EIGENVECTORS FROM EIGENVALUES

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ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

Let A be a $n \times n$ Hermitian matrix with eigenvalues $\lambda_i(A)$ and normed eigenvectors v_i. The elements of each eigenvector are denoted $v_{i,j}$. Let M_j be the $n-1 \times n-1$ submatrix of A that results from deleting the j^{th} column and the j^{th} row, with eigenvalues $\lambda_k(M_j)$.

First we prove a useful Cauchy-Binet type formula.

Lemma 1. *Let one eigenvalue of A be zero, WLOG we can set $\lambda_n(A) = 0$. Then,*

$$\prod_{i=1}^{n-1} \lambda_i(A) |\det(B \quad v_n)|^2 = \det(B^* A B)\,, \tag{1}$$

for any $n \times n-1$ matrix B.

Proof. If we diagonalize $A = VDV^*$ where $D \equiv \operatorname{diag}(\lambda_1(A), \ldots, \lambda_{n-1}(A), 0)$ and make the replacements $B \to V^* B$ and $v_n \to V^* v_n = e_n$, we can assume that $A = D$ and $v_n = e_n$. Write $B = \binom{B'}{X}$ where B' is the upper $n-1 \times n-1$ submatrix and X is some $1 \times n-1$ vector, then we find that both sides of eq. 1 are equal to $\prod_{i=1}^{n-1} \lambda_i(A) |\det(B')|^2$. $\square$

Now we are prepared to state and prove our main result.

Lemma 2. *The norm squared of the elements of the eigenvectors are related to the eigenvalues and the submatrix eigenvalues,*

$$|v_{i,j}|^2 \prod_{k=1;k\neq i}^{n} (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n-1} (\lambda_i(A) - \lambda_k(M_j))\,. \tag{2}$$

This result was noted in [DPZ19] *and is related to a result in* [ESY07, TV11].

Proof. WLOG we take $j = 1$ and $i = n$. We shift A by $\lambda_n(A) I_n$ so that $\lambda_n(A) = 0$; this also shifts all the remaining eigenvalues of A as well as those of M_j, then eq. 2

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becomes,

$$|v_{n,1}|^2 \prod_{k=1}^{n-1} \lambda_k(A) = \prod_{k=1}^{n-1} \lambda_k(M_1)\,. \tag{3}$$

Note that the RHS of eq. 3 is $\det(M_1)$.

Next, we apply Lemma 1 for the case where $B = \binom{0}{I_{n-1}}$. We find that the LHS of eq. 1 is $\prod_{i=1}^{n-1} \lambda_i(A)|v_{n,1}|^2$ and the RHS of eq. 1 is $\det(M_1)$ giving the result. □

We provide an alternate proof of Lemma 2 using adjugate matrices.

Proof. For any λ not an eigenvalue of A,

$$\operatorname{adj}(\lambda I_n - A) = \det(\lambda I_n - A)(\lambda I_n - A)^{-1}\,, \tag{4}$$

which leads to

$$\operatorname{adj}(\lambda I_n - A)v_j = \det(\lambda I_n - A)(\lambda - \lambda_j(A))^{-1}v_j = \prod_{k=1;k\neq j}^{n} (\lambda - \lambda_k(A))v_j\,, \tag{5}$$

for $j \in [1, n]$. Thus the v_j provide an orthonormal eigenbasis for $\operatorname{adj}(\lambda I_n - A)$. Then,

$$\operatorname{adj}(\lambda I_n - A) = \sum_{j=1}^{n} \prod_{k=1;k\neq j}^{n} (\lambda - \lambda_k(A))v_j v_j^*\,. \tag{6}$$

By taking the limit $\lambda \to \lambda_i(A)$ all but one of the summands on the RHS vanishes,

$$\operatorname{adj}(\lambda_i(A)I_n - A) = \prod_{k=1;k\neq i}^{n} (\lambda_i(A) - \lambda_k(A))v_i v_i^*\,. \tag{7}$$

The diagonal elements on the RHS of eq. 7 provide the LHS of eq. 2. By the definition of the adjugate, the diagonal elements on the LHS of eq. 7 are the determinants of the submatrices of $\lambda_i(A)I_n - A$ which is the RHS of eq. 2 completing the proof. □

Lemma 2 leads to the following corollary in a straightforward fashion.

Corollary 3. *If one element of an eigenvector vanishes, $v_{i,j} = 0$, then one of the eigenvalues of M_j must match $\lambda_i(A)$.*

Proof. The proof follows directly from eq. 2. In addition, if $v_{i,j} = 0$, then the eigenvector equation of A collapses to an eigenvector equation of M_j. □

Discussion. The form of Lemma 2 with the norm squared of the elements of the eigenvectors is expected in that any determination of the eigenvectors from the eigenvalues is insensitive to the phases since one can multiply any eigenvector by a phase $e^{i\theta}$ while leaving A, M_j, and the eigenvalues unchanged.

We note that computing the norm of every element of every eigenvector (n^2 numbers) requires all the eigenvalues plus all the submatrix eigenvalues which is $n + n(n-1) = n^2$ numbers. In addition, the adjugate proof of Lemma 2 provides a mechanism for computing the phases, $v_{i,j}\overline{v_{i,k}}$, although it is less simple than eq. 2.

As a consistency check, we note the fact that $|v_{i,j}| \leq 1$ follows from Lemma 2 due to the Cauchy interlacing theorem. Moreover, we can confirm that the eigenvectors are correctly normalized by summing eq. 3 over the elements in v_n. Then we have,

$$\sum_{j=1}^{n} |v_{n,j}|^2 \prod_{k=1}^{n-1} \lambda_k(A) = \sum_{j=1}^{n} \det(M_j)\,. \tag{8}$$

The RHS of eq. 8 is the $n-1$ symmetric function of the eigenvalues, s_{n-1}. Since $\lambda_n(A) = 0$, $s_{n-1} = \prod_{k=1}^{n-1} \lambda_k(A)$ thus the eigenvectors are properly normed as expected. This also follows from the adjugate proof of Lemma 2.

References

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