbb{R}^n)$. Let $\alpha, \beta \in \mathbb{N}^n$ be such that $|\alpha|, |\beta| \leq p$. We have

$$y^\alpha \partial_y^\beta \psi(y) = (-1)^{|\beta|}(y-x+x)^\alpha (\partial^\beta \varphi)(x-y) = (-1)^{|\beta|} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} x^\gamma (y-x)^{\alpha-\gamma} (\partial^\beta \varphi)(x-y)$$

Thus

$$N_p(\psi) \leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |x^\gamma| N_p(\varphi) \leq C(1+|x|)^p N_p(\varphi), \tag{5.6.12}$$

and $g : x \mapsto \langle T_y, \varphi(x-y)\rangle$ is well defined.

Now we want to show that $g \in \mathcal{C}^\infty(\mathbb{R}^n)$. Once more it is a matter of differentiating under the bracket, but $(x, y) \mapsto \varphi(x-y)$ is not compactly supported with respect to y - even if we suppose that x stays in a compact set, which we can do since we want to prove some regularity property for g.

Thus, let $K \subset \mathbb{R}^n_x$ be a compact set, and (φ_j) a sequence of functions in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ which converges to φ in $\mathcal{S}(\mathbb{R}^n)$. From the theorem of differentiation under the bracket, it follows that the function $g_j(x) = \langle T_y, \varphi_j(x-y)\rangle$ is $\mathcal{C}^\infty$ on K and that, for all $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha g_j(x) = \langle T_y, (\partial^\alpha \varphi_j)(x-y)\rangle$$

By (5.6.12), it follows that the function $y \mapsto (\partial^\alpha \varphi_j)(x-y)$ tends to $y \mapsto (\partial^\alpha \varphi)(x-y)$ in $\mathcal{S}(\mathbb{R}^n)$ uniformly with respect to $x \in K$. Therefore

$$\partial^\alpha g_j \to \langle T_y, (\partial^\alpha \varphi)(x-y)\rangle$$

uniformly on K, and g is smooth on K. Moreover

$$\partial^\alpha g(x) = \langle T_y, (\partial^\alpha \varphi)(x-y)\rangle.$$

□

Definition 5.6.4 For $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the convolution product of T and φ is the smooth function given by $T * \varphi(x) = \langle T_y, \varphi(x-y)\rangle$.

Exercise 5.6.5 Show that if $T \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then

$$\langle T * \varphi, \psi\rangle = \langle T, \check{\varphi} * \psi\rangle.$$

Notice that, since $T * \varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, we have $T * \varphi \in \mathcal{D}'(\mathbb{R}^n)$ and the R.H.S. $\langle T, \check{\varphi} * \psi\rangle$ does have a meaning. Then it suffices to write

$$\begin{aligned}\langle T * \varphi, \psi\rangle =& \langle\langle T_y, \varphi(x-y)\rangle, \psi(x)\rangle = \langle \psi(x) \otimes T_y, \varphi(x-y)\rangle \\ =& \langle T_y, \langle \psi(x), \varphi(x-y)\rangle\ \rangle = \langle T_y, \int \psi(x)\varphi(x-y)dx\rangle = \langle T, \check{\varphi} * \psi\rangle.\end{aligned}$$

Proposition 5.6.6 For $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $T * \varphi \in \mathcal{S}'(\mathbb{R}^n)$. Moreover $\widehat{T * \varphi} = \hat{\varphi}(\xi)\hat{T}$.

Proof.— Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Since $T \in \mathcal{S}'(\mathbb{R}^n)$, there are $C > 0$ and $p \in \mathbb{N}$ such that

$$|\langle T * \varphi, \psi\rangle| \leq |\langle T, \check{\varphi} * \psi\rangle| \leq CN_p(\check{\varphi} * \psi) \leq CN_{p+n+1}(\varphi)N_p(\psi),$$

where the last inequality is (5.2.5). Thus $T * \varphi \in \mathcal{S}'(\mathbb{R}^n)$.

Eventually, using (ii) in Proposition 5.2.9, we get for $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\langle \widehat{T * \varphi}, \psi\rangle = \langle T * \varphi, \hat{\psi}\rangle = \langle T, \check{\varphi} * \hat{\psi}\rangle = (2\pi)^{-n}\langle T, \widehat{\hat{\varphi}} * \hat{\psi}\rangle = \langle \hat{T}, \hat{\varphi}\psi\rangle = \langle \hat{\varphi}\hat{T}, \psi\rangle.$$

□

Exercise 5.6.7 Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Show that if $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$, then $T * \varphi_j \to T * \varphi$ in $\mathcal{S}(\mathbb{R}^n)$.

Chapter 6

The wave equation

On s'intéresse ici à l'équation des ondes $\Box u = 0$ dans $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$, où le D'Alembertien $\Box$ est l'opérateur différentiel

$$\Box u(t,x) = \partial^2_{tt} u(t,x) - \sum_{j=1}^{n} \partial^2_{jj} u(t,x) = \partial^2_{tt} u(t,x) - \Delta u(t,x).$$

Les physiciens considèrent que les solutions de cette équation décrivent correctement les ondes qui se déplacent à vitesse 1 au cours du temps t dans l'espace $\mathbb{R}^n$. On va voir que le matériel développé dans les paragraphes précédents permet d'établir des propriétés particulièrement importantes (des solutions) de l'équation des ondes.

Cette équation fait partie de la famille des équations d'évolution: la variable t joue un rôle particulier. On commence par introduire une transformation de Fourier adaptée à ce genre d'équations.

6.1 The partial Fourier transform

Definition 6.1.1 Soit $\varphi \in \mathcal{S}(\mathbb{R}^p \times \mathbb{R}^q)$. La transformée de Fourier partielle (par rapport aux q dernières variables) de φ est la fonction $\widetilde{\varphi}(t,\xi) = \mathcal{F}_{x\to\xi}(\varphi(t,x))$ définie sur $\mathbb{R}^p \times \mathbb{R}^q$ par

$$\widetilde{\varphi}(t,\xi) = \int e^{-ix\cdot\xi} \varphi(t,x) dx.$$

En raisonnant à $t \in \mathbb{R}^q$ fixé, la Proposition 5.2.5 donne immédiatement la

Proposition 6.1.2 La transformation de Fourier partielle $\mathcal{F}_{x\to\xi}$ est un isomorphisme de l'espace vectoriel $\mathcal{S}(\mathbb{R}^p \times \mathbb{R}^q)$, d'inverse $\mathcal{F}^{-1}_{\xi\to x}$ donné par

$$\mathcal{F}^{-1}_{\xi\to x}(\varphi(t,\xi)) = \frac{1}{(2\pi)^q} \int e^{-ix\cdot\xi} \varphi(t,\xi) d\xi.$$

De plus la transformation de Fourier partielle est continue, dans le sens où, pour tout $k \in \mathbb{N}$, il existe $C_k > 0$ telle que

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^p \times \mathbb{R}^q),\ N_k(\widetilde{\varphi}) \leq C N_{k+q+1}(\varphi).$$

On définit la transformation de Fourier partielle des distributions tempérées par dualité:

Definition 6.1.3 Soit $T \in \mathcal{S}'(\mathbb{R}^p \times \mathbb{R}^q)$. La transformée de Fourier partielle de T est la forme linéaire $\widetilde{T} = \mathcal{F}_{\xi \to x}(T)$ sur $\mathcal{S}(\mathbb{R}^p \times \mathbb{R}^q)$ définie par

$$\langle \mathcal{F}_{\xi \to x}(T), \varphi \rangle = \langle T, \mathcal{F}_{x \to \xi} \varphi \rangle.$$

Proposition 6.1.4 La transformation de Fourier partielle $\mathcal{F}_{\xi \to x}$ est un isomorphisme de l'espace vectoriel $\mathcal{S}'(\mathbb{R}^p \times \mathbb{R}^q)$, d'inverse

$$\mathcal{F}_{\xi \to x}^{-1} = \frac{1}{(2\pi)^q} \widecheck{\mathcal{F}_{\xi \to x}}.$$

De plus si $T_j \to T$ dans $\mathcal{S}'(\mathbb{R}^p \times \mathbb{R}^q)$, alors $\widetilde{T_j} \to \widetilde{T}$ dans $\mathcal{S}'(\mathbb{R}^p \times \mathbb{R}^q)$.

On a enfin les propriétés

$$\widetilde{D_x^\alpha T} = \xi^\alpha \widetilde{T},\ \widetilde{x^\alpha T} = (-D_\xi)^\alpha \widetilde{T},\ \text{et } \widetilde{D_t^\alpha T} = D_t^\alpha \widetilde{T}.$$

Exercice 6.1.5 $\mathcal{F}_{\xi \to x}(\delta_{t=0,\xi=0}) = \delta_{t=0} \otimes 1_x$. En effet, pour $\varphi \in \mathcal{S}(\mathbb{R}^p \times \mathbb{R}^q)$,

$$\langle \mathcal{F}_{\xi \to x}(\delta_{t=0,\xi=0}), \varphi \rangle = \langle \delta_{t=0,\xi=0}, \mathcal{F}_{x \to \xi}(\varphi) \rangle = \langle \delta_{t=0,\xi=0}, \int e^{-ix\cdot\xi} \varphi(t,x) dx \rangle = \int \varphi(0,x) dx.$$

6.2 A fundamental solution of the wave equation

On cherche $E \in \mathcal{S}'(\mathbb{R}^{1+n})$ telle que $\Box E = \delta$. Pour des raisons qui vont apparaître dans la discussion, on cherche E à support dans le futur, c'est-à-dire telle que

(6.2.1) $$\operatorname{supp} E \subset \mathbb{R}^+ \times \mathbb{R}^n.$$

Supposons que E satisfait ces conditions. On doit avoir $\widetilde{\Box E} = \delta_{t=0} \otimes 1$, ou encore

(6.2.2) $$\partial_{tt}^2 \widetilde{E} - |\xi|^2 \widetilde{E} = \delta_{t=0} \otimes 1_\xi.$$

Sur $\{(t,\xi) \in \mathbb{R}^{1+n}, t \neq 0\}$, on doit donc avoir

$$\partial_{tt}^2 \widetilde{E} - |\xi|^2 \widetilde{E} = 0.$$

Pour chaque ξ fixé, les solutions réelles de cette équation sont les fonctions

$$t \mapsto a_\xi \cos(t|\xi|) + b_\xi \sin(t|\xi|),$$

où $a_\xi, b_\xi \in \mathbb{R}$. Compte tenu de la condition (6.2.1), on cherche donc $\widetilde{E}$ sous la forme

$$\widetilde{E} = (a(\xi)\cos(t|\xi|) + b(\xi)\sin(t|\xi|))H(t).$$

Pour une telle distribution $\widetilde{E}$, on calcule facilement

$$\begin{aligned}\partial_t \widetilde{E} =&(a(\xi)\cos(t|\xi|) + b(\xi)\sin(t|\xi|))\delta_{t=0} + (-a(\xi)|\xi|\sin(t|\xi|) + b(\xi)|\xi|\cos(t|\xi|))H(t)\\ =&\delta_{t=0}\otimes a(\xi) + (-a(\xi)|\xi|\sin(t|\xi|) + b(\xi)|\xi|\cos(t|\xi|))H(t),\end{aligned}$$

et

$$\partial^2_{tt}\widetilde{E} = \delta'_{t=0}\otimes a(\xi) + \delta_{t=0}\otimes b(\xi)|\xi| - |\xi|^2(a(\xi)\cos(t|\xi|) + b(\xi)\sin(t|\xi|))H(t).$$

Donc, à condition que les calculs précédents aient un sens, $\widetilde{E}$ vérifie (6.2.1) et (6.2.2) pour $a = 0$ et $b(\xi) = 1/|\xi|$, c'est-à-dire quand

$$\widetilde{E} = \frac{\sin t|\xi|}{|\xi|}H(t).$$

La fonction $(t,\xi) \mapsto \frac{\sin t|\xi|}{|\xi|}$ est $\mathcal{C}^\infty$, donc les calculs qui précèdent sont valides pour ce choix de $\widetilde{E}$. Elle est aussi bornée, donc l'expression ci-dessus définit bien une distribution tempérée. On a donc prouvé une bonne partie de la

Proposition 6.2.1 L'équation des ondes admet une unique solution élémentaire dans $\mathcal{S}'(\mathbb{R}^{1+n})$ supportée dans le futur. Il s'agit de la distribution

$$E(t,x) = \mathcal{F}^{-1}_{x\to\xi}\big(\frac{\sin t|\xi|}{|\xi|}H(t)\big).$$

Preuve.— Il reste à montrer l'unicité. Supposons que E_1 et E_2 soient deux telles distributions, et posons $G = E_1 - E_2$. On a $\Box G = 0$ et supp $G \subset \mathbb{R}^+ \times \mathbb{R}^n$. Or

$$\Box G = 0 \Longleftrightarrow \partial^2_{tt}\widetilde{G} - |\xi|^2\widetilde{G} = 0 \Longleftrightarrow (\partial_t - i|\xi|)(\partial_t + i|\xi|)\widetilde{G} = 0 \Longleftrightarrow \partial_t(e^{2it|\xi|}\partial_t(e^{-it|\xi|})\widetilde{G}) = 0.$$

On a donc $e^{2it|\xi|}\partial_t(e^{-it|\xi|}\widetilde{G}) = Cste = 0$ compte tenu de la condition sur le support de G, puis $e^{-it|\xi|}\widetilde{G} = Cste = 0$ encore une fois parce que supp $G \subset \{t > 0\}$. Donc $\widetilde{G} = 0$, et $G = 0$. □

Exercice 6.2.2 Reprendre ce qui précède pour

i) l'équation de la chaleur $\partial_t u - \Delta u = 0$. On trouve

$$E(t,x) = \frac{H(t)}{(4\pi t)^{n/2}}e^{-|x|^2/4t}.$$

ii) l'équation de Schrödinger $i\partial_t u - \Delta u = 0$. On trouve

$$E(t,x) = \frac{H(t)}{(4\pi t)^{n/2}}e^{i|x|^2/4t}.$$

On notera que dans ces deux cas on a une expression explicite grâce au résultat sur la transformée de Fourier des gaussiennes.

6.3 The support of the fundamental solution

Pour $n = 3$, la Proposition 5.5.2 donne

$$\widehat{\sigma}_t(\xi) = 4\pi t \frac{\sin(t|\xi|)}{|\xi|},$$

où σ_t désigne la mesure de surface sur la sphère $\mathbb{S}^2_t \subset \mathbb{R}^3$ de rayon t. On en déduit une expression plus simple de la solution élémentaire E. Pour $\varphi = \widetilde{\psi} \in \mathcal{S}(\mathbb{R}^{1+3})$, on a en effet

$$\begin{aligned}
\langle E^{(3)}, \varphi(t,\xi)\rangle &= \langle \widetilde{E}_3, \psi(t,\xi)\rangle = \int_{R^{1+3}} \frac{\sin t|\xi|}{|\xi|} H(t)\psi(t,\xi)dtd\xi \\
&= \int_0^{+\infty} \frac{1}{4\pi t}\Big(\int_{\mathbb{R}^3} \widehat{\sigma}_t(\xi)\psi(t,\xi)d\xi\Big)dt \\
&= \int_0^{+\infty} \frac{1}{4\pi t}\Big(\int_{\mathbb{R}^3} \sigma_t(\xi)\widetilde{\psi}(t,\xi)d\xi\Big)dt \\
&= \int_0^{+\infty} \frac{1}{4\pi t}\langle \sigma_t, \varphi(t,\cdot)\rangle dt. \qquad (6.3.3)
\end{aligned}$$

Sur cette expression, on voit en particulier qu'en dimension 3 d'espace

$$\operatorname{supp} E^{(3)}(t,x) = \{(t,x), |x| = t\}.$$

Exercice 6.3.1 Pour $n = 2$, montrer que

$$E^{(2)}(t,x) = \begin{cases} \dfrac{1}{2\pi}\dfrac{1}{\sqrt{t^2 - x^2 - y^2}} & \text{pour } x^2 + y^2 \le t^2, \\ 0 & \text{sinon.} \end{cases}$$

On notera qu'en dimension 2 d'espace,

$$\operatorname{supp} E^{(2)}(t,x) = \{(t,x), |x| \le t\}.$$

De manière générale, on ne peut donc pas faire mieux que la

Proposition 6.3.2 Soit $E^{(n)}$ la solution élémentaire de l'équation des ondes dans $\mathcal{S}'(\mathbb{R}^{1+n})$ à support dans le futur. Pour tout $n \ge 1$, on a

$$\operatorname{supp} E^{(n)} \subset \{(t,x), |x| \le t\}.$$

Preuve.— C'est une conséquence du théorème de Paley-Wiener-Schwartz! On sait en effet que $E = \mathcal{F}^{-1}_{\xi\to x}(\frac{\sin(t|\xi|)}{|\xi|})$. Soit alors $F : \mathbb{C}^n \to \mathbb{C}$ la fonction définie par

$$F(z) = \frac{\sin(t|z|)}{|z|}.$$

On remarque que

$$F(z) = \sum_{k\geq 0} (-1)^k \frac{h(z)^k t^{2k+1}}{(2k+1)!},$$

où $z \mapsto h(z) = |z|^2 = \sum_{1\leq j\leq n} z_j^2$ est holomorphe sur $\mathbb{C}^n$. Donc F est holomorphe sur $\mathbb{C}^n$. De plus un calcul simple montre qu'il existe $C = C_t > 0$ tel que

$$|F(z)| \leq (1 + C(t))e^{t|\operatorname{Im} z|}.$$

Le point (ii) de la proposition 5.5.6 donne donc supp $E_n \subset \{(t, x), |x| \leq t\}$. □

6.4 The Cauchy problem

On s'intéresse enfin aux solutions éventuelles du problème de Cauchy pour l'équation des ondes. En raison encore une fois du rôle particulier que joue la variable t, on est conduit à la

Definition 6.4.1 Soit $(T_t)_{t\in\mathbb{R}}$ une famille de distributions tempérées sur $\mathbb{R}^n$. On dit que $(T_t) \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))$ lorsque pour toute fonction $\varphi \in \mathcal{S}(\mathbb{R}^n)$, la fonction

$$t \mapsto \langle T_t, \varphi\rangle$$

est $\mathcal{C}^\infty$ sur $\mathbb{R}$. On note alors $(\dot{T}_t)$ et $(\ddot{T}_t)$ les familles de distributions tempérées sur $\mathbb{R}^n$ définies par

$$\langle \dot{T}_t, \varphi\rangle = \partial_t\langle T_t, \varphi\rangle,\ \langle \ddot{T}_t, \varphi\rangle = \partial^2_{tt}\langle T_t, \varphi\rangle, \ldots\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Soit $E = \mathcal{F}^{-1}_{\xi\to x}\big(\frac{\sin t|\xi|}{|\xi|}H(t)\big)$ la solution fondamentale de l'équation des ondes dans $\mathbb{R}^{1+n}$. E définit clairement une famille (E_t). Puisque $E_t = \mathcal{F}^{-1}(\frac{\sin t|\xi|}{|\xi|}H(t))$, E_t appartiennent à $\mathcal{S}'(R^n)$ pour tout $t \in \mathbb{R}$. Pour $\varphi \in \mathcal{S}(\mathbb{R}^n)$ on a

$$\langle E_t, \varphi\rangle = \langle \widehat{E_t}, \mathcal{F}^{-1}(\varphi)\rangle = H(t)\int_{\mathbb{R}^n} \frac{\sin t|\xi|}{|\xi|}\mathcal{F}^{-1}(\varphi)(\xi)d\xi,$$

donc $t \mapsto \langle E_t, \varphi\rangle$ est $\mathcal{C}^\infty$ sur $[0, +\infty[$ (et aussi bien sûr sur $]-\infty, 0]$, mais pas sur $\mathbb{R}$). Autrement dit $E \in \mathcal{C}^\infty([0, +\infty[, \mathcal{S}'(\mathbb{R}^n))$, et

$$\dot{E}_t = \mathcal{F}^{-1}_{\xi\to x}(\cos(t|\xi|)),\ \ddot{E}_t = \mathcal{F}^{-1}_{\xi\to x}(-|\xi|\sin(t|\xi|)).$$

En particulier $E_0 = 0$, $\dot{E}_0 = \delta_{x=0}$, et $\ddot{E}_0 = 0$. Notons enfin que pour $\varphi \in \mathcal{S}(\mathbb{R}^{1+n})$,

$$\langle E, \varphi\rangle = \langle H(t)\frac{\sin t|\xi|}{|\xi|}, \widetilde{\varphi}(t,\xi)\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}^n} H(t)\frac{\sin t|\xi|}{|\xi|}\widetilde{\varphi}(t,\xi)d\xi dt = \int_{\mathbb{R}}\langle E_t, \varphi(t,\cdot)\rangle dt.$$

De manière générale, on a la

Proposition 6.4.2 Soit (T_t) une famille de distributions dans $\mathcal{C}^\infty(I, \mathcal{S}'(\mathbb{R}^n))$, où I est un intervalle de $\mathbb{R}$. La forme linéaire T sur $\mathcal{C}_0^\infty(I \times \mathbb{R}^n)$ définie par

(6.4.4) $$\langle T, \varphi \rangle = \int_I \langle T_t, \varphi(t, .) \rangle dt$$

est une distribution de $\mathcal{D}'(I \times \mathbb{R}^n)$.

On revient à l'équation des ondes. Puisqu'elle est invariante par renversement du temps, on peut se concentrer sur la résolution du problème de Cauchy pour les temps $t \geq 0$. On cherche donc les distributions T associées à des familles $(T_t) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{S}'(\mathbb{R}^n))$ telles que, pour $F, G \in \mathcal{S}'(\mathbb{R}^n)$ données,

(6.4.5) $$\begin{cases} \Box T = 0 \text{ dans } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n), \\ T_0 = F, \\ \dot{T}_0 = G. \end{cases}$$

Proposition 6.4.3 Le problème (6.4.5) admet une unique solution T dans $\mathcal{C}^\infty(\mathbb{R}^+, \mathcal{S}'(\mathbb{R}^n))$. Elle est donnée par (6.4.4) avec

(6.4.6) $$T_t = E_t * G + \dot{E}_t * F,$$

où $E = (E_t)$ est la solution élémentaire du D'Alembertien définie dans la Proposition 6.2.1.

Preuve.— Compte tenu de l'exemple ci-dessus, on a bien $T_0 = E_0 * G + \dot{E}_0 * F = F$. De plus $\dot{T}_0 = \dot{E}_0 * G + \ddot{E}_0 * F = G$. On a aussi, pour $t > 0$, $\ddot{T}_t = \ddot{E}_t * G + \dddot{E}_t * F$. Donc

$$\widehat{\ddot{T}}_t = -|\xi| \sin(t|\xi|)\hat{G} - |\xi|^2 \cos(t|\xi|)\hat{F} = -|\xi|^2 \widehat{T}_t,$$

d'où $\ddot{T}_t = \Delta T_t$. On veut montrer pour toute $\varphi(t, x) \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$, on a $\langle \Box T, \varphi \rangle = 0$. Or

$$\begin{aligned} \langle \Box T, \varphi \rangle =& \langle T, \Box \varphi \rangle = \int_0^{+\infty} \langle T_t, \Box \varphi(t, \cdot) \rangle dt = \int_0^{+\infty} \langle T_t, \partial_{tt}^2 \varphi(t, \cdot) \rangle dt - \int_0^{+\infty} \langle T_t, \Delta \varphi(t, \cdot) \rangle dt \\ =& \int_0^{+\infty} \langle T_t, \partial_{tt}^2 \varphi(t, \cdot) \rangle dt - \int_0^{+\infty} \langle \ddot{T}_t, \varphi(t, \cdot) \rangle dt. \end{aligned}$$

Pour conclure, on utilise le résultat suivant

Lemme 6.4.4 Soit $(T_t) \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))$, et $\psi(t, x) \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$. La fonction $t \mapsto \langle T_t, \psi(t, \cdot) \rangle$ est $\mathcal{C}^\infty$ sur $\mathbb{R}$, et

$$\partial_t(\langle T_t, \psi(t, \cdot) \rangle) = \langle \dot{T}_t, \psi(t, \cdot) \rangle + \langle T_t, \partial_t \psi(t, \cdot) \rangle.$$

On a donc

$$\int_0^{+\infty}\langle T_t,\partial_{tt}^2\varphi(t,\cdot)\rangle dt=\int_0^{+\infty}\partial_t\langle T_t,\partial_t\varphi(t,\cdot)\rangle dt-\int_0^{+\infty}\langle\dot T_t,\partial_t\varphi(t,\cdot)\rangle dt=-\int_0^{+\infty}\langle\dot T_t,\partial_t\varphi(t,\cdot)\rangle dt,$$

puisque φ est à support compact. En répétant le même argument on trouve

$$\int_0^{+\infty}\langle T_t,\partial_{tt}^2\varphi(t,\cdot)\rangle dt=\int_0^{+\infty}\langle\ddot T_t,\varphi(t,\cdot)\rangle dt,$$

d'où $\Box T=0$.

Il reste à démontrer l'unicité. Supposons donc que $U=(U_t)$ et $V=(V_t)$ soient deux solutions du problème (6.4.5), et posons $G_t=H(t)(U_t-V_t)$. On a $\Box G=0$, et $\operatorname{supp} G\subset\{t\geq 0\}$. En particulier, compte tenu de ce que l'on sait du support de E, E et G sont convolables comme distributions de $\mathcal{D}'(\mathbb{R}^{1+n})$, et

$$0=E*\Box G=\Box(E*G)=(\Box E)*G=\delta_0*G=G,$$

Donc $U=V$. □

Proposition 6.4.5 Soit $T\in\mathcal{C}^\infty(\mathbb{R},\mathcal{S}'(\mathbb{R}^n))$ la solution du problème de Cauchy (6.4.5), avec $F,G\in\mathcal{C}_0^\infty(\mathbb{R}^n)$. Alors $T_t\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ pour tout $t>0$, et la quantité

$$E(t)=\|\partial_tT_t\|_{L^2(\mathbb{R}^n)}^2+\|\nabla T_t\|_{L^2(\mathbb{R}^n)}^2$$

ne dépend pas de t.

Preuve.— Le fait que $T_t\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ se lit directement sur (6.4.6). On note $u=(T_t)\in\mathcal{C}^\infty(\mathbb{R}^{1+n})$, et on calcule

$$\begin{aligned}\partial_tE(t)=&\partial_t(\partial_tu,\partial_tu)_{L^2}+\partial_t\sum_{j=1}^n(\partial_ju,\partial_ju)_{L^2}=2\,\mathrm{Re}(\partial_{tt}^2u,\partial_tu)_{L^2}+2\,\mathrm{Re}\sum_{j=1}^n(\partial_t\partial_ju,\partial_ju)_{L^2}\\&=2\,\mathrm{Re}(\partial_{tt}^2u,\partial_tu)_{L^2}-2\,\mathrm{Re}\sum_{j=1}^n(\partial_t\partial_{jj}^2u,\partial_tu)_{L^2}=2\,\mathrm{Re}(\Box u(t,x),\partial_tu(t,x))_{L^2(\mathbb{R}^n)}=0.\end{aligned}$$

□

Pour conclure, on énonce quelques propriétés de la solution du problème de Cauchy pour l'équation des ondes. Tout d'abord les ondes se propagent à vitesse finie, et vérifient le célèbre Principe de Huygens (fort) en dimension 3 d'espace. Précisément

Proposition 6.4.6 Si F,G sont des fonctions $\mathcal{C}^\infty$ à support dans $B(0,R)$, alors, pour $t>0$,

i) $\operatorname{supp}T_t\subset\overline{B(0,R+|t|)}$.

ii) De plus, en dimension 3, T_t est nulle dans $\{(t,x)\in\mathbb{R}^{1+3},|t|>R,|x|<t-R\}$.

Preuve.—(i) se lit sur la formule (6.4.6), compte tenu de la Proposition 6.3.2. On a par exemple

$$\text{supp}\, E_t * G \subset \text{supp}\, E_t + \text{supp}\, G \subset \overline{B(0,R)} + \overline{B(0,|t|)} \subset \overline{B(0,R+|t|)}$$

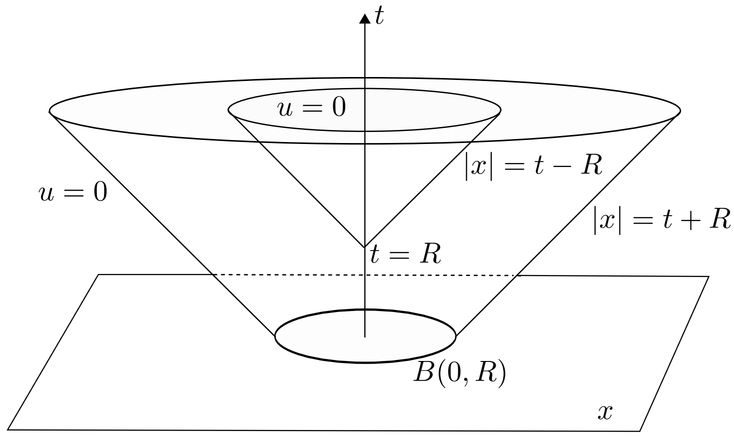
(ii) En dimension 3 d'espace, on sait que supp $E_t = \{|x| = t\}$. Supposons $t > R$, et $|x| < t - R$. Pour $\omega \in \mathbb{S}^2$, on a

$$|x - t\omega| \geq t - |x| > R,$$

donc $G(x - t\omega) = 0$. De ce fait $\int_{\mathbb{S}^2} G(x - t\omega) d\omega = 0$ et

$$E_t * G(x) = \langle E_t, G(x - \cdot)\rangle = \frac{1}{4\pi t}\langle \sigma_t, G(x - \cdot)\rangle = 0.$$

On a aussi $\dot{E}_t * F(x) = \langle \dot{E}_t, F(x - \cdot)\rangle = \partial_t \langle E_t, F(x - \cdot)\rangle = 0$ par le même raisonnement. □



Finalement, on énonce une conséquence de la propagation à vitesse finie:

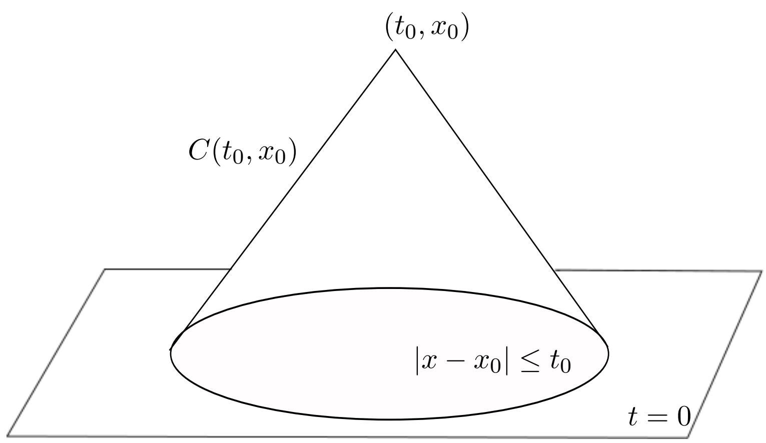
Proposition 6.4.7 Soit F, G des fonctions $\mathcal{C}_0^\infty$, et $u \in \mathcal{C}^\infty(\mathbb{R}^{1+n})$ la solution du problème de Cauchy 6.4.5 correspondant. La valeur de u en $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ ne dépend que des valeurs de F et G dans $\{t = 0\} \cap C(t_0, x_0)$, où $C(t_0, x_0)$ est le "cône rétrograde" défini par

$$C(t_0, x_0) = \{(t, x),\ t < t_0, |x - x_0| \leq t_0 - t\}.$$

Preuve.— Il s'agit de montrer que si $F = G = 0$ dans $C(t_0, x_0) \cap \{t = 0\}$, alors $u(t_0, x_0) = 0$. Or dans ce cas, supp $G \subset \{|x - x_0| \geq t_0\}$, donc

$$\text{supp}\, E_t * G \subset \{|x| \leq t\} + \{|x - x_0| \geq t_0\} \subset \{|x - x_0| \geq t_0 - t\}.$$

On a la même inclusion pour supp $\dot{E}_t * F$, donc u est nulle dans $C(t_0, x_0)$. Par continuité, on a donc bien $u(t_0, x_0) = 0$. □


(t_0, x_0)
$C(t_0, x_0)$
$|x - x_0| \leq t_0$
$t = 0$

Chapter 7

Sobolev spaces

We would like to be able to distinguish amongst (tempered) distributions, for example solutions of a PDE, those that are regular - for example given by $\mathcal{C}^k$ functions. The idea here is to take advantage of the following fact: the regularity of a distribution is encoded in the behavior of its Fourier transform at high frequencies. In a more mathematical way, one would say that the more f is regular, the faster $\hat{f}$ decreases at infinity, as indicated for example by the equality

$$\|\xi^\alpha \hat{f}\|_{L^2} = \|D^\alpha f\|_{L^2}.$$

We may also have written $\|\xi^\alpha \hat{f}\|_{L^\infty} \leq \|D^\alpha f\|_{L^1}$, but we will use in an essential manner the Hilbert space structure of $L^2(\mathbb{R}^n)$. The reader may give a look at Appendix B for essential facts on Hilbert space theory.

7.1 Sobolev spaces on $\mathbb{R}^n$

7.1.1 Definitions

For $\xi \in \mathbb{R}^n$, we denote $\langle\xi\rangle = \sqrt{1 + \|\xi\|^2}$. The function $\xi \mapsto \langle\xi\rangle$ is smooth on $\mathbb{R}^n$, and there is a constant $C > 0$ such that, for $\|\xi\| \geq 1$,

$$\frac{1}{C}\|\xi\| \leq \langle\xi\rangle \leq C\|\xi\|.$$

Thus, $\langle\xi\rangle$ is a regularized version of $\|\xi\|$, in the sense that it has the same behavior at infinity.

Definition 7.1.1 Let $s \in \mathbb{R}$. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^s(\mathbb{R}^n)$ when $\hat{u}$ is a function in $L^1_{loc}(\mathbb{R}^n)$, and $\langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)$.

Remark 7.1.2 Notice that the condition on $\hat{u}$ to be in $L^1_{loc}(\mathbb{R}^n)$ is necessary to give a meaning to the second one: it means that $\hat{u}$ is a regular distribution T_g for some $g \in L^1_{loc}$, and the second one then says that $\langle\xi\rangle^s g \in L^2$. In many textbooks, this is simply stated as "$\langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)$", with a silent identification of the distribution $\hat{u}$ to a function.

Example 7.1.3 *i)* $\delta_0 \in H^s(\mathbb{R}^n)$ if and only if $s < \frac{-n}{2}$. Indeed $\hat{\delta}_0 = 1$, so that $\langle\xi\rangle^s\hat{\delta}_0 \in L^2(\mathbb{R}^n)$ if and only if $2s > -n$.

ii) Constant functions do not belong to $H^s(\mathbb{R}^n)$, since $\hat{C} = C\delta_0$ is not in L^1_{loc}.

Exercise 7.1.4 Let $\alpha \in \mathbb{R}$ be given, and u_α be the function on $\mathbb{R}$ given by

$$u_\alpha(x) = |x|^\alpha e^{-|x|}.$$

For what values of $s \in \mathbb{R}$ do we have $u_\alpha \in H^s(\mathbb{R})$? $u_\alpha \in \mathcal{C}^0(\mathbb{R})$? $u_\alpha \in \mathcal{C}^1(\mathbb{R})$?

Proposition 7.1.5 The bilinear form $(\cdot,\cdot)_s$ defined on $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ by

$$(u,v)_s = (\langle\xi\rangle^s\hat{u}, \langle\xi\rangle^s\hat{v})_{L^2} = \int \hat{u}(\xi)\overline{\hat{v}(\xi)}\langle\xi\rangle^{2s}d\xi$$

is an hermitian scalar product, which makes $H^s(\mathbb{R}^n)$ a Hilbert space. We denote

$$\|u\|_s = \sqrt{(u,u)_s} = \|\langle\xi\rangle^s\hat{u}\|_{L^2}$$

the associated norm.

Proof.— Let (u_j) be a Cauchy sequence in $H^s(\mathbb{R}^n)$. The sequence $(\langle\xi\rangle^s\hat{u})$ is Cauchy in L^2, thus converges to some $v \in L^2$. Then let u be the tempered distribution given by $u = \mathcal{F}^{-1}(\langle\xi\rangle^{-s}\hat{v})$. We have $\hat{u} = \langle\xi\rangle^{-s}v$ where $v \in L^2$, so that $u \in H^s(\mathbb{R}^n)$ and

$$\|u_j - u\|_s = \|\langle\xi\rangle^s\hat{u}_j - v\|_{L^2} \to 0 \text{ quand } j \to +\infty.$$

Thus (u_j) converges in $H^s(\mathbb{R}^n)$. □

It is worthwhile to notice that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, where the equality holds between Hilbert spaces. We also have

$$s_1 \leq s_2 \Rightarrow H^{s_2}(\mathbb{R}^n) \hookrightarrow H^{s_1}(\mathbb{R}^n)$$

since $\langle\xi\rangle^{s_1} \leq \langle\xi\rangle^{s_2}$. Here, the symbol $\hookrightarrow$ denotes a continuous injection. To sum up, we can say that the Sobolev spaces $H^s(\mathbb{R}^n)$ form a continuous, decreasing family of Hilbert spaces. In particular, for $s \geq 0$, we have $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. We even have the

Proposition 7.1.6 (Interpolation) Let $s_0 \leq s \leq s_1$ be real numbers. If $u \in H^{s_0}(\mathbb{R}^n) \cap H^{s_1}(\mathbb{R}^n)$, then $u \in H^s(\mathbb{R}^n)$ and

$$\|u\|_s \leq \|u\|_{s_0}^{(1-\theta)}\|u\|_{s_1}^\theta,$$

where $\theta \in [0,1]$ is given by $s = (1-\theta)s_0 + \theta s_1$.

Proof.— It suffices to write

$$\|u\|_s^2 = \int \langle\xi\rangle^{2s}|\hat{u}|^2 d\xi = \int \big(\langle\xi\rangle^{2(1-\theta)s_0}|\hat{u}|^{2(1-\theta)}\big)\big(\langle\xi\rangle^{2\theta s_1}|\hat{u}|^{2\theta}\big)d\xi,$$

and to apply Hölder's inequality for $p = 1/(1-\theta)$ and $q = 1/\theta$. □

The following statement is the first step towards the regularity analysis that we are looking for. The more we differentiate a distribution, the more we go down in the scale of Sobolev spaces.

Proposition 7.1.7 If $u \in H^s(\mathbb{R}^n)$, then $\partial^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$.

Proof.— Let $u \in H^s(\mathbb{R}^n)$. We have $\widehat{\partial_j u} = \xi_j \hat{u}$, thus $\widehat{\partial_j u}$ is a function in L^1_{loc}. Moreover

$$\|\langle\xi\rangle^{s-1}\widehat{\partial_j u}\|_{L^2} = \|\langle\xi\rangle^{s-1}\xi_j\hat{u}\|_{L^2} \leq C\|\langle\xi\rangle^s\hat{u}\|_{L^2},$$

and this shows that $\partial_j u \in H^{s-1}(\mathbb{R}^n)$. The general case is obtained by induction on $|\alpha|$. □

Here follows another illustration of the fact that H^s contains elements that are more and more singular as s decreases.

Proposition 7.1.8 Let $T \in \mathcal{E}'(\mathbb{R}^n)$ be a compactly supported distribution, with order $m \geq 0$. Then $T \in H^s(\mathbb{R}^n)$for any $s < -m - \frac{n}{2}$.

Proof.— For $T \in \mathcal{E}'(\mathbb{R}^n)$, we know that $\hat{T} \in \mathcal{C}^\infty \subset L^1_{loc}$. Moreover

$$|\langle\xi\rangle^s\hat{T}(\xi)| = |\langle\xi\rangle^s\langle T_x, e^{-ix\cdot\xi}\rangle| \leq C\langle\xi\rangle^s \sum_{|\alpha|\leq m} \sup|\partial_x^\alpha(e^{-ix\cdot\xi})| \leq C\langle\xi\rangle^{s+m}$$

Thus $T \in H^s(\mathbb{R}^n)$ when $2(s+m) > -n$, as stated. □

7.1.2 Density of smooth functions

Proposition 7.1.9 For any $s \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Proof.— First of all, it is important to notice that the map $\Lambda_s : \mathcal{S}(\mathbb{R}^n) \ni u \mapsto \langle\xi\rangle^s\hat{u} \in \mathcal{S}(\mathbb{R}^n)$ is a bijection for all $s \in \mathbb{R}$.

The fact that Λ_s maps $\mathcal{S}(\mathbb{R}^n)$ to itself implies that if $u \in \mathcal{S}(\mathbb{R}^n)$, $\langle\xi\rangle^s\hat{u} \in \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, so that $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$.

Then suppose that $u \in H^s(\mathbb{R}^n)$ is such that $u \in \mathcal{S}(\mathbb{R}^n)^\perp$. For all function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$0 = (u, \varphi)_s = (\langle\xi\rangle^s\hat{u}, \langle\xi\rangle^s\hat{\varphi})_{L^2}.$$

Thus, since Λ_s is surjective, for all $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have $(\langle\xi\rangle^s\hat{u}, \psi)_{L^2} = 0$. By density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ (cf. Corollary 5.1.7), this implies $u = 0$. Therefore

$$\overline{\mathcal{S}(\mathbb{R}^n)} = (\mathcal{S}(\mathbb{R}^n)^\perp)^\perp = \{0\}^\perp = H^s(\mathbb{R}^n).$$

□

Remark 7.1.10 We have thus $\mathcal{S}(\mathbb{R}^n) \subset \cap_{s\in\mathbb{R}} H^s(\mathbb{R}^n)$, but there is no equality. For example, in dimension 1, for $u(x) = \frac{1}{1+x^2}$, we have $\hat{u}(\xi) = e^{-|\xi|}$, so that $u \in H^s(\mathbb{R})$ for all $s \in \mathbb{R}$ though $u \notin \mathcal{S}(\mathbb{R}^n)$.

Proposition 7.1.11 For all $s \in \mathbb{R}$, $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Proof.— Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$, it suffices to show that $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ for the $H^s(\mathbb{R}^n)$ norm. We proceed by troncation: let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a plateau function over $\overline{B(0,1)}$. For $k \in \mathbb{N}$, we set $\chi_k(x) = \chi(x/k)$, $\varphi_k = \chi_k\varphi$, and we have

$$\begin{aligned}\|\varphi_k - \varphi\|_s \leq & \Big(\int \langle\xi\rangle^{2s}|\hat{\varphi}_k(\xi) - \hat{\varphi}(\xi)|^2 d\xi\Big)^{1/2} \\ & \leq \sup\big(\langle\xi\rangle^{s+(n+1)/2}|\hat{\varphi}_k(\xi) - \hat{\varphi}(\xi)|\big)\Big(\int \langle\xi\rangle^{-(n+1)}d\xi\Big)^{1/2} \\ & \leq CN_p(\widehat{\varphi_k - \varphi}) \leq CN_{p+n+1}(\varphi_k - \varphi),\end{aligned}$$

where $p \in \mathbb{N}$ is such that $p \geq s + (n+1)/2$. Then, as in the proof of Proposition 5.1.8, we can see that for all q, $N_q(\varphi_k - \varphi) \to 0$ as $k \to +\infty$, and this finishes the proof. □

7.1.3 Multipliers of $H^s(\mathbb{R}^n)$

We know that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $S \in CS'(\mathbb{R}^n)$, then φS is a tempered distribution. We show now that Sobolev spaces are stable under the multiplication by $\varphi \in \mathcal{S}(\mathbb{R}^n)$. More precisely

Proposition 7.1.12 Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The multiplication by φ is a continuous operation on $H^s(\mathbb{R}^n)$.

Proof.— For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n)$, we have $\varphi u \in \mathcal{S}'(\mathbb{R}^n)$ and, using Proposition 5.6.6,

$$\widehat{\hat{\varphi} * \hat{u}} = \hat{\hat{\varphi}}\hat{\hat{u}} = (2\pi)^{2n}\check{\varphi}\check{u}.$$

Thus, applying $\mathcal{F}^{-1} = (2\pi)^{-n}\check{\mathcal{F}}$ and multiplying by $\langle\xi\rangle^s$, we get

$$\langle\xi\rangle^s\widehat{\varphi u} = (2\pi)^{-n}\langle\xi\rangle^s\hat{\varphi} * \hat{u}.$$

Therefore, for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$\langle\langle\xi\rangle^s\widehat{\varphi u}, \psi\rangle = (2\pi)^{-n}\langle\hat{\varphi} * \hat{u}, \langle\xi\rangle^s\psi\rangle = (2\pi)^{-n}\langle\hat{u}, \check{\hat{\varphi}} * (\langle\xi\rangle^s\psi)\rangle.$$

But $\langle\eta\rangle^s\hat{u}$ belongs to $L^2(\mathbb{R}^n)$ and $\langle\eta\rangle^{-s}(\check{\hat{\varphi}} * (\langle\xi\rangle^s\psi)$ is a function in $\mathcal{S}(\mathbb{R}^n)$, so that

$$\langle\langle\xi\rangle^s\widehat{\varphi u}, \psi\rangle = (2\pi)^{-n}\int \langle\eta\rangle^s\hat{u}(\eta)\Big(\int \langle\eta\rangle^{-s}\check{\hat{\varphi}}(\xi - \eta)\langle\xi\rangle^s\psi(\xi)d\xi\Big)d\eta. \tag{7.1.1}$$

Bow we want to exchange the integrals with respect to ξ and η. We have to show that the function

$$g : (\xi, \eta) \mapsto \langle\eta\rangle^s\hat{u}(\eta)\langle\eta\rangle^{-s}\check{\hat{\varphi}}(\xi - \eta)\langle\xi\rangle^s\psi(\xi)$$

belongs to $L^1(\mathbb{R}^{2n})$. We need the following

Lemma 7.1.13 (Peetre's Inequality) Pour $(\xi,\eta)\in\mathbb{R}^{2n}$, et pour tout $s\in\mathbb{R}$, on a

$$\langle\xi\rangle^s \leq 2^{|s|/2}\langle\xi-\eta\rangle^{|s|}\langle\eta\rangle^s.$$

Proof.— (of Peetre's Inequality) We can exchange the variables ξ and η, so that it suffices to prove the inequality for $s\geq 0$. In that case

$$\langle\xi\rangle^s = (1+|\xi|^2)^{s/2} = (1+|\xi-\eta+\eta|^2)^{s/2} \leq (1+2|\xi-\eta|^2+2|\eta|^2)^{s/2} \leq 2^{s/2}\langle\xi-\eta\rangle^s\langle\eta\rangle^s,$$

where the last inequality can be obtained expanding the R.H. S. □

We go back to the proof of the proposition. With Peetre's inequality, we get

$$|g(\xi,\eta)| \leq 2^{|s|/2}\langle\eta\rangle^s|\hat{u}(\eta)|\langle\xi-\eta\rangle^{|s|}|\hat{\varphi}(\xi-\eta)|\,|\psi(\xi)|.$$

Thus

$$\iint |g(\xi,\eta)|d\xi d\eta \leq 2^{|s|/2}\int |\psi(\xi)|(\langle\eta\rangle^s|\hat{u}| * \langle\eta\rangle^{|s|}|\hat{\varphi}|)(\xi)d\xi. \tag{7.1.2}$$

Since $\langle\eta\rangle^s|\hat{u}|\in L^2$ and $\langle\eta\rangle^{|s|}|\hat{\varphi}|\in L^1$, Young's inequality says that their convolution product belongs to L^2. Since $\psi\in L^2$ we do have that $g\in L^1(\mathbb{R}^{2n})$.

By Fubini, (7.1.1) thus gives

$$\langle\langle\xi\rangle^s\widehat{\varphi u},\psi\rangle = (2\pi)^{-n}\int\psi(\xi)\Big(\int\langle\eta\rangle^{-s}\hat{u}(\eta)\langle\xi\rangle^s\langle\eta\rangle^{-s}\hat{\varphi}(\xi-\eta)d\eta\Big)d\xi,$$

and

$$\langle\xi\rangle^s\widehat{\varphi u}(\xi) = \int\langle\eta\rangle^{-s}\hat{u}(\eta)\langle\xi\rangle^s\langle\eta\rangle^{-s}\hat{\varphi}(\xi-\eta)d\eta,$$

which we have shown to be a L^2 function. Thus $\varphi u\in H^s(\mathbb{R}^n)$, and one can easily get from (7.1.2) that

$$\|\varphi u\|_s \leq 2^{|s|/2}\|\langle\eta\rangle^{|s|}\hat{\varphi}\|_{L^1}\|u\|_s.$$

□

7.1.4 Some Sobolev Embedings

The results of this Section can be seen as a (partial) answer to the question "what is not in $H^s(\mathbb{R}^n)$", or as a step forward in the description of the regularity of tempered distributions.

We denote $\mathcal{C}^k_{\to 0}(\mathbb{R}^n)$ the space of $\mathcal{C}^k$ functions on $\mathbb{R}^n$ that tends to 0 at infinity, as well as all their derivatives of order $\leq k$.

Proposition 7.1.14 If $s>\frac{n}{2}+k$, then $H^s(\mathbb{R}^n)\hookrightarrow\mathcal{C}^k_{\to 0}(\mathbb{R}^n)$.

Proof.– Let $u \in H^s(\mathbb{R}^n)$. For $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$, we have $\xi^\alpha \hat{u} \in L^1$. Indeed,

$$|\xi^\alpha \hat{u}(\xi)| \leq \frac{|\xi|^{|\alpha|}}{\langle\xi\rangle^s} \langle\xi\rangle^s |\hat{u}(\xi)| \leq \langle\xi\rangle^{k-s} \langle\xi\rangle^s |\hat{u}(\xi)|,$$

and $\langle\xi\rangle^{k-s} \in L^2(\mathbb{R}^n)$ since $-2(k-s) > n$. By Cauchy-Schwartz inequality, we thus get

$$\|\xi^\alpha \hat{u}\|_{L^1} \leq C_{s,n} \|u\|_s. \tag{7.1.3}$$

Therefore $D^\alpha u = \mathcal{F}^{-1}(\xi^\alpha \hat{u}) \in \mathcal{C}^0_{\to 0}$ by Proposition 5.4.10, and the fact that the identity from $H^s(\mathbb{R}^n)$ to $\mathcal{C}^k_{\to 0}(\mathbb{R}^n)$ is continuous is just a way to read the inequalities

$$\forall |\alpha| \leq k, \ \|D^\alpha u\|_{L^\infty} \leq \|\xi^\alpha \hat{u}\|_{L^1} \leq C_{s,n} \|u\|_s.$$

□

Of course, the product of two distributions in $H^s(\mathbb{R}^n)$ is well defined when $s > \frac{n}{2}$, since they are continuous functions. But there is more: for $s > \frac{n}{2}$, $H^s(\mathbb{R}^n)$ is a normed algebra (even a Banach algebra):

Proposition 7.1.15 Let $s > \frac{n}{2}$. If $u, v \in H^s(\mathbb{R}^n)$, then $uv \in H^s(\mathbb{R}^n)$ and there is a constant $C_s > 0$, such that, for all $u, v \in H^s(\mathbb{R}^n)$,

$$\|uv\|_s \leq C_s \|u\|_s \|v\|_s.$$

Proof.– First of all $u, v \in L^2 \cap L^\infty$, since $s \geq 0$, and u and v are continuous functions that goes to 0 at infinity. Therefore $f = uv$ belongs to $L^1 \cap L^\infty$, and we have $\hat{f} = (2\pi)^{-n} \hat{u} * \hat{v}$. Thus

$$\|f\|_s^2 = (2\pi)^{-2n} \int \langle\xi\rangle^{2s} |\hat{u} * \hat{v}(\xi) d\xi \leq (2\pi)^{-2n} \int \Big(\int \langle\xi\rangle^s |\hat{u}(\xi - \eta)| \, |\hat{v}(\eta)| d\eta \Big)^2 d\xi. \tag{7.1.4}$$

But since $s > 0$, it holds that $(a+b)^s \leq 2^s(a^s + b^s)$ for any $(a, b) \in \mathbb{R}^+$. Then, writing the triangular inequality, we easily obtain

$$\langle\xi\rangle^s \leq 2^s (\langle\xi - \eta\rangle^s + \langle\eta\rangle^s).$$

Then (7.1.4) gives

$$\begin{aligned}
\|f\|_s^2 &\leq (2\pi)^{-2n} 2^{2s} \int \Big(\int \langle\xi-\eta\rangle^s |\hat{u}(\xi-\eta)| \, |\hat{v}(\eta)| + |\hat{u}(\xi-\eta)| \langle\eta\rangle^s |\hat{v}(\eta)| d\eta \Big)^2 d\xi \\
&\leq (2\pi)^{-2n} 2^{2s+1} \int \Big(\int \langle\xi-\eta\rangle^s |\hat{u}(\xi-\eta)| \, |\hat{v}(\eta)| d\eta \Big)^2 + \Big(\int |\hat{u}(\xi-\eta)| \langle\eta\rangle^s |\hat{v}(\eta)| d\eta \Big)^2 d\xi \\
&\leq (2\pi)^{-2n} 2^{2s+1} \big(\|\langle\eta\rangle^s |\hat{u}| * |\hat{v}|\|_{L^2}^2 + \||\hat{u}| * \langle\eta\rangle^s |\hat{v}|\|_{L^2}^2 \big)
\end{aligned}$$

Now Young's inequality state that, for example for the first term,

$$\|\langle\eta\rangle^s |\hat{u}| * |\hat{v}|\|_{L^2}^2 \leq \|\langle\eta\rangle^s |\hat{u}|\|_{L^2}^2 \|\hat{v}\|_{L^1}^2 \leq C_s \|u\|_s^2 \|v\|_s^2,$$

using also (7.1.3). The second term can be handled the exact same way, and we obtain

$$\|f\|_s^2 \leq C \|u\|_s^2 \|v\|_s^2,$$

as stated. □

The following result give some insight on the nature of the element of $H^s(\mathbb{R}^n)$ when s is non-negative, but not larger than $n/2$ as in Proposition 7.1.15. Its proof is a bit long and tedious, and we omit it here.

Proposition 7.1.16 Let $p \in \mathbb{N}$ and $s \in \mathbb{R}$ be such that

$$0 \le s < \frac{n}{2},\ 2 \le p \le \frac{2n}{n-2s}.$$

Then $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. More precisely, there exists a constant $C_{n,s,p} > 0$ such that

$$\forall u \in H^s(\mathbb{R}^n), \|u\|_{L^p} \le C_{n,s,p}\|u\|_s.$$

Remark 7.1.17 In many senses, this theorem is the best possible one. In particular it does not cover the case $p = +\infty$, but it is not true that $H^{n/2}(\mathbb{R}^n)$ is included in $L^\infty(\mathbb{R}^n)$.

7.1.5 The $H^s(\mathbb{R}^n)/H^{-s}(\mathbb{R}^n)$ duality

We consider now Sobolev spaces of negative order. A convenient way to handle elements in $H^{-s}(\mathbb{R}^n)$ for $s > 0$, consists in considering them as continuous linear forms on $H^s(\mathbb{R}^n)$. We have indeed the following

Proposition 7.1.18 Let $s \in \mathbb{R}$, and $u \in H^{-s}(\mathbb{R}^n)$. The linear form L_u given on $\mathcal{S}(\mathbb{R}^n)$ by

$$L_u(\varphi) = \langle u, \varphi\rangle,$$

can be extended in a unique way to a continuous linear form on $H^s(\mathbb{R}^n)$. Moreover, the map $L : u \mapsto L_u$ is a bicontinuous isomorphism from $H^{-s}(\mathbb{R}^n)$ to $(H^s(\mathbb{R}^n))'$.

Proof.— First of all, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it holds that

(7.1.5) $$|L_u(\varphi)| = |\langle u, \varphi\rangle| \le (2\pi)^{-n}\|u\|_{-s}\|\varphi\|_s.$$

This proves the first point, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Now let us show that L is a bijection. It is 1 to 1, since, using the bijectivity of the map $\Lambda_s : \mathcal{S} \to \mathcal{S}$,

$$\begin{aligned}\forall \varphi \in \mathcal{S}(\mathbb{R}^n), L_u(\varphi) = 0 &\Longleftrightarrow \forall \varphi \in \mathcal{S}(\mathbb{R}^n), \int \langle\xi\rangle^{-s}\hat{u}(\xi)\langle\xi\rangle^s\hat{\varphi}(\xi)d\xi = 0\\ &\Longleftrightarrow \forall \psi \in \mathcal{S}(\mathbb{R}^n), \int \langle\xi\rangle^{-s}\hat{u}(\xi)\psi(\xi)d\xi = 0\\ &\Longleftrightarrow u = 0.\end{aligned}$$

Now we prove that L is onto. Let $\ell \in (H^s(\mathbb{R}^n))'$. We look for $u \in H^{-s}(\mathbb{R}^n)$ such that $L_u = \ell$. Let us denote Ψ the linear form on $L^2(\mathbb{R}^n)$ given by

$$\Psi(f) = \ell(\mathcal{F}^{-1}(\langle\xi\rangle^{-s}f)).$$

For any $f \in L^2(\mathbb{R}^n)$, we have

$$|\Psi(f)| \leq C\|\mathcal{F}^{-1}(\langle\xi\rangle^{-s} f)\|_s \leq C\|f\|_{L^2},$$

so that Ψ is continuous on $L^2(\mathbb{R}^n)$. By Riesz's theorem, there is a function $g \in L^2(\mathbb{R}^n)$ such that, for all $f \in L^2(\mathbb{R}^n)$, $\Psi(f) = (g, \bar{f})_{L^2}$, and we set $u = \mathcal{F}(\langle\xi\rangle^s g)$. Then

$$\langle\xi\rangle^{-s}\hat{u} = (2\pi)^n \langle\xi\rangle^{-s}\langle\xi\rangle^s \check{g} \in L^2(\mathbb{R}^n),$$

so that $u \in H^{-s}(\mathbb{R}^n)$. Moreover, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$L_u(\varphi) = \langle u, \varphi\rangle = \langle \mathcal{F}^{-1}u, \mathcal{F}\varphi\rangle = \int \langle\xi\rangle^s g(\xi)\hat{\varphi}(\xi)d\xi = \Psi(\langle\xi\rangle^s\hat{\varphi}) = \ell(\varphi).$$

Eventually, the continuity of $L : u \mapsto L_u$ can be read on (7.1.5):

$$\|L_u\| = \sup_{\varphi \in H^s, \|\varphi\|_s = 1} |L_u(\varphi)| \leq (2\pi)^{-n}\|u\|_{-s},$$

and that of L^{-1} is automatic since we are working in Banach spaces. □

7.1.6 Trace of an element in $H^s(\mathbb{R}^n)$, $s > 1/2$

We have now in mind the notion of a Cauchy problem for a PDE, where the unknown is supposed to satisfy the PDE and is prescribed on some initial hypersurface. When one look to smooth solutions, there is no difficulty to define their restriction to an hypersurface, for example using a parametrization: the restriction of a smooth function f to the hypersurface $x_n = 0$ in $\mathbb{R}^n$ is simply the function $\gamma(f) : \mathbb{R}^{n-1} \to \mathbb{C}$ given by

$$\gamma(f)(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 0). \tag{7.1.6}$$

There is a priori nothing like this for a function in L^p say, since they are defined only almost everywhere and a hypersurface has measure 0. However, when u is in a Sobolev space of sufficiently large order, even without being continuous, we can give a meaning to such a restriction.

Proposition 7.1.19 For any $s > \frac{1}{2}$, the map $\gamma : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$ given by (7.1.6) extends in a unique way to a continuous linear operator from $H^s(\mathbb{R}^n)$ onto $H^{s-1/2}(\mathbb{R}^{n-1})$.

Proof.— We want to show that there is a constant $C > 0$ such that, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\gamma(\varphi)\|_{H^{s-1/2}(\mathbb{R}^{n-1})} \leq C\|\varphi\|_{H^s(\mathbb{R}^n)}. \tag{7.1.7}$$

Indeed, the existence of a unique continuous extension of γ will then follow from the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we can write

$$\begin{aligned}\gamma(\varphi)(x') = \varphi(x', 0) &= \mathcal{F}^{-1}_{(\xi',\xi_n)\to(x',0)}(\hat{\varphi}(\xi', \xi_n)) = (2\pi)^{-n}\iint e^{ix'\cdot\xi'}\hat{\varphi}(\xi', \xi_n)d\xi' d\xi_n \\ &= (2\pi)^{-(n-1)}\int e^{ix'\cdot\xi'}\left(\frac{1}{(2\pi)}\int \hat{\varphi}(\xi', \xi_n)d\xi_n\right)d\xi'.\end{aligned}$$

Thus, in $\mathcal{S}(\mathbb{R}^{n-1})$,

$$\widehat{\gamma(\varphi)}(\xi') = \frac{1}{(2\pi)}\int \hat{\varphi}(\xi',\xi_n)d\xi_n.$$

In particular

$$|\widehat{\gamma(\varphi)}(\xi')|^2 \leq \frac{1}{(4\pi)^2}\int \langle\xi\rangle^s|\hat{\varphi}(\xi',\xi_n)|\langle\xi\rangle^{-s}d\xi_n \leq \frac{1}{(4\pi)^2}\int \langle\xi\rangle^{2s}|\hat{\varphi}(\xi',\xi_n)|^2 d\xi_n \times \int\langle\xi\rangle^{-2s}d\xi_n.$$

But setting $\xi_n = (1+|\xi'|^2)^{1/2}t$, we get

$$\begin{aligned}\int\langle\xi\rangle^{-2s}d\xi_n =& \int\frac{1}{(1+|\xi'|^2+|\xi_n|^2)^s}d\xi_n\\ =& \int\frac{1}{(1+t^2)^s(1+|\xi'|^2)^s}(1+|\xi'|^2)^{1/2}dt\\ =&\langle\xi'\rangle^{-2s+1}\int\frac{dt}{(1+t^2)^s} = C_s\langle\xi'\rangle^{-2s+1}.\end{aligned} \tag{7.1.8}$$

Therefore

$$\int\langle\xi'\rangle^{2s-1}|\widehat{\gamma(\varphi)}(\xi')|^2d\xi' \leq \frac{C_s}{(4\pi)^2}\int\langle\xi\rangle^{2s}|\hat{\varphi}(\xi)|^2d\xi,$$

and this is (7.1.7).

To prove that this operator is onto, we will show that γ has a right inverse R. For $v\in H^{s-1/2}(\mathbb{R}^{n-1})$, we set

$$u(x) = Rv(x) = \mathcal{F}^{-1}_{\xi\to x}\left(K_N\frac{\langle\xi'\rangle^{2N}}{\langle\xi\rangle^{2N+1}}\hat{v}(\xi')\right),$$

where $N\in\mathbb{N}$ and $K_N>0$ will be fixed later on.

We have

$$\begin{aligned}\|u\|_s^2 =& \int\langle\xi\rangle^{2s}K_N^2\frac{\langle\xi'\rangle^{4N}}{\langle\xi\rangle^{4N+2}}|\hat{v}(\xi')|^2d\xi \leq K_n^2\int\langle\xi'\rangle^{4N}|\hat{v}(\xi')|^2(\int\langle\xi\rangle^{2s-4N-2}d\xi_n)d\xi'\\ \leq& K_n^2C\int\langle\xi'\rangle^{2s-1}|\hat{v}(\xi')|^2d\xi' \leq K_n^2C\|v\|^2_{H^{s-1/2}(\mathbb{R}^{n-1})},\end{aligned}$$

where we have used (7.1.8), choosing $N>s/2-1/4$ so that the integral converges. Thus R sends $H^{s-1/2}(\mathbb{R}^{n-1})$ into $H^s(\mathbb{R}^n)$.

Finally we compute

$$\begin{aligned}\widehat{\gamma(Rv)}(\xi') =& \frac{1}{(2\pi)}\int\widehat{R\varphi}(\xi',\xi_n)d\xi_n = \frac{K_N}{(2\pi)}\int\frac{\langle\xi'\rangle^{2N}}{\langle\xi\rangle^{2N+1}}\hat{v}(\xi')d\xi_n\\ =&\hat{v}(\xi')\frac{K_N}{(2\pi)}\langle\xi'\rangle^{2N}\int\langle\xi\rangle^{-(2N+1)}d\xi_n = \frac{CK_N}{2\pi}\hat{v}(\xi') = \hat{v}(\xi'),\end{aligned}$$

and we choose $K_N = 2\pi/C_N$, with C_N given in (7.1.8).Thus $\gamma\circ R = Id$. □

7.2 Sobolev spaces on Ω

7.2.1 Sobolev spaces of integer order on $\mathbb{R}^n$

Let us start by some simple remarks. For $k \in \mathbb{N}$ the elements of $H^k(\mathbb{R}^n)$ can be characterized by

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k, \partial^\alpha u \in L^2(\mathbb{R}^n).$$

Indeed, we have by the multinomial formula (see Exercise 2.1.6),

$$\begin{aligned}\|\langle \xi\rangle^k \hat{u}(\xi)\|_{L^2}^2 &= \int (1+|\xi|^2)^k |\hat{u}(\xi)|^2 d\xi = \sum_{|\alpha|\leq k} \frac{k!}{\alpha!} \int \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi \\ &= \sum_{|\alpha|\leq k} \frac{k!}{\alpha!} \int \xi^\alpha \hat{u}(\xi) \overline{\xi^\alpha \hat{u}(\xi)} d\xi \\ &= \sum_{|\alpha|\leq k} \frac{k!}{\alpha!} \|\widehat{D^\alpha u}\|_{L^2}^2 = \sum_{|\alpha|\leq k} \frac{k!}{\alpha!} \|D^\alpha u\|_{L^2}^2 \qquad (7.2.9)\end{aligned}$$

Thus $\langle \xi\rangle^k \hat{u} \in L^2(\mathbb{R}^n)$ if and only if $\|D^\alpha u\|_{L^2} < +\infty$ for all $|\alpha| \leq k$.

The equality (7.2.9) says more:

Proposition 7.2.1 For $k \in \mathbb{N}$, the Hilbert space $(H^k(\mathbb{R}^n), (\cdot,\cdot)_s)$ is equal to the space

$$\{u \in \mathcal{S}'(\mathbb{R}^n), \, \forall \alpha \in \mathbb{N}^n, \partial^\alpha u \in L^2\mathbb{R}^n\}$$

endowed with the scalar product

$$((u,v))_k = \sum_{|\alpha|\leq k} (\partial^\alpha u, \partial^\alpha v)_{L^2}.$$

We denote $\|u\|_{H^k} = \sqrt{((u,u))_k}$ the associated norm, which is thus equivalent to the norm $\|\cdot\|_k$.

For negative integers, using Proposition 7.1.18 and the density of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ in $H^k(\mathbb{R}^n)$, we obtain the following characterization:

Proposition 7.2.2 Let $k \in \mathbb{N}$. The space $H^{-k}(\mathbb{R}^n)$ is the space of linear forms u on $H^k(\mathbb{R}^n)$ such that there is a constant $C > 0$ for which

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n), |\langle u, \varphi\rangle| \leq C\|\varphi\|_{H^k}.$$

7.2.2 Sobolev spaces of positive integer order on Ω

Thanks to these remarks, which give a description of Sobolev spaces without using the Fourier transform, we can define a scale of Hilbert spaces of distributions on an open set $\Omega \subset \mathbb{R}^n$.

Definition 7.2.3 Let $\Omega \subset \mathbb{R}^n$ be an open set, and $k \in \mathbb{N}$. The distribution $u \in \mathcal{D}'(\Omega)$ belongs to $H^k(\Omega)$ when for all $|\alpha| \leq k$, it holds that $\partial^\alpha u \in L^2(\Omega)$. We denote $(\cdot,\cdot)_k$ the bilinear form defined on $H^k(\Omega) \times H^k(\Omega)$ by

$$(u,v)_k = \sum_{|\alpha|\leq k} (\partial^\alpha u, \partial^\alpha v)_{L^2}.$$

Proposition 7.2.4 The bilinear form $(\cdot,\cdot)_k$ is a hermitian scalar product, which makes $H^k(\Omega)$ a Hilbert space.

Proof.— Let (u_j) be a Cauchy sequence in $H^k(\Omega)$. For all $|\alpha| \leq k$, the sequence $(\partial^\alpha u_j)$ is Cauchy in L^2, thus converges to a $v_\alpha \in L^2$. In particular, $u_j \to v_0$ in $\mathcal{D}'(\Omega)$, so that $\partial^\alpha u_j \to \partial^\alpha v_0 = v_\alpha \in L^2(\Omega)$, and $(u_j) \to v_0$ in $H^k(\Omega)$ □

When $\Omega \neq \mathbb{R}^n$, the space of test functions $\mathcal{C}_0^\infty(\Omega)$ is not dense in $H^k(\Omega)$. We are thus lead to the following natural

Definition 7.2.5 The Hilbert space $H_0^k(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^k(\Omega)$.

It is worthwhile to give a look at a simple case, where Ω is a open interval in $\mathbb{R}$. Then, we have a rather explicit description of $H^1(I)$ and $H_0^1(I)$.

Proposition 7.2.6 Let $I =]-a,a[\subset \mathbb{R}$. If $f \in H^1(I)$, then f is a continuous function on $[-a,a]$. The set $H_0^1(I)$ is the subset of f's in $H^1(I)$ such that $f(-a) = f(a) = 0$.

Proof.— For $f \in \mathcal{H}^1(I)$, we have $f' \in L^2(I) \subset L^1(I)$. Thus the function $g : I \to \mathbb{C}$ given by

$$g(x) = \int_{-a}^{x} f'(t)dt$$

is continuous. Morever $g' - f' = 0$ so that $g - f$ is a constant function. Since g can be extended as a continuous function on $[-a,a]$, f too.

The function $x \mapsto |f(x)|$ is continuous on $[-a,a]$, therefore it has a minimum at a point $b \in [-a,a]$. Since

$$2a|f(b)|^2 = \int_{-a}^{a} |f(b)|^2 dt \leq \int_{-a}^{a} |f(t)|^2 dt,$$

we have $\sqrt{2a}|f(b)| \leq \|f\|_{L^2}$. At last, since

$$f(x) = f(b) + \int_b^x f'(t)dt,$$

we get

$$|f(x)| \leq \frac{1}{2\sqrt{a}}\|f\|_{L^2} + \sqrt{2a}\|f'\|_{L^2} \leq C\|f\|_{H^1}.$$

In particular, the linear form δ_x is continuous on $H^1(I)$ for any $x \in [-a, a]$.

Now we have seen that the linear forms $\delta_{\pm a}$ are continuous on $H^1(I)$, and vanishes on $\mathcal{C}_0^\infty(I)$. Thus if $f \in H_0^1(I)$, we have $f(-a) = f(a) = 0$. Conversely, let $f \in \mathcal{H}^1(I)$ such that $f(a) = f(-a) = 0$. Let also g be the function which is equal to f on $[-a, a]$ and is 0 everywhere else on $\mathbb{R}$. We have $g' = f'1_[-a, a]$, so that $g' \in L^2(\mathbb{R})$, and $g \in H^1(\mathbb{R})$. For $\lambda < 1$, the sequence $g_\lambda = g(x/\lambda)$ tends to f in $H^1(I)$ when $\lambda \to 1$, and the support if g_λ is contained in $[-a\lambda, a\lambda] \subset I$. If (χ_ε) is a standard mollifier, $g_\lambda * \chi_\varepsilon$ belongs to $\mathcal{C}_0^\infty(I)$ for any $\varepsilon > 0$ small enough, and converges to g_λ in $H^1(\mathbb{R})$. Thus $g_\lambda \in \mathcal{H}_0^1(I)$ and $f \in \mathcal{H}_0^1(I)$. □

The orthogonal F of $H_0^1(I)$ in $H^1(I)$ is the subspace of functions u such that

$$-u'' + u = 0 \tag{7.2.10}$$

in what is usually called the weak sense. Indeed, when f is continuous, a classical solution of this equation is a function $u \in \mathcal{C}^2(I)$ such that for all $x \in I$, $-u''(x) + u(x) = f(x)$. Obviously, when $f \in L^2$ is not continuous, this can not hold for any $u \in \mathcal{C}^2(I)$. We are thus lead to change to another notion of solution.

Definition 7.2.7 A function $u \in \mathcal{C}^1(I)$ is a (classical) weak solution of (7.2.10) when

$$\forall \varphi \in \mathcal{C}_0^1(I),\ \int_I u'(x)\varphi'(x)dx + \int_I u(x)\varphi(x)dx = \int_I f(x)\varphi(x)dx. \tag{7.2.11}$$

Notice that this integral formulation can be obtained for $\mathcal{C}^2$ functions by multiplying the differential equation in (7.2.10) by $\varphi(x)$ and integrating by parts: a classical solution is of course a weak solution. As a matter of fact, since $\mathcal{C}_0^\infty(I)$ is dense in $\mathcal{C}_0^1(I)$, we may, and we will, replace $\mathcal{C}_0^1$ by $\mathcal{C}_0^\infty$ in the definition of weak solutions. We can further extend the notion of weak solution to H^1 functions. Indeed if u belongs to $H^1(I)$, there exists $v \in L^2(I)$ such that, for all $\varphi \in \mathcal{C}_0^\infty(I)$,

$$\int_I u(x)\varphi'(x)dx = -\int_I v(x)\varphi(x)dx.$$

The function v is often called the weak derivative of u, and we have written it u' as a distribution. Therefore, for $u \in H^1(I)$ we can read (7.2.11) as

$$\forall \varphi \in \mathcal{C}_0^\infty(I),\ \int_I v(x)\varphi'(x)dx + \int_I u(x)\varphi(x)dx = \int_I f(x)\varphi(x)dx,$$

and we say that u is an H^1-weak solution.

Going back to $F = (H_0^1(I))^\perp$, the function u in $H^1(I)$ belongs to F if and only if for all $\varphi \in \mathcal{H}_0^1(I)$, therefore, by density, if and only if for all $\varphi \in \mathcal{C}_0^\infty(I)$,

$$0 = (u, \bar{\varphi})_{H^1} = \int_I u\varphi dx + \int_I u'\varphi' dx,$$

so that u is a $H^1(I)$-weak solution of (7.2.10).

7.2.3 Sobolev spaces of negative integer order on Ω

Now we turn to Sobolev spaces with negative integer order on Ω. In view of Section 7.1.5, the following definition should not be too surprising.

Definition 7.2.8 Let $k \in \mathbb{N}$. The space $H^{-k}(\Omega)$ is the space of linear forms u on $H_0^k(\Omega)$ such that there is a constant $C > 0$ for which

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega), |\langle u, \varphi \rangle| \leq C\|\varphi\|_{H^k}.$$

The smallest possible C in the above inequality is denoted $\|u\|_{H^{-k}}$.

Example 7.2.9 If $f \in L^2(\Omega)$, we have $\partial_j f \in H^{-1}(\Omega)$. Indeed, for $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$|\langle \partial_j f, \varphi \rangle| = |\langle f, \partial_j \varphi \rangle| \ \leq \int |f| \ |\partial_j \varphi| dx \leq \|f\|_{L^2} \|\varphi\|_{H^1}.$$

Notice also that $\|\partial_j f\|_{H^{-1}} \leq \|f\|_{L^2}$.

As a matter of fact, it is not very difficult to prove the following structure result about $H^{-k}(\Omega)$.

Proposition 7.2.10 Let $k \in \mathbb{N}$. A distribution $u \in \mathcal{D}'(\Omega)$ belongs to $H^{-k}(\Omega)$ if and only if there are functions $f_\alpha \in L^2(\Omega)$ such that

$$u = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha.$$

7.2.4 Poincaré's Inequality

Proposition 7.2.11 (Poincaré's inequality) Let $\Omega \subset \mathbb{R}^n$ an open subset, bounded in one direction. There exists a constant $C > 0$ such that

$$\forall u \in H_0^1(\Omega), \int_\Omega |u|^2 dx \leq C \int_\Omega |\nabla u|^2 dx.$$

Proof.— The assumption means that there is an $R > 0$ such that, for example $\Omega \subset \{|x_n| < R\}$. For $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we get

$$\varphi(x', x_n) = \int 1_{[-R,x_n]}(t)\partial_n\varphi(x', t)dt.$$

Using Cauchy-Schwartz inequality we then have,

$$|\varphi(x', x_n)|^2 \leq 2R\int_{-R}^{R} |\partial_n\varphi(x', t)|^2dt.$$

We integrate this inequality on Ω, and we get

$$\begin{aligned}\int_\Omega |\varphi(x', x_n)|^2dx &\leq 2R\int_{-R}^{R}\int_{\mathbb{R}^{n-1}}\int_{-R}^{R} |\partial_n\varphi(x', t)|^2dtdx_ndx' \\ &\leq 4R^2\int |\partial_n\varphi(x)|^2dx \leq 4R^2\int |\nabla\varphi(x)|^2dx.\end{aligned}$$

The results in $H_0^1(\Omega)$ follows by density. □

Remark 7.2.12 Poincaré's inequality is not true for constant u's. Notice that these functions do not belong to $H_0^1(\Omega)$ for Ω bounded (at least in one direction).

Corollary 7.2.13 If $\Omega \subset \mathbb{R}^n$ is bounded, the quantity

$$|||u||| = (\int_\Omega \|\nabla u\|^2dx)^{1/2}$$

is a norm on $H_0^1(\Omega)$, equivalent to the H^1 norm.

7.3 The Dirichlet Problem

Finally, we study the general Dirichlet problem in $\mathbb{R}^n$, $n \geq 2$. Let Ω be an open, regular bounded subset of $\mathbb{R}^n$. Let also $(a_{ij}(x))_{1\leq i,j\leq n}$ a family of functions in $L^\infty(\Omega, \mathbb{C})$. We suppose that there exists a constant $c > 0$ such that

$$\forall x \in \Omega, \forall\xi \in \mathbb{C}^n,\ c|\xi|^2 \leq \text{Re}(\sum_{i,j} a_{ij}(x)\xi_i\overline{\xi_j}) \leq \frac{1}{c}|\xi|^2$$

Then we denote Δ_a the differential operator defined, for $\varphi \in \mathcal{C}^\infty(\Omega)$, by

$$\Delta_a(\varphi) = \sum_{i,j=1}^{n} \partial_i(a_{i,j}(x)\partial_j\varphi)$$

Notice that when $A = Id$, Δ_a is nothing else than the usual Laplacian.

The Dirichlet problem on Ω can be stated as follows: for $f \in L^2(\Omega)$, find $u \in L^2(\Omega)$ such that

$$\begin{cases} -\Delta_a u = f, \\ u_{|\partial\Omega} = 0. \end{cases}$$

The case of the equation $-\Delta_a u + Vu = f$ for a non-negative, bounded potential V can be handled the same way, but we choose $V = 0$ for the sake of clarity.

We shall prove the

Proposition 7.3.1 Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. For any $f \in L^2(\Omega)$, the equation $-\Delta_a u = f$ has a unique weak solution in $H_0^1(\Omega)$.

In order to do so, we will use a general result in Hilbert space theory

Proposition 7.3.2 (Lax-Milgram) Let $\mathcal{H}$ be a Hilbert space on $\mathbb{C}$, and $a(x, y)$ a sesquilinear form on $\mathcal{H}$. We assume that

i) The sesquilinear form a is continuous , i.e. there exists $M > 0$ such that $|a(x, y)| \leq M\|x\| \, \|y\|$ for all $x, y \in \mathcal{H}$.

ii) The sesquilinear form a is coercive , i.e. there exists $c > 0$ such that $|a(x, x)| \geq c\|x\|^2$ for all $x \in \mathcal{H}$.

Then, for any continuous linear form ℓ on $\mathcal{H}$, there exists a unique $y \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H},\ \ell(x) = a(x, y).$$

Moreover $\|y\| \leq \|\ell\|/c$.

Proof.— For any $y \in \mathcal{H}$, the linear form $x \mapsto a(x, y)$ is continuous. Thanks to Riesz theorem, there exists a unique $A(y) \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H},\ a(x, y) = \langle x, A(y) \rangle.$$

The map $A : y \mapsto A(y)$ is linear, since, for all $x \in \mathcal{H}$,

$$\langle x, A(\alpha_1 y_1 + \alpha_2 y_2 \rangle = a(x, \alpha_1 y_1 + \alpha_2 y_2) = \overline{\alpha_1} a(x, y_1) + \overline{\alpha_2} a(x, y_2) = \langle x, \alpha_1 A(y_1) + \alpha_2 A(y_2) \rangle.$$

The map A is also continuous since we have $\langle A(y), A(y) \rangle = a(A(y), y) \leq M\|A(y)\|\|y\|$, so that

$$\|A(y)\| \leq M\|y\|.$$

Now let ℓ be a continuous linear form on $\mathcal{H}$. There exists $z \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H},\ \ell(x) = \langle x, z \rangle.$$

Therefore we are left with the equation $A(y) = z$ for a given $z \in \mathcal{H}$, and we are going to show that it has a unique solution, namely that A is a bijection on $\mathcal{H}$.

Since a is coercive, one has

$$c\|y\|^2 \leq |a(y, y)| \leq |\langle y, A(y) \rangle| \leq \|A(y)\| \, \|y\|,$$

so that

$$\|A(y)\| \geq c\|y\|, \tag{7.3.12}$$

and A is 1 to 1.

Moreover Ran A is a closed subspace of $\mathcal{H}$. Indeed if $(v_j) \in$ Ran A converges to v in $\mathcal{H}$, setting $v_j = Au_j$, we obtain thanks to (7.3.12),

$$c\|u_p - u_q\| \leq \|v_p - v_q\|.$$

So (u_j) is a Cauchy sequence, and converges to some $u \in \mathcal{H}$. Since A is continuous, one has

$$v = \lim_{j\to+\infty} v_j = \lim_{j\to+\infty} A(u_j) = A(\lim_{j\to+\infty} u_j) = Au,$$

and $v \in$ Ran A.

Eventually if $x \in (\text{Ran}\, A)^\perp$, we have $0 = |\langle A(x), x\rangle| \geq c\|x\|^2$, so that $(\text{Ran}\, A)^\perp = \{0\}$, and Ran $A = \overline{\text{Ran}\, A} = ((\text{Ran}\, A)^\perp)^\perp = \mathcal{H}$. □

Armed with this result, we can easily solve the Dirichlet problem. Indeed, in the weak sense in $H_0^1(\Omega)$, the equation $-\Delta_a u = f$ means that

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega),\ \sum_{i,j} \int_\Omega a_{ij}(x)\partial_i u(x)\partial_j\varphi(x)dx = \int_\Omega f\varphi dx.$$

Let us denote $a(v,u)$ the sesquilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ given by

$$a(v,u) = \sum_{i,j} \int_\Omega a_{ij}(x)\partial_i v(x)\overline{\partial_j u(x)}dx,$$

and ℓ the linear form on $H_0^1(\Omega)$ given by $\ell(v) = \int fvdx$. The above equation can be written

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega),\ a(\varphi, \overline{u}) = \ell(\varphi),$$

and we want to prove that it has a unique solution $\overline{u} \in \mathcal{H}_0^1(\Omega)$. Thanks to Lax-Milgram's theorem, we only need to prove that a is continuous and coercive.

The continuity comes from the boundedness of the functions a_{ij}, and Cauchy-Schwarz inequality

$$|a(v,u)| \leq \sum_{i,j} \int_\Omega |a_{ij}(x)|\ |\partial_i v(x)||\partial_j u(x)|dx \leq C \sum_{i,j} \|\partial_i v\|_{L^2}\|\partial_j v\|_{L^2} \leq C\|v\|_{H^1}\|u\|_{H^1}.$$

Concerning the coercivity, we have, first for $u \in \mathcal{C}_0^\infty(\Omega)$, then by density for $u \in \mathcal{H}_0^1(\Omega)$,

$$|a(u,u)| \geq |\operatorname{Re} a(u,u)| = \operatorname{Re} \int_\Omega \Big(\sum_{i,j} a_{i,j}\partial_i u\overline{\partial_j u} \Big) dx \geq c \int_\Omega \sum_j |\partial_j u|^2 dx.$$

It remains to prove that

$$\|\nabla u\|_{L^2}^2 := \int_\Omega \sum_j |\partial_j u|^2 dx \geq \|u\|_{H^1}^2,$$

which is a consequence of the Poincaré inequality.

7.4 An introduction to the finite elements method

In the previous section, we have seen that Lax-Milgram's theorem permits us to obtain, under suitable assumptions, existence and uniqueness for the solution to the partial differential equation $-\Delta_a u + Vu = f$. As a matter of fact, Lax-Milgram theorem can also be used to obtain approximations of the solution for this equation.

In order to introduce the main ideas, we only consider the 1d case, and the Dirichlet problem for the equation

$$-u'' + V(x)u = f(x).$$

on a bounded interval $I = [0, 1] \subset \mathbb{R}$. The main idea consists in applying Lax-Milgram's theorem on a finite dimensional subspace G of $H^1_0(]0, 1[)$, and construct the corresponding solution. Of course one can expects that the quality of this approximate solution should improve as the dimension of G grows.

We want to find an accurate approximation of the solution $v \in \mathcal{H}^1_0(]0, 1[)$ of the problem

(7.4.13) $$\forall u \in \mathcal{H}^1_0(]0, 1[),\ a(u, v) = \ell(v),$$

where the sesquilinear form

(7.4.14) $$a(u, v) = \int_0^1 u'\bar{v}' + Vu\bar{v}dx$$

is continuous:

(7.4.15) $$|a(u, v)| \le M\|u\|\ \|v\|$$

and coercive :

(7.4.16) $$c\|u\|^2 \le |a(u, u)|,$$

on $H^1_0(]0, 1[) \times H^1_0(]0, 1[)$.

Let $n \in \mathbb{N}$, and denote $x_0 = 0, x_1 = 1/n, \dots, x_{n-1} = (n-1)/n, x_n = 1$ the regular subdivision of $[0, 1]$ with step $1/n$. We define $n + 1$ functions in $\mathcal{C}^0([0, 1])$, piecewise linear, by

$$\begin{cases} g_0(0) = 1,\ g_0(x) = 0 \text{ for } x \ge 1/n, \\ g_1(0) = 0,\ g_1(1/n) = 1,\ g_1(x) = 0 \text{ for } x \ge 2/n, \\ \vdots \\ g_j(x) = 0 \text{ for } \ x \le (j-1)/n,\ g_j(j/n) = 1,\ g_j(x) = 0 \text{ for } x \ge (j+1)/n,\ j = 2, \dots, n-1 \\ \vdots \\ g_n(x) = 0 \text{ for } \ x \le (n-1)/n,\ g_n(1) = 1. \end{cases}$$

It is easy to see that the finite elements g_j are linearly independent. Thus they form a basis of the space G_n that they generate, which is the space of continuous functions on $[0, 1]$ that are linear on each interval of the form $[j/n, (j+1)/n]$, $j = 0 \dots n$.

The functions in G_n belong to $H^1(I)$. Indeed they are in $L^2(I)$ since they are continuous, and they are differentiable on $]0, 1[$ but perhaps on the set $\{j/n,\ j = 1, \dots, n-1\}$, which is of measure 0.

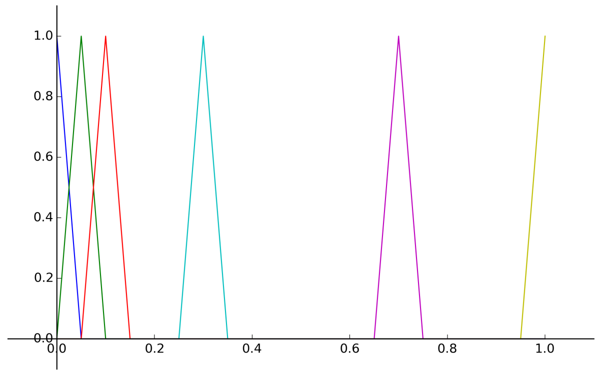


Figure 7.1: Some finite elements g_j for $n = 20$

Moreover their derivative is piecewise constant, therefore belongs to $L^2(]0,1[)$. Using the characterization of $H_0^1(]0,1[)$ in Proposition 7.2.6, we see that the space G_n^0 generated by the finite elements $g_1, g_2 \dots, g_{n-1}$ is included in $H_0^1(]0,1[)$. In particular, the sesquilinear form a is still continuous and coercive on $G_n^0 \times G_n^0$. Therefore, the problem of finding v such that

(7.4.17)
$$\forall u \in G_n^0,\ a(u,v) = \int_0^1 fudx$$

has one and only one solution v_n in G_n^0. What makes this discussion non-void is twofold. First, v_n is a good approximation of the solution to the original problem.

Proposition 7.4.1 (Céa's Lemma) Let v be the solution of (7.4.17) in $H_0^1(]0,1[)$, and v_n the solution of (7.4.17) in G_n^0. With the constants $M > 0$ and $c > 0$ given in (7.4.15) and (7.4.16) we have
$$\|v - v_n\| \leq \frac{M}{c} \inf_{y \in G_n^0} \|v - y\|.$$

Proof.— For any $z \in G_n^0$, we have
$$a(z, v - v_n) = a(z,v) - a(z,v_n) = \ell(z) - \ell(z) = 0.$$

Thus for any $y \in G_n^0$,
$$M\|v-y\|\ \|v-v_n\| \geq |a(v-y, v-v_n)| \geq |a(v-y+y-v_n, v-v_n)| \geq c\|v-v_n\|^2,$$

which proves the lemma. □

This lemma states that, up to the loss $M/c \geq 1$, v_n is the best approximation of u in G_n^0. As a matter of fact since the R.H.S. is not known in general, this result does not seem to give any interesting information. But the idea is, that we may have some a priori estimate on $\|v - y_0\|$ for some well chosen $y_0 \in G_n^0$. A good choice is the function y_0 defined by $y_0(x_j) = v(x_j)$: using this function we can obtain by elementary computations the

Proposition 7.4.2 Let $f \in L^2(I)$, and v the unique solution in H_0^1 of the problem (7.4.13). Let also $v_n \in G_n^0$ the solution of the problem (7.4.17). Then $\|v - v_n\|_{H^1} = \mathcal{O}(\frac{1}{n})$ as $n \to +\infty$.

Second, it is fairly easy to compute v_n (at least with a computer)! Since G_n^0 is spanned by the $(g_j)_{j=1,\dots n-1}$, it is clear that the problem (7.4.17) is equivalent to that of finding $v \in G_n^0$ such that

$$\forall j \in \{1, \dots, n-1\},\ a(g_j, v) = \int_I f g_j dx$$

Now since v_n belongs to G_n^0, we can write

$$v_n = \sum_{k=1}^{n-1} v^k g_k,$$

so that

$$a(g_j, v_n) = \sum_{k=1}^{n-1} \overline{v^k} a(g_j, g_k).$$

Therefore, to compute the coordinates (v_k) of v_n, we only have to solve the $(n-1) \times (n-1)$ linear system

$$AX = B, \text{ with } A = (a(g_j, g_k))_{j,k}, \text{ and } B = (\int_I f g_j dx)_j.$$

Notice that, since supp $g_j \cap$ supp $g_k = \emptyset$ when $|j-k| > 1$, the matrix A is sparse, and in particular tridiagonal.

We have inserted below a small chunk of code in Python that solves the the 1d, second order equation $-u'' + V(x)u = f(x)$ with Dirichlet boundary conditions on $[0, 1]$ using the finite elements method.

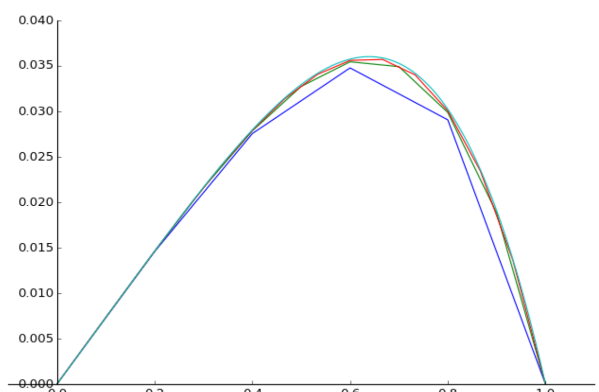


Figure 7.2: Exact and approximate solutions for $-u'' + u = x^2$ on [0,1] with Dirichlet boundary conditions, using P1 finite elements.

```
#----------------------------------------------------------------------
#----------------------------------------------------------------------
# We solve the equation -u''+Vu=f on [0,1]
# with Dirichlet boundary conditions u(0)=u(1)=0
# using P1 finite elements
# T. Ramond, 2014/06/15
#----------------------------------------------------------------------

from pylab import *
import numpy as np
from scipy.integrate import quad
from scipy import linalg as la

#----------------------------------------------------------------------
# Finite elements on [0,1]. Only those that are 0 at 0 and 1.
# numbered from 0 to numpoints-2.

def fe(j,x):
    #print 'j= ', j, ', x= ', x
    N=float(numpoints)
    if (x<j/N):
        z = 0
    if ((x>=j/N) and (x<=(j+1)/N)):
        z = x*N-j
    if ((x>(j+1)/N) and (x<=(j+2)/N)):
        z =2+j-x*N
    if  (x>(j+2)/N):
        z = 0
    return z

def dfe(j,x):
    N=float(numpoints)
    if (x<j/N):
        z = 0
    if ((x>=j/N) and (x<=(j+1)/N)):
        z = N
    if ((x>(j+1)/N) and (x<=(j+2)/N)):
        z =-N
    if  (x>(j+2)/N):
        z = 0
    return z

#----------------------------------------------------------------------
# The coefficients of the equation

def f(x):
    return x**2

def V(x):
    return 1

#----------------------------------------------------------------------
# Some true solutions

# for V(x)=1, f(x)=x**2
def truesolution1(x):
    e=np.exp(1)
```

```
    a=(2/e-3)/(e-1/e)
    b=-2-a
    return a*np.exp(x)+(b/np.exp(x)) + x**2+2

# for V(x)=0, f(x)=x**2
def truesolution0(x):
    return x*(1-x**3)/12

#--------------------------------------------------------------
# Figure

figure(figsize=(10,6), dpi=80)

#axis
ax = gca()
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
ax.xaxis.set_ticks_position('bottom')
ax.spines['bottom'].set_position(('data',0))
ax.yaxis.set_ticks_position('left')
ax.spines['left'].set_position(('data',0))
xlim(-.1, 1.1)


#uncomment each line below to draw the finite elements

# Create a new subplot from a grid of 1x2
# subplot(1,2,1)

#plot (X, [fe(0,x) for x in X])
#plot (X, [fe(1,x) for x in X])
#plot (X, [fe(2,x) for x in X])
#plot (X, [dfe(2,x) for x in X])
#plot (X, [fe(6,x) for x in X])
#plot (X, [fe(numpoints-1,x) for x in X])
#plot (X, [fe(numpoints,x) for x in X])

#numpoints=11
#X=linspace(0,1,(numpoints-1)*10)

#for k in range(numpoints-1):
#    plot (X, [fe(k,x) for x in X])

#subplot(1,2,2)

#--------------------------------------------------------------
# Solution

# We try different numbers of finite elements.
# For numpoints>16 there seem to be numerical instabilities (?)

for numpoints in range(6,20,5):

    # the matrix A

    A=np.zeros((numpoints-1,numpoints-1))

```

```
    def integrand(x,*args):
        return dfe(args[0],x)*dfe(args[1],x)+V(x)*fe(args[0],x)*fe(args[1],x)

    for i in range(numpoints-1):
        for j in range(numpoints-1):
            A[i,j],errA = quad(integrand,0,1, args=(i,j),limit=100)

    # the right-hand side B

    B=np.zeros(numpoints-1)

    def secondmembre(x,i):
        return fe(i,x)*f(x)

    for j in range(numpoints-1):
        B[j],errB = quad(secondmembre,0,1,args=j)

    # Compute the coordinates of the approximate solution
    # in the finite elements basis

    u=la.solve(A,B)

    # Build the appsolution
    # appsolution(x) = sum u_j*fe_j(x)

    def appsolution(x):
        s=0
        for j in range(numpoints-1):
            s=s+u[j]*fe(j,x)
        return s

    # Plot the approximate solution

    X = np.linspace(0, 1, numpoints, endpoint=True)
    appsolutiongraph=[appsolution(x) for x in X]
    plot (X, appsolutiongraph)

#-------------------------------------------------------------------
# For comparison: plot the true solution if it is known
# comment lines below if not

Y=linspace(0,1,400)
truesolutiongraph=[truesolution1(x) for x in Y]
plot (Y, truesolutiongraph)

#-------------------------------------------------------------------
savefig("finite_elements_1d.png",dpi=80)
show()
#-------------------------------------------------------------------
#-------------------------------------------------------------------
```

Appendix A

Lebesgue Integration

A.1 Axioms

There exists a mapping from $\mathcal{E} = \{f : \mathbb{R}^n \to [0, +\infty]\}$ to $[0, +\infty]$, that we denote

$$f \mapsto \int f = \int f(x)dx,$$

which satisfies the following properties:

i) For $f, g \in \mathcal{E}$ and $\lambda, \mu \in \mathbb{R}^+$, $\int \lambda f + \mu g = \lambda \int f + \mu \int g$.

ii) If $f \leq g$ then $\int f \leq \int g$.

iii) For any $A = \prod_{j=1}^n [a_j, b_j] \subset \mathbb{R}^n$, if we denote 1_A the characteristic function of A, given by$1_A(x) = 0$ if $x \notin A$ and $1_A(x) = 1$, we have

$$\int 1_A(x)dx = \prod_{j=1}^n (b_j - a_j).$$

iv) (Monotone convergence, or Beppo-Levi's lemma) If (f_j) is an increasing sequence of functions in $\mathcal{E}$, then

$$\int \lim_{j \to +\infty} f_j = \lim_{j \to +\infty} \int f_j.$$

Definition A.1.1 Let A be a (measurable...) subset of $\mathbb{R}^n$. We denote $\mu(A)$, and we call measure of A, the non-negative number

$$\mu(A) = \int 1_A = \int 1_A(x)dx = \int 1_A d\mu.$$

One says that the set A is negligeable when $\mu(A) = 0$.

Une propriété $\mathcal{P}(x)$ portant sur les $x \in \mathbb{R}^n$ est dite vraie presque partout (p.p.) lorsque l'ensemble A des x où elle est fausse est de mesure nulle.

Exemple A.1.2 Soit $f \in \mathcal{E}$. On a l'équivalence

$$\int f = 0 \Longleftrightarrow f = 0 \ p.p.$$

Supposons d'abord que $f = 0 \ p.p.$. Soit

$$A = \{x \in \mathbb{R}^n, f(x) \neq 0\}.$$

On sait que $\int 1_A(x)dx = \mu(A) = 0$. Or

$$f(x) \leq \lim_{j\to+\infty} j 1_A(x),$$

puisque les deux membres sont nuls quand $x \notin A$, et que pour $x \in A$, il existe $j \in \mathbb{N}$ tel que $f(x) \leq j$. D'après la propriété de Beppo-Lévi, et en utilisant la linéarité de l'intégrale,

$$\int f(x)dx = \int \lim_{j\to+\infty} j 1_A(x)dx = \lim_{j\to+\infty} j \int 1_A(x) = 0.$$

Réciproquement, si $\int f = 0$, puisque

$$1_A(x) \leq \lim_{j\to+\infty} j f(x),$$

le même argument donne $\mu(A) = \int 1_A = 0$.

Pour les suites de fonctions positives qui ne sont pas monotones, on a la très utile

Proposition A.1.3 (Lemme de Fatou) Soit $(f_j)_j$ une suite de fonctions de $\mathcal{E}$. On a toujours

$$\int \liminf_{j\to+\infty} f_j(x)dx \leq \liminf_{j\to+\infty} \int f_j(x)dx$$

Preuve.— Soit $g_j = \inf_{j\leq k} f_k$. La suite (g_j) est une suite croissante d'éléments de $\mathcal{E}$, et par définition

$$\liminf_{j\to+\infty} f_j = \lim_{j\to+\infty} g_j.$$

La propriété de Beppo-Lévi donne donc

$$\int \liminf_{j\to+\infty} f_j(x)dx = \int \lim_{j\to+\infty} g_j = \lim_{j\to+\infty} \int g_j = \lim_{j\to+\infty} \int \inf_{j\leq k} f_k.$$

Or pour tout $k \geq j$, on a

$$\int \inf_{j\leq k} f_k \leq \int f_k,$$

donc

$$\int \inf_{j\leq k} f_k \leq \inf_{j\leq k} \int f_k.$$

Finalement on a bien

$$\int \liminf_{j\to+\infty} f_j(x)dx \leq \liminf_{j\to+\infty} \int f_j.$$

□

A.2 The dominated convergence theorem

Definition A.2.1 On dit qu'une fonction $f : \mathbb{R}^n \to \mathbb{C}$ est sommable lorsque

$$\int |f| < +\infty.$$

On note $L^1(\mathbb{R}^n)$ l'espace des classes d'équivalences de fonctions sommables pour la relation d'équivalence

$$f \sim g \Longleftrightarrow f = g \ p.p.$$

Proposition A.2.2 (Théorème de Convergence Dominée) Soit (f_j) une suite de fonctions de $\mathbb{R}^n$ dans $\mathbb{C}$ telles que

i) $f_j(x) \to f(x)$ pour presque tout x,

ii) il existe $\varphi : \mathbb{R}^n \to \mathbb{R}^+$, sommable, telle que $\forall j \in \mathbb{N}, |f_j(x)| \leq \varphi(x)$ pour presque tout x,

Alors $\displaystyle\int |f_j - f| dx \to 0$ quand $j \to +\infty$.

Preuve.— On se ramène d'abord au cas où $f_j(x) \to f(x)$ pour tout x et où $|f_j(x)| \leq g(x)$ pour tout x, de sorte que l'on pourra utiliser la propriété de Beppo-Levi. Soit $A_j = \{x \in \mathbb{R}^n,\ |f_j(x)| > g(x)\}$. L'ensemble A_j est de mesure nulle, donc $A = \bigcup_j A_j$ aussi. Soit aussi $B = \{x \in \mathbb{R}^n,\ f_j(x) \not\to f(x)\}$. B est de mesure nulle, et $N = A \cup B$ aussi. Soit alors $\tilde{f}_j$ et $\tilde{f}$ les fonctions définies par $\tilde{f}_j(x) = f_j(x)$ et $\tilde{f}(x) = f(x)$ pour $x \notin N$, et $\tilde{f}_j(x) = \tilde{f}(x) = 0$ pour $x \in N$. On a bien sûr $\tilde{f}_j(x) \to \tilde{f}(x)$ pour tout $x \in \mathbb{R}^n$, et $|\tilde{f}_j(x)| \leq \varphi(x)$ pour tout j et tout $x \in \mathbb{R}^n$. De plus

$$\int |f_j(x) - f(x)| dx = \int |\tilde{f}_j(x) - \tilde{f}(x)| dx,$$

donc il suffit de montrer que $\displaystyle\int |\tilde{f}_j(x) - \tilde{f}(x)| dx \to 0$.

Soit $g_j(x) = |\tilde{f}_j(x) - \tilde{f}(x)|$, et $h_j(x) = \sup_{j \leq k} g_k(x)$. La suite (h_j) est décroissante, et puisque

$$0 \leq g_j(x) \leq |\tilde{f}_j(x)| + |\tilde{f}(x)| \leq 2\varphi(x),$$

les fonctions $2\varphi - h_j$ forment une suite croissante de fonctions positives. La propriété de Beppo-Levi entraine donc

$$\begin{aligned} 2\int \varphi(x) dx \ - \lim_{j \to +\infty} \int h_j(x) dx &= \lim_{j \to +\infty} \int (2\varphi(x) - h_j(x)) dx \\ &= \int \lim_{j \to +\infty} (2\varphi(x) - h_j(x)) dx = 2\int \varphi(x) dx, \end{aligned}$$

ce qui montre que $\displaystyle\int |f_j(x) - f(x)|dx \to 0.$ □

Proposition A.2.3 La quantité $\|f\|_{L^1} = \displaystyle\int |f|$ est une norme sur $L^1(\mathbb{R}^n)$ qui en fait un espace de Banach. De plus si la suite $(f_j) \to f$ dans $L^1(\mathbb{R}^n)$, on peut en extraire une sous-suite (f_{j_k}) qui converge vers f presque partout.

Preuve.— Le fait qu'il s'agit d'une norme est très facile. On se contente de montrer que celle-ci fait de l'espace vectoriel $L^1(\mathbb{R}^n)$ un espace complet, et pour cela que toute série absolument convergente est convergente. Soit donc $\sum f_j$ une série normalement convergente dans $L^1(\mathbb{R}^n)$. On considère la fonction g de $\mathcal{E}$ définie par $g(x) = \displaystyle\sum_{j\geq 0} |f_j(x)|$. La fonction g est sommable puisque, grâce à Beppo-Lévi,

$$\int g(x)dx = \int \lim_{k\to+\infty} \sum_{j=0}^{k} |f_j(x)|dx = \lim_{k\to+\infty} \sum_{j=0}^{k} \int |f_j(x)|dx = \sum_{j\geq 0} \|f_j\|_{L^1} < +\infty.$$

En particulier l'ensemble $A = \{x \in \mathbb{R}^n, g(x) = +\infty\}$ est de mesure nulle. Soit alors (S_p) la suite des sommes partielles de la série $\sum f_j$.

i) Pour tout p, on a

$$|S_p(x)| \leq g(x)$$

ii) Pour tout $x \notin A$, donc presque partout, on a

$$|S_{p_1}(x) - S_{p_2}(x)| \leq \sum_{j=p_1+1}^{p_2} |f_j(x)| \to 0 \text{ quand } p_2 > p_1 \to +\infty.$$

La suite $(S_p(x))$ est donc une suite de Cauchy, et converge vers un certain $S(x)$.

On peut donc appliquer le Théorème de Convergence Dominée (TCD):

$$\lim_{p\to+\infty} \int |S_p(x) - S(x)|dx = 0,$$

c'est-à-dire

$$\|S_p(x) - S(x)\|_{L^1} \to 0 \text{ quand } p \to +\infty.$$

Soit enfin (f_j) une suite de fonctions qui converge vers f dans $L^1(\mathbb{R}^n)$. Il existe une sous-suite (f_{j_k}) telle que

$$\forall k \in \mathbb{N},\ \|f_{j_k} - f\|_{L^1} \leq 2^{-k}.$$

On peut écrire

$$f_{j_k} = f_{j_0} + \sum_{\ell=0}^{k-1} (f_{j_{\ell+1}} - f_{j_\ell}),$$

et l'on sait que la série $\sum_{\ell=0}^{\infty} \|f_{j_{\ell+1}} - f_{j_\ell}\|_{L^1}$ converge, puisque

$$\|f_{j_{\ell+1}} - f_{j_\ell}\|_{L^1} \leq \|f_{j_{\ell+1}} - f\|_{L^1} + \|f_- f_{j_\ell}\|_{L^1} \leq 2^{-(\ell+1)} + 2^{-\ell} \leq 2^{-\ell+1}.$$

Puisque L^1 est complet, la série $\sum_{\ell=0}^{+\infty}(f_{j_{\ell+1}} - f_{j_\ell})$ converge, ce qui montre que la suite (f_{j_k}) converge.

□

A.3 Functions given by integrals with parameters

Proposition A.3.1 Soit $f : \Omega \times \mathbb{R}^n \to \mathbb{C}$ une fonction, où Ω est un ouvert d'un espace métrique (X, d). Si

i) $\forall \lambda \in \Omega$, $x \mapsto f(\lambda, x)$ est sommable sur $A \subset \mathbb{R}^n$,

ii) $\forall x \in A$, $\lambda \mapsto f(\lambda, x)$ est continue,

iii) il existe une fonction g sommable à valeurs positives, telle que

$$\forall x \in A, \forall \lambda \in \Omega,\ |f(\lambda, x)| \leq g(x),$$

alors la fonction $F : \lambda \mapsto \int_A f(\lambda, x)dx$ est continue sur Ω.

Preuve.— Soit $\lambda_0 \in \Omega$, et (λ_j) une suite de Ω qui tend vers λ_0 dans (X, d). On pose $f_j : x \mapsto f(\lambda_j, x)$, et on applique le TCD à la suite (f_j). □

On notera que les hypothèses de cette proposition peuvent n'être satisfaites que sur $A \setminus N$, où N est un ensemble de mesure nulle, sans que la conclusion ne soit modifée.

Proposition A.3.2 Soit $f : I \times \mathbb{R}^n \to \mathbb{C}$ une fonction, où $I \subset \mathbb{R}$ est un intervalle ouvert. Si

i) $\forall t \in I$, $x \mapsto f(t, x)$ est sommable sur $A \subset \mathbb{R}^n$,

ii) $\forall x \in A$, $t \mapsto f(t, x)$ est $\mathcal{C}^1(I)$,

iii) il existe une fonction g sommable à valeurs positives, telle que

$$\forall x \in A,\ |\partial_t f(t, x)| \leq g(x),$$

alors la fonction $F : t \mapsto \int_A f(t, x)dx$ est $\mathcal{C}^1$ sur I, et

$$F'(t) = \int_A \partial_t f(t, x)dx.$$

Preuve.— On montre d'abord que F est dérivable en chaque $t \in I$ en considérant le taux d'accroissement

$$\tau(h) = \frac{F(t+h) - F(t)}{h} = \int_A \frac{f(t+h,x) - f(t,x)}{h} dx.$$

Les hypothèses donnent

- Pour tout $x \in A$, $\dfrac{f(t+h,x) - f(t,x)}{h} \to \partial_t f(t,x)$ quand $h \to 0$,
- Pour tout $x \in A$ et $h \in]0,1]$, $|\dfrac{f(t+h,x) - f(t,x)}{h}| \leq \sup_{s \in [0,1]} |\partial_t f(t+sh,x)| \leq g(x)$,

et l'on conclut à l'aide du TCD. La continuité de F' découle directement de la proposition précédente.
□

Definition A.3.3 Soit $f : \mathbb{R}^2 \to \mathbb{C}$ une fonction, et $\Omega \subset \mathbb{R}^2$ un ouvert. On dit que f est holomorphe sur Ω lorsque $\overline{\partial} f(x,y) = 0$, où $\overline{\partial}$ (lire "d-barre") est l'opérateur différentiel défini par

$$\overline{\partial} f(x,y) = \frac{1}{2}(\partial_x f(x,y) + i\partial_y f(x,y)).$$

Proposition A.3.4 Soit $f : \Omega \times \mathbb{R}^n \to \mathbb{C}$ une fonction, où $\Omega \subset \mathbb{R}^2$ est un intervalle ouvert. Si

i) $\forall z \in \Omega$, $x \mapsto f(z,x)$ est sommable sur $A \subset \mathbb{R}^n$,

ii) $\forall x \in \mathbb{R}^n$, $z \mapsto f(z,x)$ est holomorphe sur Ω,

iii) Pour tout compact $K \subset \Omega$, il existe une fonction g_K sommable à valeurs positives, telle que

$$\forall z \in K,\ \forall x \in A,\ |f(z,x)| \leq g_K(x),$$

alors la fonction $F : z \mapsto \displaystyle\int_A f(z,x)dx$ est holomorphe sur Ω, et

$$F'(z) = \int_A \partial_z f(z,x)dx.$$

Exercice A.3.5 Soit $\varphi \in \mathcal{C}^0([0,1], \mathbb{R})$, et $f : [0,1] \times [0,1] \to \mathbb{R}$ définie par

$$f(t,x) = \begin{cases} \varphi(x) & \text{pour } x \leq t, \\ 0 & \text{pour } x > t. \end{cases}$$

On note $F(t) = \displaystyle\int f(t,x)dx$. On a $F'(t) = \varphi(t)$, mais $\displaystyle\int \partial_t f(t,x)dx = 0$. Pourquoi ne peut-on pas appliquer la Proposition A.3.2?

A.4 Fubini-Tonelli

Proposition A.4.1 Soit $f : \mathbb{R}^p \times \mathbb{R}^q \to \overline{\mathbb{R}}$.

i) Si $f \in \mathcal{E}$ on a l'égalité, dans $[0, +\infty]$,

$$\int_{\mathbb{R}^{p+q}} f(x)dx = \int_{\mathbb{R}^p} \big(\int_{\mathbb{R}^q} f(x_1, x_2)dx_2 \big) dx_1 = \int_{\mathbb{R}^q} \big(\int_{\mathbb{R}^p} f(x_1, x_2)dx_1 \big) dx_2.$$

ii) Si $f \in L^1(\mathbb{R}^{p+q})$, les trois termes ci-dessus sont finis et sont égaux.

A titre d'application immédiate, on définit le produit de convolution de deux fonctions:

Lemme A.4.2 Soient f et g deux fonctions de $L^1(\mathbb{R}^n)$. La fonction $y \mapsto f(y)g(x - y)$ est sommable pour presque tout x, et la fonction $f * g : \mathbb{R}^n \to \mathbb{C}$ définie par

$$f * g(x) = \int f(y)g(x - y)dy,$$

appartient à $L^1(\mathbb{R}^n)$. Enfin

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \ \|g\|_{L^1}.$$

Preuve.— A l'aide de la proposition précédente, on peut écrire

$$\begin{aligned} \int |f * g(x)|dx &\leq \int \big(\int |f(y)||g(x - y)|dy \big) dx \\ &\leq \int |f(y)| \big(\int |g(x - y)|dx \big) dy \leq \|f\|_{L^1} \ \|g\|_{L^1}. \end{aligned}$$

On en déduit que $f * g$ est sommable, donc finie presque partout, ce qui montre aussi que $y \mapsto f(y)g(x - y)$ est sommable pour presque tout x. □

Exercice A.4.3 Montrer de la même manière que $f * g$ est bien définie comme fonction de $L^r(\mathbb{R}^n)$ lorsque $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ avec $1/p + 1/q = 1 + 1/r$, et que l'on a alors l'inégalité de Young:

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \ \|g\|_{L^q}.$$

On ne traite que le cas où $p = 1$, et donc $r = q$. On évalue

$$\int |f * g(x)|^q dx = \int \big(\int |f(y)| \ |g(x - y)|dy \big)^q dx,$$

mais la puissance q nous empêche d'utiliser Fubini-Tonelli comme dans le cas $p = q = 1$. On remarque alors que

$$\begin{aligned} \int |f(y)| \ |g(x - y)|dy &= \int |f(y)|^{1/q'} \ \big(|f(y)|^{1-1/q'} \ |g(x - y)| \big) dy \\ &\leq \big(\int |f(y)|dy \big)^{1/q'} \times \big(\int |f(y)|^{q(1-1/q')} \ |g(x - y)|^q dy \big)^{1/q}, \end{aligned}$$

par Hölder, où $1/q + 1/q' = 1$. Ainsi

$$\int \Big(\int |f(y)|\,|g(x-y)|dy\Big)^q dx \leq \int \left(\int |f(y)|dy \right)^{q/q'} \left(\int |f(y)|^{q(1-1/q')}\,|g(x-y)|^q dy \right) dx$$
$$\leq \|f\|_{L^1}^{q/q'} \iint |f(y)|\,|g(x-y)|^q dy\,dx.$$

Pour conclure, on utilise Fubini-Tonelli comme prévu:

$$\|f*g\|_{L^q}^q \leq \|f\|_{L^1}^{q/q'} \int |f(y)|\,\Big(\int |g(x-y)|^q dx\Big) dy \leq \|f\|_{L^1}^{q/q'+1} \|g\|_{L^q}^q \leq \|f\|_{L^1}^q \|g\|_{L^q}^q.$$

A.5 Change of variable

Proposition A.5.1 Soient Ω_1 et Ω_2 deux ouverts de $\mathbb{R}^n$, et $\varphi : \Omega_1 \to \Omega_2$ un $\mathcal{C}^1$-difféomorphisme. On note $J_\varphi : \Omega_1 \to \mathbb{R}^+$ le jacobien de φ, c'est-à-dire la fonction définie par $J_\varphi(x) = |\det(\nabla J_\varphi(x))|$.

i) Pour $f : \Omega_2 \to [0, +\infty]$, on a

$$\int_{\Omega_2} f(y)dy = \int_{\Omega_1} f(\varphi(x))J_\varphi(x)dx.$$

ii) Soit $f : \Omega_2 \to \mathbb{C}$. La fonction f est sommable sur Ω_2 si et seulement si la fonction $x \mapsto f(\varphi(x))J_\varphi(x)$ est sommable sur Ω_1. Dans ce cas, les deux termes de l'égalité ci-dessus sont finis et sont égaux.

Appendix B

Hilbert spaces

B.1 Scalar Products

Let $\mathcal{H}$ be a vector space on $\mathbb{C}$.

Definition B.1.1 A linear form [resp. anti-linear form] ℓ on $\mathcal{H}$ is a mapping $\ell : \mathcal{H} \to \mathbb{C}$ such that

$$\forall x, y \in \mathcal{H}, \forall \lambda \in \mathbb{C},\ \ell(x+y) = \ell(x) + \ell(y),\ \text{and } \ell(\lambda x) = \lambda \ell(x)\ [\text{resp. } \ell(\lambda x) = \bar{\lambda}\ell(x)].$$

Definition B.1.2 A sesquilinear form on $\mathcal{H}$ is a mapping $s : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that for all $y \in \mathcal{H}$, $x \mapsto s(x,y)$ is linear and $x \mapsto s(y,x)$ is anti-linear. If moreover $s(x,y) = \overline{s(y,x)}$, the sesquilinear form s is said to be Hermitian .

Notice that when the sesquilinear form s is Hermitian, $s(x,x) \in \mathbb{R}$ for any $x \in \mathcal{H}$. Using the following identity, we can easily see that it is a necessary and sufficient condition:

Proposition B.1.3 (Polarization identity) Let s be a Hermitian sesquilinear form on $\mathcal{H}$. For all $(x,y) \in \mathcal{H} \times \mathcal{H}$,

$$4s(x,y) = s(x+y, x+y) - s(x-y, x-y) + is(x+iy, x+iy) - is(x-iy, x-iy).$$

Notice in particular that a Hermitian sesquilinear form is completely determined by its values on the diagonal of $\mathcal{H} \times \mathcal{H}$.

Remark B.1.4 For a real symmetric bilinear form b, the polarization identity reads

$$4b(x,y) = b(x+y,x+y) - b(x-y,x-y).$$

Definition B.1.5 A (Hermitian) scalar product is a Hermitian sesquilinear form s such that $s(x,x) \geq 0$ for all $x \in \mathcal{H}$, and $s(x,x) = 0 \Leftrightarrow x = 0$.

Proposition B.1.6 When s is a Hermitian scalar product, the Cauchy-Schwarz inequality holds:

$$\forall x, y \in \mathcal{H},\ |s(x,y)| \leq \sqrt{s(x,x)}\sqrt{s(y,y)},$$

as well as the triangular (or Minkowski's) inequality:

$$\forall x, y \in \mathcal{H},\ \sqrt{s(x+y,x+y)} \leq \sqrt{s(x,x)} + \sqrt{s(y,y)}.$$

Proof.— Let $x, y \in \mathcal{H}$. Denote θ the argument of the complex number $s(x,y)$, so that $|s(x,y)| = e^{-i\theta}s(x,y)$. For any $\lambda \in \mathbb{R}$, we have

$$s(x+\lambda e^{i\theta}y\,,\, x+\lambda e^{i\theta}y) \geq 0.$$

Therefore, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
0 &\leq s(x,x) + s(x,\lambda e^{i\theta}y) + s(\lambda e^{i\theta}y, x) + s(\lambda e^{i\theta}y, \lambda e^{i\theta}y) \\
&\leq s(x,x) + 2\lambda\,\mathrm{Re}(e^{-i\theta}s(x,y)) + \lambda^2 s(y,y) \\
&\leq s(x,x) + 2\lambda|s(x,y)| + \lambda^2 s(y,y).
\end{aligned}$$

Since this 2nd order polynomial has constant sign, its discriminant is negative, that is

$$|s(x,y)|^2 - s(x,x)\,s(y,y) \leq 0,$$

which is the Cauchy-Schwarz inequality.

Minkowski's inequality is then a simple consequence of the Cauchy-Schwarz inequality

$$\begin{aligned}
s(x+y,x+y) &= s(x,x) + 2\,\mathrm{Re}\,s(x,y) + s(y,y) \\
&\leq s(x,x) + 2|\,\mathrm{Re}\,s(x,y)| + s(y,y) \\
&\leq s(x,x) + 2\sqrt{s(x,x)}\sqrt{s(y,y)} + s(y,y) \\
&\leq (\sqrt{s(x,x)} + \sqrt{s(y,y)})^2.
\end{aligned}$$

□

In particular the map $\|\cdot\| : x \mapsto \sqrt{s(x,x)}$ is a norm on $\mathcal{H}$, and for all $x, y \in \mathcal{H}$, we have

$$|s(x,y)| \leq \|x\|\,\|y\|.$$

Thus the scalar product is a continuous map from $\mathcal{H}\times\mathcal{H}$ to $\mathbb{C}$ for the topology defined by its associated norm.

Definition B.1.7 A Hilbert space is a pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ where $\mathcal{H}$ is a vector space on $\mathbb{C}$, and $\langle \cdot, \cdot \rangle$ is a Hermitian scalar product on $\mathcal{H}$, such that $\mathcal{H}$ is complete for the associated norm $\| \cdot \|$.

Example B.1.8 – The space $\mathbb{C}^n$, equipped with the scalar product

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$$

is a Hilbert space.

– The space $\ell^2(\mathbb{C})$ of sequences (x_n) such that $\sum |x_n|^2 < +\infty$, equipped with the scalar product $\langle (x_n), (y_n) \rangle = \sum_n x_n \overline{y_n}$ is a Hilbert space.

– The space $L^2(\Omega)$ of square integrable functions on the open set $\Omega \subset \mathbb{R}^n$, equipped with the scalar product

$$\langle f, g \rangle_{L^2} = \int f(x) \overline{g(x)} dx,$$

is a Hilbert space. This is one of the main achievement of Lebesgue's integration theory.

Exercise B.1.9 Prove that $\ell^2(\mathbb{C})$ is a Hilbert space: Let (x^n) be a Cauchy sequence in $\ell^2(\mathbb{C})$. Denote $x^n = (x_i^n)_{i \in \mathbb{N}}$.

1. Show that the sequence $(x_i^n)_{n \in \mathbb{N}}$ is a Cauchy sequence of $\mathbb{C}$. Denote x_i its limit.
2. Show that the sequence $x = (x_i)$ belongs to $\ell^2(\mathbb{C})$, and that (x^n) converges to x.

B.2 Orthogonality

Definition B.2.1 Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and A a subset of $\mathcal{H}$. The orthogonal complement to A is the set $A^\perp$ given by

$$A^\perp = \{x \in \mathcal{H},\ \forall a \in A,\ \langle x, a \rangle = 0\}.$$

In the case where $A = \{x\}$, $A^\perp$ is the set of vectors that are orthogonal to x.

Proposition B.2.2 For any subset A of $\mathcal{H}$, $A^\perp$ is a closed subspace of $\mathcal{H}$. Moreover $A^\perp = (\bar{A})^\perp$.

Proof.– For each $a \in A$, the set $\{a\}^\perp$ is closed, since the map $x \mapsto \langle x, a \rangle$ is continuous. Thus $A^\perp$ is the intersection of a family of closed set, therefore a closed set. Now $0 \in A^\perp$, and if $x_1, x_2 \in A^\perp$, we have $\langle \lambda_1 x_1 + \lambda_2 x_2, a\rangle = \lambda_1 \langle x_1, a\rangle + \lambda_2 \langle x_2, a\rangle = 0$ for any $a \in A$, so that $A^\perp$ is indeed a subspace of $\mathcal{H}$.

Since $A \subset \bar{A}$, we have $(\bar{A})^\perp \subset A^\perp$. On the other hand let $b \in A^\perp$. For $a \in \bar{A}$, there exists a sequence (a_n) of vectors in A such that $(a_n) \to a$. Now

$$\langle a, b\rangle = \lim_{n\to+\infty} \langle a_n, b\rangle = 0,$$

so that $b \in (\bar{A})^\perp$. □

Lemma B.2.3 (Pythagore's theorem) Let $\{x_1, x_2, \dots, x_n\}$ be a family of pairwise orthogonal vectors. Then

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2.$$

Proof.— Indeed

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \langle \sum_{j=1}^n x_j, \sum_{k=1}^n x_k\rangle = \sum_{j=1}^n \sum_{k=1}^n \langle x_j, x_k\rangle = \sum_{j=1}^n \|x_j\|^2.$$

□

Lemma B.2.4 (Parallelogram's law) Let x_1 and x_2 be two vectors of the Hilbert space $\mathcal{H}$. Then

$$2\|x_1\|^2 + 2\|x_2\|^2 = \|x_1 + x_2\|^2 + \|x_1 - x_2\|^2.$$

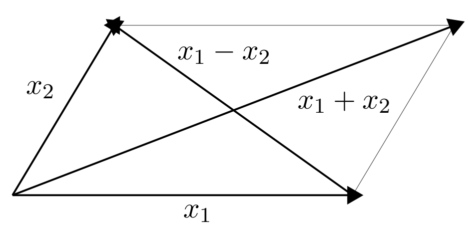


Figure B.1: Parallelogram's law

The proof of this lemma is straightforward, but it is interesting to notice that the parallelogram identity holds if and only if the norm $\|\cdot\|$ comes from a scalar product, that is the map

$$(x_1, x_2) \mapsto \frac{1}{4}(\|x_1 + x_2\|^2 - \|x_1 - x_2\|^2 + i\|x_1 + ix_2\|^2 - i\|x_1 - ix_2\|^2)$$

is a scalar product whose associated norm is $\|\cdot\|$.

Exercise B.2.5 Prove it.

Proposition B.2.6 (Orthogonal Projection) Let $\mathcal{H}$ be a Hilbert space, and F a closed subspace of $\mathcal{H}$. For any x in $\mathcal{H}$, there exists a unique vector Πx in F such that

$$\forall f \in F,\ \|x - \Pi x\| \leq \|x - f\|.$$

This element Πx is called the orthogonal projection of x onto F, and it is characterized by the property

$$\Pi x \in F \text{ and } \forall f \in F,\ \langle x - \Pi x, f\rangle = 0.$$

Moreover the map $\Pi : x \mapsto \Pi x$ is linear, $\Pi^2 = \Pi$, and $\|\Pi x\| \leq \|x\|$.

Notice that the proposition states in particular that Πx is the only element of F such that $x - \Pi x$ belongs to $F^\perp$.

Proof.— – First of all we suppose only that F is a convex, closed subset of $\mathcal{H}$. Let $x \in \mathcal{H}$ be fixed, and denote $d = \inf_{f\in F} \|x - f\|$ the distance between x and F.

If f_1 and f_2 are two vectors in F, then, since F is convex, $(f_1 + f_2)/2$ also belongs to F. Therefore $\|(f_1 + f_2)/2\| \geq d$. On the other hand the parallelogram law says that

$$\|\frac{(f_1 + f_2)}{2}\|^2 + \|\frac{(f_1 - f_2)}{2}\|^2 = \frac{1}{2}(\|f_1\|^2 + \|f_2\|^2),$$

so that

$$0 \leq \|\frac{(f_1 - f_2)}{2}\|^2 \leq \frac{1}{2}(\|f_1\|^2 + \|f_2\|^2) - d^2.$$

Now for $n \in \mathbb{N}$, we define

$$F_n = \{f \in F,\ \|x - f\|^2 \leq d^2 + \frac{1}{n}\}.$$

The sets F_n are closed, and non-empty by the definition of d, and they form a decreasing sequence of sets. Moreover, if f_1, f_2 belong to F_n, then

$$\|\frac{(f_1 - f_2)}{2}\|^2 \leq \frac{1}{2}(\|f_1 - x\|^2 + \|f_2 - x\|^2) - d^2 \leq \frac{1}{n}\cdot$$

Thus the diameter of the F_n tends to 0, and their intersection, which is the set of points in F at distance d of x contains at most one point.

At last, for all $n \in \mathbb{N}$ we pick $x_n \in F_n$. For all $p < q$ in $\mathbb{N}$, we have $F_q \subset F_p$ therefore

$$\|x_p - x_q\| \leq \frac{1}{p},$$

which proves that (x_n) is a Cauchy sequence, therefore converges to some $\Pi x \in F$, such that $\|x - \Pi x\| = d$.

Concerning the characterization of Πx, we notice that for all $t \in [0, 1]$ and all $f \in F$, we have $(1 - t)\Pi x + tf \in F$ by the convexity assumption on F. Thus

$$\|\Pi x - x\|^2 \leq \|((1 - t)\Pi x + tf) - x\|^2 \leq \|(\Pi x - x) + t(f - \Pi x))\|^2.$$

Thus, for all $f \in F$ and all $t \in [0,1]$,

$$0 \le t^2\|f-\Pi x\|^2 + 2t\,\mathrm{Re}\langle \Pi x - x, f-\Pi x\rangle.$$

Dividing by t and choosing $t=0$ gives

$$\mathrm{Re}\langle x-\Pi x, f-\Pi x\rangle \le 0.$$

Reciprocally if Re$(x-y, f-y) \le 0$ for all $f \in F$, then, for all $f \in F$

$$\|x-f\|^2 = \|x-y+y-f\|^2 \ge \|x-y\|^2,$$

so that $y = \Pi x$.

– Now we make the assumption that F is a closed vector space. Since a vector space is convex, the previous proof holds. Moreover, in the last part, we have now

$$0 \le |t|^2\|f-\Pi x\|^2 + 2\,\mathrm{Re}\,\bar{t}\langle \Pi x - x, f-\Pi x\rangle.$$

for all $t \in \mathbb{C}$ since the line $\{(1-t)\Pi x + tf,\ t\in\mathbb{C}\}$ belongs to F. Therefore, in that case, Πx is characterized by the property (noticing that $f - \Pi x$ describe F as f describes F),

$$\forall f \in F,\ \langle x-\Pi x, f\rangle = 0.$$

The linearity of the map Π then follows: for $x_1, x_2 \in \mathcal{H}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ we have, for all $f \in F$,

$$\lambda_1\langle x_1 - \Pi x_1, f\rangle = 0 \quad \text{and} \quad \lambda_2\langle x_2-\Pi x_2, f\rangle = 0$$

so that, for all $f \in F$,

$$\langle \lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1\Pi x_1 + \lambda_2 \Pi x_2), f\rangle = 0,$$

and $\Pi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \Pi x_1 + \lambda_2\Pi x_2$.

Since $\Pi x = x$ when $x \in F$, it is clear that $\Pi^2 = \Pi$, therefore we are left with the proof that $\|\Pi x\| \le \|x\|$. But this is obvious since for all $f \in F$, and in particular for $f = 0$, we have, by Pythagore's theorem

$$\|x-f\|^2 = \|x-\Pi x\|^2 + \|\Pi x - f\|^2$$

and thus $\|x-f\| \ge \|\Pi x - f\|$. □

Corollary B.2.7 If F is a closed subspace of $\mathcal{H}$, then

$$F \oplus F^{\perp} = \mathcal{H}.$$

Proof.— For $x \in \mathcal{H}$, we can write $x = \Pi x + (I-\Pi)x = x_1 + x_2$. Since $x_1 \in F$ and $x_2 = x - \Pi x$ is orthogonal to F, we have $\mathcal{H} = F + F^{\perp}$, and it remains to show that the sum is direct. If $0 = x_1 + x_2$ with $x_1 \in F$ and $x_2 \in F^{\perp}$, then Pythagore's theorem give

$$0 = \|x_1\|^2 + \|x_2\|^2,$$

so that $x_1 = x_2 = 0$. □

Corollary B.2.8 A subspace F of $\mathcal{H}$ is dense in $\mathcal{H}$ if and only if $F^\perp = \{0\}$. Moreover $(F^\perp)^\perp = \bar{F}$.

Proof.— Since $\bar{F}$ is a closed subspace of $\mathcal{H}$, we have

$$H = \bar{F} \oplus (\bar{F})^\perp = \bar{F} \oplus F^\perp,$$

and the first statement follows.

It is clear that $F \subset (F^\perp)^\perp$. Since $(F^\perp)^\perp$ is a closed set, this implies that $\bar{F} \subset (F^\perp)^\perp$. On the other hand, let $x \in (F^\perp)^\perp$, and denote Πx its projection onto $\bar{F}$. We have

$$\|x - \Pi x\|^2 = \langle x - \Pi x, x - \Pi x\rangle = \langle x - \Pi x, x\rangle - \langle x - \Pi x, \Pi x\rangle = 0.$$

Indeed, $\langle x - \Pi x, \Pi x\rangle = 0$ since $\Pi x \in \bar{F}$, and $\langle x - \Pi x, x\rangle = 0$ since $x - \Pi x \in (\bar{F})^\perp = F^\perp$. Thus $x = \Pi x \in \bar{F}$. □

B.3 Riesz's theorem

A linear form $\ell : \mathcal{H} \to \mathbb{C}$ is continuous if there exists $C > 0$ such that

$$\forall x \in \mathcal{H},\ |\ell(x)| \leq C\|x\|.$$

Proposition B.3.1 (Riesz's representation Theorem) Let ℓ be a continuous linear form on $\mathcal{H}$. There exists a unique $y = y(\ell) \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H},\ \ell(x) = \langle x, y\rangle.$$

Moreover

$$|||\ell||| := \sup_{x\in\mathcal{H}, x\neq 0} \frac{|\ell(x)|}{\|x\|} = \|y(\ell)\|.$$

Proof.— The uniqueness part of the statement is easy, and we concentrate on the existence part. We denote $\mathrm{Ker}\,\ell = \{x \in \mathcal{H},\ \ell(x) = 0\}$ the kernel of ℓ. Since ℓ is continuous, it is a closed subspace of $\mathcal{H}$, and we denote by Π the orthogonal projection onto $\mathrm{Ker}\,\ell$. If $\ell = 0$, we can take $y(\ell) = 0$. Otherwise, there exists $z \in \mathcal{H}$ such that $\ell(z) \neq 0$, which means that $w = z - \Pi z \neq 0$. Therefore we can set

$$y = y(\ell) = \frac{\overline{\ell(w)}}{\|w\|^2} w.$$

Notice in particular that $\ell(y) = \|y\|^2$. As a matter of fact, y spans $(\mathrm{Ker}\,\ell)^\perp$. Indeed if $x \in (\mathrm{Ker}\,\ell)^\perp$, we have

$$\ell(x - \frac{\ell(x)}{\ell(y)} y) = 0$$

therefore $x - \dfrac{\ell(x)}{\ell(y)} y \in \text{Ker}\,\ell \cap \big(\text{Ker}\,\ell\big)^{\perp} = \{0\}$, so that $x = \dfrac{\ell(x)}{\ell(y)} y$.

Thus, again since $H = \text{Ker}\,\ell \oplus \big(\text{Ker}\,\ell\big)^{\perp}$, any $x \in \mathcal{H}$ can be written

$$x = \Pi x + \lambda y,$$

for some $\lambda \in \mathbb{C}$. Then $\ell(x) = \lambda \ell(y)$ and

$$\langle x, y \rangle = \langle \Pi x + \lambda y, y \rangle = \langle \Pi x, y \rangle + \lambda \frac{\ell(w)^2}{\|w\|^2} = \lambda \|y\|^2 = \ell(x).$$

□